Low-Sensitivity, Low-Power Active-*RC* Allpole Filters Using Impedance Tapering

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***Abstract*—A procedure for the design of allpole filters with low sensitivity to component tolerance is presented. The filters are based on resistance–capacitance *(RC)* ladder structures combined with single operational amplifiers. It is shown that by the use of impedance tapering, in which L-sections of the *RC* ladder are successively impedance-scaled upwards, from the driving source to the amplifier input, the sensitivity of the filter characteristics to component tolerances can be significantly decreased. Impedance tapering is achieved by the appropriate choice of component values. *The design procedure, therefore, adds nothing to the cost of conventional circuits;* component count and topology remain unchanged, whereas the component values selected for impedance tapering account for the considerable decrease in component-tolerance sensitivity.**

***Index Terms*— Allpole filters, biquadratic active filters, impedance tapering, low-sensitivity active filters, third-order active filters.**

I. Introduction

IN SPITE of the large variety of available modern filter techniques there are situations in which *discrete-component active resistance–capacitance (RC) filters* have a distinct edge over their hi-tech competitors. These situations are characterized, among other things, by the following requirements: 1) fast turn-around time for design and manufacture; 2) low power (e.g., only one opamp per filter) and low cost (i.e., no need for analog-to-digital or digital-to-analog converters, anti-aliasing filters, etc.); 3) moderate frequency selectivity, i.e., pole Q's less than, say, five; 4) moderate size, i.e., smaller than inductance–capacitance *(LC)* filters but larger than integrated circuit (IC) chips; and 5) relatively small quantities and a diversity of filter specifications, which prohibit a high-runner modular approach to the design of the filters. For situations such as these, single-amplifier allpole filters of varying order should be considered. However, to maintain their cost effectiveness relative to other filter techniques, such discrete-component active *RC* filters must be manufacturable with relatively wide-tolerance *RC* components, and yet with no need for filter tuning. In other words, their filter characteristics, e.g., frequency response, must be as insensitive to component tolerances as possible.

Since the application of discrete-component active *RC* filters is generally limited to systems in which power and cost are at a premium, and the required filter selectivity and precision is relatively modest, one very useful filter candidate is the *single-amplifier nth-order* allpole filter. Unfortunately, the design equations for filters of higher than third order become unwieldy, if not unsolvable. Thus, the combination of cascadable low-sensitivity second- and third-order filter sections ("bi-quads" and, for lack of a better word, "bi-triplets") can be used to obtain low-power tolerance-insensitive high-order active *RC* filters. This is the type of filter discussed here.

It is known (and briefly shown here) that the amplitude and phase sensitivity to coefficient variations is directly proportional to the pole Q's and, therefore, to the passband ripple specified by the filter requirements. The smaller the required ripple, the lower the pole Q's. Similarly, the highest pole Q of a filter will increase with the filter order. Thus, the conventional (and economical) wisdom of keeping the filter order as low as the specifications will permit (rather than, as might have seemed reasonable, overdesign in order to minimize sensitivity) is entirely justified, also from a sensitivity point of view. Note, however, that this last statement is true only for the type of active *RC* filters considered here. For *LC* and simulated *LC* ladder filters, increasing the order decreases the component sensitivity. This is a consequence of what has come to be known as Orchard's Theorem [1].

Using the allpole low-pass filter as the most representative and important of the allpole filters, we discuss the sensitivity of the transfer-function coefficients to variations of the components (i.e., resistors, capacitors, and amplifier gain). We demonstrate that, whereas the amplitude and phase sensitivity to coefficient variations depends entirely on the transfer function itself, the coefficient sensitivity to component variations can be influenced directly by the design of the filter circuit. Introducing the concept of *impedance tapering* for the input *RC* ladder network, we demonstrate that the larger the impedance-tapering factor ρ can be made, the less sensitive the circuit will be to component tolerances. However, the impedance-tapering factor cannot be made arbitrarily large. It is shown that the maximum possible degree of impedance tapering depends directly on the value of the transfer function coefficients. Bounds on the impedance-tapering factor as a function of the transfer function coefficients are given, both for second- and third-order low-pass filters. Sensitivity expressions and design equations are also given for these filters, and Monte Carlo simulations demonstrate the effectiveness of impedance tapering as a means of reducing filter sensitivity to component tolerances. This desensitization comes at no additional cost; it

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simply requires a judicious choice of component values. Since, as is initially shown, the component sensitivity also depends on the specified ripple, order, and pole Q of the filters, it is demonstrated here that the filter designer essentially has three factors to consider, when designing low sensitivity active *RC* allpole filters. These are the following.

1) The component sensitivity increases with inband ripple. Thus, the specifications should be geared toward a low ripple (or, if possible, maximally-flat) amplitude response. This, in turn, decreases the pole Q's.
2) The component sensitivity increases with the filter order; the latter should, therefore, be held as low as possible. Fortunately, this is standard practice in filter design and minimizes filter cost.
3) By using the newly introduced concept of impedance tapering, the circuit can be directly and significantly desensitized with respect to component tolerances at no extra cost. Nothing but an appropriate choice of component values is required.

In this paper, particularly the third step is described in detail for second- and third-order low-pass filters. Since the impedance-tapering factor ρ cannot be chosen arbitrarily, but depends on the transfer function coefficients, a detailed and exact design procedure is required, and is presented here. The extension to high-pass and bandpass filters is straightforward and will be published shortly.

Because of the complexity of the design equations (which are nonlinear) for anything higher than second-order filters (i.e., "biquads"), even conventional third-order filters have, in the past, been considered only for special cases, e.g., unity gain [2] and equal-valued resistors or capacitors [3], with little attention given to sensitivity considerations. Sensitivity to component tolerances has been dealt with only in rather general terms, and with a view to the initial transfer function, e.g., [4]. In what follows, we demonstrate that impedance tapering will decrease the sensitivity to component tolerances also of higher-than-third-order allpole filters. Unfortunately, analytical design equations for these filters become intractable and are, in fact, mostly unobtainable in closed form. Nevertheless, tabulated values for special-case allpole low-pass filters for up to the sixth order are available [5], [6]. Starting out with these values, a design-optimization routine can be used to find impedance-tapered components that meet the permissible amplitude and phase tolerances specified. However, describing such routines goes beyond the scope of this paper.

II. Sensitivity to Coefficient Variations

Consider the transfer function $T(s)$ of an nth-order, allpole low-pass filter

$$T(s) = \frac{N(s)}{D(s)} = \frac{\beta a_0}{a_n s^n + a_{n-1} s^{n-1} + \cdots a_i s^i + \cdots a_1 s + a_0}. \quad (2.1)$$

The frequency response of the filter depends on the coefficients a_i of the polynomial $D(s)$. These are available from any filter handbook or CAD program and determine the location of the poles [i.e., roots of $D(s)$] in the s plane. For example, the poles of a sixth-order Butterworth low-pass filter will lie on a semicircle about the origin in the left half plane, and those of a Chebyshev filter on an ellipse. The larger the ripple of the Chebyshev filter, the smaller the eccentricity ε, i.e., the closer the poles will be to the $j\omega$ axis, and *the higher the corresponding pole* Q's [7]. Note, however, that the slope of the asymptotic response of the two allpole filters of equal order n will be the same, irrespective of the inband ripple.

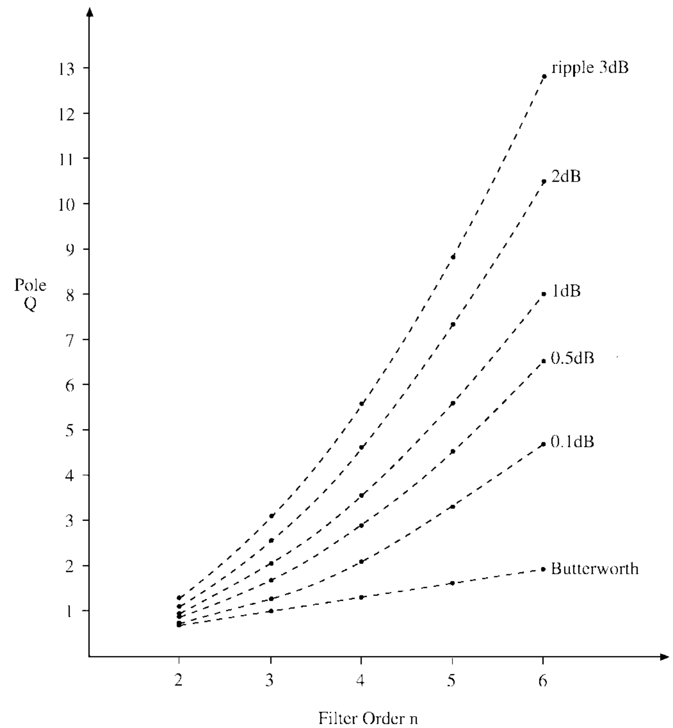


Fig. 1. Highest pole Q for Butterworth filters of increasing order n, and Chebyshev filters of increasing order n and ripple, in decibels.

With the *relative sensitivity* of a function $F(x)$ to variations of a variable x defined as

$$S_x^{F(x)} = \frac{dF/F}{dx/x} = \frac{dF(x)}{dx} \cdot \frac{x}{F(x)} = \frac{d[\ln F(x)]}{d[\ln x]} \quad (2.2)$$

we obtain the relative change of $T(s)$ as given in (2.1)—to the variations of its coefficients a_i, as

$$\frac{\Delta T(s)}{T(s)} = \sum_{i=0}^{n} S_{a_i}^{T(s)} \frac{\Delta a_i}{a_i}. \quad (2.3)$$

Thus, with (2.1)

$$S_{a_0}^{T}(s) = 1 - \frac{a_0}{D(s)} \quad (2.4)$$

and for $i \neq 0$

$$S_{a_i}^{T(s)} = \frac{-a_i s^i}{D(s)}. \quad (2.5)$$

Letting

$$F_i(s) = \frac{-a_i s^i}{D(s)}, \qquad i = 1, 2, \cdots, n \quad (2.6)$$

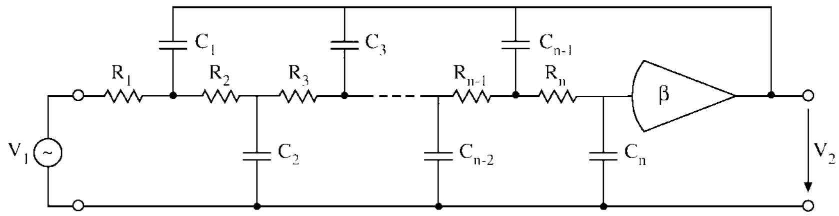


Fig. 2. General nth-order single-amplifier low-pass filter.

and, with (2.4)

$$F_0(s) = \frac{a_n s^n + a_{n-1}s^{n-1} + \cdots a_1 s}{D(s)} \tag{2.7}$$

we obtain

$$\Delta\alpha(\omega) = \mathrm{Re}\left\{\sum_{i=0}^{n} F_i(j\omega)\frac{\Delta a_i}{a_i}\right\} \tag{2.8}$$

where

$$T(s)|_{s=j\omega} = |T(j\omega)| \cdot e^{j\phi(\omega)} \tag{2.9}$$

and

$$\ln T(j\omega) = \ln|T(j\omega)| + j\phi(\omega) = \alpha(\omega) + j\phi(\omega). \tag{2.10}$$

$\alpha(\omega)$ is the amplitude response in Nepers, and $\phi(\omega)$ the phase response, e.g., in degrees.

Recalling two important properties of the coefficients a_i of $D(s)$, namely: 1) the coefficients a_i of $D(s)$ [and $N(s)$] must be real and 2) the coefficients a_i of $D(s)$ must be positive, it follows that the $\Delta a_i/a_i$ terms in (2.8) are real, so that we can rewrite this expression as

$$\Delta\alpha(\omega) = \sum_{i=0}^{n} \mathrm{Re}\{F_i(j\omega)\}\frac{\Delta a_i}{a_i} = \sum_{i=0}^{n} f_i(\omega)\frac{\Delta a_i}{a_i} \tag{2.11}$$

where, for $i = 1, 2, \cdots, n$

$$f_i(\omega) = \mathrm{Re}\left\{\frac{-a_i s^i}{D(s)}\right\}_{s=j\omega} \tag{2.12}$$

and

$$f_0(\omega) = \mathrm{Re}\left\{1 - \frac{a_0}{D(s)}\right\}_{s=j\omega}. \tag{2.13}$$

The functions $f_i(\omega)$ are frequency-dependent multiplicands of the coefficient variations $\Delta a_i/a_i$ which cause the amplitude deviation $\Delta\alpha(\omega)$. They depend only on the initial transfer function $T(s)$ of a given filter, i.e., on the filter specifications and on the required filter order, and demonstrate a direct dependence of sensitivity on the Q's of the transfer function poles: the higher the pole Q's the higher the sensitivity. This dependence will appear again in the next section when we discuss coefficient-to-component sensitivity. We therefore can already conclude here that for low sensitivity of a filter to its component tolerances, the filter with the lowest possible pole Q's (consistent with the filter specifications) should be used. Thus, for example, with respect to sensitivity, a Butterworth filter is always preferable to a Chebyshev filter and, likewise, a low-ripple Chebyshev filter is always preferable to a Chebyshev filter with higher ripple. Unfortunately, this preference frequently conflicts with cost, since the lower the ripple, for a given filter specification, the higher the required order will be. A Butterworth filter, with its "maximally flat" amplitude response corresponds to the limit case of no ripple in the filter passband and, compared to a Chebyshev filter of equal order, invariably has lower pole Q's. This is shown in Fig. 1, where the highest pole Q of second- to 6th-order Butterworth and Chebyshev filters of varying ripple is shown. The figure clearly indicates that in order to keep the pole Q's at a minimum for the sake of low filter sensitivity to coefficient variations, it is desirable to design the filter with as low ripple and as low order as consistent with the filter specifications. Whereas this is common practice in conventional filter design, it may not be obvious that an infringement of this practice violates not only the requirements of economy and performance (in terms of the inband ripple) but in terms of filter sensitivity as well.

III. Coefficient Sensitivity to Component Tolerances

As pointed out in the introduction, for reasons of low cost, low power, and fast turnaround time, single-amplifier allpole active *RC* filters designed with discrete components are often used in high-tech signal-processing and communications-oriented applications, even when the brunt of the signal-processing is carried out by "megatransistor" integrated system chips. Our discussion is therefore focused on such filters, and in particular on those with relatively low order, i.e., "$n = 2$" and "3." As we shall see, it is possible to extrapolate from these results, and to derive design guidelines for single-amplifier allpole active *RC* filters of arbitrary order n. However, a more practical, and immediately applicable method of designing a filter of any order is to cascade low-sensitivity single-amplifier filters of second and third order. As will be shown, the required power is still low, and the individual second- and third-order filters have a low sensitivity to component tolerances.

The representative nth-order general allpole single-amplifier filter structure to be used for our discussion is shown in Fig. 2 [5], [6]. This is a low-pass filter, but any other (e.g., high-pass or bandpass) applies equally well. The transfer function of this filter has the form of (2.1). The amplitude variation due to coefficient variations is given by (2.8). The coefficients a_i are functions of the r resistors, the c capacitors and the gain β. Thus, with (2.2), the coefficient variations can be expressed in the form

$$\frac{\Delta a_i}{a_i} = \sum_{\mu=1}^{r} S_{R_\mu}^{a_i}\frac{\Delta R_\mu}{R_\mu} + \sum_{\nu=1}^{c} S_{C_\nu}^{a_i}\frac{\Delta C_\nu}{C_\nu} + S_{\beta}^{a_i}\frac{\Delta\beta}{\beta}. \tag{3.1}$$

In general the individual resistors R_μ, capacitors C_ν, and gain-determining resistors will be characterized by their mean

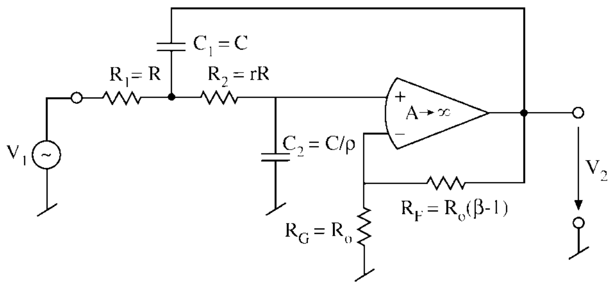


Fig. 3. Second-order low-pass filter with ideal opamp and voltage gain β.

μ_x, and standard deviation σ_x,, where x in turn represents each of the component types (e.g., resistors or capacitors). The coefficient variations $\Delta a_i/a_i$ will then be random variables whose statistical behavior is a function of the components on which they depend.

In what follows, we derive the coefficient-to-component sensitivity analytically for the second-, third-, and nth-order allpole filters. The resulting expressions provide insight into methods of deterministically reducing this sensitivity. The efficacy of these methods can be tested by Monte Carlo analysis using given component statistics. It is shown that the methods introduced to minimize the sensitivity of filters of the kind shown in Fig. 2 are very effective in reducing frequency response variations due to component tolerances.

A. Second-Order Allpole Filters

Consider the second-order low-pass filter shown in Fig. 3. The voltage gain β is obtained with an ideal noninverting opamp. The voltage transfer function for this circuit (known as class 4 or Sallen and Key [6]) expressed in terms of the coefficients a_i [see (2.1)] is given by

$$T(s) = \frac{Ka_0}{s^2 + a_1 s + a_0} \tag{3.2}$$

and in terms of the pole frequency ω_p and pole Q q_p by

$$T(s) = \frac{K\omega_p^2}{s^2 + \dfrac{\omega_p}{q_p}s + \omega_p^2} \tag{3.3}$$

where $K = \beta$ and

$$\begin{aligned} a_0 &= \omega_p^2 = \frac{1}{R_1R_2C_1C_2} \\ a_1 &= \frac{\omega_p}{q_p} = \frac{R_1(C_1+C_2) + R_2C_2 - \beta R_1C_1}{R_1R_2C_1C_2} \\ q_p &= \frac{\sqrt{R_1R_2C_1C_2}}{R_1C_1 + R_2C_2 + R_1C_2 - \beta R_1C_1}. \end{aligned} \tag{3.4}$$

Considering the overall variation of coefficient a_o, we readily obtain

$$\frac{\Delta a_0}{a_0} = -\left(\sum_{\mu=1}^{2}\frac{\Delta R_\mu}{R_\mu} + \sum_{\nu=1}^{2}\frac{\Delta C_\nu}{C_\nu}\right). \tag{3.5}$$

Note that $\Delta\omega_p/\omega_p = 0.5\Delta a_0/a_0$. Furthermore, the mean of $\Delta a_0/a_0$ will equal the negative sum of the means of all R_μ and C_ν, and the variance will be the positive sum of their

TABLE I
SENSITIVITY OF a_1 TO COMPONENT VARIATIONS OF SECOND-ORDER LOW-PASS FILTER

x	$-\frac{1}{q_p}S_x^{a_1}$		
		$R_1 = R; C_1 = C$ $R_2 = rR; C_2 = \frac{C}{\rho}$	$r_i = \rho$
R_1	$\sqrt{\frac{R_2C_2}{R_1C_1}}$	$\sqrt{\frac{r}{\rho}}$	1
R_2	$\sqrt{\frac{R_1C_1}{R_2C_2}}\left(1+\frac{C_2}{C_1}-\beta\right)$	$\sqrt{\frac{\rho}{r}}(1-\beta)+\frac{1}{\sqrt{\rho r}}$	$1-\beta+\frac{1}{\rho}$
C_1	$\sqrt{\frac{R_1C_2}{R_2C_1}}+\sqrt{\frac{R_2C_2}{R_1C_1}}$	$\sqrt{\frac{r}{\rho}}+\frac{1}{\sqrt{\rho r}}$	$1+\frac{1}{\rho}$
C_2	$\sqrt{\frac{R_1C_1}{R_2C_2}}(1-\beta)$	$\sqrt{\frac{\rho}{r}}(1-\beta)$	$(1-\beta)$

variances. Since the sensitivity of a_0 to all *RC* components is -1 (and to the gain β, it is zero), $\Delta a_0/a_0$ can be decreased only technologically, i.e., by prescribing the quality, precision, temperature coefficient, aging behavior, etc., of the resistors and capacitors. This is true for all filters of the type shown in Fig. 2, irrespective of their order n.

For the sensitivity of a_1 to the tolerances of the passive components, we readily obtain the expressions given in the first column of Table I. Furthermore, since ω_p is independent of β, we obtain

$$S_\beta^{a_1} = -S_\beta^{q_p} = -\left(\frac{q_p}{\hat{q}} - 1\right) \tag{3.6}$$

where we denote the pole Q of the passive network (i.e., $\beta = 0$) by $\hat{q}$.

Note that the coefficient sensitivities are all proportional to the pole Q, q_p. Thus, as mentioned earlier, in deciding on a filter type for a given application, one does well to select the one yielding the lowest pole Q's. This means a preference for low ripple or maximally flat filters if possible, and as low a filter degree n as the specifications will allow.

From (3.6) it follows that the coefficient (or q_p-) sensitivity to the gain is inversely proportional to the passive *RC* pole Q, $\hat{q}$. Thus $\hat{q}$ should be as large as possible. Since a passive *RC* network can have only negative-real single poles, it follows that $\hat{q}$ is limited to less than 0.5 [8]. It can be shown [6] that $\hat{q}$ of a two-section *RC* ladder structure can be maximized by impedance scaling the second L-section of the ladder such as to minimize the loading on the first. For example, referring to Fig. 3, the second L-section comprising R_2 and C_2 can be impedance-scaled upwards such as to minimize the loading on the first, i.e., R_1 and C_1. Referring to Fig. 3 and letting

$$R_1 = R; \quad C_1 = C; \quad R_2 = r \cdot R; \quad C_2 = C/\rho \tag{3.7}$$

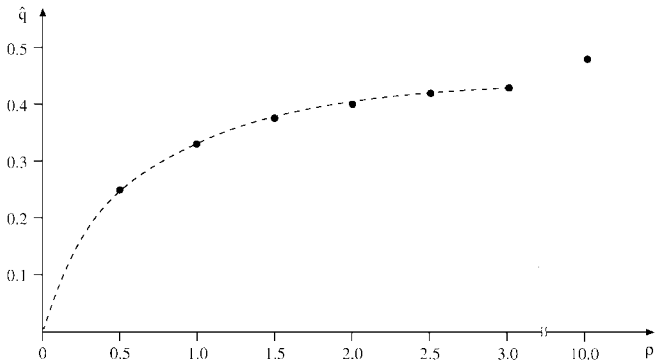


Fig. 4. Plot of $\hat{q}$ versus impedance-scaling factor ρ.

we obtain with q_p, given in (3.4)

$$\hat{q} = q_p(\beta = 0) = \frac{\sqrt{r\rho}}{1 + r + \rho} \tag{3.8a}$$

and for $r = \rho$

$$\hat{q} = \left. \frac{\rho}{1 + 2\rho} \right|_{\rho \to \infty} = 0.5. \tag{3.8b}$$

Thus, impedance scaling R_2 and C_2 by a value of $\rho \gg 1$, $\hat{q}$ will approach 0.5 and the sensitivity of a_1 (or q_p) to β will, according to (3.6), be minimized. Actually, ρ does not have to be that much larger than unity to be effective, as the plot of $\hat{q}$ versus ρ shows in Fig. 4. This is fortunate since, in practice, a large ρ may cause C_2 to decrease into the range of the parasitic capacitances of the circuit. From Fig. 4, it is apparent that a value of ρ between 2 and 3 will already bring $\hat{q}$ close to its upper boundary of 0.5, i.e., between 0.4 and 0.43, respectively.

The question is now whether impedance scaling with a factor $\rho > 1$ *will also decrease the other coefficient sensitivities, given in the first column of Table I*. Inserting the expressions in (3.7), we obtain the sensitivity relations given in the second and third column of Table I. Although proper impedance scaling requires that $r = \rho$, this may not always be possible, since a given design may require two independent degrees of freedom i.e., ρ and r. Taking this into account, both sets of expressions, namely those for $r = \rho$ and $r \neq \rho$, are given. It is apparent from these expressions that impedance scaling (in which case $\rho = r > 1$) also reduces the coefficient sensitivities to the other components, as well as to the gain β. If r is required to be unequal to ρ (for reasons of design flexibility), then increasing only the capacitor ratio ρ will also reduce the sensitivities. Although some of the expressions include a term $(\rho/r)^{1/2}(1 - \beta)$, this term will be small since the gain β will generally be in the range between unity and, say, 2.5.

To demonstrate the effect of *impedance tapering*, i.e., impedance scaling by $\rho = r > 1$, Fig. 5 shows Monte Carlo runs of the circuit in Fig. 3 for ρ values ranging from 0.1 to 10, and q_p values from 1 to 5. Comparing Fig. 5(a)–(c) for $\rho = r = 1$, the influence of the pole Q on component sensitivity for nonimpedance-scaled circuits is shown (note the vertical scale). As the pole Q is increased, the circuits become increasingly sensitive to component variations. The latter are uniformly distributed with zero mean and 5% tolerance. Impedance scaling by a factor of three (i.e., $\rho = r = 3$), the sensitivities are already decreased significantly, and for $\rho = r = 10$, even more so. Conversely, when $\rho = r = 0.1$, the high sensitivity of the circuits renders them practically useless. We do not show curves here for $r \neq \rho$ since, in general, second-order circuits do not require this added design flexibility. However, in Section IV, we shall deal with this more general case, as required for various special situations.

Incidentally, it can be shown that the sum of the sensitivities of a_1 to all resistors and capacitors must equal minus one, respectively, i.e.,

$$\sum_{\mu=1}^{2} S_{R_\mu}^{a_1} = \sum_{\nu=1}^{2} S_{C_\nu}^{a_1} = -1. \tag{3.9}$$

Expressions of this kind are often referred to as *sensitivitiy invariants*. They are a result of the so-called *homogeneity* of the function in question, the function in this case being the coefficient $a_1(R_i, C_i)$ [8].

B. Third-Order Allpole Filters

The third-order version of the nth-order low-pass filter shown in Fig. 2 is shown in Fig. 6. The voltage transfer function is given by

$$T(s) = \frac{\beta a_0}{s^3 + a_2 s^2 + a_1 s + a_0} \tag{3.10}$$

or, in terms of the pole frequencies ω_p and γ and the pole Q, q_p

$$T(s) = \frac{\beta \omega_p^2 \gamma}{(s + \gamma)\left(s^2 + \dfrac{\omega_p}{q_p} s + \omega_p^2\right)}. \tag{3.11}$$

Note that γ is equal to ω_p for a Butterworth, and to the eccentricity $\varepsilon\ times\ \omega_p$ for a Chebyshev third-order low-pass filter. The coefficients of $T(s)$ in terms of the circuit components are given by

$$a_0 = \gamma \omega_p^2 = \frac{1}{R_1 R_2 R_3 C_1 C_2 C_3} \tag{3.12}$$

where in terms of the eccentricity $\varepsilon, \gamma = \varepsilon \omega_p$, and a_1, and a_2 are shown in (3.13) and (3.14), at the bottom of the next page.

With (3.1) and (3.12) we obtain

$$\frac{\Delta a_0}{a_0} = -\left(\sum_{\mu=1}^{3} \frac{\Delta R_\mu}{R_\mu} + \sum_{\nu=1}^{3} \frac{\Delta C_\nu}{C_\nu}\right). \tag{3.15}$$

For the case of a Butterworth filter, where $\gamma = \omega_p$, it follows that $\Delta\omega_p/\omega_p = \Delta a_0/3a_0$. The sensitivity of a_1 and a_2 to all the circuit components follows from (3.1), and we obtain the expressions given in Table II.

These analytical expressions for the coefficient sensitivity to individual component tolerances indicate that impedance scaling the ladder network of an nth-order filter, as shown in Fig. 2, with a scaling factor such as to increase the impedance level from left to right, L-section by L-section, has an effect similar to that in the simple second-order case. It reduces the overall sensitivity of the network to all component tolerances.

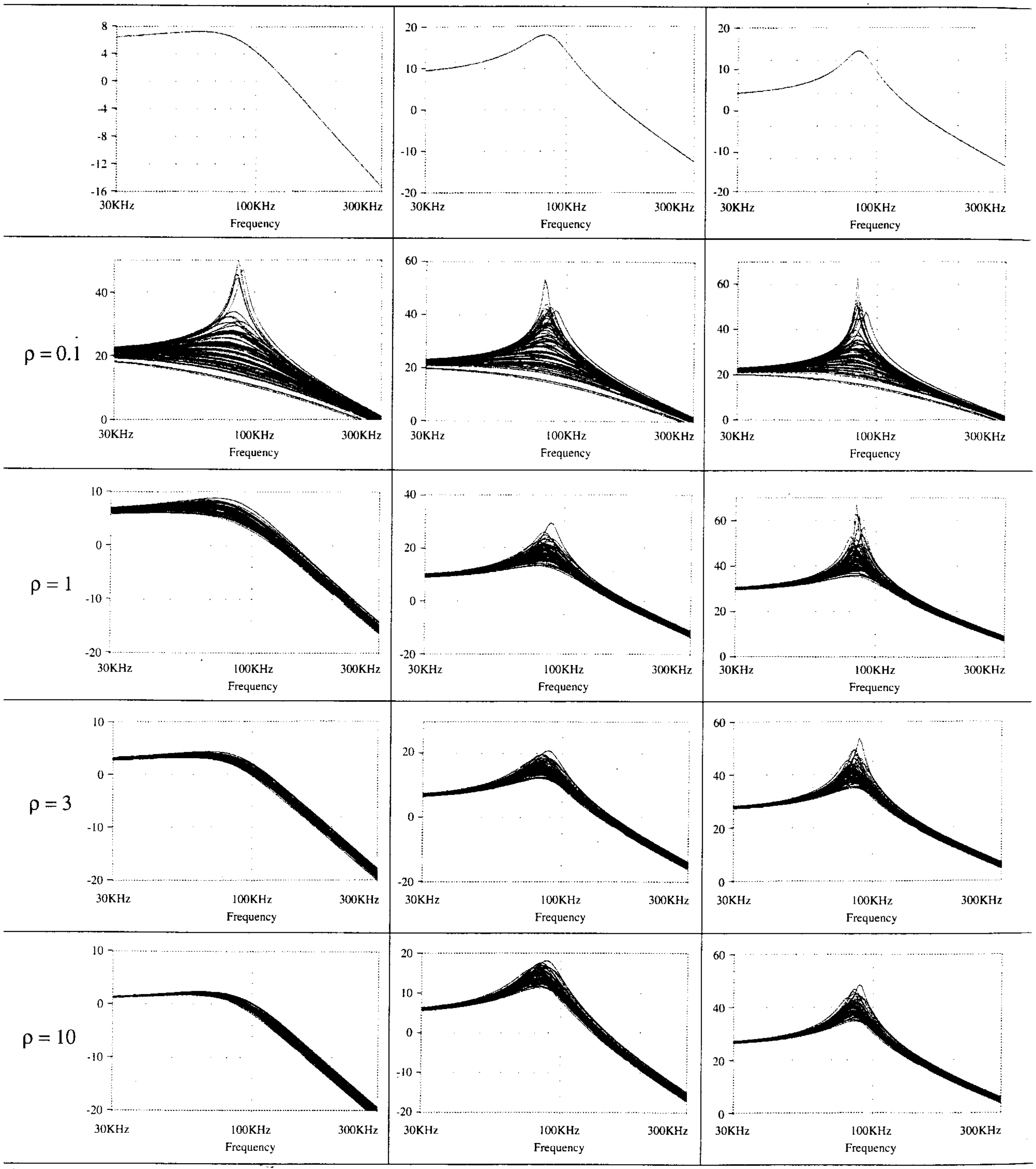


Fig. 5. Monte Carlo response plots of impedance-tapered second-order low-pass filters, with tapering factor ρ varying from 0.1 to 10 and pole Q's from 1 to 5. (a) $q = 1$. (b) $q = 3$. (c) $q = 5$.

For the third-order network, this results in the circuit shown in Fig. 6, where we increase each resistor R_i from left to right by a factor r_i, and decrease each capacitor C_i by a factor ρ_i^{-1}. Ideally, r_i should be equal to ρ_i for proper impedance

$$a_1 = \omega_p^2\left(1 + \frac{\varepsilon}{q_p}\right) = \frac{R_1(C_1 + C_2 + C_3) + R_2(C_2 + C_3) + R_3C_3 - \beta C_2(R_1 + R_2)}{R_1R_2R_3C_1C_2C_3} \tag{3.13}$$

and

$$a_2 = \omega_p\left(\frac{1}{q_p} + \varepsilon\right) = \frac{R_1R_2C_1(C_2 + C_3) + R_1R_3C_3(C_1 + C_2) + R_2R_3C_2C_3 - \beta R_1R_2C_1C_2}{R_1R_2R_3C_1C_2C_3} \tag{3.14}$$

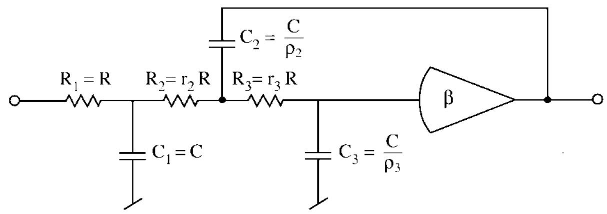


Fig. 6. Impedance tapering a third-order low-pass filter by r_i and $\rho_i, i = 2, 3$.

TABLE II
SENSITIVITY OF a_1 AND a_2 TO COMPONENT VARIATIONS OF THIRD-ORDER LOWPASS FILTER

x	$-\frac{a_1}{a_0}\frac{1}{R_1C_1}S_x^{a_1}$	$-\frac{a_2}{a_0}\frac{1}{R_1C_2C_1C_2}S_x^{a_2}$
R_1	$\frac{R_2C_2}{R_1C_1}\left(1+\frac{C_3}{C_2}+\frac{R_3C_3}{R_2C_2}-\beta\right)$	$\frac{R_3C_3}{R_1C_1}$
R_2	$\left[1+\frac{C_2}{C_1}(1-\beta)+\frac{C_3}{C_1}\left(1+\frac{R_3}{R_1}\right)\right]$	$\frac{R_3C_3}{R_2}\left(\frac{C_1+C_2}{C_1C_2}\right)$
R_3	$\left[1+\left(1+\frac{R_2}{R_1}\right)\left(\frac{C_2}{C_1}(1-\beta)+\frac{C_3}{C_1}\right)\right]$	$(1-\beta)+\frac{C_3}{C_2}$
C_1	$\left[\left(1+\frac{R_2}{R_1}\right)\left(\frac{C_2}{C_1}(1-\beta)+\frac{C_3}{C_1}\right)+\frac{R_3C_3}{R_1C_1}\right]$	$\frac{R_3C_3}{C_1}\left(\frac{R_1+R_2}{R_1R_2}\right)$
C_2	$\left[1+\frac{C_3}{C_1}\left(1+\frac{R_2+R_3}{R_1}\right)\right]$	$\frac{C_3(R_2+R_3)}{R_2C_2}$
C_3	$\left[1+\left(1+\frac{R_2}{R_1}\right)\left(\frac{C_2}{C_1}(1-\beta)\right)\right]$	$(1-\beta)$
β	$\frac{\beta C_2}{C_1}\left(1+\frac{R_2}{R_1}\right)$	β

TABLE III
SENSITIVITY OF a_1 TO COMPONENT VARIATIONS OF THIRD-ORDER LOW-PASS FILTER

x	$-\frac{a_1}{a_0}\frac{1}{RC}S_x^{a_1}$		
	General Impedance Scaling	**Separate Impedance Tapering**	**Ideal Impedance Tapering**
	r_i, ρ_i $i=2,3$	$r_i = r^{i-1}$; $\rho_i = \rho^{i-1}$ $i=2,3$	$r_i = \rho_i = \rho^{i-1}$ $i=2,3$
R_1	$\frac{r_2}{\rho_2}\left[1+\frac{\rho_2}{\rho_3}\left(1+\frac{r_3}{r_2}\right)-\beta\right]$	$\frac{r}{\rho}\left[1-\beta+\frac{1}{\rho}(1+r)\right]$	$2-\beta+\frac{1}{\rho}$
R_2	$1+\frac{1}{\rho_2}(1-\beta)+\frac{1}{\rho_3}(1+r_3)$	$1+\frac{1}{\rho}(1-\beta)+\frac{1}{\rho^2}(1+r^2)$	$2+\frac{1}{\rho}(1-\beta)+\frac{1}{\rho^2}$
R_3	$1+(1+r_2)\left[\frac{1}{\rho_2}(1-\beta)+\frac{1}{\rho_3}\right]$	$1+(1+r)\left[\frac{1-\beta}{\rho}+\frac{1}{\rho^2}\right]$	$(2-\beta)\left[1+\frac{1}{\rho}\right]+\frac{1}{\rho^2}$
C_1	$(1+r_2)\left[\frac{1}{\rho_2}(1-\beta)+\frac{1}{\rho_3}\right]+\frac{r_3}{\rho_3}$	$\frac{1+r}{\rho}\left(1-\beta+\frac{1}{\rho}\right)+\frac{r^2}{\rho^2}$	$\left(1+\frac{1}{\rho}\right)\left(1+\frac{1}{\rho}-\beta\right)+1$
C_2	$1+\frac{1}{\rho_3}(1+r_2+r_3)$	$1+\frac{1}{\rho^2}(1+r+r^2)$	$2+\frac{1}{\rho}+\frac{1}{\rho^2}$
C_3	$\left(\frac{1+r_2}{\rho_2}\right)(1-\beta)+1$	$\frac{1+r}{\rho}(1-\beta)+1$	$\left(1+\frac{1}{\rho}\right)(1-\beta)+1$
β	$\frac{\beta}{\rho_2}(1+r_2)$	$\beta\left(\frac{1+r}{\rho}\right)$	$\beta\left(1+\frac{1}{\rho}\right)$

scaling. However, for third and higher order filters, in order to maintain a sufficient number of degrees of freedom, this condition cannot, in general, be satisfied exactly. Nevertheless, because we are still gradually increasing the impedance level, L-section by L-section, from left to right, we still refer here to "impedance tapering of the ladder network." By impedance tapering, we may be increasing the impedance of only the resistors, capacitors, or both, from left to right. Referring to Fig. 6 for ideal tapering, we require that

$$r_2 = \rho_2 = \rho; \quad r_3 = \rho_3 = \rho^2 \tag{3.16}$$

and for the general nth-order network in Fig. 2, we require that

$$r_i = \rho_i = \rho^{i-1}, \qquad i = 2, 3, \cdots, n. \tag{3.17}$$

Inserting the general impedance-scaling factors r_i and ρ_i as in Fig. 6, i.e.,

$$\begin{aligned} &R_1 = R; \quad R_2 = r_2R; \quad R_3 = r_3R \\ &C_1 = C; \quad C_2 = C/\rho_2; \quad C_3 = C/\rho_3 \end{aligned} \tag{3.18}$$

into the sensitivity expressions given in Table II, we obtain the expressions listed in the first column of Tables III and IV. Tapering the resistors and capacitors separately by $r_i = r^{i-1}$ and $\rho_i = \rho^{i-1}$, respectively, the sensitivity expressions in column 2 (*separate impedance tapering*) of the two tables result. Finally, for *ideal impedance tapering*, we have the condition of (3.17), which results in the third column of Tables III and IV. Before interpreting these results, it is useful to point out their adherence to the property of the coefficient sensitivity invariants as given for the second-order network in (3.9). It can be shown that for an nth-order network

$$\sum_{\mu=1}^{r} S_{R_\mu}^{a_{n-i}} = \sum_{\nu=1}^{c} S_{C_\nu}^{a_{n-i}} = -i \tag{3.19}$$

as can readily be verified for the expressions given in Tables III and IV. Furthermore, in general, the coefficient sensitivity of the impedance-tapered filter has the form

$$S_x^{a_i} = -\frac{a_0}{a_i}(\mathrm{RC})^i f(r_\mu, \rho_\nu) \tag{3.20}$$

where x is any resistor, capacitor, or gain element of the circuit. For ideal impedance tapering, it follows that $a_0(\mathrm{RC})^i = 1$ and $S_x^{a_i} = f(\rho)/a_i$.

Whereas the coefficients a_i are given by the filter specifications, the function $f(\rho)$ can be minimized by making $\rho > 1$. Here again, as with the second-order filter, a value of ρ between two and five will already reduce the coefficient sensitivity appreciably. The same applies to the nonideally tapered circuits for which the coefficient sensitivities are given in the first two columns of Tables III and IV.

TABLE IV
SENSITIVITY OF a_2 TO COMPONENT VARIATIONS FOR THIRD-ORDER LOW-PASS FILTER

x	$-\frac{a_2}{a_0}\frac{1}{R^2C^2}S_x^{a_2}$		
	General Impedance Scaling	Separate Impedance Tapering	Ideal Impedance Tapering
	r_i, ρ_i i=2,3	$r_i = r^{i-1}$; $\rho_i = \rho^{i-1}$ i=2,3	$r_i = \rho_i = \rho^{i-1}$ i=2,3
R_1	$\frac{r_2 r_3}{\rho_2 \rho_3}$	$\frac{r^3}{\rho^3}$	1
R_2	$\frac{r_3}{\rho_3}\left(1+\frac{1}{\rho_2}\right)$	$\frac{r^2}{\rho^2}\left(1+\frac{1}{\rho}\right)$	$1+\frac{1}{\rho}$
R_3	$r_2\left[\frac{1}{\rho_2}(1-\beta)+\frac{1}{\rho_3}\right]$	$\frac{r}{\rho}\left(1+\frac{1}{\rho}-\beta\right)$	$1+\frac{1}{\rho}-\beta$
C_1	$\frac{r_3}{\rho_2\rho_3}(1+r_2)$	$\frac{r^2}{\rho^3}(1+r)$	$1+\frac{1}{\rho}$
C_2	$\frac{1}{\rho_3}(r_2+r_3)$	$\frac{r}{\rho^2}(1+r)$	$1+\frac{1}{\rho}$
C_3	$\frac{r_2}{\rho_2}(1-\beta)$	$\frac{r}{\rho}(1-\beta)$	$1-\beta$
β	$\beta\frac{r_2}{\rho_2}$	$\beta\frac{r}{\rho}$	β

For third-order filters, the specifications generally do not permit ideal impedance tapering, in which case just the capacitors may be tapered and the resistor values will be determined by the filter design equations. By inspection of the "separate impedance tapering" columns of Tables III and IV, it follows that the resistor tapering factor should actually be held as small as possible, or typically close to unity, while the capacitor tapering factor ρ should be as large as possible, in order to minimize coefficient sensitivity. We shall later see that when ideal tapering for filter orders higher than two is not possible, the optimum solution is to make the r-values equal, (i.e., for the third-order case: $r_2 = r_3$) and ρ as large as possible. In general a ρ value between 2 and 5 is sufficient to provide a significant degree of insensitivity to all component tolerances.

In Fig. 7, amplitude response curves for a third-order Butterworth and Chebyshev filter are shown. The capacitor scaling factor ρ_c was varied from 1 to 5. The resistor scaling factor could not be freely chosen; it was determined by the design equations. Just how the design equations constrain the resistive scaling factor will be discussed in Section IV. Monte Carlo runs with 5% flat-distribution, zero-mean resistors and capacitors were carried out. Clearly, the circuits with tapered capacitors (i.e., $\rho_c = 3$ and $\rho_c = 5$) are considerably less sensitive to component tolerances than the nontapered circuits.

IV. DESIGN EQUATIONS FOR SECOND- AND THIRD-ORDER IMPEDANCE-TAPERED ALLPOLE FILTERS

In this section we present the design equations for tapered second- and third-order filters. Unfortunately, the equations for higher-order filters still defy a satisfactory closed-form solution. However, the combination of impedance-tapered second-and third-order sections, which can be cascaded to provide any desired filter order, is already a great improvement in terms of component sensitivity, compared to a nontapered equivalent filter, while remaining modest in terms of required power. Thus, for example, up to sixth-order filters can be realized with two, up to ninth-order with three, twelfth-order with four, fifteenth-order with five amplifiers, and so on.

A. Second-Order Allpole Filters

We start out with the second-order low-pass filter shown in Fig. 3. Its transfer function is given by (3.2) to (3.4). With the tapering factors in (3.7) and with

$$\omega_0 = \frac{1}{RC} \tag{4.1}$$

we obtain for the coefficients of $T(s)$

$$\begin{aligned} a_0 &= \omega_p^2 = \frac{\rho}{r}\cdot\omega_0^2; \quad a_1 = \frac{\omega_p}{q_p} = \omega_0\frac{\rho}{r}\left(1+\frac{1+r}{\rho}-\beta\right) \\ \beta &= 1+\frac{1+r}{\rho}-\frac{1}{q_p}\sqrt{\frac{r}{\rho}}. \end{aligned} \tag{4.2}$$

In practice, K, a_0, and a_1 or, equivalently, K, ω_ρ and q_p will be given by the filter specifications. From these quantities, and possibly some additional constraints, such as input resistance level, maximum or minimum acceptable capacitor values, etc., we must determine ω_0, ρ, r, and β.

From (4.2), we obtain

$$r = \frac{\omega_0^2}{(a_1-\omega_0)\omega_0 + a_0(\beta-1)} \tag{4.3}$$

and

$$\rho = \frac{ra_0}{\omega_0^2} = \frac{a_0}{(a_1-\omega_0)\omega_0 + a_0(\beta-1)}. \tag{4.4}$$

r and ρ must both be positive, and β, which is, in general, the gain of a noninverting opamp, must obey the constraint

$$\beta \geq 1. \tag{4.5}$$

Thus, the denominator of (4.3) and (4.4) must be larger than zero, resulting in the constraint that

$$\omega_0 < \frac{a_1}{2} + \sqrt{\frac{a_1^2}{4} + a_0(\beta-1)}. \tag{4.6}$$

Because of (4.5), the expression under the square root will always be positive.

As we shall see below, impedance tapering may be only one consideration necessary for the minimization of active filter sensitivity, namely that of minimizing for sensitivity to *passive* components. Minimizing sensitivity to *active* elements in the filter circuit has been shown to require the minimization of the *gain-sensitivity product* [6], [7]. This and other considerations

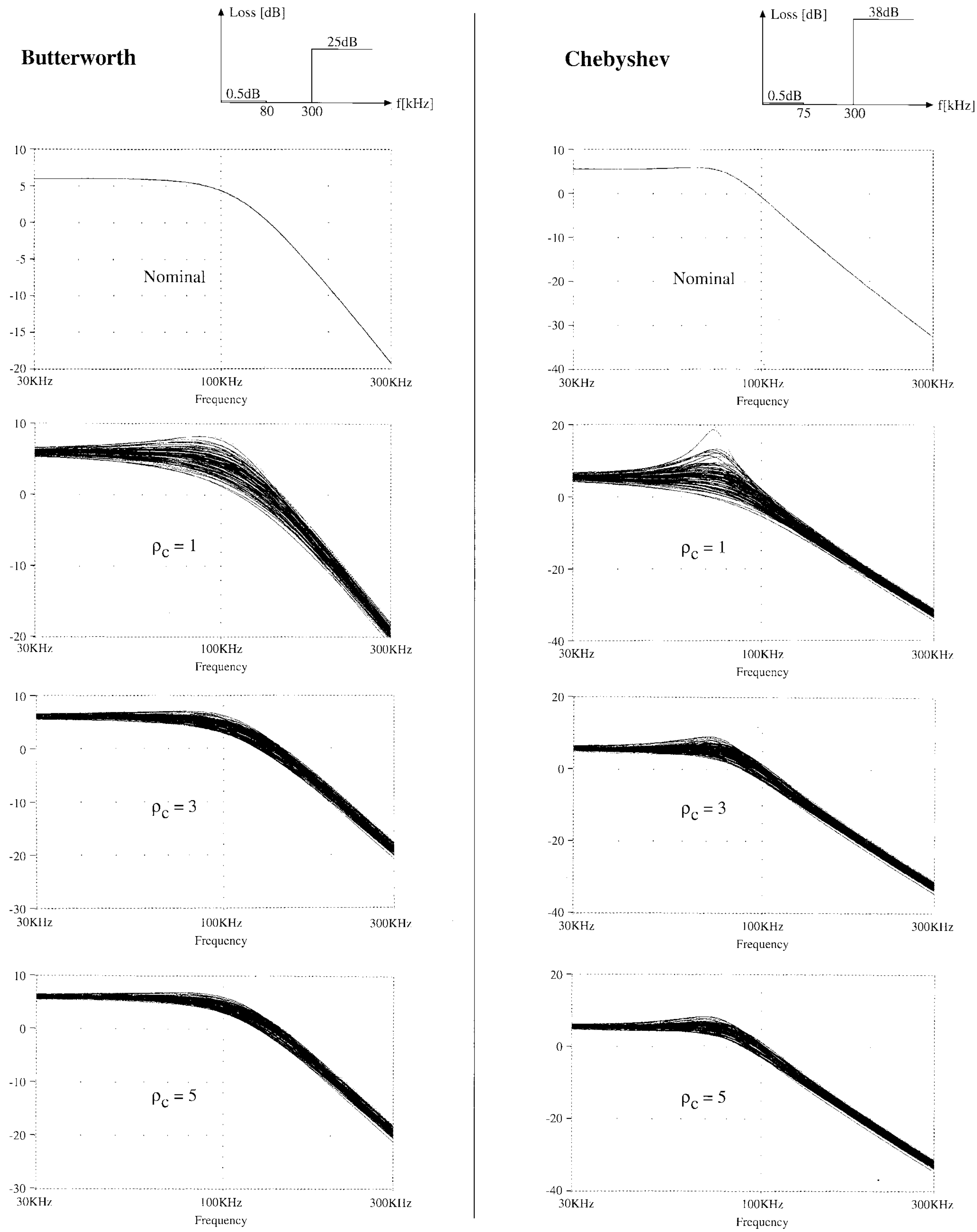


Fig. 7. Monte Carlo response plots of impedance-tapered third-order Butterworth and Chebyshev low-pass filters. Capacitors are tapered by tapering factor ρ_c; resistor values follow from design equations.

may result in only partial, as opposed to ideal, impedance tapering being the best strategy for the comprehensive sensitivity minimization of a filter circuit. The resulting different cases are discussed briefly in what follows.

1) Ideal Impedance Tapering: In this case $r = \rho$. From (4.1) and (4.2), we obtain

$$\omega_0 = \omega_p = \frac{1}{\mathrm{RC}} = \sqrt{a_0} \tag{4.7}$$

and

$$\beta = 2 + \frac{1}{\rho} - \frac{1}{q_p}. \tag{4.8}$$

Thus, the required gain β increases with increasing q_p and decreases with increasing ρ.

In general, rather than having the amplifier gain β in the numerator of $T(s)$, a more general factor K is used [see (3.2)] which, in the case of a low-pass filter, is essentially the dc forward gain. Clearly, K will be proportional to β, but may well have a different value. Thus, if the specified gain factor K does differ from the β value obtained in (4.8), then additional circuit techniques, discussed in 4) below, must be applied to obtain the required K.

Consider the following practical example. Suppose that

$$\omega_p = \omega_0 = 2\pi \cdot 86 \text{ kHz}; \quad q_p = 5; \quad C = 500 \text{ pF}. \tag{4.9}$$

From (4.7) we obtain $R = 3.7$ kΩ. Assuming an impedance-tapering factor $\rho = 4$, we obtain from (4.8) $\beta = 2.05$. Referring to Fig. 3, the resulting filter has the values given in line 1 of Table V. For the equivalent circuit with $\rho = 1$ and, consequently, with much higher sensitivity to component tolerances, we obtain $\beta = 2.8$ and the values given in line 2.

2) Combining Gain-Sensitivity-Product Minimization with Partial Impedance Tapering: It is well known that in order to minimize the sensitivity of filter characteristics to tolerances of the *gain* elements, the *gain-sensitivity product* (GSP) of the filter should be minimized [6], [7]. In [7], well-proven biquadratic filter circuits ("biquads") and the corresponding design flow-chart listings are given. In the design programs, the specifications in terms of pole and zero frequencies and Q's is admitted as input, and the circuit with the minimum GSP results as output. In most of the circuits, one additional degree of freedom is available, namely the values and ratio of two (or three) capacitors. In the present context this permits the implementation of *partial impedance tapering* (namely with respect to the capacitors of the circuit) while, at the same time, minimizing the GSP. The latter will rarely, if ever, permit ideal impedance tapering (i.e., $\rho = r$); since the resistor values are selected such as to minimize the GSP they will only rarely coincide with the condition for ideal impedance tapering. Thus, the resulting circuits will, in general, have $\rho > 1$ and $r \neq \rho$. Let us now reexamine the previous example from this point of view.

Referring to [7], we use the so-called low-pass medium-Q (LP-MQ) circuit, which essentially corresponds to the circuit in Fig. 3 (see [7, p. 52]). With the specifications (4.9), we let $\rho = 4$ and obtain (with the program in the handbook) the circuit component values given in line 3 of Table V. The value of r is equal to 1.85. Referring to Fig. 3, we obtain

TABLE V
COMPONENT VALUES OF SECOND-ORDER FILTERS AS IN FIG. 3 (RESISTORS IN kΩ, CAPACITORS IN pF)

	R_1	R_2	r	C_1	C_2	ρ	R_0	β
1) Impedance Tapered	3.7	14.8	4	500	125	4	10	2.05
2) Non-tapered	3.7	3.7	1	500	500	1	10	2.8
3) C-tapered and Min.GSP	5.4	10	1.85	500	125	4	10	1.58
4) Partially tapered (r=1)	7.4	7.4	1	500	125	4	10	1.4

$$\begin{aligned}
\omega_0 &= \frac{1}{R_1 C_1} = 3.70 \cdot 10^5;\ a_0 = \omega_p^2 = 29.198 \cdot 10^{10}\\
\frac{\rho}{r} &= \frac{a_0}{\omega_0^2} = 2.16\\
a_1 &= \frac{\omega_p}{q_p} = \omega_0 \frac{\rho}{r}\left(1 + \frac{1+r}{\rho} - \beta\right) = 1.1 \cdot 10^5\\
\beta &= 1 + \frac{1+r}{\rho} - \frac{1}{q_p}\sqrt{\frac{r}{\rho}} = 1.58.
\end{aligned} \tag{4.10}$$

These values correspond to those given in the third line of Table V.

Monte Carlo runs were carried out for this circuit and compared with those of an ideally impedance-tapered circuit designed for the same specifications (i.e., line 1, Table V). The results for the latter, i.e., the circuit with capacitive impedance tapering *and* minimum gain sensitivity product, were somewhat better than those of the former. However, the difference is not very large, particularly when compared with the much inferior nontapered circuit (Table V, line 2). From this and other examples, it appears that ideal impedance tapering, and capacitive impedance tapering with GSP minimization, produce approximately the same improvement, although the latter is to be preferred if the choice is available.

3) Partial Impedance Tapering with Equal Resistors ($r = 1$):

In some cases, it may be desirable simply to use equal resistors in the circuit of Fig. 3, i.e., $r = 1$. For this case, we have from (4.2)

$$\begin{aligned}
&\omega_0 = \frac{1}{RC};\ \ a_0 = \omega_p^2 = \rho\omega_0^2;\ \ a_1 = \frac{\omega_p}{q_p} = \rho\omega_0\left(1 + \frac{2}{\rho} - \beta\right)\\
&\beta = 1 + \frac{2}{\rho} - \frac{1}{q_p}\sqrt{\frac{1}{\rho}}.
\end{aligned} \tag{4.11}$$

From (4.3), we obtain the value of ω_0 by setting $r = 1$, namely

$$\omega_0 = \frac{a_1}{4} + \sqrt{\frac{a_1^2}{16} + \frac{a_0}{2}(\beta - 1)}. \tag{4.12}$$

A comparison with (4.6) shows that (4.12) is guaranteed to remain below the upper bound. With the specifications in (4.9), and with $\rho = 4$, we obtain

$$a_0 = 29.198 \cdot 10^{10}; \quad a_1 = 10.8 \cdot 10^4$$
$$\omega_0 = \frac{1}{\sqrt{\rho}} \omega_p = 2.70 \cdot 10^5; \quad \beta = 1.4 \tag{4.13}$$

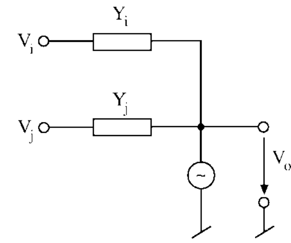
and with $C = 0.5$ nF we obtain $R = (\omega_0 C)^{-1} = 7.4$ kΩ. The resulting circuit values are given in line 4 of Table V. Comparing Monte Carlo runs for this circuit with those defined by the values in line 3, very little difference was found. In fact, if anything, the $r = 1$-circuit was found to be somewhat superior to that of line 3. This is not surprising, if we reconsider the general sensitivity expressions given in Table I. Since some of the sensitivities are proportional to r, others to r^{-1}, setting $r = 1$ is an optimum compromise. Thus, *in summary, for the general second-order allpole low-pass filter of Fig. 3, capacitive impedance tapering with either equal resistors* ($r = 1$), *or resistor values selected for GSP-minimization, provide circuits with minimum sensitivity to the component tolerances of the circuit*. It can be shown that the same strategy for desensitization to component tolerances holds also for all other allpole filters, e.g., high-pass and bandpass filters. It also holds for biquads with finite zeros; there, however, the tapering must be carried out on so-called *potentially-symmetrical* bridged-T or twin-T circuits [6], [8].

4) The Gain Factor K: As pointed out above, the dc forward gain K of the filter transfer function, as in (3.2), will generally not coincide with the amplifier gain β required to obtain ω_p and q_p. The gain factor K may be specified by the filter designer, but the amplifier gain is determined by the general expression for β [see (4.2)] or, for ideal impedance tapering (i.e., $\rho = r$), by (4.8). Thus, the value of β cannot be freely chosen; it depends on the scaling factors r and ρ, and on the specified pole Q, q_p whereas the overall dc filter gain K may very likely be required to have a different value. Fortunately, there are various schemes for the decoupling of K and β [6], one of which will be presented in what follows.

Consider the partial output circuit shown in Fig. 8(a). The output voltage level V_o is determined by the output voltage source, which also determines the voltages V_i and V_j at the ends of two arbitrary admittances Y_i and Y_j. Assume now that we wish to have a μ-times larger output voltage, while leaving the terminal voltages V_i and V_j unchanged. In terms of our filter, this implies that

$$\mu = \frac{K}{\beta} \tag{4.14}$$

where $\mu > 1$. A glance at Fig. 8(b) shows that with the addition of two admittances this can readily be achieved; the loading on terminals V_i and V_j remains unchanged, whilst the resulting voltage dividers require a μ-times larger output voltage. By definition, $\mu > 1$ and the additional components are positive. Applying this transform to the *RC* output configuration shown in Fig. 9(a), we obtain the configuration in Fig. 9(b). Note that, although the number of capacitors is increased by one, the total capacitance remains unchanged.



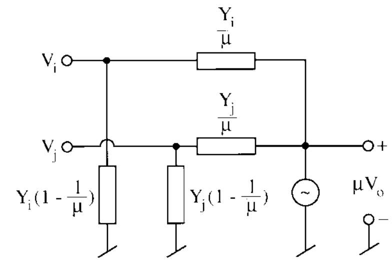


Fig. 8. Output voltage-level transform, such that input levels remain unchanged, while output voltage is increased by μ.

The two resistors are larger than the original R, but their value can be reduced by introducing a resisitive T into the network, such that only a part of R is transformed (see [6]). Consider now the impedance-tapered filter in Fig. 3, whose gain β is equal to 2.05 (see Table V, line 1). Assuming that we desire a dc gain $K = 4$, i.e., $\mu = K = 4/2.05 = 1.95$ we obtain the corresponding circuit shown in Fig. 10. Note that $C_1' = C_1(1 - 1/\mu) = 244$ pF; $C_1'' = C_1/\mu = 256$ pF, and $C_{1tot} = 500$ pF.

Another closely related method of obtaining $\mu > 1$ is by simply using a reasonably low-impedance voltage divider at the amplifier output such that $1/\mu$ times the output voltage can be tapped off and fed back into the circuit [see Fig. 9(c)]. Since the amplifier's output impedance is roughly equal to the Thevenin-equivalent output impedance divided by the loop gain, the driving capability of the amplifier will not be significantly affected as long as the loop gain is sufficiently large. However, the open loop gain is now decreased by a factor of $1/\mu$, so that the error introduced by finite amplifier gain is increased. Consequently, this scheme should be used only for relatively small μ values, corresponding to a gain enhancement of, say, less than 10 dB. If the desired value μ is less than unity, i.e., $\beta > K$, then a resistive voltage divider can be inserted at the input of the network, as shown in Fig. 11. In this case

$$\mu = \frac{R_1'}{R_1' + R_1''} \quad \text{and} \quad \frac{R_1' R_1''}{R_1' + R_1''} = R_1$$

i.e., $R_1' = R_1/\mu$ and $R_1'' = R_1/(1 - \mu)$. Since μ is, in this case, less than unity, R_1'' is always positive.

B. Third-Order Allpole Filters

Here we consider a third-order low-pass filter as shown in Fig. 6. The transfer function is given by (3.10) and (3.11); in general the amplifier gain β in the numerator should be replaced by the more general K. The coefficients in terms of

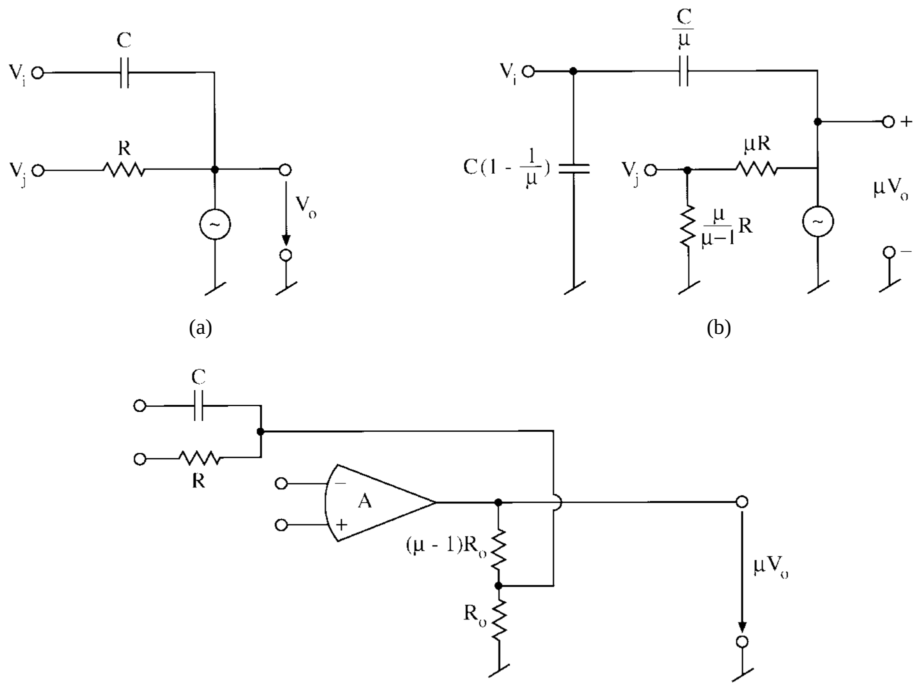


Fig. 9. Voltage-level transform applied to *RC* output feedback network. (a) Original circuit. (b) Transformed circuit. (c) Alternative circuit with output voltage divider.

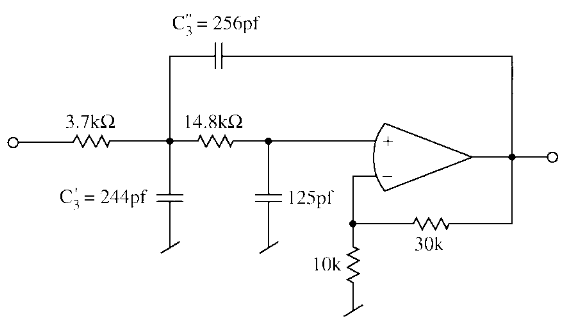


Fig. 10. Circuit of Table V, line 1, transformed for loop gain $\beta = 2.05$, dc gain $K = 4$, and $\mu = 1.95$.

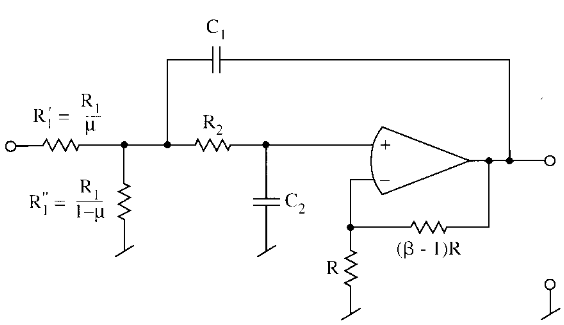


Fig. 11. Biquad circuit with loop gain β, dc gain K, and $\mu = K/\beta < 1$.

the circuit components are given by (3.12)–(3.14). With the tapering expressions shown in Fig. 6 and with $\omega_0 = (RC)^{-1}$, we obtain the following relations between the transfer-function coefficients, tapering factors, design frequency ω_0, and gain

$$\begin{aligned} a_0 =& \omega_0^3 \cdot \frac{\rho_2\rho_3}{r_2 r_3} \\ a_1 =& \omega_0^2 \frac{\rho_2\rho_3}{r_2 r_3}\left[\left(1+\frac{1}{\rho_2}+\frac{1}{\rho_3}\right)+r_2\left(\frac{1}{\rho_2}+\frac{1}{\rho_3}\right)+\frac{r_3}{\rho_3}\right. \\ &\left. -\beta\frac{(1+r_2)}{\rho_2}\right] \\ a_2 =& \omega_0 \frac{\rho_2\rho_3}{r_2 r_3}\left[r_2\left(\frac{1}{\rho_2}+\frac{1}{\rho_3}\right)+\frac{r_3}{\rho_3}\left(1+\frac{1}{\rho_2}\right)+\frac{r_2 r_3}{\rho_2\rho_3}\right. \\ &\left. -\beta\frac{r_2}{\rho_2}\right]. \end{aligned} \tag{4.15}$$

As we shall see below, ω_0 is an important design parameter, whose value determines the realizability of a third-order filter.

Before going to the ideal impedance-tapering case, i.e., the case for which $r_2 = \rho$, $r_3 = \rho^2$, $\rho_2 = \rho$, and $\rho_3 = \rho^2$, we shall solve the general case in which R, C, ρ_2, ρ_3, a_0, a_1, a_2, and K are given, and r_2, r_3, and β must be found. It is useful to normalize the coefficients a_i with respect to the design frequency $\omega_0 = (RC)^{-1}$, thus

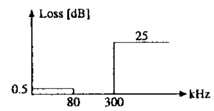
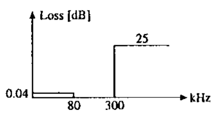
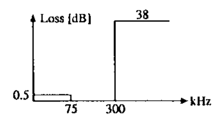
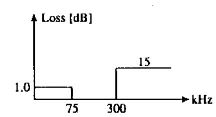
$$\alpha_0 = \frac{a_0}{\omega_0^3}; \quad \alpha_1 = \frac{a_1}{\omega_0^2}; \quad \alpha_2 = \frac{a_2}{\omega_0} \tag{4.16}$$

and in general

$$\alpha_i = \frac{a_i}{\omega_0^{n-i}}, \qquad i = 0, 1, \cdots, n-1 \tag{4.17}$$

where n is the order of the filter transfer function $T(s)$. After some calculation, we obtain the following three equations for

TABLE VI
SUMMARY OF SPECIFICATIONS AND COEFFICIENTS FOR FILTER DESIGN EXAMPLES

	a_0 $\cdot 10^{18}$	a_1 $\cdot 10^{12}$	a_2 $\cdot 10^{6}$	ω_a $\cdot 10^{5}$	ω_{DI} $\cdot 10^{5}$	$\omega_{0\,\max}$ $\cdot 10^{5}$
1) Butterworth (Fig. 3-6) 	0.364	1.02	1.43	7.154	7.154	7.154
2) Chebyshev 	0.33	0.91	1.2	6.0	6.0	6.0
3) Chebyshev (Fig. 3-6) 	0.0749	0.341	0.59	2.949	2.949	2.949
4) Bessel 	1.36	3.02	2.69	1.0457	1.1231	1.0457

the unknown quantities, namely for r_2

$$ar_2^2 + br_2 + c = 0 \tag{4.18}$$

where

$$a = \alpha_0 + \alpha_2 - \alpha_1 - 1; \quad b = \alpha_2 - 2; \quad c = -(1+\rho_2).$$

For r_3, we obtain

$$r_3 = \frac{\rho_2\rho_3}{r_2\alpha_0} \tag{4.19}$$

and for β

$$\beta = 1 + \frac{\rho_2}{\rho_3} - \frac{r_3}{\rho_3}\left[(\alpha_2 - 1) - \frac{1+\rho_2}{r_2}\right]. \tag{4.20}$$

Since r_2 must be positive and real, it follows that the discriminant D of the quadratic equation (4.18) must be greater than zero, thus

$$\begin{aligned} D =& b^2 - 4ac \\ =& (\alpha_2 - 2)^2 + 4(\alpha_0 + \alpha_2 - \alpha_1 - 1)(1+\rho_2) > 0. \end{aligned} \tag{4.21}$$

In [9], it is shown that the necessary and sufficient conditions for r_2 to be real and positive result from (4.21) in the form of an upper—and in some rare cases also lower—bound on the design frequency ω_0. Thus

$$\frac{1}{RC} = \omega_0 < \omega_{0\,\max}. \tag{4.22}$$

The upper bound $\omega_{0\,\max}$ depends mainly on the polarity of the coefficient a in (4.18), and to some extent also on the polarity of the coefficient b. The main results of the realizability conditions for an impedance-tapered third-order low-pass filter are summarized in [9, Table I]. Two main conditions are derived in [9]. The first (Condition I) requires that the coefficient $a = \alpha_0 + \alpha_2 - \alpha_1 - 1 > 0$. This results in the requirement that the design frequency $\omega_0 = (RC)^{-1}$ is less than an upper-bound frequency ω_a. Furthermore, ω_0 must also be less than a second boundary frequency ω_{D_I}. Thus the upper bound on ω_0 for Condition I is the smaller of the two design frequencies, ω_a and ω_{D_I}

$$\omega_{0\,\max} = \min\{\omega_a, \omega_{D_I}\}. \tag{4.23}$$

Should, for some reason, ω_0 be required to be larger than ω_a, then the coefficient a is negative, and Condition II is valid. As shown in [9], this condition is far more confining than Condition I and should, if possible, be avoided.

A realizability condition for the product r_2r_3 is also derived in [9]. This product is shown to also depend on $\omega_{0\,\max}$, namely

$$\frac{r_2r_3}{\rho_2\rho_3} < \frac{\omega_{0\,\max}^3}{a_0}. \tag{4.24a}$$

This condition implies that for ideal impedance tapering, in which case $r_2r_3 = \rho_2\rho_3$, we must have

$$\frac{\omega_{0\,\max}^3}{a_0} > 1. \tag{4.24b}$$

Whether, and how, these conditions can be fulfilled depends entirely on the coefficients a_i of the specified filter.

Finally, in [9], bounds on r_3 guaranteeing both that r_2 be real and positive, and that the gain β be larger than or equal to unity, are shown to be

$$\rho_2\rho_3\frac{r_2}{1+\rho_2}\left[\frac{\alpha_2 - 1}{r_2\alpha_0} - \frac{1}{\rho_3}\right] < r_3 < \frac{1}{r_2\alpha_0}\rho_2\rho_3 \tag{4.25}$$

where it is understood that the ω_0 contained in α_0 and α_2 obeys the inequality in (4.22).

Design Examples: In what follows, we go through some filter design examples that are summarized in Table VI.

Consider the amplitude tolerance limits of the third-order Chebyshev low-pass filter, and the coefficients of the third-order transfer function satisfying these specifications (see Table VI, line 3).

TABLE VII
COMPONENT VALUES OF THIRD-ORDER FILTERS AS IN FIG. 6 (RESISTORS IN kΩ, CAPACITORS IN pF)

1) Specs. given in Table 4.2	R_1	R_2	R_3	r_2	r_3	C_1	C_2	C_3	ρ_2	ρ_3	R_0	β	$\omega_0 \cdot 10^5$
2) Chebyshev: line 3*	5.79	5.41	15.76	0.93	2.72	900	300	100	3	9	10	2	1.916
3) Butterworth: line 1	10	30	90	3	9	213	20	23	10.56	8.99	10	4	4.69
4) Chebyshev: line 2	10	30	90	3	9	235	21	23	11.14	10.4	10	4	4.25
5) Chebyshev: line 3	8.88	3.4	16.36	3	9	900	300	100	3	9	10	2.7	1.25

*The line number refers to Table VI.

For reasons given in [9], we restrict ourselves to using Condition I wherever possible. In doing so, we obtain essentially the same value for $\omega_a$and ω_{D_I}, so that the upper bound on $\omega_a \approx \omega_{D_I} = \omega_{0\,\max} = 4a_0/(4a_1 - a_2^2) = 294.91 \cdot 10^3$ rad. Note that $\omega_{0_{\max}}^3/a_0 = 0.342$. Thus, with (4.24b), it follows that for the filter given by the coefficients in Table VI, (line 3), ideal impedance tapering is not possible. This is because ideal impedance tapering requires that $r_2 r_3 = \rho_2 \rho_3$, in which case, according to condition (4.24b), $(\omega_{0\,\max})^3/a_0$ must be larger than unity.

We now go through the step-by-step design for the third-order Chebyshev low-pass filter satisfying the specifications given in Table VI, line 3, while at the same time obeying different types of impedance-tapering criteria.

1) Impedance Tapering of the Capacitors: For a capacitively impedance-tapered filter, the step-by-step design proceeds as follows:

i) Calculate $\omega_{0\,\max}$: From [9, Table I], we obtain $\omega_a \approx \omega_{D_1} = \omega_{0\,\max} = 4a_0/(4a_1 - a_2^2) = 2.9491 \cdot 10^5$ rad/s.

ii) Select ρ_2, ρ_3, ω_0: The selection of these values is influenced by the upper bound given by (4.24a), thus $r_2 r_3/\rho_2 \rho_3 < \omega_{0\,\max}^3/a_0 = 0.342$. Letting $\rho_2 = 3$ and $\rho_3 = 9$ it follows that $r_2 r_3 < \rho_2 \rho_3 \omega_{0\,\max}^3/a_0 = 9.246$. For practical reasons (e.g., in terms of component values), we chose $\omega_0 = 1.916 \cdot 10^5$.

iii) Calculate α_0, α_1, and α_2 and a, b, and c of (4.18): With $\omega_0 = 1.916 \cdot 10^5$, we obtain $\alpha_0 = a_0/\omega_0^3 = 10.649$, $\alpha_1 = a_1/\omega_0^2 = 9.289$, $\alpha_2 = a_2/\omega_0 = 3.079$, and therefore

$$a = \alpha_0 + \alpha_2 - \alpha_1 - 1 = 3.439; \quad b = \alpha_2 - 2 = 1.079$$
$$c = -1 - \rho_2 = -4$$

iv) Calculate r_2 and r_3: Solving the quadratic equation (4.18) for r_2, we obtain $r_2 = 0.933$. From (4.19), we have $r_2 r_3 = \rho_2 \rho_3/\alpha_0 = 2.535$ and with (4.30) $r_3 = 2.718$.

v) Select C_1 and compute R_1, R_2, and R_3: We select $C_1 = 900$ pF, thus $R_1 = (\omega_0 C_1)^{-1} = 5.799$ kΩ and, with (4.30) and (4.31), $R_2 = r_2 \cdot R_1 = 5.41$ kΩ and $R_3 = r_3 \cdot R_1 = 15.76$ kΩ

vi) Compute $\beta = \mathrm{K}$: From (4.20), we obtain $\beta = 1.999 \approx 2$.

Referring to Fig. 6, the component values for the resulting circuit are given in line 2 of Table VII. A simple first-order check for the correctness of these results is to use (3.12) in order to verify that $a_0 = (R_1 R_2 R_3 C_1 C_2 C_3)^{-1}$ and that the obtained gain $\beta = K$.

2) Impedance Tapering of the Resistors: In the preceding examples, we considered impedance tapering of the capacitors by starting out with values of ρ_c (i.e., ρ_2 and ρ_3 in Fig. 6) and then calculating the resistor values by computing r_2 and r_3. These depend on the choice of the design frequency $\omega_0 = (R_1 C_1)^{-1}$, which must always be smaller than $\omega_{0\,\max}$. The latter depends on the desired filter transfer function and its coefficients. As was pointed out earlier, impedance tapering both the capacitors and the resistors for filters of higher than second order is possible only in rare cases, since the degrees of freedom necessary to satisfy a given set of filter specifications permit only the capacitors or the resistors to be tapered. The preceding examples have demonstrated that impedance tapering only the capacitors provides a significant improvement of the insensitivity to component tolerances. We shall now show that the alternative procedure, i.e., tapering only the resistors, is effective in the same way. To show this, we consider the Butterworth filter specified in line 1 of Table VI. The frequency response of this filter for various capacitive-impedance-tapering values and component tolerances was given in Fig. 7. For this example we select $K = 4$. In what follows, we show how to design this filter, but with tapering of the resistors rather than the capacitors. The degree of desensitization to component tolerances obtained with this resistively-tapered filter is similar to that of the capacitively-tapered version shown in Fig. 7.

i) Calculate $\omega_{0\,\max}$: From [9], we have $\omega_{o\,\max} = 4a_0/(4a_1 - a_2^2)$; thus $\omega_{0\,\max} = 7.1544 \cdot 10^5$.

ii) Select r_2, r_3, and ω_0: From (4.24a), we obtain $r_2 r_3/\rho_2 \rho_3 < 1.006$. Letting $r_2 = 3$ and $r_3 = 9$, we find $\rho_2 \rho_3 > 27/1.006 \approx 27$. For practical reasons (in terms of component values), we select $\omega_0 = (R_1 C_1)^{-1} = 4.695 \cdot 10^5$.

iii) Calculate α_0, α_1, and α_2, and a, b, and c of (4.18): From (4.16), we obtain $\alpha_0 = 3.5176$, $\alpha_1 = 4.6276$, $\alpha_2 = 3.046$, and therefore $a = \alpha_0 + \alpha_2 - \alpha_1 - 1 = 0.936$, $b = \alpha_2 - 2 = 1.046$, $c = -1 - \rho_2$

iv) Calculate $\rho_2 \rho_3$: Solving (4.18) for ρ_2, where we insert $c = -(1 + \rho_2)$, we obtain

$$\rho_2 = a r_2^2 + b r_2 - 1 \tag{4.26}$$

which is a first-order equation, in contrast to the quadratic equation for r_2 [see (4.18)] which must be solved for capacitance tapering. With the values given in step iii) above, we obtain $\rho_2 = 10.562$ and, with (4.19), $\rho_3 = r_2 r_3 \alpha_0/\rho_2 = 8.99$. Note that the condition in (4.24a) is automatically satisfied if $\omega_0 < \omega_{0\ \max}$.

v) Select R_1 and compute C_1, C_2, and C_3: We select $R_1 = 10$ kΩ, thus $C_1 = (\omega_0 R_1)^{-1} = 2.13$ pF, $C_2 = C_1/\rho_2 = 20.1$ pF, and $C_3 = C_1/\rho_3 = 23.7$ pF.

vi) Compute $\beta = K$: With (4.20) we obtain $\beta = 3.98 \approx 4$.

The resulting component values are summarized in Table VII line 3. Similarly, for the filter characteristics depicted in line 2 of Table VI, we obtain the Chebyshev filter coefficients $a_0 = 3.30 \cdot 10^{17}$, $a_1 = 0.91 \cdot 10^{12}$, $a_2 = 1.2 \cdot 10^6$. The resulting component values are given in Table VII, line 4.

Finally, consider a third-order Bessel low-pass filter with a loss of 1 dB at 75 kHz, and 15 dB at 300 kHz (see line 4, Table VI).[1] With the filter coefficients given in Table VI, we proceed as follows:

i) Calculate $\omega_{0\ \max}$: From [9, Condition I], we obtain $\omega_a = 1.046 \cdot 10^6$ and $\omega_{D_I} = 1.123 \cdot 10^6$. Thus, it follows that $\omega_{0\ \max} = \omega_a = 1.046 \cdot 10^6$.

ii) Select r_2, r_3, and ω_0: With (4.24a) we obtain $r_2 r_3/\rho_2\rho_3 < 0.842$. Letting $r_2 = 3$ and $r_3 = 9$, we find $\rho_2 > 27/0.842 \approx 32$ and select $\omega_0 = 0.8 \cdot 10^6$.

iii) Calculate α_0, α_1, α_2 and a, b, and c of (4.18): We obtain $\alpha_0 = 2.6563$, $\alpha_1 = 4.7188$, and $\alpha_2 = 3.3625$, and therefore, $a = \alpha_0 + \alpha_2 - \alpha_1 - 1 = 0.3$, $b = \alpha_2 - 2 = 1.3625$, and $c = -1 - \rho_2$.

iv) Calculate ρ_2 and ρ_3: We obtain $\rho_2 = ar_2^2 + br_2 - 1 = 5.788$ and, with step iii) above, $\rho_3 = r_2 r_3 \alpha_0/\rho_2 = 12.39$.

v) Select R_1 and compute C_1 C_2, and C_3: We select $R_1 = 5$ KΩ, thus $C_1 = (\omega_0 R_1)^{-1} = 250$ pF, $C_2 = 43.2$ pF and $C_3 = 20.2$ pF.

vi) Compute $\beta = K$: With (4.20), $\beta = 1.39$.

Note that for this filter $\omega_a < \omega_{D1}$. Had we selected an ω_0 value larger than ω_a (but smaller than ω_{D1}) we would not have obtained a realizable filter.

3) Influence of the Design Frequency ω_0: We have shown that in order to find a realizable third-order filter circuit capable of satisfying given filter requirements we must select $\omega_0 = (R_1 C_1)^{-1} < \omega_{0\ \max}$ where $\omega_{0\ \max}$ is given in terms of the filter coefficients as in (4.23). The question to be answered now is: what influence does the choice of ω_0 have on the filter design, assuming, of course, that it is chosen less than $\omega_{0\ \max}$?

To find the influence of the choice of ω_0 on the component sensitivity of a given circuit, we proceed as follows. Using the coefficients of the Chebyshev third-order transfer function given in line 3, Table VI as an illustrative example, the procedure can be summarized by the following steps:

i) Select ρ_2, ρ_3 and $\omega_0 < \omega_{0\ \max}$: For the given coefficients, we have $\omega_{0\ \max} = 2.9491 \cdot 10^5$ rad/s. Select $\rho_2 = 3$, $\rho_3 = 9$, $\omega_0 = 1.25 \cdot 10^5$.

ii) Calculate α_0, α_1, α_2 and a, b, and c: $\alpha_0 = a_0/\omega_0^3 = 38.3488$; $\alpha_1 = 21.824$; $\alpha_2 = 4.72$; $a = \alpha_0 + \alpha_2 - \alpha_1 - 1 = 20.248$; $b = \alpha_2 - 2 = 2.72$; $c = -\rho_2 - 1 = -4$.

[1] Note that with an equivalent third-order Chebyshev low-pass filter, a loss of less than 0.5 dB at 75 kHz, and of at least 38 dB at 300 kHz, can be obtained (see line 3, Table VI).

TABLE VIII
DEPENDENCE OF DESIGN PARAMETERS r_2, r_3, r_3/r_2 AND β ON SELECTION OF ω_0

Filter No.	$\omega_0 \cdot 10^5$	C_1 [pf]	R_1 [kΩ]	ρ_2	ρ_3	r_2	r_3	r_3/r_2	β
①	1	900	11.11	3	9	0.256	1.4	5.47	3.0
②	1.25	900	8.88	3	9	0.382	1.84	4.81	2.71
③	1.5	900	7.4	3	9	0.544	2.24	4.12	2.43
④	1.75	900	6.35	3	9	0.754	2.56	3.4	2.17
⑤	2.0	900	5.6	3	9	1.04	2.77	2.66	1.92
⑥	2.25	900	4.94	3	9	1.455	2.82	1.94	1.69
⑦	2.5	900	4.44	3	9	2.15	2.62	1.22	1.48
⑧	2.6	900	4.27	3	9	2.594	2.44	0.94	1.41
⑨	2.75	900	4.04	3	9	3.806	1.97	0.52	1.35
⑩	2.9	900	3.83	3	9	8.58	1.03	0.12	1.27

iii) Calculate r_2 and r_3: From (4.18), we have 20.2448 $r_2^2 + 2.72 r_2 - 4 = 0$, which gives the positive real root $r_2 = 0.3824$ and, with (4.19) $r_3 = \rho_2\rho_3/r_2\alpha_0 = 1.84$.

iv) Select C_1 and calculate R_1, R_2, R_3, C_2, C_3: We select $C_1 = 900$ pF, thus $R_1 = (\omega_0 C_1)^{-1} = 8.88$ kΩ, $R_2 = r_2 R_1 = 3.4$ kΩ, $R_3 = r_3 R_1 = 16.36$ kΩ, $C_2 = C_1/\rho_2 = 300$ pF, and $C_3 = C_1/\rho_3 = 100$ pF.

We can verify these results to a first order [see (3.12)] since $a_0 = (R_1 R_2 R_3 C_1 C_2 C_3)^{-1} = 74.9 \cdot 10^{15}$.

v) Calculate $\beta = K$: With (4.19) and (4.20), we have

$$\beta = 1 + \frac{\rho_2}{\rho_3} - \underbrace{\frac{\rho_2}{\alpha_0 r_2}}_{\frac{r_3}{\rho_3}}\left[\alpha_2 - 1 - \frac{(1+\rho_2)}{r_2}\right] = 2.712. \quad (4.27)$$

This results in the circuit values given in Table VII, line 5.

Going through the five design steps above for ten different values of ω_0 (all of which must, of course, be less than $\omega_{0\ \max}$), we obtained the ten different third-order circuits determined by the design values listed in Table VIII. The resulting functions of r_2, r_3, (r_3/r_2), and β versus the ten values of ω_0 are plotted in Fig. 12. Monte Carlo runs of the ten resulting third-order low-pass filters showed that the deviation from the ideal filter response becomes smallest for the value of ω_0 in the region of $2.6 \cdot 10^5$. A glance at Fig. 12 shows that this corresponds to a ratio of $r_3/r_2 \approx 1$, i.e., for the case that $r_2 = r_3$. *Interestingly enough, this is similar to the conclusion arrived at with second-order networks.* For example, the second-order filter with $r = 1$ seemed equally good, and possibly even slightly better than the filters designed for ideal impedance tapering (i.e., $r = \rho$). The reason given there, which is also valid here, is that because some of the sensitivities are proportional to r and others to r^{-1}, setting $r = 1$ provides an optimum compromise. This corresponds, in the third-order case, to making $r_2 = r_3 = r$. *Thus, we conclude from this and other examples that the ω_0 value to be selected is the one for which* $r_2 = r_3$. Whether this is always true is difficult to tell. Nevertheless, experiments with numerous other third-order low-pass filters with capacitively tapered impedances, and the design frequency ω_0 selected such that $r_2 = r_3$, have invariably produced similar results. Presumably this is true only when "ideal-tapering" cannot take place, which is generally the case (i.e., whenever $\omega_{0\ \max}^3 < a_0$). In any event, whether the one case (i.e., ideal tapering) is superior to the other does not seem very relevant, since the difference between

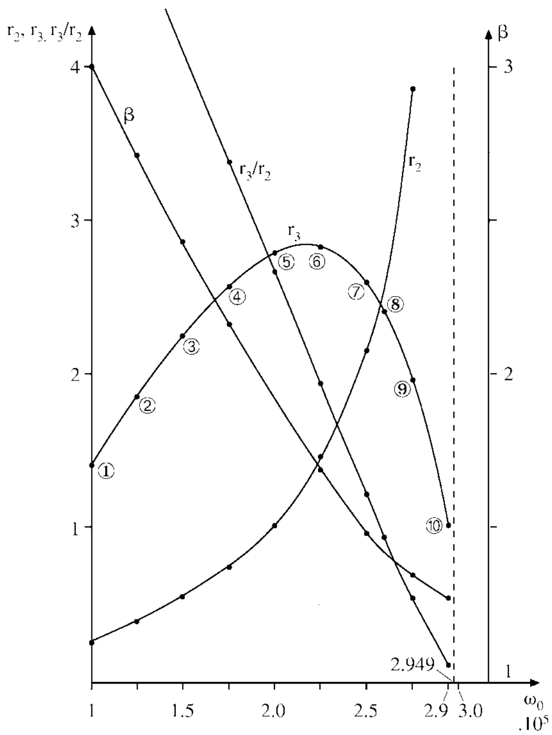


Fig. 12. Design parameters r_2, r_3, r_3/r_2, and β as a function of ω_o.

the two appears to be minimal, and ideal tapering is rarely possible. Thus, the obvious choice is to take the simpler of the two alternatives. *This choice which, more to the point, is generally realizable, is to make* $r_2 = r_3 = r$.

4) Impedance Tapering of the Capacitors with $r_2 \approx r_3 = r$*:* As discussed above, the design equations (4.18)–(4.20) are implemented by selecting the design parameters ω_0, ρ_2, and ρ_3 and then computing the remaining design parameters r_2, r_3, and β. This is done by following the six-step design procedure outlined under 1) and 2) above. If we now let $r_2 = r_3 = r$, and select tapering values for ρ_2 and ρ_3, we can use (4.18)–(4.20) to compute ω_0, r, and β. We cannot compute these values explicitly, however, because the resulting polynomial equations in ω_0 or r are of sixth order, and therefore, are not directly solvable. Instead, we can compute the bounds on ω_0 and then iteratively solve the equations until we have, more or less, $r_2 \approx r_3 = r$. "More or less" is quite sufficient here, since the optimum for which the component sensitivities are minimum is relatively broad. The resulting design procedure is summarized in the following design steps. As in one of the previous illustrative examples, we shall again use the specifications for the third-order Chebyshev filter specified in line 3 of Table VI.

i) Calculate $\omega_{0\ \max}$*:* From [9], we obtain

$$\omega_{0\ \max} = \frac{4a_0}{4a_1 - a_2^2} = 2.9491 \cdot 10^5 \text{ rad/s.} \tag{4.28}$$

ii) Select ρ_2*,* ρ_3*, and calculate* $\omega_{0\ \min}$*:* As in the previous examples, we select $\rho_2 = 3$ and $\rho_3 = 9$. With $r_2 = r_3 = r$, we have from (4.16) and (4.19)

$$\omega_0 = \left(a_0 \frac{r^2}{\rho_2 \rho_3}\right)^{1/3}. \tag{4.29}$$

In order to guarantee a minimum degree of resistive tapering, we require that $r > 1$. The lower bound on ω_0, i.e., $\omega_{0\ \min}$, is then obtained for $r = 1$, i.e.,

$$\omega_{0\ \min} = \left(\frac{a_0}{\rho_2 \rho_3}\right)^{1/3}. \tag{4.30}$$

Thus, with Table VI, line 3, we have $\omega_{0\ \min} = (0.749 \cdot 10^{17}/27)^{1/3} = 1.405 \cdot 10^5$ rad/s.

iii) Select ω_0 *between* $\omega_{0\ \min}$ *and* $\omega_{0\ \max}$ *and calculate* r_2 *and* r_3 *from (4.18) and (4.19), respectively.*

iv) Repeat Step iii) with new ω_0 *until* $r_2 \approx r_3 = r$ *is found:* These two steps were already carried out for this example under 3) above. The value of $\omega_0 = 2.6 \cdot 10^5$ rad/s was found, for which $r_2 = 2.594$ and $r_3 = 2.44$.

The results of this design are listed in Table VIII under (8). It was found that this design yielded the best results, i.e., the lowest sensitivity to component tolerances.

v) Select C_1 *and calculate* R_1, R_2, R_3, C_2, C_3*:* We select $C_1 = 900$ pF and obtain $R_1 = (\omega_0 C_1)^{-1} = 4.27$ kΩ, $R_2 = r_2 R_1 = 11.07$ kΩ, $R_3 = r_3 R_1 = 10.42$ kΩ, $C_2 = C_1/\rho_2 = 300$ pF, $C_3 = C_1/\rho_3 = 100$ pF.

vi) Calculate β*:* With (4.20), we obtain $\beta = 1.41$.

Note that for the component calculations, we must use the actual r_2 and r_3 values obtained in step iv). The fact that they are close to each other is important and determines the value of ω_0 used in the final circuit design.

The steps outlined above can readily be carried out by computer. A MATLAB program going through these steps has been developed. Using this program (TAPERCALC),[2] the values of r_2, r_3, and β are calculated for a variety of filters. For each filter, $\omega_{0\ \min}$ and $\omega_{0\ \max}$ are calculated according to (4.30) and (4.23), respectively, and the ω_0 value found for which $r_2 \approx r_3 = r$. This is obtained at the intersection of the $r_2(\omega_0)$ and $r_3(\omega_0)$ curves, a typical example of which is plotted as a function of ω_0 in Fig. 13.

Monte Carlo runs for four ω_0 values for each of the filters listed in Table VI were carried out. The first and second (i.e., lowest and highest) value of ω_0, designated by ω_{0L} and ω_{0H}, respectively, was within 10% of $\omega_{0\ \min}$ and $\omega_{0\ \max}$, respectively. The third ω_0 value, designated ω_0 $(r_2 = 1)$, was the ω_0 value for which r_2 is equal to unity. This ω_0 value is readily calculated from (4.18). Thus, for $r_2 = 1$, we obtain

$$1 + \frac{b}{a} - \frac{1}{a} - \frac{\rho_2}{a} = 0. \tag{4.31}$$

With (4.17) and (4.18), this can be expressed as a third order polynomial in ω_0, namely

$$\omega_0^3(4 + \rho_2) - 2a_2\omega_0^2 + a_1\omega_0 - a_0 = 0 \tag{4.32}$$

the real root of which is ω_0 $(r_2 = 1)$. The fourth ω_0 value used for the Monte Carlo analysis was our "optimum" value designated $\hat{\omega}_0 = \omega_0(r_2 = r_3)$, i.e., the one for which $r_3/r_2 \approx 1$, as found by the TAPERCALC program. For all the examples, the Monte Carlo runs for $\hat{\omega}_0$ showed the smallest sensitivity to component tolerances. These were followed

[2] This program was written by H. P. Schmid of the Institute for Signal Processing, Zürich, Switzerland.

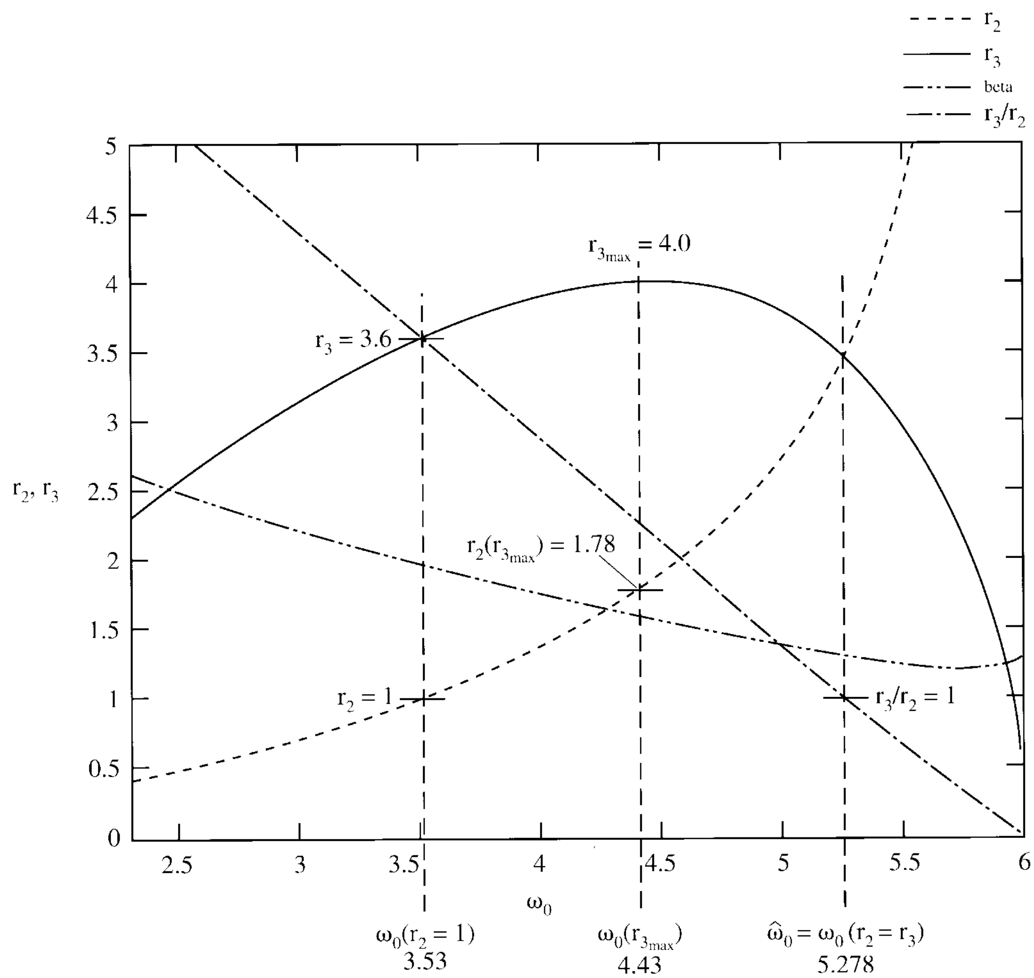


Fig. 13. Implicit and graphical method of finding ω_o for the case that $r_2 \approx r_3$. Example: Chebyshev low pass (0.04/25 dB; 80/300 kHz).

by those for $\hat{\omega}_{0H}$ which provided similarly good results. As pointed out above, the optimum range for ω_0 seems to be relatively wide, i.e., in the range between $\hat{\omega}_0$ and ω_{0H}. However, since the radian frequency $\hat{\omega}_0 = \omega_0(r_2 = r_3)$ is invariably smaller than ω_{0H}, it is the recommended value to use. This is because it is generally desirable for the design value ω_0 to be as small as possible, since this permits a larger input *RC* product, i.e., $R_1 C_1$, to be used. A larger R_1 provides a higher input impedance to the filter, which is generally advantageous; a larger C_1 permits larger tapering factors ρ_2 and ρ_3 to be used, without $C_3 = C_1/\rho_3$ becoming so small that it is comparable in value to the parasitic capacitance of the circuit.

V. Conclusion

A procedure for the design of allpole filters with low sensitivity to component tolerances has been presented. The filters are based on *RC* ladder structures combined with single operational amplifiers. The filter amplifier provides a low output impedance and supplies positive feedback in order to obtain pole Q's larger than 0.5, i.e., complex-conjugate poles. It is shown that by the use of impedance tapering, in which L-sections of the *RC* ladder are successively impedance-scaled upwards, from the driving source to the amplifier input, the sensitivity of the filter characteristics to component tolerances can be significantly decreased. Various schemes for this newly introduced concept of impedance tapering are presented. Detailed design equations for second- and third-order low-pass filters are given. The extension to other types of allpole filters, i.e., high-pass and bandpass, follows precisely the same principles as those presented here, and will be reported on shortly. Although the concept of impedance tapering for allpole filters of the same topology, but higher than third order, is perfectly valid and will have the same beneficial results, closed-form design equations cannot be derived for higher than third-order-filters. Equi-valued resistor and capacitor filter circuits for Butterworth and Chebyshev filters up to sixth-order have, in fact, been published, but their desensitization to component tolerances by impedance tapering can be accomplished only by iterative procedures. However, with the design methods presented here, higher order low-sensitivity allpole filters can be obtained by cascading second- and third-order low sensitivity (i.e., impedance-tapered) filter circuits. Since practical experience has shown that in a large segment of applications, the required filter order is between two and, say, five, the design methods outlined here should be broadly applicable. This all the more, since the circuit topology of the filters dealt with are basically conventional. Indeed, it is merely by the judicious choice of component values such that impedance tapering is achieved that the significant desensitization to component tolerances is obtained. The design procedure therefore adds nothing to the cost of conventional circuits; component count and topology remain unchanged, whereas the component values, selected for impedance tapering, account for the considerable decrease in component-tolerance sensitivity.

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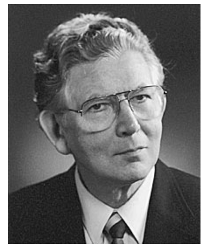

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