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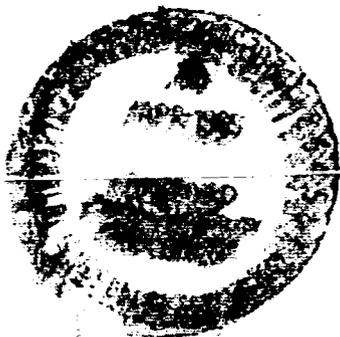
# LOW-GRAVITY FUEL SLOSHING IN AN ARBITRARY AXISYMMETRIC RIGID TANK

by

Wen-Hwa Chu

**TECHNICAL REPORT NO. 8**  
**Contract No. NAS 8-20290**  
**Control No. DCN 1-9-75-10030(IF)**  
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Prepared for  
**George C. Marshall Space Flight Center**  
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# **LOW-GRAVITY FUEL SLOSHING IN AN ARBITRARY AXISYMMETRIC RIGID TANK**

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**Huntsville, Alabama**

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**Approved:**



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**H. Norman Abramson, Director**  
**Department of Mechanical Sciences**

## ABSTRACT

Solutions to free and forced oscillations have been found in terms of an auxiliary set of eigenfunctions. The slosh force and moment for an arbitrary axisymmetric rigid tank at arbitrary Bond number have been derived for both pitching and translation and expressed in terms of characteristics of an equivalent spring-mass system. Numerical examples have been constructed which compare favorably with available theories and experiments.

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## NOTATION

$a$	a reference length, say, maximum radius of the ullage
$dA$	$rdrd\theta$
$\underline{dA}$	$dA/a^2$
$dS$	3-D surface element, e. g., $rd\theta drd\phi$
$\underline{dS}$	$dS/a^3$ , nondimensional surface element
$F$	equilibrium (mean) interface or $f/a$
$F_e$	instantaneous interface
$F_H$	horizontal force defined by Eq (19)
$F_R$	$\frac{dF}{dR}$ , slope of $F$ in the generatrix plane
$F_x$	x - component of force on the tank
$f$	equilibrium (mean) interface elevation
$g$	gravitational acceleration
$H$	amplitude of $h/a$ , nondimensional slosh height
$h$	interface perturbation
$h_0$	a reference length, say depth of liquid at center of tank
$M_0, I_0$	rigid mass and moment of inertia of the mechanical model
$M_F$	liquid mass
$M_y$	pitching moment about y-axis
$m_k$	$k^{\text{th}}$ slosh mass
$N_F$	Bond number
$n$	outer normal
$n_0$	$n/a$ , nondimensional normal distance

$p$	pressure
$p_I$	equilibrium liquid pressure at origin - a constant
$p_u$	ullage pressure
$p_{uI}$	equilibrium ullage pressure at origin - a constant
$R$	$r/a$ , nondimensional radius
$r, \theta, z$	tank fixed cylindrical coordinates
$t$	time
$V$	volume of the liquid divided by $a^3$
$V_L$	liquid volume (lower fluid)
$W$	wall wetted by liquid
$W_e$	instantaneous wetted wall below instantaneous interface, $F_e$
$x_o$	translational amplitude in $x_s$ -direction
$x_s, y_s, z_s$	space-fixed rectangular coordinates
$\Gamma$	$\gamma a$ , nondimensional hysteresis coefficient
$\gamma$	hysteresis coefficient
$\Delta\rho$	density difference, $\rho - \rho_u$
$\delta_{ij}$	Kronecker delta
$\epsilon_1$	sign of $n \cdot z$ , $\cos(n, z)$ , or $\frac{\partial z}{\partial n}$
$\theta_y$	amplitude of pitching about $y$ -axis
$\kappa$	the mean curvature
$\kappa'$	perturbation of the mean curvature
$\lambda_j$	$j^{\text{th}}$ eigenvalue ( $m = 1$ )
$\lambda_{mj}$	$j^{\text{th}}$ eigenvalue corresponds to $m^{\text{th}}$ circumferential mode
$\rho$	lower fluid density

$\rho_u$	density of ullage fluid (vapor or gas)
$\sigma$	surface tension
$\Phi$	amplitude of nondimensional velocity potential, $\phi/\omega a^2$
$\Phi_j$	velocity potential of the auxiliary eigenfunctions
$\phi_k$	amplitude of nondimensional potential $\phi_k/\omega a^2$
$\Phi_N$	see Equation (15)
$\phi^*$	amplitude of the nondimensional potential $\phi^*/\omega^* a$
$\phi$	velocity potential
$\phi_k$	velocity potential of the $k^{\text{th}}$ natural mode
$\phi'$	additional velocity potential due to interface movement
$\phi^o$	velocity potential of liquid with a frozen interface
$\omega$	frequency of oscillation
$\omega_k$	$k^{\text{th}}$ natural frequency
$\Omega^2$	$\rho a^3 \omega^2 / \sigma$ , product of Bond number and frequency parameter

### Subscripts

( ) <sub>I</sub>	( )	at the vertex of the equilibrium interface (origin)
( ) <sub>II</sub>	( )	at the contact point in the generatrix plane
( ) <sub>C.G.</sub>	( )	related to center of gravity
( ) <sub>e</sub>		effective value of ( )
( ) <sub>F</sub>	( )	on F
( ) <sub>m</sub>	( )	associated with $\cos(m\theta)$ mode
( ) <sub>p</sub>	( )	related to pitching

- ( )<sub>T</sub> ( ) related to translation
- ( )<sub>W</sub> ( ) on W
- ( )<sub>u</sub> ( ) related to the ullage
- ( )<sub>-</sub> ( ) just below the interface
- ( )<sub>+</sub> ( ) just above the interface

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## INTRODUCTION

The behavior and consequences of fuel sloshing in rockets under a high effective gravity were recognized problems which have been quite well understood (Refs. 1, 2, and 3). The problem of low-gravity fuel sloshing, characterized by the significant role of interfacial tension, is now a subject of importance for application to coasting rockets or orbital stations.

The equilibrium behavior of fluids at zero and/or low gravity has been studied in References 4 through 7. The theoretical determination of an equilibrium interface shape is nonlinear and requires a trial and error procedure for a given contact angle (Refs. 5 and 6).

Satterlee and Reynolds (Ref. 8) have successfully solved the free sloshing problem in cylindrical containers under low gravity and formulated a variational principle for this purpose. Yeh (Ref. 9), using a similar approach, solved the free and forced sloshing problem under low-gravity conditions, without force and moment or an equivalent mechanical model. Dodge and Garza (Refs. 10 and 11) performed force measurements under simulated low-gravity conditions and predicted forces of moment for circular cylindrical tanks under lateral (translational) motion. The equivalent spring-mass model was given in Reference 10. Additional work by Dodge and Garza for other special tanks was given in References 12 and 13. A finite difference approach with application to a hemispherically bottomed cylindrical tank was given by Concus, Crane, and Satterlee in Reference 14.

These investigations indicate a need of a program for a general axisymmetric tank. A preliminary study on liquid sloshing in an arbitrary axisymmetric tank was reported in Reference 15, but it is limited to translational oscillations. It is the object of the present paper to present a semi-numerical approach for an arbitrary axisymmetric tank with simplified force and moment calculations and the resultant mechanical model for both pitching and translational oscillations. A general computer program will be completed to obtain sloshing frequencies, slosh mass, and mass-height, for which a brief description is given in the Appendix.

### Governing Equations

Assuming irrotational incompressible flow, there is a space-fixed velocity potential  $\phi$  satisfying the Laplace equation

$$\nabla^2 \phi = 0 \quad (1)$$

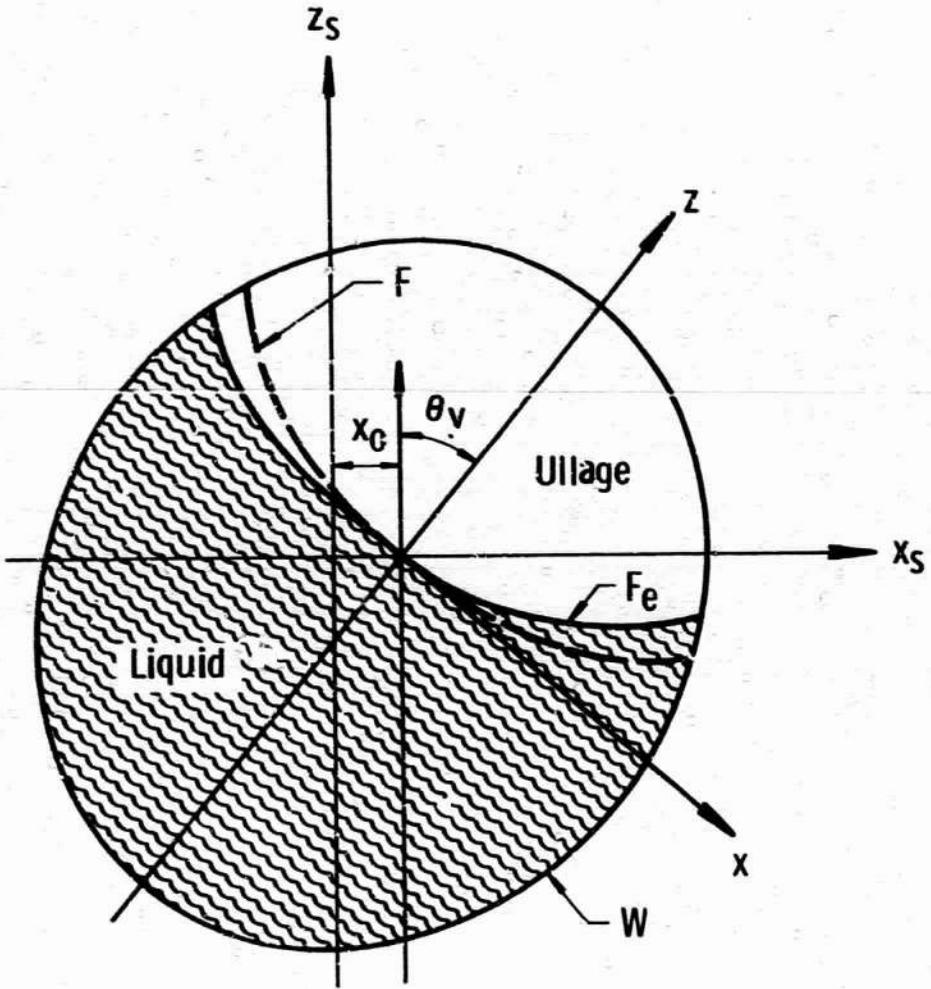
As in thin airfoil theory, the velocity potential can be obtained by imposing boundary conditions on the initial or mean position, but the hydrostatic pressure due to gravity possesses components along both the tank axis  $z$  and the lateral axis  $x$  (Fig. 1) for pitching oscillations. The linearized Bernoulli's equation states

$$p - p_I + \rho \frac{\partial \phi}{\partial t} + \rho g (z - x\theta_y) = 0 \quad (2)$$

and

$$p - p_{u_I} + \rho_u \frac{\partial \phi_u}{\partial t} + \rho_u g (z - x\theta_y) = 0 \quad (3)$$

for the liquid and the ullage, respectively, and  $p_I$ ,  $p_{u_I}$  are constants.



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Figure 1. Some Nomenclatures

### Boundary Conditions

The linearized interface kinematic condition states

$$\frac{\partial h}{\partial t} \cong \frac{\partial \phi}{\partial n} \sqrt{1 + \left(\frac{\partial f}{\partial r}\right)^2} \epsilon_1 \quad (4)$$

where

$$\epsilon_1 = \text{sgn}(\underline{n} \cdot \underline{z}) \quad (4a)$$

The interface dynamic condition states

$$p_- - p_+ = \sigma \kappa = \sigma \kappa_0 + \sigma \kappa' \quad (5)$$

For the "mean" interface location  $f$  (in general,  $p_I = p_I^{\circ} + p_I'$ ,  $p_I^{\circ}$ ,  $p_I'$  being constants),

$$\sigma \kappa_0 + (\rho - \rho_u) g f - (p_I^{\circ} - p_{uI}^{\circ}) = 0 \quad (6)$$

where the curvature of the mean interface,  $\kappa_0$ , is axisymmetric and

$$\kappa_0 = -\frac{1}{r} \frac{\partial}{\partial r} \left\{ \frac{r \frac{\partial f}{\partial r}}{\sqrt{1 + \left(\frac{\partial f}{\partial r}\right)^2}} \right\} \quad (6a)$$

Equation (6) holds for  $r = 0$ , thus

$$p_I^{\circ} - p_{uI}^{\circ} = -2 \left( \frac{\partial^2 f}{\partial r^2} \right)_I$$

The linearized interface dynamic condition is then

$$-(p'_u - p'_{u_I}) + \sigma \kappa' + \rho \frac{\partial \phi}{\partial t} - \rho_u \frac{\partial \phi_u}{\partial t} + (\rho - \rho_u) gh - (\rho - \rho_u) g x \theta_y = 0 \quad (7)^\dagger$$

where the perturbation curvature for  $\cos(m\theta)$  variation

$$\kappa' = - \left\{ \frac{1}{r} \frac{\partial}{\partial r} \left[ \frac{r \frac{\partial h}{\partial r}}{\sqrt{1 + \left(\frac{\partial f}{\partial r}\right)^2}} \right] - \frac{m^2}{r^2} \frac{h}{\sqrt{1 + \left(\frac{\partial f}{\partial r}\right)^2}} \right\} \quad (7a)$$

$m$  being unity for lateral excitation of a rigid tank. At point I, the origin,

$h = 0$ ,  $\phi = 0$ ,  $\kappa' = 0$ , and thus  $p'_I = p'_{u_I}$ . For most analyses,  $\rho_u = 0$  was

assumed. We shall assume the impulsive pressure in the ullage is negligible,

i. e.,  $\phi_u \cong 0$ . Then for sinusoidal oscillations, Equations (7) and (4) yield

$$- \left\{ \frac{1}{R} \frac{\partial}{\partial r} \left[ R \frac{\partial H}{\partial r} (1 + F_R^2)^{-3/2} - \frac{m^2 H}{R^2} (1 + F_R^2)^{-1/2} \right] \right\} + N_{B_e} H + \Omega^2 \Phi = 0 \text{ on } F \quad (7b)^\ddagger$$

The boundary condition on the wall is that the relative normal

velocity be zero, i. e., with  $\cos(n, x) = \frac{\partial x}{\partial n}$  and  $\cos(n, z) = \frac{\partial z}{\partial n}$ ,

$$\frac{\partial \psi}{\partial n} = \dot{x}_c \frac{\partial x}{\partial n} \quad (8)$$

<sup>†</sup>For sinusoidal oscillations and  $m = 1$ ,  $h = 0$ ,  $\phi = \phi_u = 0$ ,  $x = 0$ , and  $\kappa' = 0$  at point I, thus  $p'_I - p'_{u_I} = 0$ . For other  $m$  values,  $p'_I - p'_{u_I} = \sigma \kappa'_I$ , which will be omitted until needed.

<sup>‡</sup>For sinusoidal oscillations, without loss of generality,  $\dot{x}_c$ ,  $\dot{\theta}_y$ ,  $\phi$  are assumed to be proportional to  $\sin(\omega t)$  while  $h$  is proportional to  $\cos(\omega t)$ .

and

$$\frac{\partial \phi}{\partial n} = \dot{\theta}_y \left( z \frac{\partial x}{\partial n} - x \frac{\partial z}{\partial n} \right) \quad (9)$$

for translational and pitching oscillations, respectively.

In addition, there is an interface contact point condition which takes the form (Refs. 8, 9 and 15)

$$\frac{\partial h}{\partial r} = \gamma h \quad \text{at point } \Pi \quad (10)$$

where  $\gamma$  may be a frequency-dependent constant. However, if the contact angle remains constant and if the not well-defined second derivative at the contact point is neglected, we can show that  $\gamma = 0$  (Ref. 15). This value has been successfully used in References 10, 11, 12, and 13.

#### Method of Solution

We shall decompose  $\phi$  into two parts,  $\phi'$  and  $\phi^o$ :  $\phi^o$  is the velocity potential corresponding to a liquid contained by a rigid mean interface and the tank walls. Therefore, it satisfies the Laplace equation and the boundary condition on the contour, Equation (8) for translation and Equation (9) for pitching on  $F_e$  and  $W_e$ . It is noted that

$$\phi_T^o = \dot{x}_0 x \quad (11)$$

while  $\phi_p^o$  can be constructed numerically.

$\phi'$  is the perturbed velocity potential due to sloshing which is governed by the interface conditions and zero normal velocity condition at the wall.

We shall employ a set of auxiliary characteristic functions, \*  $\psi_j$  orthogonal on the curved interface and vanishing on the walls, instead of constructing natural modes directly. The natural modes and frequencies are then calculated in terms of a truncated series satisfying the free sloshing ( $\phi^\circ = 0, \theta_y = 0$ ) inter-surface condition by the Galerkin method (Ref. 17).

The velocity potential  $\phi'$  for forced oscillations is then calculated by expansion into normal modes and the interface condition is again satisfied by the Galerkin method.

The force and moment are obtained by integration of pressure, not only on the wall, but also on the interface since the direct surface tension force and moment on the tank is equivalent to those on the interface due to pressure, assuming the interface inertia is negligible as well as the interface mass. To put results in the mechanical model form, the divergence theorem has been most useful (with some easy manipulations).

## Analytical Results

### Free Oscillations

For free oscillations, the natural mode  $\phi_k$  is expanded into a truncated series of the auxiliary eigenfunctions, i. e.,

$$\phi_k = \frac{\phi_k}{\omega_a^2} = \sum_{j=1}^{j_{mx}} c_{kj} \psi_{mj} \cos(m\theta) \quad \psi_j = \psi_{mj} \cos(m\theta) \quad (12a, b)$$

---

\*For direct application of the Winslow method (Ref. 16), we impose the simpler normal derivation condition,  $\partial\psi_j/\partial n_0 = \lambda\psi_j$ , on F and used the well-known influence coefficient technique to determine the eigenvector  $\psi_j$  on the inter-surface, the eigenvalue  $\lambda_j$ , and  $\psi_j$  on the wall.

$c_{kj}$  is the  $k$ th eigenvector of the following matrix equation obtained by the Galerkin method from integrating the nondimensional Equation (7b) with weighting function  $\psi_{m_i}$

$$\left\{ -\Gamma[\nu_{m_{ij}}] + [\gamma_{m_{ij}}] + m^2 [\epsilon_{m_{ij}}] + \frac{\Delta\rho}{\rho} N_B [\beta_{m_{ij}}] - \Omega^2 [\Delta_{m_{ij}}] \right\} \{c_j\} = 0 ; i, j = 1 \text{ to } J_{mx} \quad (13)$$

where

$$\beta_{m_{ij}} = \frac{\lambda_{m_j}}{a_{m_i}^2} \int_F \psi_{m_i} \psi_{m_j} dS = \lambda_{m_j} \delta_{ij} \quad (13a)^*$$

$$\epsilon_{m_{ij}} = \frac{\lambda_{m_j}}{a_{m_i}^2} \int \frac{\psi_{m_i} \psi_{m_j}}{F R^2 \sqrt{1 + F_R^2}} dS \quad (13b)$$

$$\nu_{m_{ij}} = \frac{2\pi\lambda_{m_j}}{a_{m_i}^2} [R \psi_{m_i} \psi_{m_j}] \frac{\epsilon_1}{(1 + F_R^2)_{II}} \quad (13c)$$

$$\gamma_{m_{ij}} = \frac{\lambda_{m_j}}{a_{m_i}^2} \left\{ \int_F [1 + F_R^2]^{-3/2} \frac{d\psi_{m_i}}{dR} \frac{d\psi_{m_j}}{dR} dS + \int \frac{F_R F_{RR}}{(1 + F_R^2)} \frac{d\psi_{m_i}}{dR} \psi_{m_j} dS \right\} \quad (13d)$$

$$\Delta_{m_{ij}} + \frac{1}{a_{m_i}^2} \int \frac{\psi_{m_i} \psi_{m_j}}{F \sqrt{1 + F_R^2}} dS \quad (13e)$$

\*The orthogonality property of  $\psi_j$ , thus,  $\psi_{m_j}$  can be easily proved (Ref. 15) as in the high-G case.

$$a_{mi}^2 = \int_F \frac{\psi_{mi}^2 dS}{\sqrt{1 + F_R^2}} \quad (13f)$$

and  $m = 1$  for lateral excitation of a rigid tank.

### Forced Oscillations

Let

$$\phi' = -\omega^2 a \sum_{k=1}^{K_{mx}} d_k \Phi_k \quad ; \quad d_k = \frac{\bar{d}_k \Omega^2}{\Omega_k^2 - \Omega^2} \quad ; \quad \Phi_k = \sum_{j=1}^{J_{mx}} c_{kj} \phi_j \quad (14a, b, c)$$

in order to satisfy the interface condition that

$$\sum_{k=1}^{K_{mx}} d_k (\Omega_k^2 - \Omega^2) \Phi_k = -\Omega^2 \Phi_0 \quad \epsilon_2 N_B \frac{\Delta \rho x}{\rho a} \theta_r \equiv \Phi_N \Omega^2 \quad (15)$$

$\epsilon_2 = 0$  for translational oscillation,  $\epsilon_2 = 1$  for pitching oscillation.

We have by the Galerkin procedure

$$\sum_{k=1}^{K_{mx}} \bar{d}_k \int_F \Phi_k (-H_\ell) dA = \int_F \Phi_N (-H_\ell) dA \quad , \quad \ell = 1 \text{ to } K_{mx} \quad (16)$$

$\bar{d}_k$  can be solved from Equation (16) by matrix inversion. There is no need of storing information of  $\phi_j$  inside the fluid domain as only the force and moment are of interest. It is noted in the limit (Refs. 8 and 9)

$$\int_F \Phi_k (-H_\ell) dA = \delta_{k\ell} \int_F \Phi_k (-H_\ell) dA \quad (17)*$$

\*This will be referred to as biorthogonal relation.

then

$$\bar{d}_l = \int_F \Phi_N(-H_l) d\bar{A} / \int_F \Phi_l(-H_l) d\bar{A} \quad (18)$$

which was utilized in proving that a unique spring-mass system exists for both pitching and translation.

### Force and Moment

The force and moment exerted by a spring mass system (Fig. 2) without damping can be written in the following form (Ref. 18)

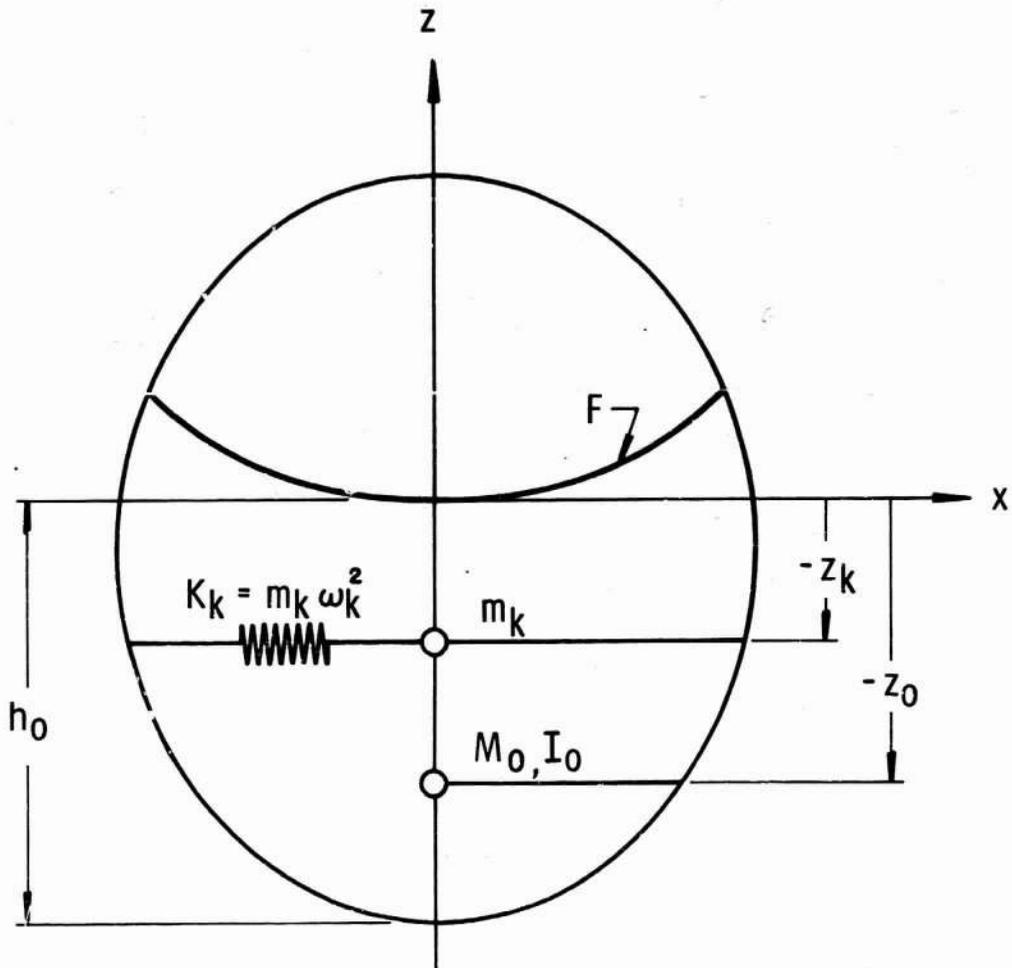
$$F_H = F_x - M_F g \theta_y \quad (19)$$

$$F_{H_T} = x_0 \omega^2 M_F \left\{ 1 + \sum_{k=1}^{\infty} \frac{m_k}{M_F} \frac{1}{\left(\frac{\omega_k^2}{\omega^2} - 1\right)} \right\} \quad (20)$$

$$M_{y_T} = x_0 \omega^2 M_F h_0 \left\{ \frac{z_{C.G.}}{h_0} + \sum_{k=1}^{\infty} \frac{m_k}{M_F} \left( \frac{z_k}{h_0} + \frac{a}{h_0 \omega^2} \right) \frac{1}{\left(\frac{\omega_k^2}{\omega^2} - 1\right)} \right\} \quad (21)$$

$$F_{H_P} = \theta_y \omega^2 M_F h_0 \left\{ \frac{z_{C.G.}}{h_0} + \sum_{k=1}^{\infty} \frac{m_k}{M_F} \left( \frac{z_k}{h_0} + \frac{a}{h_0 \omega^2} \right) \frac{1}{\left(\frac{\omega_k^2}{\omega^2} - 1\right)} \right\} \quad (22)$$

$$M_{y_P} = \theta_y \omega^2 M_F h_0^2 \left\{ \frac{I_F}{M_F h_0^2} + \sum_{k=1}^{\infty} \frac{m_k}{M_F} \left( \frac{z_k}{h_0} + \frac{a}{\omega^2 h_0} \right)^2 \frac{1}{\left(\frac{\omega_k^2}{\omega^2} - 1\right)} \right\} \\ + \theta_y g M_F h_0 \left( \frac{z_{C.G.}}{h_0} \right) \quad (23)$$



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Figure 2. Equivalent Mechanical Model

with rigid mass  $m_0$ , its location  $z_0$ , and moment of inertia  $I_0$  given by

$$\frac{m_0}{M_F} = 1 - \sum_{k=1}^{\infty} \frac{m_k}{M_F} \quad (24)$$

$$\frac{z_0}{h_0} = \frac{1}{\frac{m_0}{M_F}} \left[ \frac{z_{C.G.}}{h_0} - \sum_{k=1}^{\infty} \frac{z_k}{h_0} \frac{m_k}{M_F} \right] \quad (25)$$

$$\frac{I_0}{M_F h_0^2} = \frac{I_F}{M_F h_0^2} - \left( \frac{m_0}{M_F} \frac{z_0^2}{h_0^2} + \sum_{k=1}^{\infty} \frac{m_k}{M_F} \frac{z_k^2}{h_0^2} \right) \quad (26)$$

Since the force due to liquid pressure,  $F_x$ , is

$$F_x = \int_{W_e + F_e} p \frac{\partial x}{\partial n} dS \quad (27)$$

and the moment due to liquid pressure  $M_y$  is

$$M_y = \int_{W_e + F_e} p \left( z \frac{\partial x}{\partial n} - x \frac{\partial z}{\partial n} \right) dS \quad (28)$$

it can be shown that

$$\frac{m_k}{M_F} = d_{kT} f'_k \frac{1}{V} \quad (29)$$

where

$$f'_k = \sum_{j=1}^{\infty} c_{kj} \int_F \lambda_j \psi_j \frac{x}{a} d\bar{S} \quad (29a)$$

$$\bar{d}_{kT} = \frac{1}{\beta_k^2} \int_{W+F} \Phi_k \frac{\partial x}{\partial n} d\bar{S} \cong \frac{1}{\beta_k^2} \sum_{j=1}^{J_{mx}} c_{kj} \lambda_j \int_F \frac{x}{a} \psi_j d\bar{S} \quad (29b)^\dagger$$

$$V = V_L / \rho a^3 \quad , \quad \beta_k^2 = \int_F \Phi_k (-H_k) d\bar{A} \quad (29c, d)$$

and

$$\frac{z_k}{a} = \frac{1}{m_k} \left( \frac{l'_k}{V} \right) \quad (30)$$

where

$$l'_k = \bar{d}_{kT} \sum_{j=1}^{\infty} c_{kj} \mu_j \quad (30a)$$

$$\mu_j = \int_{F+W} \psi_j \left( \frac{z}{a} \frac{\partial x}{\partial n} - \frac{x}{a} \frac{\partial z}{\partial n} \right) d\bar{S} \quad (30b)$$

and that

$$I_F = M_F a^2 I_F^* \quad , \quad I_F^* \cong \frac{1}{V} \int_{W+F} \Phi_p^\circ \left( \frac{z}{a} \frac{\partial r}{\partial n} - \frac{x}{a} \frac{\partial z}{\partial n} \right) d\bar{S} \quad (31a, b)$$

In deriving the mechanical model,  $\rho_u$  has been set to zero. A simple modification can be made for small ullage density by using

$$N_{B_e} = \frac{\Delta \rho}{\rho} N_B$$

<sup>†</sup> For finite  $J_{mx}$ , it was found that  $\bar{d}_{kT}$ , determined by matrix inversion of Equation (16) without using biorthogonal relation, yields results in better agreement with Dodge's theory (Ref. 12) than Equation (29b) which is correct in the limit.

the effective Bond number instead of the Bond number based on the density of the liquid, provided that the dynamic pressure due to ullage motion is negligible.

### Numerical Examples

The computer program has been checked out by the following examples, using the cylindrical tank results given in Reference 12 for comparison purposes.

#### Flat Interface with High Bond Number

$$N_B = 1000 \quad , \quad \frac{h_0}{a} = 2.34 \quad , \quad 12 \times 18 \text{ mesh yielded}$$

$$\frac{\omega_1^2 a}{g} = 1.85 \text{ compared with } 1.847 \text{ from exact theory (Ref. 1, p 415).}$$

$$\frac{m_1}{M_F} = 0.193 \text{ compared with } 0.194 \text{ from high-G theory (Ref. 18).}$$

$$z_1 = -0.729''^* \text{ compared with } -0.724'' \text{ from high-G theory (Ref. 18).}$$

#### Flat Interface with Low Bond Numbers

$$N_B = 10 \quad , \quad \frac{h_0}{a} = 2.34 \quad , \quad 12 \times 18 \text{ mesh yielded}$$

$$\frac{\omega_1^2 a}{g} = 2.15 \text{ compared with } 2.46 \text{ from exact theory.}$$

A finer mesh is required for better agreement.

#### Curved Interface with Low Bond Number and Zero Contact Angle

$$N_B = 100 \quad , \quad \frac{h_0}{a} = 2.34 \quad , \quad 12 \times 18 \text{ mesh}^\dagger \text{ yielded}$$

\*Here, the origin is at the vertex of the meniscus.

†12 net points on the interface and 18 net points on the "side" wall.  
(See Appendix)

$$\frac{\omega_1^2 a}{g} = 1.810 \quad , \quad \frac{m_1}{\rho a^3} = 0.442 \quad , \quad z_1 = -0.734'' \quad (a = 0.68'').$$

compared with theoretical values of  $\frac{\omega_1^2 a}{g} = 1.777$  ,  $\frac{m_1}{\rho a^3} = 0.438$

from Reference 12.

The experimental value of  $\frac{\omega_1^2 a}{g}$  lies between 1.78 to 1.80.

With a  $23 \times 34$  mesh, the present method yielded  $\frac{\omega_1^2 a}{g} = 1.789$  ,

$\frac{m_1}{\rho a^3} = 0.445$  ,  $z_1 = -0.732''$ . For the  $12 \times 18$  mesh, the

CDC-6600 central process time is 2 min, while for the  $23 \times 34$  mesh

it is 21 min. Most of the computing time was expended for the generation of influence coefficients, each of which is a Neumann problem.

However, the influence coefficient method may be more convenient

than the inversion of a large matrix if not faster. No computer

running time was reported in Reference 14, which finds natural modes

by (partial) matrix inversion.

## Conclusion

It seems that the present method yielded a practical way of computing the fundamental natural frequency, the first slosh mass, and its location. Higher masses and locations are usually not needed for design purposes and can be obtained by using finer meshes and longer machine time. A computer program utilizing triangular meshes and Winslow method (Ref. 16) has been successfully employed and is expected to be completed in the near future for the titled problem. However, the present logical diagram may be limited to a convex axisymmetric tank for good accuracy.

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**APPENDIX**

## BRIEF DESCRIPTION OF A COMPUTER PROGRAM

The following steps of a computer program are briefly described:

### Construction of a Triangular Mesh

The triangular mesh is generated as described in Reference 16 except a simple parallelogram is used as the logical diagram (Fig. 3). For a cylindrical tank of Bond number 100, the physical diagram is shown in Figure 4. The lengths of the edge of the parallelogram can be adjusted for each individual case to yield "near" uniform triangular meshes. A continuous wall needs to be broken into two parts for the logical diagram. This only affects the local distribution of the triangular mesh and has shown to yield equally good results for a half full spherical tank at high-G as well as a cylindrical tank.

### Construction of the Auxiliary Characteristic Functions

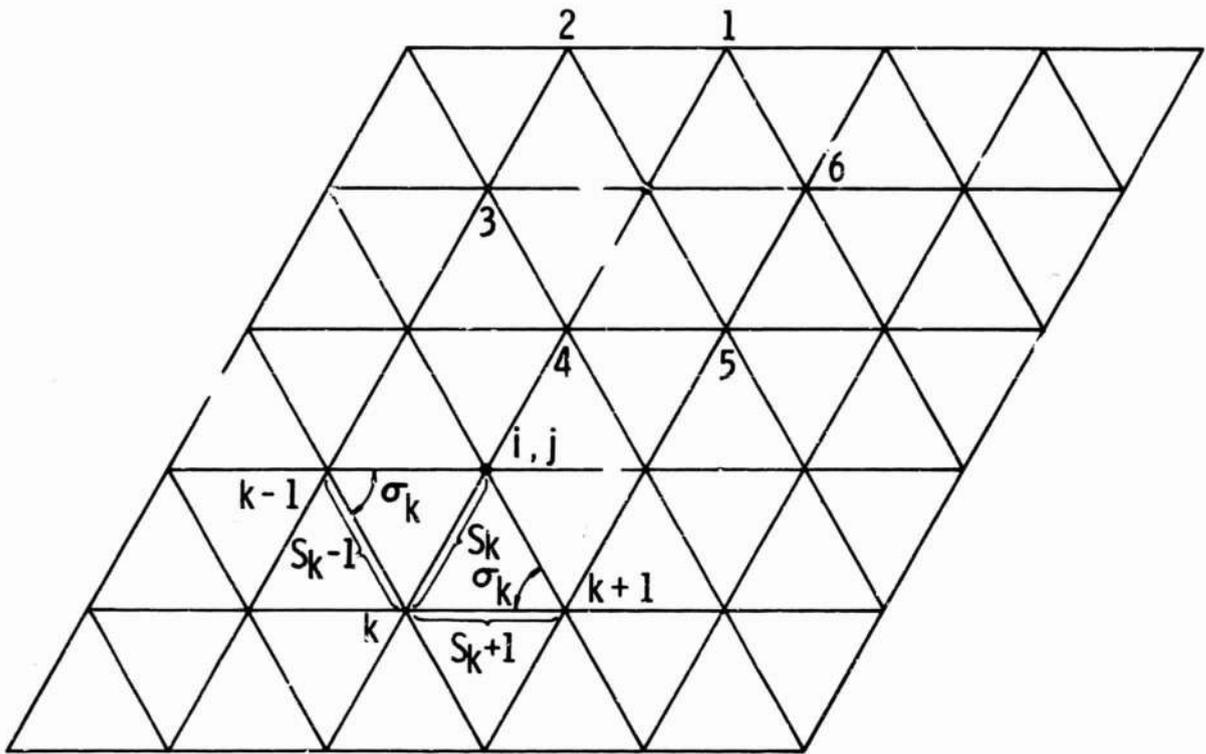
The characteristic functions  $\phi$  satisfy

$$\nabla^2 \phi = 0 \quad (A-1)$$

$$\frac{\partial \phi}{\partial n_0} = 0 \quad \text{on } W \quad (A-2)$$

$$\frac{\partial \phi}{\partial n_0} = \lambda \phi \quad \text{on } F \quad (A-3)$$

$\phi$  can be solved numerically with the constructed triangular mesh by Winslow method (Ref. 16). Contact point is treated as one of the mesh points as are the other boundary points. Hence,  $\frac{\partial \phi}{\partial n}$  may be discontinuous at the contact point. Zero contact angle cannot be constructed graphically but results of



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Figure 3. A Simple Logical Diagram For Triangular Mesh

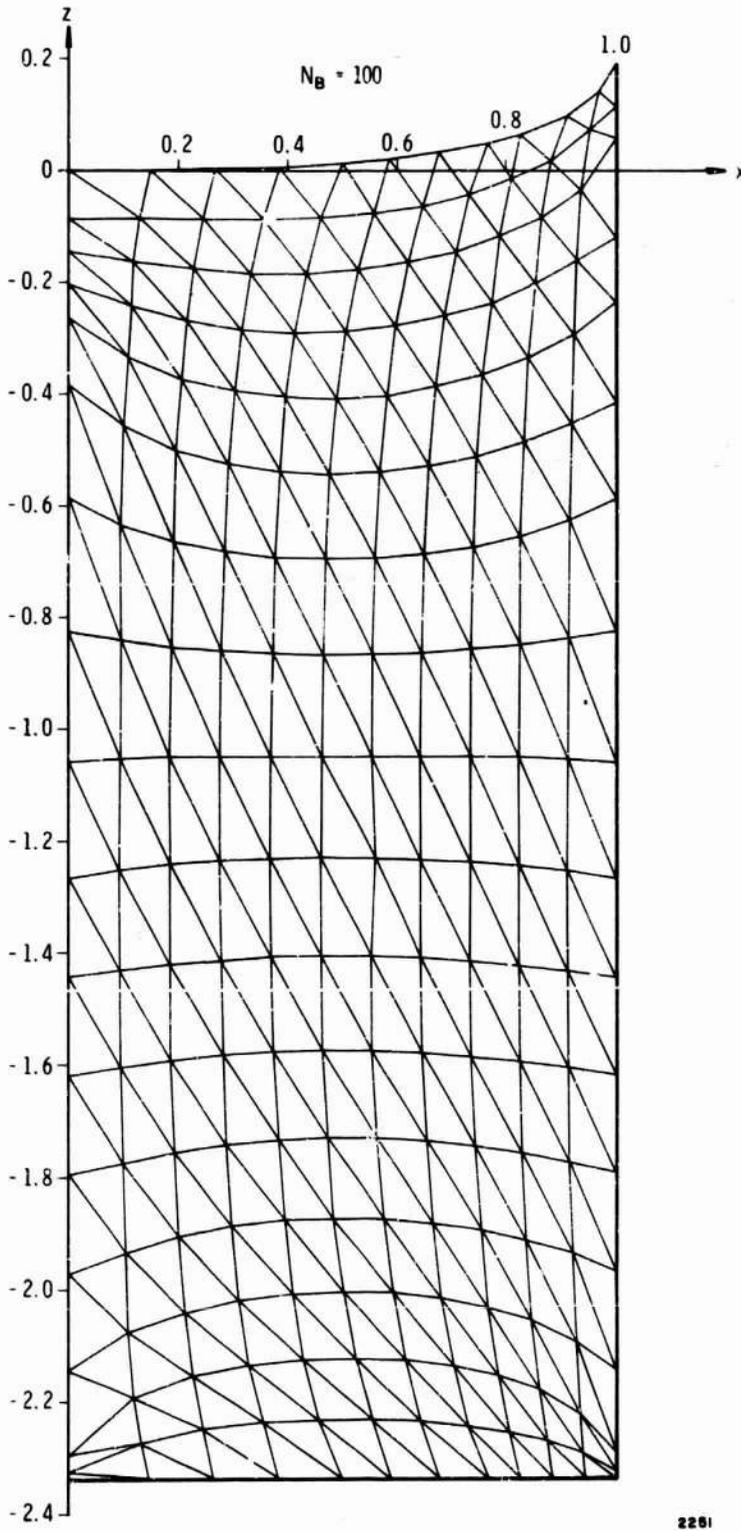


FIGURE 4. A PHYSICAL DIAGRAM OF TRIANGULAR MESH-CYLINDRICAL TANK

decrease mesh size give closer and closer approximations to the interface and would probably lead to the correct limiting value.

For an interior joint,  $ij$ ,  $[\phi = \phi_{ij}, \phi_k = \phi_k(i, j), r_k = r_k(i, j), r = r_{ij}]$

$$\sum_{k=1}^6 \omega_k (\phi_k - \phi) - \frac{m^2}{r_{ij}} A_{ij} \phi = 0 \quad (\text{A-4})$$

where

$A_{ij}$  is the area of the  $ij^{\text{th}}$  dodecagon (see Ref. 16)

$r_{ij}$  is the radius of the  $ij^{\text{th}}$  point

$$\omega_k = \frac{1}{2} (\lambda_k \bar{r}_k \cot \theta_k + \lambda_{k-1} \bar{r}_{k-1} \cot \sigma_k) \quad k = 1 \text{ to } 6 \quad (\text{A-4a})$$

$$\bar{r}_k = \frac{1}{3} (r_{ij} + r_k + r_{k+1}) \quad \lambda_k = 1 \quad (\text{A-4b, c})$$

$\theta_k, \sigma_k$  (see Fig. 3) can be expressed in terms of  $t_k, s_{k+1}, t_{k-1},$

$s_{k-1},$  and  $s_k.$

For interface point,

$$\sum_{k=1}^6 \omega_k (\phi_k - \phi) - \frac{m^2}{r_{ij}} A_{ij} \phi + \left( \frac{\partial \phi}{\partial n} \right)_{1,j} \left[ \frac{1}{2} s_3 + \frac{1}{2} s_6 \right] r_{ij} = 0 \quad (\text{A-5})$$

where

$$\lambda_6 = \lambda_1 = \lambda_2 = 0, \quad \lambda_3 = \lambda_4 = \lambda_5 = 1$$

Note:  $(\lambda_j - 1/2)$  Ref. 16 =  $\lambda_j - 1, (\lambda_j + 1/2)$  Ref. 16 =  $\lambda_j.$

To solve for the eigenfunctions on the interface, we use influence coefficient method in which  $\left(\frac{\partial \phi}{\partial n}\right)_{1,i} = 0$  except  $\left(\frac{\partial \phi}{\partial n}\right)_{1,j} = 1$  for the  $j^{\text{th}}$  column of the influence matrix. A standard eigenvalue problem involving only the interface points, excluding  $\phi_{1,1}$  at  $r = 0$ , is needed to obtain the eigenvalues  $\lambda_j$  and eigenvectors  $\psi_j$ . Knowing the  $j^{\text{th}}$  eigenvector on the intersurface, the corresponding value of  $\psi_j$  on the wall can be easily solved numerically again by the method of over-relaxation.

For  $ij^{\text{th}}$  point on the tank wall

$$\sum_{k=1}^6 \omega_k (\phi_k - \phi) - \frac{m^2}{r_{ij}} A_{ij} \phi = 0 \quad (\text{A-6})$$

$\lambda_3 = \lambda_4 = \lambda_5 = 0$  and  $\lambda_1 = \lambda_2 = \lambda_6 = 1$  on the bottom wall

$\lambda_4 = \lambda_5 = \lambda_6 = 0$  and  $\lambda_1 = \lambda_2 = \lambda_3 = 1$  on the side wall

On centerline,  $r = 0$ ,

$\phi = 0$  for  $m \geq 1$

$$\frac{\partial \phi}{\partial r} = 0 \text{ for } m = 0 \quad (\text{A-7})$$

At contact point  $i = 1, j = j_{\text{mx}}$ .

$$\sum_{k=1}^6 \omega_k (\phi_k - \phi) - \frac{m^2}{r_{ij}} A_{ij} \phi + \left(\frac{\partial \phi}{\partial n}\right)_{1,j_{\text{mx}}} \left(\frac{1}{2} s_3\right) r_{ij} = 0 \quad (\text{A-9})$$

$\lambda_3 = \lambda_4 = 1$  ,  $\lambda_1 = \lambda_2 = \lambda_5 = \lambda_6 = 0$

### Calculation of Natural Frequencies, Slosh Masses, and Their Location

The remaining steps are relatively routine and therefore will not be described, except it is remarked that trapezoidal rule was employed conveniently in evaluating the integrals.