Low-gain Integral Control for Multi-Input, Multi-Output Linear Systems with Input Nonlinearities

Chris Guiver, Hartmut Logemann and Stuart Townley

***Abstract*—We consider the inclusion of a static anti-windup component in a continuous-time low-gain integral controller in feedback with a multi-input multi-output stable linear system subject to an input nonlinearity (from a class of functions that includes componentwise diagonal saturation). We demonstrate that the output of the closed-loop system asymptotically tracks every constant reference vector which is "feasible" in a natural sense, provided that the integrator gain is sufficiently small. Robustness properties of the proposed control scheme are investigated and three examples are discussed in detail.**

***Index Terms*—Anti-windup methods, Constrained control, Robust control, Stability of NL systems**

I. Introduction

Integral control is a classical control engineering technique for robustly regulating the measured variables of a stable linear system to a prescribed constant reference. The theoretical development of low-gain integral control can be traced back to the 1970s and contributors include [1]–[5]. Integral control is one of the three facets of celebrated PID-control which has been described as one of the "success stories in control" [6, p. 103].

Low-gain integral control of continuous-time, stable linear systems pertains to the situation whereby the transfer function G of the system is connected in series, as depicted in Figure 1, with an integrator gK/s, where the matrix K and the positive scalar gain g are design parameters.

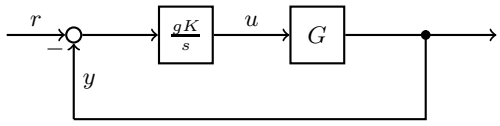

Figure 1. Block diagram of a low-gain integral control scheme. Here u, y and r denote the input, output and reference signal, respectively.

The resulting closed-loop system is known to be globally exponentially stable if: (i) $-KG(0)$ is a Hurwitz matrix, and; (ii) the gain $g > 0$ is sufficiently small, in which case the output of the closed loop system asymptotically tracks every constant reference r.

Low-gain integral control has been further developed by the present authors in, for example [7]–[10], to address discrete-time systems, sampled-data systems, classes of distributed parameter systems, to allow the gain parameter g to be determined adaptively and to include input and output nonlinearities. One situation not addressed to date is low-gain integral control for multi-input, multi-output (MIMO) systems in the presence of input nonlinearities (such as saturation).

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Chris Guiver and Hartmut Logemann are with the Department of Mathematical Sciences, University of Bath, Claverton Down, Bath BA2 7AY, UK, email: c.guiver@ex.ac.uk, h.logemann@bath.ac.uk.

Chris Guiver is the corresponding author.

Stuart Townley is with the Environment & Sustainability Institute, College of Engineering, Mathematics and Physical Sciences, University of Exeter, Penryn Campus, TR10 9FE, UK, email: s.b.townley@ex.ac.uk.

It is known that input saturation may lead to an undesirable degradation of tracking performance of MIMO integral controlled systems, or even destabilise them; a phenomenon often called *integrator windup* [11]. Integrator windup is a consequence of basing controller design on the assumption of linearity, when in reality input saturation is an archetypal nonlinear effect. Anti-windup control refers to the study of mechanisms to alleviate or remove integrator windup and, owing to its importance in applications, is a well-studied topic. The chronological bibliography of the 1995 paper [12] contains already 250 references, for instance. We refer the reader to the tutorial [13], survey [14] or monograph [15] and the references therein for a thorough overview of anti-windup control. Briefly, as described there, many anti-windup mechanisms are designed under the assumption that the unsaturated system has the desired closed-loop stability and performance properties and an anti-windup compensator is subsequently included — a static or dynamical system driven by the error $z-\phi(z)$, where z is the state or output of the controller and ϕ denotes the input nonlinearity.

We present a low-gain integral controller that includes a (direct linear, in the terminology of [13]) anti-windup component and prove that, for a large class of input nonlinearities, it achieves global exponential tracking for all feasible references provided that the integrator gain is sufficiently small. The class of nonlinearities is assumed to satisfy a global Lipschitz type assumption. The anti-windup component contains a matrix parameter that is required to be close, in a sense to be described, to the matrix $KG(0)$, and does not require the solution of an LMI. We recall that if the steady-state gain $G(0)$ is subject to uncertainty, then estimates of $G(0)$ can be obtained by step-response experiments (see, for example, [16] or [17]). Integrator windup is particularly acute in the low-gain integral control of MIMO systems as issues arise that are absent from SISO systems, see Remark 11 and Example 12. Further, we emphasise that the SISO case is well-studied, and [18]–[21] all propose solutions which do not include anti-windup components.

The closed-loop feedback system under consideration in the paper can be re-written in form of a Lur'e system, and we invoke absolute stability arguments to derive our results. The reader is referred to, for example, [22]–[25], for more background on Lur'e systems and absolute stability theory. We demonstrate that the closed-loop system has several robustness properties: with respect to uncertainty in $G(0)$ and with respect to additive disturbances. To establish the latter, we make use of recent input-to-state-stability (ISS) results for Lur'e systems [26]. Additional background on ISS may be found in [27] or [28].

Finally, the results reported here extend, generalise and refine those in [29] where, in the context of ecological management, low-gain PI control of linear discrete-time positive systems (see, for example, [30], [31]) subject to input saturation is considered.

Notation and terminology. The space of all rational $p \times m$-matrices which are bounded on the half plane $\mathrm{Re}\, s > 0$ is denoted by H^∞,

endowed with the sup norm given by

$$\|H\|_\infty := \sup_{\mathrm{Re}\, s>0} \|H(s)\| = \sup_{s\in \mathrm{i}\mathbb{R}} \|H(s)\|,$$

where $\|\cdot\|$ is the operator norm induced by the 2-norm. As usual, a square complex matrix M is said to be Hurwitz if every eigenvalue of M has a negative real part. We let $\mathrm{rk}\, M$ denote the rank of M. The symbol I_q denotes the $q \times q$ identity matrix, although the subscript shall be omitted when the dimension is clear from the context. Finally, for a function $f : \mathbb{R}^p \to \mathbb{R}^m$ and a set $S \subseteq \mathbb{R}^m$, $f^{-1}(S)$ denotes the pre-image of S under f, that is, $f^{-1}(S) = \{z \in \mathbb{R}^p : f(z) \in S\}$. If $S = \{s\}$ is a singleton, then we write $f^{-1}(s) := f^{-1}(\{s\}) = f^{-1}(S)$. Finally, for a vector v, v_k denotes the k-th component of v.

II. Low-gain integral control with input nonlinearities

We focus on the linear control system with input nonlinearity

$$\dot{x} = Ax + B\phi(u), \quad x(0) = x^0, \quad y = Cx\,, \tag{1}$$

where $A \in \mathbb{R}^{n\times n}$ is Hurwitz, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$ and $\phi : \mathbb{R}^m \to \mathbb{R}^m$ is locally Lipschitz continuous. As usual, x and y denote the state and output variables, respectively, whilst u is the control signal. There are m, n and p input, state and output variables, respectively. We let G denote the transfer function of the linear system specified by the triple (A, B, C), that is, the matrix-valued function of the complex variable s given by $G(s) := C(sI - A)^{-1}B$.

We seek to apply low-gain integral control to (1), with the aim that the output $y(t)$ converges to a prescribed constant reference vector $r \in \mathbb{R}^p$ as $t \to \infty$. We say that r is feasible, if the set

$$U^r := \{w \in \mathbb{R}^m : G(0)\phi(w) = r\},$$

is non-empty. Obviously, if $m = p$ and $G(0)$ is invertible, then $U^r = \phi^{-1}(G(0)^{-1}r)$. If the control signal u in (1) is such that $\phi(u)$ has a limit λ, that is, $\phi(u(t)) \to \lambda$ as $t \to \infty$, then, as A is Hurwitz, it is well-known that for any initial state x^0 the output y of (1) has limit

$$\lim_{t\to\infty} y(t) = G(0) \lim_{t\to\infty} \phi(u(t)) = G(0)\lambda.$$

Trivially, if r is feasible, then there exists a control signal u such that $y(t) \to r$ (for example, $u(t) \equiv w$ with $w \in U^r$). On the other hand, if r is not feasible, then there does not exist a bounded u such that $\phi(u(t))$ converges and $y(t) \to r$ as $t \to \infty$. Moreover, under the additional assumption that $m = p$ and $G(0)$ is invertible, then, if r is not feasible, $y(t)$ does not converge to r whenever the control signal u is bounded.

We say that a set $R \subseteq \mathbb{R}^p$ of reference vectors is feasible if every $r \in R$ is feasible. Given a feasible set $R \subseteq \mathbb{R}^p$, we introduce the following assumption:

(F) there exists $L > 0$ such that

$$\|\phi(w + \tilde{w}) - \phi(\tilde{w})\| \le L\|w\| \quad \forall\, w \in \mathbb{R}^m,\ \forall\, \tilde{w} \in \bigcup_{r\in R} U^r.$$

Assumption (F) is certainly satisfied if ϕ is globally Lipschitz with Lipschitz constant L.

A function $\phi : \mathbb{R}^m \to \mathbb{R}^m$ with components ϕ_i is called diagonal if, for all $i = 1, 2, \ldots, m$ and all $v \in \mathbb{R}^m$,

$$\phi_i(v) = \phi_i(P_i v)\,, \quad \text{where} \quad (P_i v)_j := \begin{cases} v_i & j = i \\ 0 & j \ne i\,. \end{cases}$$

The next example illustrates the feasibility property for the familiar diagonal saturation function.

Example 1. For given $v^1 < v^2$, set $V := \{v^1, v^2\}$ and define the function sat_V by

$$\mathrm{sat}_V : \mathbb{R} \to \mathbb{R}, \quad \mathrm{sat}_V(w) := \max\left\{v^1, \min\left\{w, v^2\right\}\right\}, \tag{2}$$

illustrated in Figure 2. The diagonal saturation function sat is defined as follows:

$$\mathrm{sat} : \mathbb{R}^m \to \mathbb{R}^m,\ \mathrm{sat}(w) := \left[\mathrm{sat}_{V_1}(w_1), \ldots, \mathrm{sat}_{V_m}(w_m)\right]^T, \tag{3}$$

for $V_k := \{v_k^1, v_k^2\}$, $k = 1, \ldots, m$. Clearly, the set

$$R := \{G(0)w \ :\ v_k^1 \le w_k \le v_k^2,\ k = 1, \ldots, m\}\,, \tag{4}$$

is feasible, and, since

$$R = \{G(0)\mathrm{sat}(w) \ :\ w \in \mathbb{R}^m\},$$

R is the maximal feasible set. Furthermore, it is straightforward to see that sat satisfies (F) with $L = 1$. □

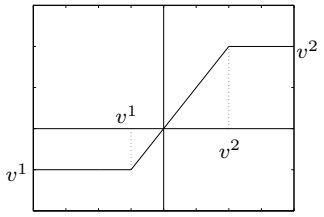


Figure 2. Graph of the saturation function sat_V defined in (2), here with $v^1 < 0 < v^2$.

Our second example contains a saturation function that satisfies (F) but is not diagonal.

Example 2. For fixed $\theta > 0$, the function $\rho : \mathbb{R}^m \to \mathbb{R}^m$ defined by

$$\rho(v) := \begin{cases} v & \|v\| \le \theta \\ \theta\dfrac{v}{\|v\|} & \|v\| > \theta\,, \end{cases}$$

satisfies (F) with $L \in [1, 2]$. For certain m and choice of norm, the upper bound 2 for L is achieved. The set

$$R := \{G(0)w \ :\ w \in \mathbb{R}^m,\ \|w\| \le \theta\}\,,$$

is the maximal feasible set. □

Given the control system (1) and a feasible set $R \subseteq \mathbb{R}^p$, let $r \in R$ and consider the control law

$$\dot{u} = gK(r - y) - g\Gamma\big(u - \phi(u) - u^r + \phi(u^r)\big), \tag{5}$$

where $g > 0$, $K \in \mathbb{R}^{m\times p}$ and $\Gamma \in \mathbb{R}^{m\times m}$ are design parameters and

$$u^r \in U^r. \tag{6}$$

The assumption that $R \subseteq \mathbb{R}^p$ is feasible implies that U^r is non-empty and hence (6) is meaningful. Note that if $\Gamma = 0$ or $\phi = \mathrm{id}$, the identity function, then (5) reduces to "pure" integral control. Of course, in the case $\phi = \mathrm{id}$ the control system (1) is in fact linear. The term $g\Gamma\big(w - \phi(w) - u^r + \phi(u^r)\big)$ is the so-called anti-windup component of the controller.

In the following, we will focus on the analysis of the feedback interconnection of (1) and (5):

$$\left.\begin{aligned} \dot{x} &= Ax + B\phi(u), \quad y = Cx, \\ \dot{u} &= gK(r - y) - g\Gamma\big(u - \phi(u) - u^r + \phi(u^r)\big), \\ x(0) &= x^0, \quad u(0) = u^0. \end{aligned}\right\} \tag{7}$$

Remark 3. Finding u^r satisfying (6) requires knowledge of $G(0)$, in general. However, in numerous applications this knowledge is not

required. For instance, when $\phi = \mathrm{sat}$, the saturation nonlinearity in (3), and R is given by (4), then, for every $r \in R$, there exists $u^r \in U^r$ such that $\phi(u^r) = u^r$ and so $-u^r + \phi(u^r) = 0$ in (5). We note that in this case (7) may be placed in the anti-windup framework as presented in [13] with static anti-windup dynamics. □

The following result is the main contribution of this note.

Theorem 4. *Given the closed-loop integral control system* (7) *with feasible $R \subseteq \mathbb{R}^p$, assume that*

(a) *A is Hurwitz,*

(b) *$-\Gamma$ is Hurwitz,*

(c) (F) *holds.*

Then, for all $g > 0$, $r \in R$, $u^r \in U^r$ and $(x^0, u^0) \in \mathbb{R}^n \times \mathbb{R}^m$, there exists a unique solution (x, u) of (7) *defined on $\mathbb{R}_+$. Furthermore, if*

$$\sup_{s \in \mathrm{i}\mathbb{R}} \left\|(sI + \Gamma)^{-1}\right\| \cdot \|\Gamma - KG(0)\| < 1/L\,, \tag{8}$$

then there exists $g^* > 0$ such that, for all $g \in (0, g^*)$, $r \in R$, $u^r \in U^r$ and $(x^0, u^0) \in \mathbb{R}^n \times \mathbb{R}^m$, the solution (x, u) of (7) *satisfies*

(C1) $u(t) \to u^r$,

(C2) $x(t) \to -A^{-1}B\phi(u^r)$,

(C3) $y(t) = Cx(t) \to r$,

(C4) $\dot{x}(t) \to 0$,

as $t \to \infty$, and the rates of convergence are exponential.

Remark 5. At first glance, the hypotheses of Theorem 4 place very few constraints on m, p and $KG(0)$. However, assuming that hypotheses (a)–(c) and inequality (8) are satisfied, it follows that, for nonlinearities ϕ with $L \geq 1$ (which is the case for the diagonal and non-diagonal saturation functions in Examples 1 and 2, respectively)

$$\|\Gamma - KG(0)\| < \frac{\rho}{L} \leq \rho, \tag{9}$$

where $\rho := 1/\sup_{s \in \mathrm{i}\mathbb{R}} \|(sI + \Gamma)^{-1}\|$. Now ρ is the (unstructured) complex stability radius (see [32]) of the Hurwitz matrix $-\Gamma$, and thus, since $-KG(0) = -\Gamma + (\Gamma - KG(0))$, it follows from (9) that $-KG(0)$ is Hurwitz. In particular, if (a)–(c) are satisfied and $L \geq 1$, then $\mathrm{rk}\, G(0) = m$ (implying that $p \geq m$) is a necessary condition for (8) to hold. The assumption $\mathrm{rk}\, G(0) = p$, and thus necessarily $m \geq p$, is typically made in output regulation problems so that in the unsaturated case ($\phi = \mathrm{id}$) every reference vector in $\mathbb{R}^p$ is feasible; see, for instance [35]. Our results do not apply when $L \geq 1$ and $m > p$. □

The proof of Theorem 4 is based on Lemma 6, stated and proven below.

Lemma 6. *Given (A, B, C) as in* (1) *with A Hurwitz, and transfer function G, let $K \in \mathbb{R}^{m \times p}, \Gamma \in \mathbb{R}^{m \times m}$ and assume that $-\Gamma$ is Hurwitz. For $g > 0$ define*

$$\mathcal{A} := \begin{bmatrix} A & 0 \\ -gKC & -g\Gamma \end{bmatrix}, \quad \mathcal{B} := \begin{bmatrix} B \\ g\Gamma \end{bmatrix}, \quad \mathcal{C} := \begin{bmatrix} 0 & I \end{bmatrix}, \tag{10}$$

and let $\mathcal{G}$ denote the transfer function of the triple $(\mathcal{A}, \mathcal{B}, \mathcal{C})$. Then, $\mathcal{G} \in H^\infty$ for every $g > 0$. Moreover, for each $\varepsilon > 0$, there exists $g^* > 0$ such that for all $g \in (0, g^*)$

$$\|\mathcal{G}\|_\infty \leq \varepsilon + \sup_{s \in \mathrm{i}\mathbb{R}} \left\|(sI + \Gamma)^{-1}\right\| \cdot \|\Gamma - KG(0)\|\,. \tag{11}$$

Proof. The matrices A and $-g\Gamma$ for $g > 0$ are Hurwitz by assumption, so that $\mathcal{A}$ in (10) is clearly Hurwitz. Hence, $\mathcal{G} \in H^\infty$ for all $g > 0$. An elementary calculation shows that

$$\mathcal{G}(s) = g(sI + g\Gamma)^{-1}(\Gamma - KG(s))\,,$$

and so,

$$\mathcal{G} = \mathcal{G}_1 + \mathcal{G}_2\,, \tag{12}$$

where $\mathcal{G}_1$ and $\mathcal{G}_2$ are given by

$$\begin{aligned} \mathcal{G}_1(s) &:= g(sI + g\Gamma)^{-1}K\big(G(0) - G(s)\big), \\ \mathcal{G}_2(s) &:= -g(sI + g\Gamma)^{-1}\big(KG(0) - \Gamma\big)\,. \end{aligned}$$

Since $-g\Gamma$ is Hurwitz for $g > 0$ and $G \in H^\infty$, it is clear that $\mathcal{G}_1, \mathcal{G}_2 \in H^\infty$. To prove (11), let $\varepsilon > 0$ be given. We proceed to estimate $\|\mathcal{G}_1\|_\infty$ and $\|\mathcal{G}_2\|_\infty$. Since $-\Gamma$ is Hurwitz it follows that there exists $M > 0$ such that

$$\left\|(I + z\Gamma)^{-1}\right\| \leq M \quad \forall\ z \in \mathrm{i}\mathbb{R}\,. \tag{13}$$

Next, define $J \in H^\infty$ by

$$J(s) := \begin{cases} \dfrac{1}{s}K\big(G(s) - G(0)\big) & s \neq 0 \\ KG'(0) & s = 0 \end{cases},$$

where $G'(0)$ denotes the derivative of G at $s = 0$. Let $s \in \mathrm{i}\mathbb{R}$. If $s \neq 0$, we use (13) to estimate

$$\|\mathcal{G}_1(s)\| \leq g\left\|(I + (g/s)\Gamma)^{-1}\right\| \cdot \|J(s)\| \leq gM\|J\|_\infty\,.$$

Since $\mathcal{G}_1(0) = 0$, it follows that

$$\|\mathcal{G}_1\|_\infty \leq gM\|J\|_\infty,$$

and thus, setting $g^* := \varepsilon/(M\|J\|_\infty) > 0$, we conclude that

$$\|\mathcal{G}_1\|_\infty \leq \varepsilon \quad \forall\, g \in (0, g^*). \tag{14}$$

To estimate $\|\mathcal{G}_2\|_\infty$, we note that

$$\mathcal{G}_2(s) = -\big((s/g)I + \Gamma\big)^{-1}\big(KG(0) - \Gamma\big)\,,$$

and hence obtain, for every $g > 0$,

$$\|\mathcal{G}_2\|_\infty \leq \sup_{s \in \mathrm{i}\mathbb{R}} \left\|(sI + \Gamma)^{-1}\right\| \cdot \left\|KG(0) - \Gamma\right\|. \tag{15}$$

Combining (12), (14) and (15) yields (11), completing the proof. □

Proof of Theorem 4. Note that by hypothesis (F), the nonlinearity ϕ is affinely linearly bounded, and thus, by [33, Proposition 4.12], it follows that, for all $g > 0$, $r \in R$, $u^r \in U^r$ and all (x^0, u^0), the unique maximally defined solution (x, u) of (7) exists on $\mathbb{R}_+$.

Let $r \in R$ and $u^r \in U^r$ and set

$$z := x + A^{-1}B\phi(u^r) \quad \text{and} \quad v := u - u^r\,. \tag{16}$$

Then

$$\begin{aligned} \dot{z} = \dot{x} &= Ax + B\phi(u) = Az + B\big[\phi(v + u^r) - \phi(u^r)\big] \\ &= Az + B\phi_{u^r}(v)\,, \end{aligned} \tag{17}$$

where $\phi_{u^r} : \mathbb{R}^m \to \mathbb{R}^m$ is defined by

$$\phi_{u^r}(w) := \phi(w + u^r) - \phi(u^r), \quad \forall\, w \in \mathbb{R}^m. \tag{18}$$

Furthermore,

$$\begin{aligned} \dot{v} = \dot{u} &= gK(r - Cx) - g\Gamma\big[u - \phi(u) - u^r + \phi(u^r)\big] \\ &= -gKCz - g\Gamma\big[v - \phi_{u^r}(v)\big]\,, \end{aligned} \tag{19}$$

where we have used that $r = G(0)\phi(u^r) = -CA^{-1}B\phi(u^r)$. We recast (17) and (19) as

$$\begin{aligned}\begin{bmatrix}\dot z\\ \dot v\end{bmatrix} &= \begin{bmatrix}A & 0\\ -gKC & -g\Gamma\end{bmatrix}\begin{bmatrix}z\\ v\end{bmatrix} + \begin{bmatrix}B\\ g\Gamma\end{bmatrix}\phi_{u^r}(v)\\ &= \mathcal{A}\begin{bmatrix}z\\ v\end{bmatrix} + \mathcal{B}\phi_{u^r}\left(\mathcal{C}\begin{bmatrix}z\\ v\end{bmatrix}\right), \end{aligned} \tag{20}$$

where $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ are as in (10). By (F), the nonlinearity ϕ_{u^r} satisfies

$$\|\phi_{u^r}(w)\| \le L\|w\| \quad \forall\, w \in \mathbb{R}^m\,. \tag{21}$$

Invoking (8), it is clear that there exists $\varepsilon > 0$ such that

$$\varepsilon + \sup_{s\in \mathrm{i}\mathbb{R}} \big\|(sI+\Gamma)^{-1}\big\| \cdot \|\Gamma - KG(0)\| < 1/L\,.$$

Combining this with Lemma 6, we see that there exists $g^* > 0$ such that, for all $g \in (0, g^*)$,

$$\|\mathcal{G}\|_\infty \le \varepsilon + \sup_{s\in \mathrm{i}\mathbb{R}} \big\|(sI+\Gamma)^{-1}\big\| \cdot \|\Gamma - KG(0)\| < 1/L\,. \tag{22}$$

Let $g \in (0, g^*)$. The claims (C1)–(C3) follow once the zero equilibrium of the Lur'e system (20) is shown to be globally exponentially stable which in turn follows from (21) and (22) and an absolute stability result, such as [32, Corollary 5.6.50] or [34, Theorem 5 (iii)] (whilst the latter result is for SISO systems, it is not difficult to show that it extends to the MIMO case). To establish (C4), we note that $\dot x = \dot z = Az + B\phi_{u^r}(v)$ and use (C1), (C2), (17) and (21) to obtain that, for all $t \ge 0$,

$$\|\dot x(t)\| \le ce^{-\gamma t}\big(\|z(0)\| + \|v(0)\|\big)$$

for suitable positive constants $c > 0$ and γ, as required. □

We continue with some remarks on Theorem 4, particularly the existence of a suitable matrices K and Γ.

Remark 7. (i) Recall from Remark 5 that $p \ge m$ and $\mathrm{rk}\, G(0) = m$ are necessary conditions for (a)–(c) and (8) to hold (in the usual case that $L \ge 1$). The rank condition on $G(0)$ implies that $G(0)$ has a left inverse: for example, $(G(0)^TG(0))^{-1}G(0)^T$. Choosing K as *any* left inverse of $G(0)$ and $\Gamma := KG(0) = I$, ensures that (b) holds and (8) is trivially satisfied for every $L > 0$. In the special case that $m = p$ and $G(0)$ is invertible, then the above choices simplify to $K := G(0)^{-1}$ and $\Gamma := I$.

(ii) Assume that the system (A, B, C) is subject to parametric uncertainty, and that the "true" linear system is given by $(\tilde A, \tilde B, \tilde C)$ with $\tilde A$ Hurwitz. If the "nominal" steady-state gain $G(0)$ is such that there exists K with $-KG(0)$ Hurwitz and

$$\sup_{s\in \mathrm{i}\mathbb{R}} \big\|(sI+KG(0))^{-1}\big\| \cdot \|K\big(G(0) - \tilde G(0)\big)\| < 1/L\,, \tag{23}$$

where $\tilde G$ is the transfer function of $(\tilde A, \tilde B, \tilde C)$, then, with the choice $\Gamma = KG(0)$, it is guaranteed that the conclusions of Theorem 4 hold in the context of the "true" system $(\tilde A, \tilde B, \tilde C)$. Assuming that $\mathrm{rk} G(0) = m$, and in light of part (i), choosing K as any left inverse of $G(0)$ and $\Gamma := I$, the condition (23) simplifies to

$$\|I - K\tilde G(0)\| < 1/L\,, \tag{24}$$

which is certainly satisfied if

$$\|G(0) - \tilde G(0)\| < \frac{1}{L\|K\|}\,. \tag{25}$$

We comment that the estimate (25) may equivalently be formulated as a ball condition (with centre $\tilde G(0)$ and known radius). To summarise the above discussion, Theorem 4 applies to all plants with (unknown) steady-state gain $\tilde G(0)$ if the design of Γ and K is based on are the (nominal) steady-state gain $G(0)$ and $\tilde G(0)$ is sufficiently close in norm to $G(0)$. In other words, the closed-loop system (7) is locally robust with respect to uncertainty in the steady-state gain, captured by the estimates (23)–(25).

(iii) We comment further that it is possible to augment (7) with an adaptation for the parameter g, replacing it by a dynamic variable (in the spirit of [10]) and obviating the requirement that it is chosen "sufficiently small". The conclusions of Theorem 4 may still be shown to hold, although a formal statement and proof of this claim is beyond the scope of the present note. □

Theorem 4 can be applied to the problem of regulating the output of a stable linear control system to a prescribed constant reference vector in the presence of input constraints. To this end, consider the linear system

$$\dot x = Ax + Bv, \quad x(0) = x^0, \quad y = Cx\,, \tag{26}$$

where $A \in \mathbb{R}^{n\times n}$ is Hurwitz, $B \in \mathbb{R}^{n\times m}$ and $C \in \mathbb{R}^{m\times n}$, together with the control objective of asymptotic tracking of constant reference vectors subject to the input constraint $v(t) \in V \subset \mathbb{R}^m$ for all $t \ge 0$, where $V := [v_1^1, v_1^2] \times \ldots \times [v_m^1, v_m^2]$. This problem has been studied in [35], where a solution is proposed that determines v in (26) adaptively. We will show that Theorem 4 provides an alternative solution.

To that end, let $r \in G(0)V$, where G denotes the transfer function matrix of (26). Defining sat as in Example 1, it follows from Theorem 4, that the control objective is achieved by the control law

$$v = \mathrm{sat}(u), \quad \dot u = gK(r-y) - g\Gamma\big(u - \mathrm{sat}(u)\big),$$

provided that $-\Gamma$ is Hurwitz, the estimate

$$\sup_{s\in \mathrm{i}\mathbb{R}} \big\|(sI+\Gamma)^{-1}\big\| \cdot \|\Gamma - KG(0)\| < 1, \tag{27}$$

holds and $g > 0$ is sufficiently small. Recall from Remark 7 that if K is equal to a left inverse of $G(0)$ (requiring that $\mathrm{rk}\, G(0) = m$) and $\Gamma = I$, then (27) holds.

Compared to [35], the controller proposed here is much simpler and easier to implement, particularly in the (high-dimensional) MIMO case. However, more information — namely of the steady-state gain $G(0)$ and a sufficiently small gain g — is required. We emphasise that the problems considered here and in [35] are related, but not identical, as we permit non-diagonal as well as unbounded input nonlinearities.

We next show that, under the assumptions of Theorem 4, the equilibrium $(-A^{-1}B\phi(u^r), u^r)$ of the system (7) is input-to-state stable (ISS) with respect to additive disturbances. We refer the reader to [27] and [28] for a detailed discussion of ISS.

Proposition 8. *Consider the closed-loop system*

$$\left.\begin{aligned} \dot x &= Ax + B\phi(w) + d_1, \quad y = Cx + d_2,\\ \dot w &= gK(r-y) - g\Gamma(w - \phi(w) - u^r + \phi(u^r)) + d_3,\\ x(0) &= x^0, \quad u(0) = u^0, \end{aligned}\right\} \tag{28}$$

where $r \in R$ for feasible $R \subseteq \mathbb{R}^p$, $u^r \in U^r$, $d_1 \in L^\infty_{\mathrm{loc}}(\mathbb{R}_+;\mathbb{R}^n)$, $d_2 \in L^\infty_{\mathrm{loc}}(\mathbb{R}_+;\mathbb{R}^p)$ and $d_3 \in L^\infty_{\mathrm{loc}}(\mathbb{R}_+;\mathbb{R}^m)$. Assume that A and $-\Gamma$ are Hurwitz and (F) *holds. Then, if* (8) *is satisfied, there exists $g^* > 0$ such that for all $g \in (0, g^*)$, there exist constants $c_1, c_2, \gamma > 0$ such that for all $(x^0, u^0) \in \mathbb{R}^n \times \mathbb{R}^m$, all $r \in R$ and all $d_1, d_2, d_3 \in L^\infty_{\mathrm{loc}}$,*

the solution (x,u) *of* (28) *and the output* y *satisfy*

$$\left\| \begin{bmatrix} x(t)+A^{-1}B\phi(u^r) \\ u(t)-u^r \\ y(t)-r \end{bmatrix} \right\| \le c_1 e^{-\gamma t} \left\| \begin{bmatrix} x^0+A^{-1}B\phi(u^r) \\ u^0-u^r \end{bmatrix} \right\| + c_2 \sum_{j=1}^{3} \|d_j\|_{L^\infty(0,t)} \quad \forall\, t \ge 0\,. \tag{29}$$

Proof. Let $K \in \mathbb{R}^{m\times p}$, $\Gamma \in \mathbb{R}^{m\times m}$ and assume that $-\Gamma$ is Hurwitz and (8) holds. Defining the variables z and v as in (16) and setting

$$\zeta := \begin{bmatrix} z \\ v \end{bmatrix}, \quad \mathcal{B}_{\rm e} := \begin{bmatrix} I & 0 & 0 \\ 0 & -gK & I \end{bmatrix} \quad \text{and} \quad d := \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix},$$

the system (28) may be rewritten as

$$\dot\zeta = \mathcal{A}\zeta + \mathcal{B}\phi_{u^r}(\mathcal{C}\zeta) + \mathcal{B}_{\rm e} d\,, \tag{30}$$

where $(\mathcal{A},\mathcal{B},\mathcal{C})$ and ϕ_{u^r} are given by (10) and (18), respectively. Equation (30) is a Lur'e system with additive forcing. The assumptions of Theorem 4 are satisfied and so there exists $g^* > 0$ such that, for all $g \in (0, g^*)$, (22) holds. Now, by (F), (21) is also satisfied, and consequently, the claim follows from [26, Theorem 3.2 and comment after the proof of Theoem 3.2]. □

Remark 9. In the undisturbed case ($d_i = 0$ for $i = 1,2,3$), $(-A^{-1}B\phi(u^r), u^r)$ is an equilibrium of system (28). Proposition 8 implies that this equilibrium is ISS (with respect to d_1, d_1 and d_3). In particular, the tracking error is "small" for "small" disturbances. An important consequence of Proposition 8 is that "small" uncertainties in ϕ and u^r (both quantities appear on the right-hand side of the control law (5)) will cause only a small deterioration of the tracking performance. □

III. Examples

In the absence of an input nonlinearity (that is, $\phi = \mathrm{id}$), the control law (5) reduces to

$$\dot u = gK(r-y), \quad u(0) = u^0 \tag{31}$$

and it is well known that if A and $-KG(0)$ are Hurwitz, then, for all sufficiently small $g > 0$, the integrator (31) in feedback connection with (1) achieves asymptotic tracking for all initial conditions (x^0, u^0) and all r in the image of $G(0)$. In Example 10 below we construct an example which demonstrates that this is in general not true when an input nonlinearity is present.

Example 10. Consider (1) with

$$A = -I_4,\ B = C = I_4,\ \phi = \mathrm{sat}, \tag{32a}$$

where sat is as in Example 1 with saturation bounds given by

$$\left.\begin{aligned} V_1 &:= \{0, 300\}, \quad V_2 := \{0, 300\} \\ V_3 &:= \{0, 356\}, \quad V_4 := \{137, 300\}. \end{aligned}\right\} \tag{32b}$$

Evidently A is Hurwitz and $G(0) = I_4$. We choose

$$K := \begin{bmatrix} 23 & 20 & 18 & 17 \\ 23 & 26 & 29 & 26 \\ 56 & 56 & 62 & 68 \\ -90 & -90 & -96 & -97 \end{bmatrix}, \tag{32c}$$

which has the property that $-KG(0) = -K$ is Hurwitz. The reference vector

$$r := 200 \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}^T, \tag{32d}$$

is feasible since it trivially satisfies $r = G(0)r = G(0)\mathrm{sat}(r)$. Defining

$$u^0 := \begin{bmatrix} 291 & 8.5 & 357 & 136 \end{bmatrix}^T \quad \text{and} \quad x^0 := \mathrm{sat}(u^0)\,, \tag{32e}$$

we note that $u_3^0 > 356 = v_3^2$ and $u_4^0 < 137 = v_4^1$. Now

$$r - \mathrm{sat}(u^0) = \begin{bmatrix} 200 \\ 200 \\ 200 \\ 200 \end{bmatrix} - \begin{bmatrix} 291 \\ 8.5 \\ 356 \\ 137 \end{bmatrix} = \begin{bmatrix} -91 \\ 191.5 \\ -156 \\ 63 \end{bmatrix},$$

and a calculation shows that, for every $g > 0$,

$$gK(r - Cx^0) = gK(r - \mathrm{sat}(u^0)) = g \begin{bmatrix} 0 \\ 0 \\ 240 \\ -180 \end{bmatrix}. \tag{33}$$

Furthermore, we have that

$$Ax^0 + B\mathrm{sat}(u^0) = -x^0 + \mathrm{sat}(u^0) = 0\,. \tag{34}$$

Defining (x,u) by

$$x(t) := x^0 = \mathrm{sat}(u^0), \quad u(t) = u^0 + gt \begin{bmatrix} 0 \\ 0 \\ 240 \\ -180 \end{bmatrix} \quad \forall\, t \ge 0,$$

we see that $\mathrm{sat}(u(t)) = \mathrm{sat}(u^0)$ for all $t \ge 0$ and therefore, it follows from (33) and (34) that (x,u) solves the integral control system given by (1) and (31). Consequently, the feedback systems is unstable for any choice of $g > 0$. In particular, $y(t) \not\to r$ as $t \to \infty$. We conclude that, in the presence of input nonlinearities, the "pure" integral controller (31) does not guarantee asymptotic tracking (actually, may fail to achieve global stability).

We now apply the controller (7) to the model data (32) with $\Gamma = KG(0) = K$. Theorem 4 then guarantees convergence of $y(t)$ to r for all sufficiently small integrator gains g. Figure 3 contains the resulting simulation for $g = 0.027$. We see that, although the performance is somewhat sluggish (not unexpected because the K matrix has been chosen rather "badly"), the inputs and states (the latter are equal to the outputs) converge and the states track the reference. □

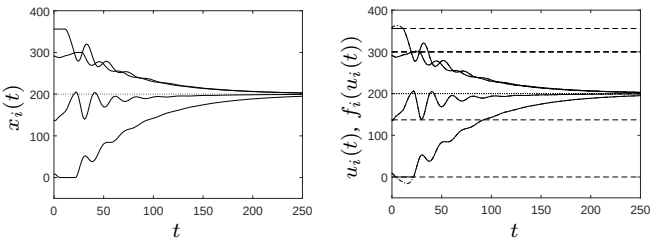


Figure 3. Simulation: low-gain controller (7) applied to model data (32). (a) State variables. (b) Saturated inputs in solid lines, inputs in dashed-dotted lines and saturation bounds in dashed lines. In both panels, dotted lines indicate the limits.

Remark 11. When $\phi = \mathrm{sat}$ and A and $-KG(0)$ are Hurwitz, then the equilibrium $(-A^{-1}B\phi(u^r), u^r)$ of the low-gain integral control feedback system (1) and (31) is known to be locally asymptotically stable for every feasible r and all sufficiently small $g > 0$. Under the additional assumptions that $KG(0)$ is symmetric, the equilibrium is globally asymptotically stable (this can be proved using absolute stability arguments and [36, Proposition 3.9]). However, the symmetry assumption is extremely non-robust to parametric uncertainty in

$G(0)$ and hence is unrealistic (except in the SISO case, where it holds trivially). Finally, note that in Example 10, the matrix $KG(0)$ is not symmetric. □

The next example relates to Remark 11. We provide a simulation which illustrates the fact that if $KG(0)$ is symmetric, then a "pure" integral controller does achieve asymptotic tracking for all sufficiently small $g > 0$. The example also shows that the rate of convergence may be worse than that obtained by using the integral/anti-windup controller (7).

Example 12. Consider (1) with

$$A = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}, \quad B = C = I_2\,, \tag{35a}$$

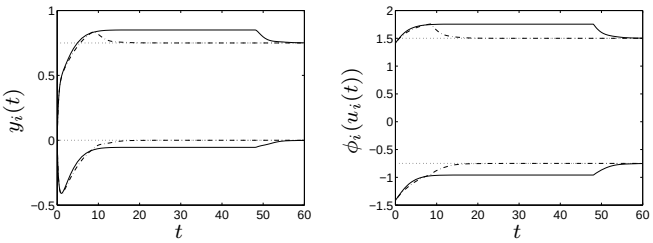
and where $\phi : \mathbb{R}^2 \to \mathbb{R}^2$ is the non-diagonal saturation function from Example 2 with $\theta = 2$ (and the usual Euclidean two-norm). Note that ϕ satisfies (F) with $L = 2$. It is readily verified that A is Hurwitz with $A = A^T$ and $B = C = C^T$, so that

$$-G(0) = CA^{-1}B = A^{-1} = -\frac{1}{3}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

is Hurwitz and symmetric. Consider the data

$$r = \begin{bmatrix} 0.75 \\ 0 \end{bmatrix}, \quad u^0 = \begin{bmatrix} 5 \\ -5 \end{bmatrix}, \quad x^0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \tag{35b}$$

and note that r is feasible since $r = G(0)u^r = G(0)\phi(u^r)$, where $u^r = \begin{bmatrix} 1.5 & -0.75 \end{bmatrix}^T$. Finally, we choose $K = I$, $g = 1$ and $\Gamma = G(0)$. Figure 4 shows simulations of the closed-loop dynamics generated by the integral controller (31) and controller (7) with anti-windup component. We observe that the rate of convergence of the saturated inputs and outputs is faster for the latter: in light of Figure 4 (b), the closed-loop dynamics "spend less time" at the saturation bounds than in the "pure" integral control scenario. □



(a) (b)

Figure 4. Simulations of the low-gain integral control systems from Example 12. The dotted lines denote reference values and the limits of the saturated inputs. In each panel the solid and dashed-dotted lines correspond to the controllers (31) and (7), respectively. (a) Outputs and reference components. (b) Saturated inputs and saturation limits.

Example 13. As a final illustrative example of the theory presented, consider the electrical circuit depicted in Figure 5 with two inductors, two resistors, a single capacitor and two external voltage sources.

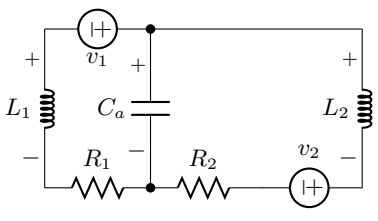


Figure 5. Sample circuit diagram from Example 13.

For $k = 1, 2$, let L_k and i_{L_k} denote the inductance and current, respectively, of the k-th inductor, and let C_a and v_{C_a} denote the capacitance and voltage across the capacitor, respectively. Let R_k denote the resistance of the k-th resistor for $k = 1, 2$. We assume that the voltage sources are subject to saturation.

Invoking Kirchoff's Laws with state variables

$$x_1 := L_1 i_{L_1}, \quad x_2 := L_2 i_{L_2}, \quad x_3 = C_a v_{C_a}\,,$$

and input and output variables

$$u_1 := v_1, \quad u_2 := v_2, \quad y_1 := i_{L_2}, \quad y_2 := v_{C_a}\,,$$

leads to the MIMO control system of the form (1) with

$$\left.\begin{aligned} A &= \begin{bmatrix} -\frac{R_1}{L_1} & 0 & \frac{1}{C_a} \\ 0 & -\frac{R_2}{L_2} & \frac{1}{C_a} \\ -\frac{1}{L_1} & -\frac{1}{L_2} & 0 \end{bmatrix}, \quad B = -\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \\ C &= \begin{bmatrix} 0 & \frac{1}{L_2} & 0 \\ 0 & 0 & \frac{1}{C_a} \end{bmatrix} \end{aligned}\right\} \tag{36a}$$

and $\phi = \mathrm{sat}$, with saturation bounds V_k to be specified. A routine calculation shows that the matrix A is Hurwitz for all $C_a, L_1, L_2, R_1, R_2 > 0$.

For the following simulation, we assume that the actual parameter values are unknown, but within 10% of the nominal values:

$$\left.\begin{aligned} &C_a = 3 \times 10^{-3} F,\ L_1 = 0.01H,\ L_2 = 0.05H, \\ &R_1 = 1\Omega,\ R_2 = 1.5\Omega\,. \end{aligned}\right\} \tag{36b}$$

Let G denote the transfer function of the triple in (36a) with the nominal values in (36b). As $-G(0)$ has two eigenvalues both with real part equal to -0.4, it is Hurwitz and so we choose $K = I$ in (5) and $\Gamma = KG(0)$. With saturation bounds

$$V_1 = \{0, 7.5\} \quad \text{and} \quad V_2 = \{-5, 7.5\}\,, \tag{36c}$$

(units in volts) it follows from (4) that

$$R := \{G(0)v \,:\, v_1 \in [0, 7.5],\ v_2 \in [-5, 7.5]\}\,,$$

is the maximal set of feasible references for the nominal system, and is depicted in Figure 6. The actual set of feasible references depends on the uncertain "true" system. For

$$r := \begin{bmatrix} -2 \\ 4 \end{bmatrix}, \quad u^0 := \begin{bmatrix} -1 \\ 4 \end{bmatrix}, \quad x^0 := 0 \quad \text{and} \quad g := 2\,, \tag{36d}$$

simulations of the dynamics generated by the integral/anti-windup controller (7) are plotted in Figure 7. We see that both the saturated inputs and outputs converge and that the outputs track the reference. The simulation shown in Figure 7 was performed by (pseudo)randomly drawing parameter values from within 10% of the nominal values. A calculation shows that the nominal and "true" steady-state gains are given by

$$G(0) = \frac{1}{R_1 + R_2}\begin{bmatrix} 1 & -1 \\ R_2 & R_1 \end{bmatrix}, \quad \tilde{G}(0) = \frac{1}{\tilde{R}_1 + \tilde{R}_2}\begin{bmatrix} 1 & -1 \\ \tilde{R}_2 & \tilde{R}_1 \end{bmatrix},$$

respectively. It is readily verified that the estimate (23) holds with $L = 1$, $K = I$ and $\Gamma = G(0)$, for all $\tilde{R}_1, \tilde{R}_2$ within 10% of the values in (36b), ensuring that Theorem 4 holds. Note that in fact no knowledge of C_a, L_1 or L_2 is required, and the imposed bounds on the parameter variation in R_1 and R_2 is sufficient to implement (7), meaning that exact knowledge of $\tilde{G}(0)$ is not required. □

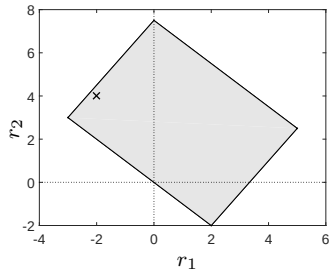


Figure 6. Maximal feasible region (shaded area) of the nominal electrical circuit example (36) and reference (cross) r in (36d).

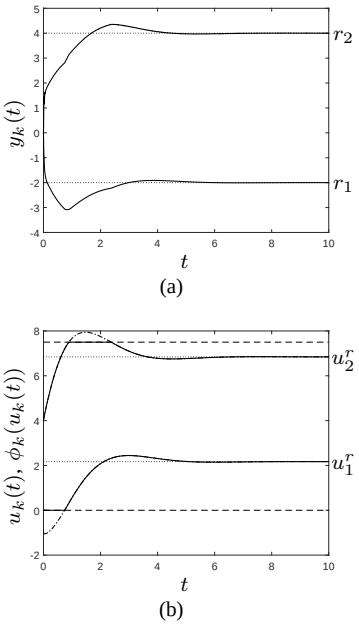


Figure 7. Simulation: circuit example controlled by (7). (a) Outputs in solid lines with dotted lines denoting reference components. (b) Saturated inputs in solid lines, the dash-dotted lines denote the unsaturated inputs for comparison purposes, the dashed lines are selected saturation bounds and the dotted lines are the input limits.

REFERENCES

[1] E.J. Davison, "Multivariable tuning regulators: the feedforward and robust control of a general servomechanism problem" *IEEE Trans. Automatic Control*, vol. AC-21, no. 1, pp. 35–47, 1976.

[2] S. Pohjolainen, "Robust multivariable PI-controller for infinite dimensional systems" *IEEE Trans. Automatic Control*, vol. AC-27, no. 1, pp. 17–30, 1982.

[3] P. Grosdidier, M. Morari, and B.R. Holt, "Closed-loop properties from steady-state gain information" *Ind. Eng. Chem. Fund.*, vol. 24, no. 2, pp. 221–235, 1985.

[4] M. Morari, "Robust stability of systems with integral control" *IEEE Trans. Automat. Control*, vol. AC-30, no. 6, pp. 574–577, 1985.

[5] H. Logemann and D.H. Owens, "Low-gain control of unknown infinite-dimensional systems: a frequency-domain approach" *Dynamics and Stability of Systems*, vol. 4, no. 1, pp. 13-29, 1989.

[6] T. Samad and A. Annaswamy, eds., *The Impact of Control Technology*. IEEE Control Systems Society, 2011.
Available at http://ieeecss.org/general/impact-control-technology.

[7] H. Logemann and S. Townley, "Discrete-time low-gain control of uncertain infinite-dimensional systems" *IEEE Trans. Automat. Control*, vol. AC-42, no. 1, pp. 22–37, 1997.

[8] R.F. Curtain, H. Logemann and O. Staffans, "Stability results of Popov-type for infinite-dimensional systems with applications to integral control" *Proc. London Mathematical Society*, vol. 86, no. 3, pp. 779–816, 2003.

[9] H. Logemann and S. Townley, "Adaptive low-gain integral control of multivariable well-posed linear systems" *SIAM J. Control Optim.*, vol. 41, no. 6, pp. 1722–1732, 2003.

[10] Z. Ke, H. Logemann, and S. Townley, "Adaptive sampled-data integral control of stable infinite-dimensional linear systems" *Systems Control Lett.*, vol. 58, no. 4, pp. 233–240, 2009.

[11] K.J. Åström and L. Rundqwist, "Integrator windup and how to avoid it" in *American Control Conference*, pp. 1693–1698, IEEE, 1989.

[12] D. S. Bernstein and A. N. Michel, "A chronological bibliography on saturating actuators" *Int. J. Robust. Nonlin.*, vol. 5, no. 5, pp. 375–380, 1995.

[13] S. Galeani, S. Tarbouriech, M. Turner and L. Zaccarian, "A tutorial on modern anti-windup design," in *European Control Conference (ECC)*, pp. 306–323, IEEE, 2009.

[14] S. Tarbouriech and M. Turner, "Anti-windup design: an overview of some recent advances and open problems" *IET Control Theory Appl.*, vol. 3, no. 1, pp. 1–19, 2009.

[15] S. Tarbouriech, G. Garcia, J.M.G da Silva Jr., & I. Queinnec. *Stability and stabilization of linear systems with saturating actuators*, Springer, London, 2011.

[16] J. Penttinen and H. Koivo, "Multivariable tuning regulators for unknown systems" *Automatica*, vol. 16, no. 4, pp. 393 – 398, 1980.

[17] J. Lunze, "Determination of robust multivariable I-controllers by means of experiments and simulation" *Systems Anal. Modelling Simulation*, vol. 2, no. 3, pp. 227–249, 1985.

[18] U. Jönsson and A. Megretski, "The Zames-Falb IQC for critically stable systems" in *American Control Conference*, pp. 3612–3616, IEEE, 1998.

[19] U. Jönsson and A. Megretski, "The Zames-Falb IQC for systems with integrators" *IEEE Trans. Automat. Control*, vol. 45, no. 3, pp.560–565, 2000.

[20] T. Fliegner, H. Logemann and E.P. Ryan, "Adaptive low-gain integral control of linear systems with input and output nonlinearities" *Systems Control Lett.*, vol. 44, no. 1, pp.1–12, 2001.

[21] T. Fliegner, H. Logemann and E.P. Ryan, "Absolute stability and integral control" *Int. J. Control*, vol. 79, no. 4, pp. 311–326, 2006.

[22] M. Vidyasagar, *Nonlinear Systems Analysis*, 2nd ed., Prentice-Hall, Englewood Cliffs, New Jersey, 1993.

[23] H. K. Khalil *Nonlinear systems*, 3rd ed., Prentice Hall, New Jersey, 2002.

[24] M.R. Liberzon, Essays on the absolute stability theory, *Autom. Remote Control*, vol. 67, no. 10, pp. 1610–1644, 2006.

[25] W.M. Haddad & V. Chellaboina, *Nonlinear Dynamical Systems and Control*, Princeton University Press, New Jersey, 2008.

[26] E. Sarkans and H. Logemann, "Input-to-state stability of Lur'e systems" *Math. Control Signal.*, vol. 27, pp. 439–465, 2015.

[27] E.D. Sontag, "Input to state stability: basic concepts and results" in *Nonlinear and Optimal Control Theory*, vol. 1932 of *Lecture Notes in Math.*, pp. 163–220, Springer, Berlin, 2008.

[28] S.N. Dashkovskiy, D.V. Efimov and E.D. Sontag, "Input-to-state stability and allied system properties", *Automation and Remote Control*, vol. 72, no. 8, pp. 1579–1614, 2011.

[29] C. Guiver, M. Mueller, D. Hodgson and S. Townley, "Robust set-point regulation for ecological models with multiple management goals", *J. Math. Biol.*, vol. 72, no. 6, pp. 1467–1529, 2016.

[30] D. Luenberger, *Introduction to Dynamic Systems: Theory, Models, and Applications*. John Wiley & Sons Inc., New York, 1979.

[31] L. Farina and S. Rinaldi, *Positive Linear Systems: Theory and Applications*. John Wiley & Sons Inc., New York, 2000.

[32] D. Hinrichsen and A.J. Pritchard, *Mathematical Systems Theory I: Modelling, State Space Analysis, Stability and Robustness*. Springer-Verlag, Berlin, 2005.

[33] H. Logemann & E.P. Ryan, *Ordinary Differential Equations*, Springer, London, 2014.

[34] B. Jayawardhana, H. Logemann, and E.P. Ryan, "The circle criterion and input-to-state stability: new perspectives on a classical result" *IEEE Control Syst. Mag.*, vol. 31, no. 4, pp. 32–67, 2011.

[35] D.E. Miller and E.J. Davison, "An adaptive tracking problem with a control input constraint", *Automatica*, vol. 29, no. 4, pp. 877–887, 1993.

[36] H. Logemann and S. Townley, "Low-gain control of uncertain regular linear systems", *SIAM J. Control Optim.*, vol. 35, no. 1, pp. 78–116, 1997.