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LOW REYNOLDS NUMBER FLOW IN A HEATED TUBE OF VARYING SECTION

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Abstract

In this paper we study the effect of forced and free convection heat transfer on flow in an axisymmetric tube whose radius varies slowly in the axial direction. Asymptotic series expansions in terms of a small parameter ε, which is a measure of the radius variation, are obtained for the velocity components, pressure and temperature on the assumption that the Reynolds number (R) is of order one. The effect of the free convection parameter or Grashof number (G) on the axial velocity, temperature distribution, shear stress and heat flux at the wall are discussed quantitatively for a locally constricted tube.

1. Introduction

The determination of the flow through a tube of varying section is a fundamental one with obvious applications in physiology and engineering. One of the initiators of the study, Manton [5], considered the steady flow in axisymmetric tubes of varying section. He obtained an asymptotic series expansion in terms of a small parameter ε for the velocity, pressure and shear stress and found that his solutions compared favourably with the numerical results of Lee and Fung [4] even for values of ε as large as 2. Hall [3] considered the unsteady flow in a slowly varying tube of small eccentricity when a pulsatile pressure difference is applied across the ends.

Recently the problem of heat transfer has been gaining importance owing to its wide application in a variety of physiological flow situations, both natural and artificial (see for example Radhakrishnamacharya and Maiti [6]). Also, from the technological standpoint, the possibility of sodium-water fires in the secondary

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heat exchangers of liquid-metal fast-breeder reactors has necessitated further study of problems in pipe flows with heat transfer. The object of this paper is to consider the combined effect of forced and free convention heat transfer to steady flow in axisymmetric tubes of slowly varying radius. The procedure adopted below is as follows.

In Section 2 the governing equations are derived. The approximate solutions to these equations are presented in Section 3. Using these solutions the expressions for the wall shear stresses and heat flux are obtained in Sections 4 and 5 respectively. In Section 6 quantitative discussion is presented for the axial velocity, temperature, shear stress and heat flux in a locally constricted tube.

2. Formulation of the problem

We consider flow in cylindrical polar coordinates (r', φ, z') with corresponding velocity components (u', v', w') such that $r' = 0$ is the axis of symmetry of the tube (Figure 1). The tube wall defined as

$$r' = a(z'; \varepsilon) = a_0 s(\varepsilon z'/a_0) \tag{2.1}$$

is maintained at a constant temperature T_w. Here ε is small and a_0 a suitable constant. The equations of continuity, momentum and energy are

$$\frac{1}{r'}\frac{\partial(ru')}{\partial r'} + \frac{1}{r'}\frac{\partial v'}{\partial \varphi} + \frac{\partial w'}{\partial z'} = 0, \tag{2.2a}$$

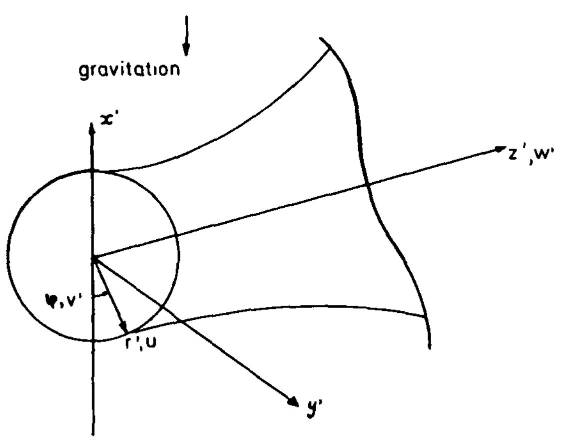

Figure 1. Physical model.

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$$\rho_\infty\left(u'\frac{\partial u'}{\partial r'}+\frac{v'}{r'}\frac{\partial u'}{\partial\varphi}-\frac{v'^2}{r'}+w'\frac{\partial u'}{\partial z'}\right)$$
$$=-\frac{\partial p'}{\partial r'}+\mu\left(\nabla'^2u'-\frac{u'}{r'^2}-\frac{2}{r'^2}\frac{\partial v'}{\partial\varphi}+\frac{\partial^2u'}{\partial z'^2}\right)+\rho g\cos\varphi, \tag{2.2b}$$

$$\rho_\infty\left(u'\frac{\partial v'}{\partial r}+\frac{v'}{r'}\frac{\partial v'}{\partial\varphi}+\frac{u'v'}{r'}+w'\frac{\partial v'}{\partial z'}\right)$$
$$=-\frac{1}{r'}\frac{\partial p'}{\partial\varphi}+\mu\left(\nabla'^2v'-\frac{v'}{r'^2}+\frac{2}{r'^2}\frac{\partial u'}{\partial\varphi}+\frac{\partial^2v'}{\partial z'^2}\right)-\rho g\sin\varphi, \tag{2.2c}$$

$$\rho_\infty\left(u'\frac{\partial w'}{\partial r'}+\frac{v'}{r'}\frac{\partial w'}{\partial\varphi}+w'\frac{\partial w'}{\partial z'}\right)=-\frac{\partial p'}{\partial z'}+\mu\left(\nabla'^2w'+\frac{\partial^2w'}{\partial z'^2}\right), \tag{2.2d}$$

$$\rho_\infty C_p\left(u'\frac{\partial T'}{\partial r'}+\frac{v'}{r'}\frac{\partial T'}{\partial\varphi}+w'\frac{\partial T'}{\partial z'}\right)=k\left(\nabla'^2T+\frac{\partial^2T'}{\partial z'^2}\right)+Q, \tag{2.2e}$$

in which

$$\nabla'^2=\frac{\partial^2}{\partial r'^2}+\frac{1}{r'}\frac{\partial}{\partial r'}+\frac{1}{r'^2}\frac{\partial^2}{\partial\varphi^2},$$

and p' is the pressure, T' the temperature, k the thermal conductivity, Q the constant heat source/sink term. The undisturbed fluid density is denoted by ρ_∞, C_p is the heat capacity while μ is the molecular viscosity and $(\rho g\cos\varphi, -\rho g\sin\varphi, 0)$ the buoyancy force terms.

In (2.2) we have assumed that the density of the fluid is constant and equal to its value in the undisturbed fluid except in the buoyancy force terms, that is, we have made the Boussinesq approximation. Also in the energy equation (2.2e) viscous dissipation is neglected since the low Reynolds number assumption necessarily entails low flow velocities. The heat source/sink term in most physiological flows through blood vessels, which is the primary concern of this investigation, stems primarily from depletion of white blood cells and consequent rise in body temperature during infection.

The boundary conditions are

$$\left.\begin{aligned}&u'=0=v'=w',\quad T'=T_w\quad\text{on } r'=a(z'),\\&\int_0^{a(z')}dr'\int_0^{2\pi}r'w'\,d\varphi=2\pi\Psi_0,\\&u',v',w',T'<\infty\quad\text{on } r'=0.\end{aligned}\right\} \tag{2.3}$$

Now in the undisturbed fluid

$$0 = -\frac{\partial p'_\infty}{\partial r'} + \rho_\infty g \cos\varphi, \qquad 0 = -\frac{1}{r'}\frac{\partial p'_\infty}{\partial \varphi} - \rho_\infty g \sin\varphi, \tag{2.4}$$

where p'_∞ is the undisturbed fluid pressure. Also the equation of state for a Boussinesq fluid is

$$\rho_\infty - \rho = \rho_\infty \beta (T' - T_\infty), \tag{2.5}$$

β being the coefficient of volume expansion. To facilitate analysis we introduce the following non-dimensional quantities

$$r = \frac{r'}{a_0}, \quad z = \frac{\varepsilon z'}{a_0}, \quad (u, v, w) = \frac{1}{\varepsilon}(u', v', \varepsilon w')\frac{a_0^2}{\Psi_0}, \quad p = \frac{(p' - p'_\infty)a_0^3}{\mu \Psi_0},$$
$$\Theta = \frac{T' - T_\infty}{T_w - T_\infty}, \quad \alpha = \frac{Q a_0^2}{k(T_w - T_\infty)},$$
$$R = \frac{\Psi_0}{a_0 \nu}, \quad P = \frac{\mu C_p}{k}, \quad G = \frac{g\beta a_0^4 (T_w - T_\infty)}{\varepsilon \nu \Psi_0}. \tag{2.6}$$

Combining (2.2), (2.4), (2.5) and (2.6) we get

$$\frac{1}{r}\frac{\partial}{\partial r}(ru) + \frac{1}{r}\frac{\partial v}{\partial \varphi} + \frac{\partial w}{\partial z} = 0, \tag{2.7a}$$

$$R\varepsilon^2\left(u\frac{\partial u}{\partial r} + \frac{v}{r}\frac{\partial u}{\partial \varphi} - \frac{v^2}{r} + w\frac{\partial u}{\partial z}\right) = -\frac{\partial p}{\partial r} + \varepsilon\left\{\nabla^2 - \frac{1}{r^2}\right\}u - \frac{2\varepsilon}{r}\frac{\partial v}{\partial \varphi} + \varepsilon^3\frac{\partial^2 u}{\partial z^2} - G\varepsilon\Theta\cos\varphi, \tag{2.7b}$$

$$R\varepsilon^2\left(u\frac{\partial v}{\partial r} + \frac{v}{r}\frac{\partial v}{\partial \varphi} + \frac{uv}{r} + w\frac{\partial v}{\partial z}\right) = -\frac{1}{r}\frac{\partial p}{\partial \varphi} + \varepsilon\left\{\nabla^2 - \frac{1}{r^2}\right\}v + \frac{2\varepsilon}{r}\frac{\partial u}{\partial \varphi} + \varepsilon^3\frac{\partial^2 v}{\partial z^2} + G\varepsilon\Theta\sin\varphi, \tag{2.7c}$$

$$R\varepsilon\left(u\frac{\partial w}{\partial r} + \frac{v}{r}\frac{\partial w}{\partial \varphi} + w\frac{\partial w}{\partial z}\right) = -\varepsilon\frac{\partial p}{\partial z} + \nabla^2 w + \varepsilon^2\frac{\partial^2 w}{\partial z^2}, \tag{2.7d}$$

$$PR\varepsilon\left(u\frac{\partial \Theta}{\partial r} + \frac{v}{r}\frac{\partial \Theta}{\partial \varphi} + w\frac{\partial \Theta}{\partial z}\right) = \nabla^2\Theta + \varepsilon^2\frac{\partial^2 \Theta}{\partial z^2} + \alpha, \tag{2.7e}$$

subject to the boundary conditions

$$\begin{aligned} &u = 0 = v = w, \quad \Theta = 1 && \text{on } r = s(z), \\ &u, v, w, \Theta < \infty && \text{on } r = 0. \end{aligned} \tag{2.8}$$

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In these equations R is the Reynolds number, α the non-dimensional heat source/sink term, P the Prandtl number and G the Grashof number or free convection parameter.

If the pressure gradients are eliminated from equations (2.7b, c) we get

$$R\varepsilon\left[\frac{1}{r}\frac{\partial}{\partial r}\left\{r\left(u\frac{\partial v}{\partial r}+\frac{v}{r}\frac{\partial v}{\partial \varphi}+\frac{uv}{r}+w\frac{\partial v}{\partial z}\right)\right\}\right.$$
$$\left.-\frac{1}{r}\frac{\partial}{\partial \varphi}\left(u\frac{\partial u}{\partial r}+\frac{v}{r}\frac{\partial u}{\partial \varphi}-\frac{v^2}{r}+w\frac{\partial u}{\partial z}\right)\right]$$
$$=\nabla^2\left\{\frac{1}{r}\frac{\partial}{\partial r}(rv)-\frac{1}{r}\frac{\partial u}{\partial \varphi}\right\}+G\left\{\frac{\partial \Theta}{\partial r}\sin\varphi+\frac{1}{r}\frac{\partial \Theta}{\partial \varphi}\cos\varphi\right\}$$
$$+\varepsilon^2\frac{1}{r^2}\frac{\partial^2}{\partial z^2}\left\{\frac{\partial}{\partial r}(rv)-\frac{\partial u}{\partial \varphi}\right\}. \tag{2.9}$$

Equation (2.9) will be found useful in subsequent analysis.

3. Approximate solutions

The mathematical statement of the problem is to solve (2.7) subject to conditions (2.8). This problem is nonlinear and coupled and not readily amenable to closed form analytical treatment. We are interested in low Reynolds number situations such that $\varepsilon R = o(1)$ rather than $\varepsilon R = O(1)$ and larger. We therefore adopt an asymptotic analysis similar to that in Bestman [1] by expanding the velocity components and temperature in the form

$$u = u^{(0)} + \varepsilon u^{(1)} + \varepsilon^2 u^{(2)} + \cdots, \quad etc. \tag{3.1a}$$

while the pressure is expanded as

$$p = \frac{1}{\varepsilon}p^{(0)} + p^{(1)} + \varepsilon p^{(2)} + \cdots. \tag{3.1b}$$

In adopting a hybrid finite-element finite-difference technique in an allied peristaltic problem, Bestman [2] discovered that approximate solutions in the form (3.1) agree well with numerical results when the Reynolds number is fairly low.

3.1 Solutions for the leading expansion

When (3.1) is substituted in (2.7) and (2.8) we find that the leading terms satisfy the equations

$$\frac{1}{r}\frac{\partial}{\partial r}(ru^{(0)}) + \frac{1}{r}\frac{\partial v^{(0)}}{\partial \varphi} + \frac{\partial w^{(0)}}{\partial z} = 0, \tag{3.2a}$$

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$$\frac{\partial p^{(0)}}{\partial r} = 0 = \frac{1}{r}\frac{\partial p^{(0)}}{\partial \varphi}, \tag{3.2b}$$

$$0 = -\frac{\partial p^{(0)}}{\partial z} + \nabla^2 w^{(0)}, \qquad 0 = \nabla^2 \Theta^{(0)} + \alpha, \tag{3.2c}$$

subject to the conditions

$$u^{(0)} = 0 = v^{(0)} = w^{(0)}, \tag{3.3a}$$

$$\Theta^{(0)} = 1 \quad \text{on } r = s(z), \tag{3.3b}$$

$$u^{(0)} = v^{(0)} = w^{(0)} = \Theta^{(0)} < \infty \quad \text{on } r = 0. \tag{3.3c}$$

We see that $p^{(0)}$ is independent of r and φ and we take solutions to (3.2c), which are regular at the origin, in the form

$$\Theta^{(0)} = A^{(0)}(z) - \tfrac{1}{4}\alpha r^2, \tag{3.4a}$$

$$w^{(0)} = B^{(0)}(z) + \tfrac{1}{4}p^{(0)\prime} r^2 \tag{3.4b}$$

where from now on in the rest of this section, a dash over a symbol or subscript z will be used to denote differentiation with respect to z. In (3.4) $A^{(0)}(z)$ and $B^{(0)}(z)$ are arbitrary functions of z.

Next we substitute (3.4b) into (3.2a) and we get

$$\frac{1}{r}\frac{\partial}{\partial r}(r u^{(0)}) + \frac{1}{r}\frac{\partial v^{(0)}}{\partial \varphi} = -B^{(0)\prime} - \tfrac{1}{4}p^{(0)\prime\prime} r^2. \tag{3.5a}$$

To obtain another equation linking $u^{(0)}$ and $v^{(0)}$ we substitute expansion (3.1a) into (2.9), the leading term of which is

$$\nabla^2\left\{\frac{1}{r}\frac{\partial}{\partial r}(r v^{(0)}) - \frac{1}{r}\frac{\partial u^{(0)}}{\partial \varphi}\right\} + G\left\{\frac{\partial \Theta^{(0)}}{\partial r}\sin\varphi + \frac{1}{r}\frac{\partial \Theta^{(0)}}{\partial \varphi}\cos\varphi\right\} = 0. \tag{3.5b}$$

Putting

$$u^{(0)} = f^{(0)}(r,z)\cos\varphi - \tfrac{1}{2}B^{(0)\prime} r - \tfrac{1}{16}p^{(0)\prime\prime} r^3, \quad v^{(0)} = g^{(0)}(r,z)\sin\varphi, \tag{3.6}$$

and substituting (3.4a) and (3.6) in (3.5b), then integrating the resulting equation we have

$$\frac{1}{r}\frac{\partial}{\partial r}(r g^{(0)}) + \frac{1}{r} f^{(0)} = \frac{1}{2}G\alpha\left(C^{(0)}(z) r + \frac{1}{8}r^3\right) \tag{3.7a}$$

while (3.5a) becomes

$$\frac{1}{r}\frac{\partial}{\partial r}(r f^{(0)}) + \frac{1}{r} g^{(0)} = 0. \tag{3.7b}$$

Solving (3.7a, b) simultaneously, the final results are

$$u^{(0)} = -\tfrac{1}{2}B^{(0)\prime} r - \tfrac{1}{16}p^{(0)\prime\prime} r^3 + \tfrac{1}{384}G\alpha(D^{(0)} + C^{(0)} r^2 - r^4)\cos\varphi, \tag{3.8a}$$

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$$v^{(0)} = -\tfrac{1}{384}G\alpha(D^{(0)} + 36^{(0)}r^2 - 5r^4). \tag{3.8b}$$

Employing the boundary conditions on $\Theta^{(0)}$ and $w^{(0)}$ we find that

$$A^{(0)} = 1 + \tfrac{1}{4}\alpha s^2, \tag{3.9a}$$

$$B^{(0)} = -\tfrac{1}{4}p^{(0)\prime}s^2, \tag{3.9b}$$

while the boundary conditions on $u^{(0)}$ and $v^{(0)}$ give

$$p^{(0)\prime\prime} + 4(s'/s)p^{(0)\prime} = 0,$$

which is reminiscent of the Reynolds equation for pressure in lubrication theory, and

$$D^{(0)} + G^{(0)}s^2 = s^4, \qquad D^{(0)} + 3C^{(0)}s^2 = 5s^4.$$

We may integrate the Reynolds equation and, choosing an arbitrary constant appropriately, we get

$$p^{(0)} = -\int^z (16/s^4)\,dz, \qquad B^{(0)} = 4/s^2,$$

so that the first term in the expansion for the pressure equation (3.9b) corresponds to that of Manton [5], while

$$C^{(0)} = 2s^2, \qquad D^{(0)} = -s^4. \tag{3.10}$$

The solutions are now complete. For reference purposes we now collect them together:

$$\begin{aligned}
&\Theta^{(0)} = 1 + \tfrac{1}{4}\alpha s^2\{1 - (r/s)^2\}, \quad w^{(0)} = (4/s^2)\{1 - (r/s)^2\},\\
&p^{(0)} = -\int^z (16/s^4)\,dz,\\
&u^{(0)} = 4(s_z/s^2)\{r/s - (r/s)^3\} - \tfrac{1}{384}G\alpha s^4\{1 - (r/s)^2\}\cos\varphi,\\
&v^{(0)} = \tfrac{1}{384}G\alpha s^4[1 - (r/s)^2][1 - 5(r/s)^2]\sin\varphi.
\end{aligned} \tag{3.11}$$

We observe that the axial velocity and pressure correspond to those of Manton [5]. The radial velocity is modified by the appearance of additional free convection terms while the presence of an azimuthal velocity is wholly accounted for by free convection currents. The buoyancy contribution to the total flow is a double longitudinal roll system with the centre line of the pipe the boundary between the rolls.

3.2 Higher approximations

The equations governing the next approximation are

$$\frac{1}{r}\frac{\partial}{\partial r}(ru^{(1)}) + \frac{1}{r}\frac{\partial v^{(1)}}{\partial \varphi} + \frac{\partial w^{(1)}}{\partial z} = 0, \tag{3.12a}$$

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$$\frac{\partial p^{(1)}}{\partial r} = 0 = \frac{1}{r}\frac{\partial p^{(1)}}{\partial \varphi}, \tag{3.12b}$$

$$R\left(u^{(0)}\frac{\partial w^{(0)}}{\partial r} + \frac{v^{(0)}}{r}\frac{\partial w^{(0)}}{\partial \varphi} + w^{(0)}\frac{\partial w^{(0)}}{\partial z}\right) = -\frac{\partial p^{(1)}}{\partial z} + \nabla^2 w^{(1)}, \tag{3.12c}$$

$$RP\left(u^{(0)}\frac{\partial \Theta^{(0)}}{\partial r} + \frac{v^{(0)}}{r}\frac{\partial \Theta^{(0)}}{\partial \varphi} + w^{(0)}\frac{\partial \Theta^{(0)}}{\partial z}\right) = \nabla^2 \Theta^{(1)}, \tag{3.12d}$$

subject to the conditions

$$\begin{aligned} &u^{(1)} = 0 = v^{(1)} = w^{(1)}, \quad \Theta^{(1)} = 0 && \text{on } r = s(z), \\ &u^{(1)}, v^{(1)}, w^{(1)}, \Theta^{(1)} < \infty && \text{on } r = 0. \end{aligned} \tag{3.13}$$

Also, an equation corresponding to (3.5b) is

$$\begin{aligned} &R\left[\frac{1}{r}\frac{\partial}{\partial r}\left\{r\left(u^{(0)}\frac{\partial v^{(0)}}{\partial r} + \frac{v^{(0)}}{r}\frac{\partial v^{(0)}}{\partial \varphi} + \frac{u^{(0)}v^{(0)}}{r} + w^{(0)}\frac{\partial v^{(0)}}{\partial z}\right)\right\}\right. \\ &\qquad\left. - \frac{1}{r}\frac{\partial}{\partial \varphi}\left(u^{(0)}\frac{\partial u^{(0)}}{\partial r} + \frac{v^{(0)}}{r}\frac{\partial u^{(0)}}{\partial \varphi} - \frac{v^{(0)2}}{r} + w^{(0)}\frac{\partial u^{(0)}}{\partial z}\right)\right] \\ &= \nabla^2\left\{\frac{1}{r}\frac{\partial}{\partial r}(rv^{(0)}) - \frac{1}{r}\frac{\partial u^{(1)}}{\partial \varphi}\right\} + G\left\{\frac{\partial \Theta^{(1)}}{\partial r}\sin\varphi + \frac{1}{r}\frac{\partial \Theta^{(1)}}{\partial \varphi}\cos\varphi\right\}. \end{aligned} \tag{3.14}$$

Again we find that $p^{(1)}$ is independent of r and φ. If we substitute (3.11) in (3.12c, d) we get

$$\begin{aligned} \nabla^2 w^{(1)} = p^{(1)\prime} &- 32R(s_z/s^5)\{1 - 2(r/s)^2 + (r/s)^4\} \\ &+ \tfrac{1}{48}GR\alpha s\{r/s - 2(r/s)^3 + (r/s)^5\}\cos\varphi, \end{aligned}$$

and

$$\begin{aligned} \nabla^2 \Theta^{(1)} = 2PR\alpha(s_z/s)&\{1 - 2(r/s)^2 + (r/s)^4\} \\ &+ \tfrac{1}{768}GPR\alpha^2 s^5\{r/s - 2(r/s)^3 + (r/s)^5\}\cos\varphi. \end{aligned}$$

If we integrate these, then on employing (3.13) we find that

$$\begin{aligned} \Theta^{(1)} = &-\tfrac{1}{36}PR\alpha s s_z\{11 - 18(r/s)^2 + 9(r/s)^4 - 2(r/s)^6\} \\ &- \tfrac{1}{4\times 96^2}GPR\alpha^2 s^7\{3 - 6(r/s)^3 + 4(r/s)^5 - (r/s)^7\}\cos\varphi, \end{aligned} \tag{3.15}$$

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and

$$w^{(1)} = -\tfrac{1}{4}p^{(1)\prime}s^2\{1-(r/s)^2\}$$
$$+Rs^2(1/s^4)_z\{2(r/s)^2-(r/s)^4+\tfrac{2}{9}(r/s)^6-\tfrac{11}{9}\}$$
$$-\tfrac{1}{48^2}GR\alpha s^3\{3-6(r/s)^3+4(r/s)^5-(r/s)^7\}\cos\varphi. \qquad (3.16)$$

Next we calculate $u^{(1)}$ and $v^{(1)}$ by substituting (3.16) into (3.12a) and (3.11) and (3.15) in (3.14). Then writing

$$u^{(1)} = f_0^{(1)}(r,z) + f_1^{(1)}(r,z)\cos\varphi + f_2^{(1)}(r,z)\cos 2\varphi, \qquad (3.17a)$$
$$v^{(1)} = g_1^{(1)}(r,z)\sin\varphi + g_2^{(1)}(r,z)\sin 2\varphi, \qquad (3.17b)$$

we get

$$\frac{1}{r}\frac{\partial}{\partial r}(rf_1^{(1)}) = \tfrac{1}{4}p^{(1)\prime\prime}s^2\{1-(r/s)^2\} + \tfrac{1}{2}p^{(1)\prime}ss_z$$
$$-Rs^2(1/s^4)_{zz}\{2(r/s)^2-(r/s)^4+\tfrac{2}{9}(r/s)^6-\tfrac{11}{9}\}$$
$$+\tfrac{1}{2}Rs^6[(1/s^4)_z]^3\{(r/s)^4-\tfrac{4}{9}(r/s)^6-\tfrac{11}{9}\}, \qquad (3.18)$$

$$\frac{1}{r}\frac{\partial}{\partial r}(rf_1^{(1)}) + \frac{1}{r}g_1^{(1)} = \frac{1}{48^2}GR\alpha s^2 s_z\{9-8(r/s)^5+4(r/s)^7\}, \qquad (3.19a)$$

$$\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}-\frac{1}{r^2}\right)\left\{\frac{1}{r}\frac{\partial}{\partial r}(rg_1^{(1)}) + \frac{1}{r}f_1^{(1)}\right\}$$
$$= -\tfrac{1}{12}GR\alpha s_z\{11(r/s)-32(r/s)^3+21(r/s)^5\}$$
$$-\tfrac{1}{3}GPR\alpha s_z\{3(r/s)-3(r/s)^3+(r/s)^5\}, \qquad (3.19b)$$

and

$$\frac{1}{r}\frac{\partial}{\partial r}(rf_2^{(1)}) + \frac{2}{r}g_2^{(1)} = 0, \qquad (3.20a)$$

$$\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}-\frac{4}{r^2}\right)\left\{\frac{1}{r}\frac{\partial}{\partial r}(rg_2^{(1)}) + \frac{2}{r}f_2^{(1)}\right\}$$
$$= \frac{1}{2}\left(\frac{G\alpha}{96}\right)^2 Rs^6\{(r/s)^2-4(r/s)^4+3(r/s)^6\}$$
$$-2\left(\frac{G\alpha}{384}\right)^2 PRs^6\{3s/r+12(r/s)^2-16(r/s)^4+6(r/s)^6\}. \qquad (3.20b)$$

First we integrate (3.18) to give

$$f_0^{(1)} = \tfrac{1}{4}p^{(1)\prime\prime}s^3\{\tfrac{1}{2}(r/s) - \tfrac{1}{4}(r/s)^3\} + \tfrac{1}{4}p^{(1)\prime}s^2s_z(r/s)$$
$$-Rs^3(1/s^4)_{zz}\{\tfrac{1}{2}(r/s)^3 - \tfrac{1}{6}(r/s)^5 + \tfrac{1}{36}(r/s)^7 - \tfrac{11}{18}(r/s)\}$$
$$+\tfrac{1}{2}Rs^7[(1/s^4)_z]^2\{\tfrac{1}{6}(r/s)^5 - \tfrac{1}{18}(r/s)^7 - \tfrac{11}{18}(r/s)\}.$$

Invoking the boundary condition $f_0^{(1)}(s, z) = 0$, we obtain the Reynolds equation for the pressure

$$\tfrac{1}{16}p^{(1)\prime\prime}s^3 + \tfrac{1}{4}p^{(1)\prime}s^2s_z + \tfrac{1}{4}Rs^3(1/s^4)_{zz} - \tfrac{1}{4}Rs^7[(1/s^4)_z]^2 = 0,$$

which can be put in the form

$$\frac{d}{dz}\{s^4p^{(1)\prime}\} = 4(s_z/s)_z.$$

We can now integrate this twice with the eventual result

$$p^{(1)} = -4R/s^4. \tag{3.21}$$

So that to terms of order ε, the free convection currents have no effect on the pressure distribution. Finally,

$$f_0^{(1)} = -Rs^3(1/s^4)_{zz}\{\tfrac{1}{2}(r/s)^3 - \tfrac{1}{6}(r/s)^5 + \tfrac{1}{36}(r/s)^7 - \tfrac{11}{18}(r/s)\}$$
$$+\tfrac{1}{2}Rs^7[(1/s^4)_z]^2\{\tfrac{1}{6}(r/s)^5 - \tfrac{1}{8}(r/s)^7 - \tfrac{11}{18}(r/s)\}, \tag{3.22}$$

and this result can be obtained if equation (4.7) in [5] is expressed in terms of velocity components.

Second, employing (3.21), (3.16) can now be written as

$$w^{(1)} = Rs^2(1/s^4)_z\{\tfrac{2}{9}(r/s)^6 - (r/s)^4 + (r/s)^2 - \tfrac{2}{9}\}$$
$$-\tfrac{1}{48^2}GR\alpha s^3\{3 - 6(r/s)^3 + 4(r/s)^5 - (r/s)^7\}\cos\varphi. \tag{3.23}$$

And third, we integrate (3.19) and (3.20) as for $u^{(0)}$ and $v^{(0)}$. The results after invoking the homogeneous boundary conditions at the tube wall, are

$$f_1^{(1)} = \tfrac{1}{576}GR\alpha s^3s_z\{-\tfrac{3}{4} + \tfrac{3}{2}(r/s) - \tfrac{9}{4}(r/s)^2 + \tfrac{11}{4}(r/s)^4 - \tfrac{39}{24}(r/s)^6 + \tfrac{3}{8}(r/s)^8\}$$
$$+\tfrac{1}{144}GPR\alpha s^3s_z\{\tfrac{43}{80} - \tfrac{44}{40}(r/s)^2 + \tfrac{3}{4}(r/s)^4 - \tfrac{1}{8}(r/s)^6 + \tfrac{1}{80}(r/s)^8\},$$
$$g_1^{(1)} = -\tfrac{1}{576}GR\alpha s^3s_z\{-\tfrac{3}{4} + \tfrac{3}{4}(r/s) - \tfrac{27}{4}(r/s)^2$$
$$+\tfrac{55}{4}(r/s)^4 - \tfrac{225}{24}(r/s)^6 + \tfrac{19}{8}(r/s)^8\}$$
$$-\tfrac{1}{144}GPR\alpha s^3s_z\{\tfrac{43}{80} - \tfrac{141}{40}(r/s)^2 + \tfrac{15}{4}(r/s)^4 - \tfrac{7}{8}(r/s)^7 + \tfrac{9}{80}(r/s)^8\}, \tag{3.24}$$

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and

$$f_2^{(1)} = -\left(\frac{G\alpha}{96}\right)^2 Rs^9\{-\tfrac{1}{960}(r/s)^3 + \tfrac{1}{384}(r/s)^5 - \tfrac{1}{480}(r/s)^7 + \tfrac{1}{1920}(r/s)^9\}$$
$$+4\left(\frac{G\alpha}{384}\right)^2 PRs^9\{\tfrac{113}{960}(r/s) - \tfrac{1}{5}(r/s)^2 + \tfrac{7}{120}(r/s)^3$$
$$+\tfrac{1}{32}(r/s)^5 - \tfrac{1}{120}(r/s)^7 + \tfrac{1}{960}(r/s)^9\},$$
$$g_2^{(1)} = \tfrac{1}{2}\left(\frac{G\alpha}{96}\right)^2 Rs^9\{-\tfrac{1}{240}(r/s)^3 + \tfrac{1}{64}(r/s)^5 - \tfrac{1}{60}(r/s)^7 + \tfrac{1}{192}(r/s)^9\}$$
$$-2\left(\frac{G\alpha}{384}\right)^2 PRs^9\{\tfrac{113}{480}(r/s) - \tfrac{3}{5}(r/s)^2 + \tfrac{7}{30}(r/s)^3$$
$$+\tfrac{3}{16}(r/s)^5 - \tfrac{1}{15}(r/s)^7 + \tfrac{1}{96}(r/s)^9\}. \quad (3.25)$$

The free convection contribution to the total flow of the present approximation is now in two parts: a double longitudinal roll system as in the previous approximation and four separate roll systems, one in each quadrant in the vertical plane.

4. The shear stresses

Knowing the velocity distributions, the local shear stresses at the tube wall $r' = a(z')$ can be computed from the equations

$$\bar{\tau}'_{rz} = \left\{\tau'_{rz}\left[1 - \left(\frac{da}{dz'}\right)^2\right] + (\tau'_{rr} - \tau'_{zz})\frac{da}{dz'}\right\} \Big/ \left\{1 + \left(\frac{da}{dz'}\right)^2\right\},$$

$$\bar{\tau}_{\varphi z} = \left\{\tau'_{\varphi r}\frac{da}{dz'} + \tau'_{\varphi z}\right\} \Big/ \left\{1 + \left(\frac{da}{dz'}\right)^2\right\}^{1/2}.$$

Introducing non-dimensional stresses by (say)

$$\tau_{rz} = a_0^3 \tau'_{rz}/\mu\Psi_0,$$

we have

$$\bar{\tau}_{rz} = \left\{\tau_{rz}\left[1 - \varepsilon^2\left(\frac{ds}{dz}\right)^2\right] + (\tau_{rr} - \tau_{zz})\varepsilon\frac{ds}{dz}\right\} \Big/ \left\{1 + \varepsilon^2\left(\frac{ds}{dz}\right)^2\right\}, \quad (4.1a)$$

$$\bar{\tau}_{\varphi z} = \left\{\tau_{\varphi r}\varepsilon\frac{ds}{dz} + \tau_{\varphi z}\right\} \Big/ \left\{1 + \varepsilon^2\left(\frac{ds}{dz}\right)^2\right\}^{1/2}. \quad (4.1b)$$

Substituting the various quantities in (4.1) it may be deduced that

$$\bar{\tau}_{rz} = -\frac{8}{s^3}\left\{1 - \tfrac{1}{3}\varepsilon R\frac{1}{s}\frac{ds}{dz}\right\} + \tfrac{5}{48^2}\varepsilon GR\alpha s^2\cos\varphi + O(\varepsilon^2), \tag{4.2}$$

$$\bar{\tau}_{\varphi z} = 0 + O(\varepsilon^2). \tag{4.3}$$

Thus to order ε, the shear stress $\bar{\tau}_{\varphi z}$ is zero, the free convection currents have a null effect.

5. The heat flux

The local heat transfer rate at the wall is, by definition,

$$q' = -\left.\frac{\partial T'}{\partial n}\right|_{r'=a(z')} = -k\nabla T'\cdot\hat{n}\,|_{r'=a(z')}.$$

Here $\hat{n}$ is a unit normal to the wall. Thus it may be deduced that

$$q' = \frac{-k'}{\left\{1 + (da/dz')^2\right\}^{1/2}}\left.\left\{\frac{\partial T'}{\partial r'} - \frac{da}{dz'}\frac{\partial T'}{\partial z'}\right\}\right|_{r'=a(z')}. \tag{5.1}$$

Introducing the non-dimensional heat transfer rate

$$q = a_0 q'/k(T_w - T_\infty), \tag{5.2}$$

and employing (2.6), we get

$$q = -\left\{\frac{\partial\Theta}{\partial r} - \varepsilon^2 s_z\frac{\partial\Theta}{\partial z}\right\}\Big/\left\{1 + \varepsilon^2 s_z^2\right\}^{1/2}|_{r=s(z)}. \tag{5.3}$$

Thus

$$q = \frac{1}{2}\alpha s - \frac{1}{3}\varepsilon RP\alpha\frac{ds}{dz} - 20\varepsilon\left(\frac{\alpha s^3}{384}\right)^2 GRP\cos\varphi. \tag{5.4}$$

6. Discussion

In the last four sections, we have formulated and solved approximately for the velocity components, pressure, shear stress and heat flux for flow in a heated tube of slowly varying section. A primary observation is the appearance of a rotational velocity component as a result of free convection heat transfer. To obtain a physical feel for the various parameters involved in the problem numerical results are presented for flow in a locally constricted tube defined as

$$s(z) = 1 - \tfrac{1}{2}e^{-z^2}. \tag{6.1}$$

Equation (6.1) is useful in simulating stenosis in arterial diseases. Since in the systemic circulation the nutrients are convected by the blood plasma in the axial direction, the discussion of the velocity distributions will be limited to the axial component. The other velocity components can be discussed similarly. In all the numerical results, ε is taken as 0.1.

In the discussion of the velocity and temperature distributions (Figures 2 and 3) the value of φ is taken as zero. Other values of φ (such as $\varphi = 30°, 120°$) show insignificant change in the presentation.

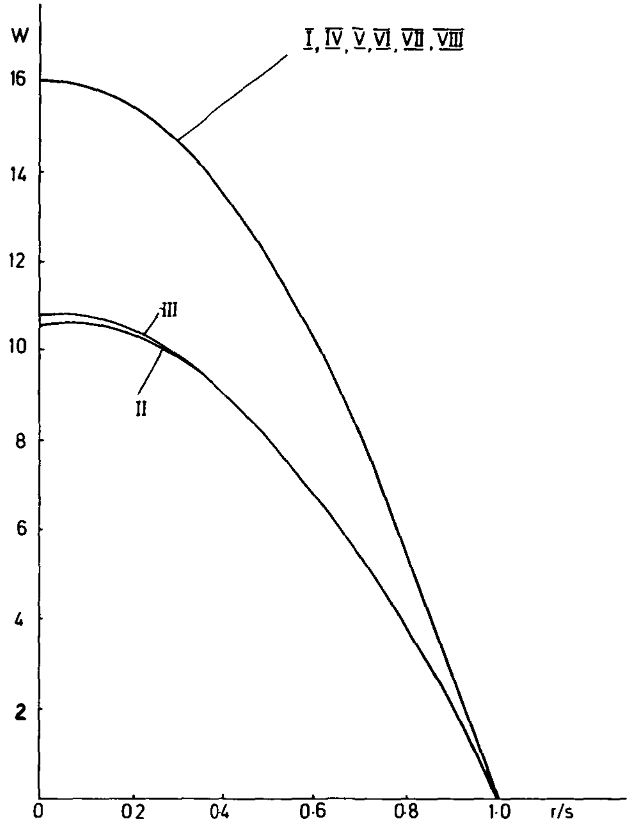

Figure 2. Axial velocity distribution in a locally constricted tube.

	R	P	G	α	z
I	0.5	7.0	5.0	1.0	0.0
II	0.5	7.0	5.0	1.0	−0.5
III	0.5	7.0	5.0	1.0	0.5
IV	0.5	7.0	10.0	1.0	0.0
V	1.0	7.0	5.0	1.0	0.0
VI	0.5	0.71	5.0	1.0	0.0
VII	0.5	7.0	−5.0	−1.0	0.0
VIII	0.5	7.0	0.0	0.0	0.0

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Figure 2 depicts the axial velocity distribution at various stations of the constriction. When the Reynolds number of the flow is small, the presence of a free convection current has little effect on the velocity whether there is external heat generation or not. Increase in the free convection current and Prandtl number causes insignificant change in the axial velocity and this situation remains the same when the source and sink interchange their roles. However the velocity distribution is highest at the throat. Upstream of the throat the axial velocity is slightly lower, near the centre, than downstream of this constriction.

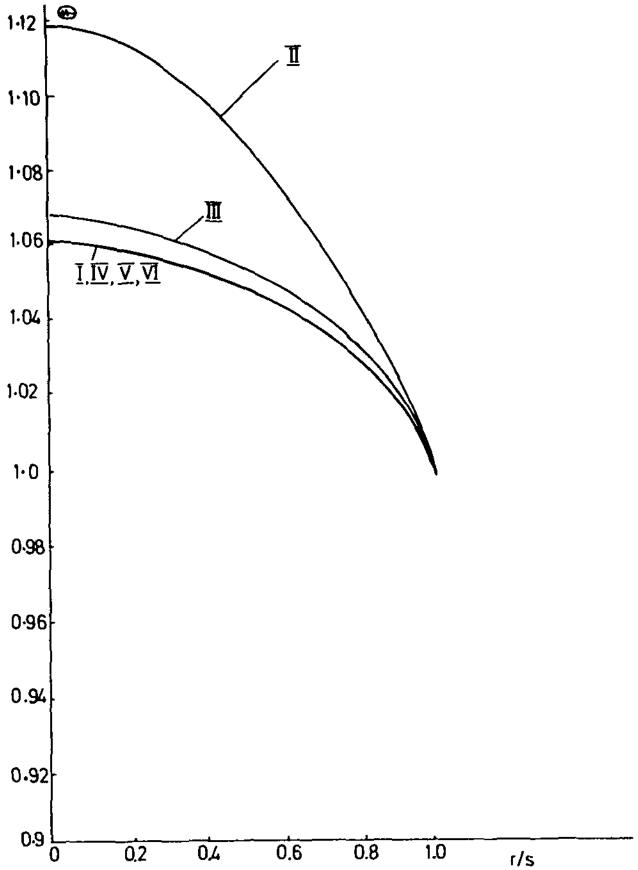

Figure 3. Temperature distribution in a locally constricted tube.

	R	P	G	α	z
I	0.5	7.0	5.0	1.0	0.0
II	0.5	7.0	5.0	1.0	−0.5
III	0 5	7.0	5.0	1.0	0.5
IV	0.5	7.0	10.0	1.0	0.0
V	1.0	7.0	5.0	1.0	0.0
VI	0.5	0.71	5.0	1.0	0.0

For the temperature distribution (Figure 3), change in the Reynolds number, Prandtl number and free convection parameter also causes little effect on the temperature. But the temperature is now higher away from the throat than at the throat, the value of the temperature downstream of the constriction being higher than that upstream of this constriction.

The heat flux at the wall is shown in Figure 4. Little change occurs as a result of change in Prandtl number, free convection parameter and angular displacement. When the Reynolds number increases, the heat flux increases upstream and decreases downstream of the constriction. Increase in the heat source causes a

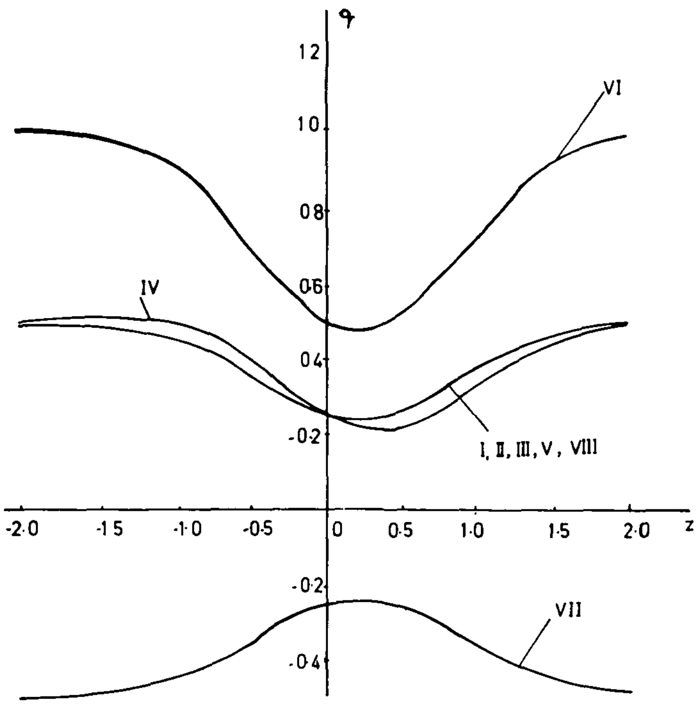

Figure 4. Heat flux distribution in a locally constricted tube.

	R	P	G	α
I	0.5	7.0	5.0	1.0
II	0.5	7.0	5.0	1.0
III	0.5	7.0	5.0	1.0
IV	1.0	7.0	5.0	1.0
V	0.5	7.0	10.0	1.0
VI	0 5	7.0	5.0	2.0
VII	0.5	7.0	−5.0	−1.0
VIII	0.5	0.71	5.0	1.0

corresponding increase in the heat flux while when the source changes to a sink the heat flux decreases and in fact becomes negative. In Figure 5 the modulus of the shear stress distribution is given for the zero heat transfer case when $R = 0.5$. The changes in the various parameters for the case of Figure 3 cause virtually no departure from this distribution.

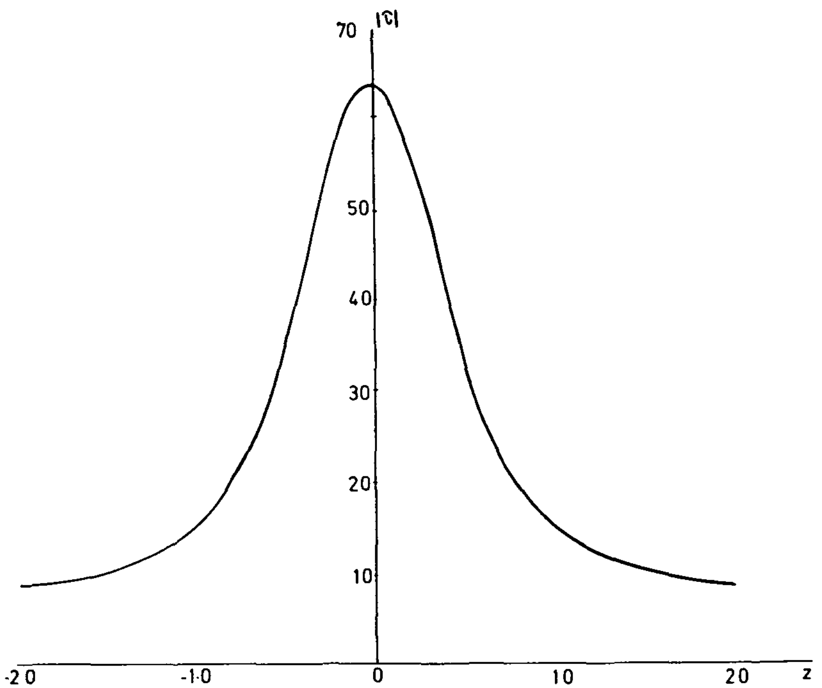

Figure 5. Shear stress distribution in a locally constricted tube for zero heat transfer for $R = 0.5$.

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