Low-Rank Semidefinite Programming: Theory and Applications

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Contents

Abstract

Finding low-rank solutions of semidefinite programs is important in many applications. For example, semidefinite programs that arise as relaxations of polynomial optimization problems are exact relaxations when the semidefinite program has a rank-1 solution. Unfortunately, computing a minimum-rank solution of a semidefinite program is an NP-hard problem. In this paper we review the theory of low-rank semidefinite programming, presenting theorems that guarantee the existence of a low-rank solution, heuristics for computing low-rank solutions, and algorithms for finding low-rank approximate solutions. Then we present applications of the theory to trust-region problems and signal processing.

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Introduction

1

Introduction

1.1 Low-rank semidefinite programming

A semidefinite program (SDP) is an optimization problem of the form

$$\begin{array}{ll} \text{minimize} & C \bullet X \\ \text{subject to} & A_i \bullet X = b_i, \quad i = 1, \ldots, m \\ & X \succeq 0. \end{array} \tag{SDP}$$

The optimization variable is $X \in \mathbf{S}^n$, where $\mathbf{S}^n$ is the set of all $n \times n$ symmetric matrices, and the problem data are $A_1, \ldots, A_m, C \in \mathbf{S}^n$ and $b \in \mathbf{R}^m$. The trace inner product of $A, B \in \mathbf{R}^{m \times n}$ is

$$A \bullet B = \mathbf{tr}(A^\mathsf{T} B) = \sum_{i=1}^{m} \sum_{j=1}^{n} A_{ij} B_{ij}.$$

The constraint $X \succeq 0$ denotes a generalized inequality with respect to the cone of positive-semidefinite matrices, and means that X is positive semidefinite: that is, $z^\mathsf{T} X z \geq 0$ for all $z \in \mathbf{R}^n$. We can write (SDP) more compactly by defining the operator $\mathcal{A} : \mathbf{S}^n \to \mathbf{R}^m$ such that

$$\mathcal{A}(X) = \begin{bmatrix} A_1 \bullet X \\ \vdots \\ A_m \bullet X \end{bmatrix}.$$

Using this notation we can express (SDP) as

$$\begin{array}{ll} \text{minimize} & C \bullet X \\ \text{subject to} & \mathcal{A}(X) = b \\ & X \succeq 0. \end{array}$$

The dual problem of (SDP) is

$$\begin{array}{ll} \text{maximize} & b^{\mathsf{T}} y \\ \text{subject to} & \sum_{i=1}^m y_i A_i + S = C \\ & S \succeq 0, \end{array} \tag{SDD}$$

where the optimization variables are $y \in \mathbf{R}^m$ and $S \in \mathbf{S}^n$. We can write (SDD) more succinctly as

$$\begin{array}{ll} \text{maximize} & b^{\mathsf{T}} y \\ \text{subject to} & \mathcal{A}^*(y) + S = C \\ & S \succeq 0, \end{array}$$

where the adjoint operator $\mathcal{A}^* : \mathbf{R}^m \to \mathbf{S}^n$ is given by

$$\mathcal{A}^*(y) = \sum_{i=1}^m y_i A_i.$$

We do not attempt to give a general exposition of the theory of semidefinite programming in this paper – an excellent survey is provided by Vandenberghe and Boyd [96]. The preceding remarks are only meant to establish our particular conventions for talking about SDPs. Additional results about SDPs are given in Appendix A, which presents those aspects of the theory that are most relevant for our purposes.

Semidefinite programs can be solved efficiently using interior-point algorithms. However, such algorithms typically converge to a maximum-rank solution [45], and in many cases we are interested in finding a low-rank solution. For example, it is well known that every polynomial optimization problem has a natural SDP relaxation, and this relaxation is exact when it has a rank-1 solution. (We include the derivation of this important result in Appendix A for completeness.) Unfortunately, finding a minimum-rank solution of an SDP is NP-hard: a special case of this problem is finding a minimum-cardinality solution of a system of linear equations, which is known to be NP-hard [36]. In this paper we review approaches to finding low-rank solutions and approximate solutions of SDPs, and present some applications in which low-rank solutions are important.

1.2 Outline

Chapter 2 discusses reduced-rank exact solutions of SDPs and theorems about rank. We give an efficient algorithm for reducing the rank of a solution. Although the algorithm may not find a minimum-rank solution, it often works well in practice, and we can prove a bound on the rank of the solution returned by the algorithm. Then we give a theorem relating the uniqueness of the rank of a solution to the uniqueness of the solution itself, and show how to use this theorem for sensor-network localization. The chapter concludes with a theorem that allows us to deduce the existence of a low-rank solution from the sparsity structure of the coefficients.

Because finding a minimum-rank solution of an SDP is NP-hard, we do not expect to arrive at an algorithm that accomplishes this task in general. However, there are many heuristics for finding low-rank solutions that often perform well in practice; we discuss these methods in Chapter 3. We also present rounding methods, in which we find a low-rank approximate solution that is close to a given exact solution in some sense. One of the rounding methods that we discuss is the famous Goemans-Williamson algorithm [39]; if the unique-games conjecture is true, then this algorithm achieves the best possible approximation ratio for the maximum-cut problem [57, 58].

The paper concludes with two chapters covering applications of the theoretical results to trust-region problems and signal processing. There are three appendices: the first gives background information, and establishes our notation; the second reviews some classical results about linear programming that we generalize to semidefinite programming in Chapter 2; and the last contains technical probability lemmas that are used in our analysis of rounding methods.

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