

Low-Temperature Specific Heat of Spin-1/2 Anisotropic Heisenberg Ring

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Low-temperature specific heat of Heisenberg-Ising ring at $|d| < 1$ is calculated by the method of non-linear integral equations. Specific heat in constant magnetic field (C_H) is proportional to temperature (T) at $|2\mu_0 H| < (1+d)J$. In particular $\lim_{H \rightarrow 0} \lim_{T \rightarrow 0} C_H/T$ is $2\theta/(3J \sin \theta)$ where $\theta = \cos^{-1} d$, $0 \leq \theta < \pi$. It is conjectured that $\lim_{T \rightarrow 0} \lim_{H \rightarrow 0} C_H/T = \lim_{H \rightarrow 0} \lim_{T \rightarrow 0} C_H/T$ from the result of numerical calculation. Low-temperature specific heat of the one-dimensional X-Y-Z model is proportional to $T^{-3/2} \exp(-\alpha/T)$ in the case of $J_z > J_y > 0$, $J_y \geq J_x > -J_y$.

§ 1. Introduction

The Heisenberg-Ising model plays a very important role in the theory of magnetism and quantum fluids. The one-dimensional system of this model can be treated by the method of Bethe's hypothesis. The Hamiltonian is

$$\mathcal{H} = J \sum_{i=1}^N \{S_i^z S_{i+1}^z + S_i^y S_{i+1}^y + d(S_i^x S_{i+1}^x - \frac{1}{4})\} - 2\mu_0 H \sum_{i=1}^N S_i^z, \quad S_{N+1} = S_1. \quad (1.1)$$

The ground state energy, magnetic susceptibility ($\lim_{H \rightarrow 0} \lim_{T \rightarrow 0} \chi$) and elementary excitations were obtained by solving linear-integral equations.

We gave a set of non-linear integral equations for the free energy at $|d| < 1$ in previous papers,¹⁾ where we used some assumptions about the distribution of quasi-momenta on the complex plane. Though we cannot prove their validity, the high-temperature expansion of free energy is calculated through the second term and we find that it coincides with a known expansion. In Appendix A we will calculate the third term, by which the coincidence is supported. One obtains that the free energy per site is

$$f(T, H) = -T \log \left(2 \operatorname{ch} \frac{\mu_0 H}{T} \right) - \frac{Jd}{4} \left(\frac{1}{\operatorname{ch}^2(\mu_0 H/T)} \right) - \frac{J^2}{4T} \left\{ \frac{1+2d^2}{4 \operatorname{ch}^2(\mu_0 H/T)} \right. \\ \left. - \frac{3d^2}{8 \operatorname{ch}^4(\mu_0 H/T)} \right\} + O\left(\frac{1}{T^2}\right) \quad (1.2)$$

by calculating $\lim_{N \rightarrow \infty} \{-T \log(\operatorname{Tr} \exp(-\mathcal{H}/T))/N\}$. It was shown in Ref. 1) that this set of equations gives known exact results at zero temperature. So it is quite possible that our assumptions about the distribution of quasi-momenta are correct and that the set of equations gives the exact free energy of one-

dimensional Heisenberg-Ising model.

In § 2 we give the set of integral equations and some other representations of this set. In § 3 we obtain a set of linear integral equations for $\lim_{T \rightarrow 0} C_H/T$ at $J(1+\Delta) > 2\mu_0 H > 0$. We will obtain

$$\lim_{H \rightarrow 0} \lim_{T \rightarrow 0} C_H/T = \frac{2\theta}{3J \sin \theta}, \quad \theta = \cos^{-1} \Delta, \quad 0 \leq \theta < \pi \quad \text{for } J > 0.$$

In § 4 we give a set of non-linear equations for the free energy at $T \ll J$ and $T \sim \mu_0 H$. We calculate numerically the value of $\lim_{T \rightarrow 0} \lim_{H \rightarrow 0} C_H/T$ and find $\lim_{H \rightarrow 0} \lim_{T \rightarrow 0} C_H/T = \lim_{T \rightarrow 0} \lim_{H \rightarrow 0} C_H/T$ within the error of numerical calculation in the case $\Delta = \cos(\pi/3), \cos(\pi/4), \dots, \cos(\pi/10)$. In § 5 the low-temperature free energy of the spin- $\frac{1}{2}$ X-Y-Z ring is calculated through the use of integral equations given in Ref. 1).

§ 2. Mathematical formulation

In this section we give notations of series of numbers and functions which will be used in this paper:

$$\begin{aligned} \cos \theta &= \Delta, \quad p_0 \equiv \pi/\theta, \quad p_1 = 1, \\ p_i &= p_{i-2} - [p_{i-2}/p_{i-1}]p_{i-1}, \quad \nu_i = [p_{i-1}/p_i], \\ m_i &= \sum_{l=1}^i \nu_l, \\ y_{-1} &= 0, \quad y_0 = 1, \quad y_1 = \nu_1, \quad y_i = \nu_i y_{i-1} + y_{i-2}, \\ n_j &= y_{i-2} + (j - m_i)y_{i-1} \quad \text{for } m_{i-1} \leq j < m_i, \\ q_j &= (-1)^{i-1} (p_{i-1} + (m_{i-1} - j)p_i) \quad \text{for } m_{i-1} \leq j < m_i, \end{aligned} \tag{2.1}$$

$$\begin{aligned} s_i(x) &= \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{e^{i\omega x}}{2 \operatorname{ch} p_i \omega} = \frac{1}{4p_i} \operatorname{sech} \frac{\pi x}{2p_i}, \\ d_i(x) &= \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{\operatorname{ch}(p_i - p_{i+1})\omega \cdot e^{i\omega x}}{2 \operatorname{ch} p_i \omega \operatorname{ch} p_{i+1} \omega}. \end{aligned} \tag{2.2}$$

We define the following functions,

$$D_{j_l}(x) = \begin{cases} (1 - 2\delta_{j, m_{i-1}})\delta_{j, l+1} s_i(x) + \delta_{j, l-1} s_i(x) & \text{for } m_{i-1} \leq j \leq m_i - 2, \\ (1 - 2\delta_{j, m_{i-1}})\delta_{j, l+1} s_i(x) + \delta_{j_l} d_i(x) + \delta_{j, l-1} s_{i+1}(x) & \text{for } j = m_i - 1, \end{cases} \tag{2.3}$$

$$\begin{aligned} \alpha_j(x) &= \int_{-\infty}^{\infty} \frac{\operatorname{ch}(p_0 - 1 - j)\omega}{\operatorname{ch}(p_0 - 1)\omega} \frac{e^{i\omega x} d\omega}{2\pi} = \frac{1}{p_0 - 1} \left\{ \operatorname{ch} \frac{\pi x}{2(p_0 - 1)} \frac{\sin \pi(p_0 - 1 - j)}{2(p_0 - 1)} \right\} \\ &\times \left\{ \operatorname{ch} \frac{\pi x}{p_0 - 1} + \cos \frac{\pi(p_0 - 1 - j)}{p_0 - 1} \right\}^{-1} \quad \text{for } j = 1, 2, \dots, \nu_i - 1, \end{aligned} \tag{2.4a}$$

$$\beta(x) = \int_{-\infty}^{\infty} \frac{e^{i\omega x}}{2 \operatorname{ch}(p_0 - 1)\omega} \frac{d\omega}{2\pi} = \frac{1}{4(p_0 - 1)} \operatorname{sech} \frac{\pi x}{2(p_0 - 1)}, \tag{2.4b}$$

$$B_{jl}(x) = \begin{cases} \alpha_{|j-l|} + 2\alpha_{|j-l|+2} + \dots + 2\alpha_{j+l-2} + \alpha_{j+l} & \text{for } j \neq l, \\ 2\alpha_2 + 2\alpha_4 + \dots + 2\alpha_{2l-2} + \alpha_{2l} & \text{for } j = l, \end{cases} \quad (2.4c)$$

$$C_j(x) = s_2^* B_{j, \nu_1-1}(x) + \delta_{j, \nu_1-1} s_2(x), \quad (2.4d)$$

$$a_j(x) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{\sinh q_j \omega}{\sinh p_0 \omega} e^{i\omega x} = \frac{\sin(\pi q_j / p_0)}{2p_0 (\text{ch}(\pi x / p_0) + \cos(\pi q_j / p_0))}. \quad (2.4e)$$

When θ/π is an irrational number and $0 < \theta < \pi/2$, the set of integral equations is

$$\begin{aligned} \ln \eta_j &= -A s_1(x) \delta_{j1} / T + \sum_{i=1}^{\infty} D_{ji}^* \ln(1 + \eta_i), \quad j=1, 2, \dots, \\ \lim_{j \rightarrow \infty} \frac{\ln \eta_j}{n_j} &= \frac{2\mu_0 H}{T}, \quad A = 2\pi J \sin \theta / \theta, \end{aligned} \quad (2.5a)$$

and the free energy per site is

$$f(T, H) = -A \int_{-\infty}^{\infty} s_1(x) a_1(x) dx - T \int_{-\infty}^{\infty} s_1(x) \ln(1 + \eta_1(x)) dx. \quad (2.5b)$$

Equations (2.5a) can be transformed as follows:

$$\begin{aligned} \ln \eta_j &= -\frac{A}{T} \alpha_j(x) + \sum_{i=1}^{\nu_1-1} B_{ji}^* \ln(1 + \eta_i^{-1}) + C_j^* \ln(1 + \eta_{\nu_1}), \quad \text{for } 1 \leq j < \nu_1, \\ \ln \eta_{\nu_1} &= \frac{A}{T} \beta(x) - \sum_{i=1}^{\nu_1-1} C_i^* \ln(1 + \eta_i^{-1}) + \sum_{i>\nu_1} D_{\nu_1 i}^* \ln(1 + \eta_i) - s_2^* C_{\nu_1-1}^* \ln(1 + \eta_{\nu_1}), \\ \ln \eta_j &= \sum_{i=\nu_1}^{\infty} D_{ji}^* \ln(1 + \eta_i) \quad \text{for } j > \nu_1, \\ \lim_{j \rightarrow \infty} \frac{\ln \eta_j}{n_j} &= \frac{2\mu_0 H}{T}. \end{aligned} \quad (2.6a)$$

The expression for free energy (2.5b) can be transformed as follows:

$$\begin{aligned} f(T, H) &= -A \int_{-\infty}^{\infty} s_1(x) (a_1(x) - \alpha_1(x)) dx - T \sum_{j=1}^{\nu_1-1} \int_{-\infty}^{\infty} \alpha_j(x) \ln(1 + \eta_j^{-1}(x)) dx \\ &\quad - T \int_{-\infty}^{\infty} \beta(x) \ln(1 + \eta_{\nu_1}(x)) dx. \end{aligned} \quad (2.6b)$$

By taking the limit in Eqs. (2.5), we obtain integral equations, when θ/π is a rational number. In this case the number of unknown functions becomes finite. Equations (2.5) and (2.6) are useful for the discussion of the low-temperature properties in the antiferromagnetic case ($J > 0$) and ferromagnetic case ($J < 0$), respectively.

§ 3. Low-temperature expansion of the free energy at $\mu_0 H \gg T$

Assume that $\ln \eta_i \gg 1$ for $J > 0$, $l \geq 2$ and $2\mu_0 H \gg T$. The term $\ln(1 + \eta_i)$ in

(2.5a) is replaced by $\ln \eta_l$ and one obtains

$$\begin{aligned} \ln \eta_j(x) = & \int_{-\infty}^{\infty} dx' \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{\text{sh } q_j \omega}{\text{sh } q_1 \omega} e^{i\omega(x-x')} \ln(1 + \eta_1(x')) \\ & + \left(n_j - \frac{q_j}{q_1}\right) \frac{2\mu_0 H}{T} + \sum_{i=2}^{\infty} O(\eta_i^{-1}), \quad j=2, 3, \dots \end{aligned} \tag{3.1a}$$

Substituting the case $j=2$ of this equation into the case $j=1$ of (2.5a), we have

$$\begin{aligned} T \ln \eta_1 = & -\frac{2\pi J \sin \theta}{\theta} s_1(x) + \frac{\pi\mu_0 H}{\pi - \theta} + \int_{-\infty}^{\infty} R_+(x-x') T \ln(1 + \eta_1(x')) dx' \\ & + O\left(T \exp\left(-\frac{2\pi\mu_0 H}{T(\pi - \theta)}\right)\right), \end{aligned} \tag{3.1b}$$

where

$$R_+(x) \equiv \int_{-\infty}^{\infty} \frac{\text{sh}(p_0 - 2)\omega e^{i\omega x}}{2\text{ch } \omega \text{ sh}(p_0 - 1)\omega} \cdot \frac{d\omega}{2\pi} \tag{3.1c}$$

At $J < 0, 1 > \Delta \geq 0$ we assume that $\ln \eta_j \gg 1$ for $j > \nu_1$. Then we have

$$\begin{aligned} T \ln \eta_j(x) = & T \int_{-\infty}^{\infty} d\omega \int_{-\infty}^{\infty} dx' \frac{\text{sh } q_j \omega}{\text{sh } q_{\nu_1} \omega} e^{i\omega(x-x')} \ln(1 + \eta_{\nu_1}(x')) \\ & + \left(n_j - \frac{q_j}{q_{\nu_1}}\right) 2\mu_0 H + \sum_{i > \nu_1}^{\infty} O(T\eta_i^{-1}) \quad \text{for } j > \nu_1. \end{aligned} \tag{3.2a}$$

In the same way as is used in the preceding paragraph, we obtain

$$\ln \eta_j = -\frac{A}{T} \alpha_j(x) + \sum_{i=1}^{\nu_1-1} B_{ji}^* \ln(1 + \eta_i^{-1}) + C_j^* \ln(1 + \eta_{\nu_1}), \tag{3.2b}$$

$$\ln \eta_{\nu_1} = +\frac{A}{T} \beta(x) - \sum_{i=1}^{\nu_1-1} C_i^* \ln(1 + \eta_i^{-1}) + \int_{-\infty}^{\infty} R_-(x-x') \ln(1 + \eta_{\nu_1}) dx' + \frac{\pi\mu_0 H}{\theta T}, \tag{3.2c}$$

where

$$R_-(x) = \int_{-\infty}^{\infty} \frac{\text{sh}(2-p_0)\omega e^{i\omega x}}{2\text{ch}(p_0-1)\omega \cdot \text{sh } \omega} \cdot \frac{d\omega}{2\pi} \tag{3.2d}$$

Considering $-A/T \gg 1$ and $C_j^* \ln(1 + \eta_{\nu_1}) \sim \mu_0 H/T$, we have $\ln \eta_j \gg 1, j=1, 2, \dots, \nu_1 - 1$.

From Eq. (2.6b) we have

$$f(T, H) - f(0, 0) = -T \int_{-\infty}^{\infty} \beta(x) \ln(1 + \eta_{\nu_1}(x)) dx + \sum_{j=1}^{\nu_1-1} O(T\eta_j^{-1}). \tag{3.2e}$$

Considering that $f(T, H)$ is invariant under the transformation $(J, \Delta) \rightarrow (-J, -\Delta)$, we have from Eqs. (3.1), (3.2) and (2.5b)

$$\epsilon_1(x) = \frac{2\pi J \sin \theta}{\theta} \frac{1}{4} \operatorname{sech} \frac{\pi x}{2} + \frac{\pi \mu_0 H}{\pi - \theta} + T \int_{-\infty}^{\infty} R(x-x') \ln(1 + \exp(\epsilon_1(x')/T)) dx', \tag{3.3a}$$

$$R(x) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{i\omega x} \frac{\operatorname{sh}(\pi - 2\theta)\omega}{2 \operatorname{ch} \omega \operatorname{sh}(\pi - \theta)\omega}, \tag{3.3b}$$

$$f(T, H) = -\frac{2\pi J \sin \theta}{\theta} \int_{-\infty}^{\infty} \frac{1}{4} \operatorname{sech} \frac{\pi x}{2} \frac{\theta}{2\pi} \frac{\sin \theta}{\cosh \theta x - \cos \theta} dx - T \int_{-\infty}^{\infty} \frac{1}{4} \operatorname{sech} \frac{\pi x}{2} \ln(1 + \exp(\epsilon_1(x)/T)) + O\left(T \exp\left(-\frac{2\pi \mu_0 H}{(\pi - \theta)T}\right)\right), \tag{3.3c}$$

where θ is determined by

$$\Delta = \cos \theta, \quad 0 \leq \theta \leq \pi \tag{3.3d}$$

for $|\Delta| \leq 1$ and $J > 0$, $2\mu_0 H \gg T$. This set of equations is equivalent to

$$\begin{aligned} \epsilon_1(x) = & -\frac{2\pi J \sin \theta}{\theta} a_1(x) + 2\mu_0 H \\ & + T \int_{-\infty}^{\infty} a_2(x-x') \ln(1 + \exp(-\epsilon_1(x')/T)) dx', \end{aligned} \tag{3.4a}$$

$$a_j(x) = \frac{\theta}{2\pi} \frac{\sin j\theta}{\operatorname{ch} \theta x - \cos j\theta}, \quad j = 1, 2, \tag{3.4b}$$

$$f(T, H) = -\mu_0 H - T \int_{-\infty}^{\infty} a_1(x) \ln(1 + \exp(-\epsilon_1(x)/T)) dx. \tag{3.4c}$$

a) Case $J > 0$, $J(1 + \Delta) > 2\mu_0 H$

We put the solution of (3.4a) at $T=0$ as $\epsilon_1^{(0)}(x)$, which satisfies

$$\epsilon_1^{(0)}(x) = -\frac{2\pi J \sin \theta}{\theta} a_1(x) + 2\mu_0 H - \int_{\epsilon_1^{(0)}(x') < 0} a_2(x-x') \epsilon_1^{(0)}(x') dx'. \tag{3.5}$$

Function $\epsilon_1^{(0)}$ is a monotonically increasing function of x^2 and is $2\mu_0 H$ in the limit $x \rightarrow \pm \infty$ because $a_1(\infty)$ and $a_2(\infty)$ are zero. It is clear that if $-(2\pi J \sin \theta / \theta) \times a_1(0) + 2\mu_0 H < 0$ (namely, $J(1 + \Delta) > 2\mu_0 H$), $\epsilon_1^{(0)}(x)$ has necessarily two zeros $\pm K$, $K > 0$. Using $\epsilon_1^{(0)}(K) = 0$ and (3.5), one can calculate $2\mu_0 H$ from K :

$$2\mu_0 H = \frac{2\pi J \sin \theta}{\theta} \frac{V(K)}{Z(K)}, \tag{3.6a}$$

where

$$V(x) + \int_{-K}^K a_2(x-x') V(x') dx' = a_1(x), \tag{3.6b}$$

$$Z(x) + \int_{-K}^K a_2(x-x') Z(x') dx' = 1, \tag{3.6c}$$

If $T \ll J(1 + \Delta) - 2\mu_0 H$, $\varepsilon_1(x)$ has two zeros even at finite T and we put these zeros $\pm K'$, $K' > 0$. From (3.4a) and (3.5) we have

$$\begin{aligned} \varepsilon_1(x) - \varepsilon_1^{(0)}(x) &+ \int_{-K}^K a_2(x-x') (\varepsilon_1 - \varepsilon_1^{(0)})(x') dx' \\ &= \int_{K'}^K a_2(x-x') \varepsilon_1(x') dx' + \int_{-K}^{-K'} \dots \\ &+ T \int_{-\infty}^{\infty} a_2(x-x') \ln\left(1 + \exp\left(-\frac{|\varepsilon_1(x')|}{T}\right)\right) dx'. \end{aligned} \tag{3.7}$$

The first and the second terms of r.h.s. are of the order of $(K - K')$. The third term is

$$\frac{\pi^2 T^2}{6\varepsilon_1'(K')} (a_2(x - K') + a_2(x + K')) (1 + O(T)). \tag{3.8}$$

From Eq. (3.7) one can see

$$K - K' = O(T^2).$$

Equation (3.7) can then be written as

$$\begin{aligned} \varepsilon_1(x) - \varepsilon_1^{(0)}(x) &+ \int_{-K}^K a_2(x-x') (\varepsilon_1(x') - \varepsilon_1^{(0)}(x')) dx' \\ &= \frac{\pi^2 T^2}{6\varepsilon_1^{(0)'(K)}} (a_2(x - K) + a_2(x + K)) + O(T^3), \end{aligned} \tag{3.9a}$$

or

$$\varepsilon_1(x) - \varepsilon_1^{(0)}(x) = \frac{\pi^2 T^2}{6\varepsilon_1^{(0)'(K)}} U(x) + O(T^3), \tag{3.9b}$$

where $U(x)$ is defined by

$$U(x) + \int_{-K}^K a_2(x-x') U(x') dx' = a_2(x - K) + a_2(x + K). \tag{3.9c}$$

Using Eq. (3.4c) we have

$$\begin{aligned} f(T, H) - f(0, H) &= -T \int_{-\infty}^{\infty} a_1(x) \ln(1 + \exp(-|\varepsilon_1(x)|/T)) \\ &+ \int_{-K'}^{K'} a_1(x) \varepsilon_1(x) dx' - \int_{-K}^K a_1(x) \varepsilon_1^{(0)}(x) dx \\ &= -\frac{\pi^2 T^2}{6\varepsilon_1^{(0)'(K)}} \left[2a_1(K) - \int_{-K}^K a_1(x) U(x) dx \right] + O(T^3). \end{aligned} \tag{3.10a}$$

The bracket of the r.h.s. is $2V(K)$, where $V(x)$ is defined in (3.6b). Differentiating (3.5) and using partial integration, we have

$$\varepsilon_1^{(0)'(x)} = \frac{2\pi J \sin \theta}{\theta} \cdot W(x), \tag{3.10b}$$

$$W(x) + \int_{-K}^K a_2(x-x') W(x') dx' = -\frac{d}{dx} a_1(x). \quad (3.10c)$$

Then we have

$$f(T; H) = f(0, H) - \frac{\pi\theta T^3}{6J \sin \theta} \cdot \frac{V(K)}{W(K)} + O(T^3). \quad (3.11)$$

In the limit $K \rightarrow \infty (H \rightarrow 0)$ we can calculate $V(K)/W(K)$.

By the Fourier transformation (3.6b) and (3.10c), we have

$$V(x) - \int_K^\infty (R(x-x') + R(x+x')) V(x') dx' = \frac{1}{4} \operatorname{sech} \frac{\pi x}{2}, \quad (3.12a)$$

$$W(x) - \int_K^\infty (R(x-x') + R(x+x')) V(x') dx' = \frac{d}{dx} \left(\frac{1}{4} \operatorname{sech} \frac{\pi x}{2} \right). \quad (3.12b)$$

Considering that the inhomogeneous terms are $\frac{1}{2}e^{-\pi x/2}$ and $e^{-\pi x/2}\pi/4$ at $x \gg 1$ and that $R(x)$ becomes zero at $x \rightarrow \infty$, we have

$$\frac{V(K)}{W(K)} = \begin{cases} \frac{2}{\pi} + O(e^{-\pi K}) + O(e^{-2\pi\theta K/(\pi-\theta)}) = \frac{2}{\pi} + O\left(\left(\frac{\mu_0 H}{J}\right)^2\right) \\ \quad + O\left(\left(\frac{2\mu_0 H}{J}\right)^{4\theta/(\pi-\theta)}\right) & \text{for } -1 < \Delta < 1, \\ \frac{2}{\pi} + O(e^{-\pi K}) + O(K^{-3}) = \frac{2}{\pi} + O\left(\left(\frac{\mu_0 H}{J}\right)^2\right) + O\left(\left(\ln\left(\frac{\mu_0 H}{J}\right)\right)^{-3}\right) & \text{for } \Delta = 1. \end{cases} \quad (3.13)$$

Using the relation $C_H = -\partial/\partial T(T^2(\partial/\partial T)(f/T))$, we have

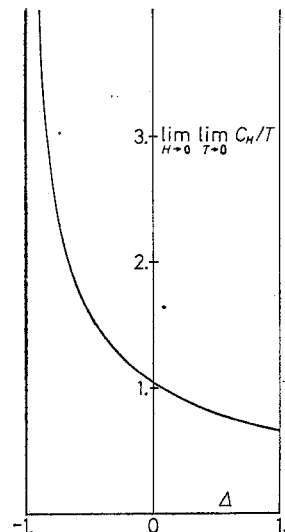
$$\lim_{H \rightarrow 0} \lim_{T \rightarrow 0} C_H/T = \frac{2\theta}{3J \sin \theta}. \quad (3.14)$$

At $\Delta = 1$, $\lim_{H \rightarrow 0} \lim_{T \rightarrow 0} C_H/T$ is $2/3J$. This coincides with Yamada's result calculated from the Fermi liquid theory, and with Johnson and McCoy's⁸⁾ calculation. At $\Delta = 0$, (3.14) is $\pi/3J$. This coincides with the known result for the X-Y model.⁹⁾ In the limit $\Delta \rightarrow -1$, (3.14) becomes infinity.

b) Case $J > 0, 2\mu_0 H \geq J(1 + \Delta), 1 \geq \Delta > -1$

The third term of (3.4a) is of the order of $T^{3/2} \exp((-2\mu_0 H + J(1 + \Delta))/T)$. Substituting (3.4a) into (3.4c), we have

Fig. 1. The coefficient of T -linear low-temperature specific heat in the limit $T \rightarrow 0$ and $H \rightarrow 0$. In the limit $\Delta \rightarrow -1$, $\lim_{H \rightarrow 0} \lim_{T \rightarrow 0} C_H/T$ diverges as $(1 + \Delta)^{-1/2}$.



$$f(T, H) = -\mu_0 H - \frac{T^{3/2}}{\pi} \sqrt{\frac{2}{J}} \int_0^\infty \ln(1 + \exp(\frac{-2\mu_0 H + J(1+\Delta)}{T})) \times \exp(-x^2) dx (1 + O(T)) + O(T^{1/2} \exp(\frac{-2\mu_0 H + J(1+\Delta)}{T})). \tag{3.15a}$$

At $2\mu_0 H - J(1 + \Delta) \gg T$, (3.15a) is

$$f(T, H) = -\mu_0 H - T^{3/2} \frac{1}{\sqrt{2\pi J}} \exp(\frac{-2\mu_0 H - J(1+\Delta)}{T}) [1 + O(T)], \tag{3.15b}$$

and we have

$$C_H = \frac{1}{\sqrt{2\pi J}} \frac{(2\mu_0 H - J(1 + \Delta))^2}{T^{3/2}} \exp(\frac{-2\mu_0 H - J(1+\Delta)}{T}) [1 + O(T)]. \tag{3.15c}$$

At $2\mu_0 H = J(1 + \Delta)$ one obtains from (3.15a)

$$f(T, H) = -\mu_0 H - T^{3/2} \frac{1}{\sqrt{2\pi J}} \zeta(\frac{3}{2}) (1 - \frac{1}{\sqrt{2}}) (1 + O(T^{1/2})), \tag{3.15d}$$

$$C_H = \frac{3}{4} \frac{1}{\sqrt{2\pi}} (1 - \frac{1}{\sqrt{2}}) \zeta(\frac{3}{2}) \sqrt{\frac{T}{J}} (1 + O(T^{1/2})), \tag{3.15e}$$

where ζ is Riemann's zeta function. It is remarkable that the low-temperature specific heat is proportional to $T^{1/2}$ and that the coefficient does not depend on Δ at $2\mu_0 H = J(1 + \Delta)$.

§ 4. Case $\mu_0 H \sim T$ and $T \ll J$

a) Case $J > 0, 1 \geq \Delta \geq 0$

Equations (2.5) are transformed as

$$\ln \eta_j(x) = -\frac{A}{4T} \operatorname{sech} \frac{\pi x}{2} \delta_{j1} + \int_0^\infty \{D_{j1}(x-x') + D_{j1}(x+x')\} \ln(1 + \eta_1(x')) dx',$$

$$\lim_{j \rightarrow \infty} \frac{\ln \eta_j}{n_j} = \frac{2\mu_0 H}{T},$$

$$f(T, H) - f(0, 0) = -2T \int_0^\infty \frac{1}{4} \operatorname{sech} \frac{\pi x}{2} \ln(1 + \eta_1(x)) dx.$$

Putting $x \rightarrow x + (2/\pi) \ln(A/2T)$ and $T \rightarrow 0$, we have a set of non-linear integral equations which determines the T^2 order term of free energy as follows:

$$\ln \eta_j = -\delta_{j1} e^{-\pi x/2} + \sum_{i=1}^\infty D_{ji}^* \ln(1 + \eta_i), \quad j=1, 2, \dots, \tag{4.1a}$$

$$\lim_{j \rightarrow \infty} \frac{\ln \eta_j}{n_j} = y, \tag{4.1b}$$

$$C(y) = \int_{-\infty}^{\infty} e^{-\pi x^2/2} \ln(1 + \eta_1(x)) dx, \tag{4.1c}$$

$$f(T, H) - f(0, 0) = -\frac{2\theta T^2}{\pi J \sin \theta} C\left(\frac{2\mu_0 H}{T}\right) + O(T^4). \tag{4.1d}$$

When π/θ is a rational number, one can reduce this set of equations to that with finite unknowns. Therefore the numerical method is available for the calculation of $C(x)$. Especially when $\pi/\theta=3$, the set of integral equations is

$$\begin{aligned} \ln \eta_1 &= -e^{-\pi x^2/2} + s_1^* \ln(1 + 2\text{ch}(3y/2) \cdot \kappa + \kappa^2), \\ \ln \kappa &= s_1^* \ln(1 + \eta_1). \end{aligned}$$

In order to determine $\lim_{T \rightarrow 0} \lim_{H \rightarrow 0} C_H/T$, it is sufficient to calculate $C(0)$. From the numerical calculation, we have $C(0) = 0.5235378$. This value is very near to $\pi/6 = 0.5235987\dots$. When $\pi/\theta = \nu$ is an integer larger than 3, we have

$$\begin{aligned} \ln \eta_1 &= -e^{-\pi x^2/2} + s_1^* \ln(1 + \eta_2), \\ \ln \eta_j &= s_1^* \ln(1 + \eta_{j-1}) (1 + \eta_{j+1}), \quad 2 \leq j < \nu - 2, \\ \ln \eta_{\nu-2} &= s_1^* \ln(1 + \eta_{\nu-3}) \left(1 + 2 \text{ch}\left(\frac{\nu y}{2}\right) \cdot \kappa + \kappa^2\right), \\ \ln \kappa &= s_1^* \ln(1 + \eta_{\nu-2}). \end{aligned}$$

The results of our numerical calculation of $C(0)$ at $\Delta = \cos(\pi/\nu)$, $\nu = 3, 4, \dots, 10$ are shown in Table I. In this calculation the number of unknown functions is $\nu - 1$. An unknown function is represented by 40 unknown numbers. This set of non-linear equations is solved by the method of iteration through the use of computer NEAC 500. These values are very near to $\pi/6$. We can regard the small deviations from $\pi/6$ as numerical errors. It is very plausible that $C(0)$ is $\pi/6$ for any value of Δ and that $\lim_{T \rightarrow 0} \lim_{H \rightarrow 0} C_H/T$ is $2\theta/3J \sin \theta$, that is, $\lim_{T \rightarrow 0} \lim_{H \rightarrow 0} C_H/T = \lim_{H \rightarrow 0} \lim_{T \rightarrow 0} C_H/T$.

Table I. Numerical calculation of $C(0)$.

ν	Δ	$C(0)$
3	0.500000	0.5235378
4	0.707107	0.5233042
5	0.809017	0.5230769
6	0.866025	0.5229287
7	0.900969	0.5228421
8	0.923880	0.5227903
9	0.939693	0.5227562
10	0.951057	0.5227310

b) Case $J < 0, 1 > \Delta \geq 0$

At $x \gg p_0 - 1$, it is clear that

$$\begin{aligned} \alpha_j(x) &= \frac{1}{p_0 - 1} \sin \frac{\pi(p_0 - 1 - j)}{2(p_0 - 1)} \cdot e^{-\pi x^2/2(p_0 - 1)}, \\ \beta(x) &= \frac{1}{2(p_0 - 1)} e^{-\pi x^2/2(p_0 - 1)}, \end{aligned}$$

from (2.4a) and (2.4b). Using these equations, we have the following integral

equations for T^2 order term of free energy at $\mu_0 H \sim T$ and $T \ll |J|$:

$$\ln \eta_j = 2 \sin \frac{\pi(p_0 - 1 - j)}{2(p_0 - 1)} e^{-\pi x/2(p_0 - 1)} + \sum_{i=1}^{j-1} B_{ji}^* \ln(1 + \eta_i^{-1}) + C_j^* n(1 + \eta_{\nu_1}) \quad \text{for } 1 \leq j \leq \nu_1 - 1, \quad (4.2a)$$

$$\ln \eta_{\nu_1} = -e^{-\pi x/2(p_0 - 1)} - \sum_{j=1}^{\nu_1 - 1} C_j^* \ln(1 + \eta_j^{-1}) - s_2^* C_{\nu_1 - 1}^* \ln(1 + \eta_{\nu_1}) + \sum_{i > \nu_1} D_{\nu_1 i}^* \ln(1 + \eta_i), \quad (4.2b)$$

$$\ln \eta_j = \sum_{i=\nu_1}^{\infty} D_{ji}^* \ln(1 + \eta_i) \quad \text{for } j > \nu_1, \quad (4.2c)$$

$$\lim_{j \rightarrow \infty} \frac{\ln \eta_j}{n_j} = y, \quad (4.2d)$$

$$C(y) = \sum_{\alpha=1}^{\nu_1 - 1} \sin \frac{\pi(p_0 - 1 - j)}{2(p_0 - 1)} \int_{-\infty}^{\infty} e^{-\pi x/2(p_0 - 1)} \ln(1 + \eta_j^{-1}) dx + \int_{-\infty}^{\infty} e^{-\pi x/2(p_0 - 1)} \ln(1 + \eta_{\nu_1}) dx, \quad (4.2e)$$

$$f(T, H) - f(0, 0) = -\frac{2(\pi - \theta) T^2}{\pi |J| \sin \theta} C\left(\frac{2\mu_0 H}{T}\right) + O(T^4). \quad (4.2f)$$

The author believes that $C(0)$ is $\pi/6$ in this case also, though numerical calculation is not yet done.

§ 5. The X-Y-Z model in zero field

In this section we consider the low-temperature specific heat of the X-Y-Z model in zero field. The Hamiltonian is

$$\mathcal{H} = \sum_{i=1}^N J_x S_i^x S_{i+1}^x + J_y S_i^y S_{i+1}^y + J_z S_i^z S_{i+1}^z, \quad S_{N+1} = S_1. \quad (5.1)$$

a) Case $0 \leq J_x \leq J_y < J_z$

Equations (4.10), (4.13) and (4.16) of Ref. 1) are written as

$$\varepsilon_1 = -A s_1(x) + T D_{12}^* \ln(1 + \eta_2) + T D_{11}^* \ln(1 + \exp(\varepsilon_1/T)), \quad (5.2a)$$

$$\ln \eta_j = D_{j1}^* \ln(1 + \exp(\varepsilon_1/T)) + \sum_{i=2}^{\infty} D_{ji}^* \ln(1 + \eta_i), \quad j \geq 2, \quad (5.2b)$$

$$\lim_{j \rightarrow \infty} \frac{\ln \eta_j}{n_j} = 0, \quad (5.2c)$$

$$f(T) = f(0) - T \int_{-Q}^Q \ln(1 + \exp(\varepsilon_1(x)/T)) s_1(x) dx, \quad (5.2d)$$

$$A = \pi J_z \operatorname{sn} 2\zeta/\zeta, \quad (5.2e)$$

$$D_{ji}(x) = \begin{cases} (1 - 2\delta_{j,m_{i-1}})\delta_{j,i+1}s_i(x) + \delta_{j,i-1}s_i(x) & \text{for } m_{i-1} \leq j < m_i - 1, \\ (1 - 2\delta_{j,m_{i-1}})\delta_{j,i+1}s_i(x) + \delta_{ji}d_i(x) + \delta_{j,i-1}s_{i+1}(x) & \text{for } j = m_i - 1. \end{cases} \quad (5.2f)$$

The first term of the r.h.s. of (5.2b) is of the order of $T \exp(-As_1(Q)/T)$. From Eqs. (5.2b) and (5.2c) we have

$$\eta_2 = 3 + O(\sqrt{T} \exp(-As_1(Q)/T)) \quad (5.3a)$$

and from (5.2a)

$$\varepsilon_1(x) = -As_1(x) + T \ln 2 + O(T^{3/2} \exp(-As_1(Q)/T)). \quad (5.3b)$$

Substituting this into (5.2d), we have

$$f(T) = f(0) - 2T^{3/2}s_1(Q) \sqrt{\frac{2\pi}{As_1''(Q)}} \exp(-As_1(Q)/T) \{1 + O(T)\}. \quad (5.4a)$$

Using

$$s_1(x) = \frac{K_{k'}}{2\pi} \operatorname{dn}(K_{k'}x; k),$$

we have

$$s_1(Q) = \frac{K_{k'}}{2\pi} k', \quad s_1''(Q) = \frac{K_{k'}^3 k^2 k'}{2\pi}, \quad (5.4b)$$

where k' is $\sqrt{1-k^2}$ and $K_{k'}$ is the complete elliptic integral of the first kind with modulus k' , k being determined by

$$\frac{K_k}{K_{k'}} = \frac{K_{k'}}{\zeta}, \quad \frac{J_y}{J_z} = \operatorname{dn}(2\zeta, l), \quad \frac{J_x}{J_z} = \operatorname{cn}(2\zeta, l), \quad (5.4c)$$

with $0 \leq \zeta \leq K_l/2$.

b) Case $0 \leq -J_x < -J_y < -J_z$

In the same way as Eqs. (2.5) were transformed into Eqs. (2.6), Eqs. (5.2) are transformed as follows:

$$\varepsilon_j(x) = -A\alpha_j(x) + \sum_{i=1}^{\nu_1-1} B_{ji}^* T \ln(1 + \exp(-\varepsilon_i/T)) + C_j^* T \ln(1 + \exp(\varepsilon_{\nu_1}/T)) \quad \text{for } 1 \leq j < \nu_1, \quad (5.5a)$$

$$\varepsilon_{\nu_1}(x) = A\beta(x) - \sum_{i=1}^{\nu_1-1} C_i^* T \ln(1 + \exp(-\varepsilon_i/T)) + \sum_{i>\nu_1} D_{\nu_1 i}^* T \ln(1 + \eta_i) - s_2^* C_{\nu_1-1}^* T \ln(1 + \exp(\varepsilon_{\nu_1}/T)), \quad (5.5b)$$

$$\ln \eta_j = \sum_{i>\nu_1} D_{ji}^* \ln(1 + \eta_i) + D_{j,\nu_1}^* \ln(1 + \exp(\varepsilon_{\nu_1}/T)) \quad \text{for } j > \nu_1, \quad (5.5c)$$

$$\lim_{j \rightarrow \infty} \frac{\ln \eta_j}{n_j} = 0, \quad (5.5d)$$

$$f(T) = f(0) - T \sum_{j=1}^{\nu_1-1} \int_{-Q}^Q \alpha_j(x) \ln(1 + \exp(-\epsilon_j/T)) dx - T \int_{-Q}^Q \beta(x) \ln(1 + \exp(\epsilon_{\nu_1}/T)) dx, \tag{5.5e}$$

where $\alpha_j, \beta, B_{j1}, C_j$ and D_{j1} are defined by

$$\alpha_j(x) = \sum_{j=-\infty}^{\infty} \alpha_j(x + 2jQ), \quad \beta(x) = \sum_{j=-\infty}^{\infty} \beta(x + 2jQ), \dots$$

and $f(0)$ is the ground state energy per site. One can show

$$\begin{aligned} \beta(x) &= \frac{K_{k'}}{2\pi(p_0-1)} \operatorname{dn}\left(\frac{K_{k'}x}{p_0-1}, k\right), \quad \alpha_j(x) = \beta(x + i(p_0-1-j)) \\ &+ \beta(x - i(p_0-1-j)) = \frac{K_{k'}}{\pi(p_0-1)} \operatorname{dn}\left(\frac{K_{k'}x}{p_0-1}, k\right) \operatorname{dn}\left(\frac{K_{k'}(p_0-1-j)i}{p_0-1}, k\right) \\ &\times \left(1 - k^2 \operatorname{sn}^2\left(\frac{K_{k'}x}{p_0-1}, k\right) \operatorname{sn}^2\left(\frac{K_{k'}(p_0-1-j)i}{p_0-1}, k\right)\right)^{-1}, \end{aligned}$$

where k is defined by

$$\frac{K_k}{K_{k'}} = \frac{K_{l'}}{K_l - \zeta}, \quad \frac{J_y}{J_z} = \operatorname{dn}(2\zeta, l), \quad \frac{J_x}{J_z} = \operatorname{cn}(2\zeta, l). \tag{5.6}$$

Functions $\alpha_j(x)$ and $\beta(x)$ have the minimum at $x=Q$ and

$$\begin{aligned} \alpha_j(Q) &= \frac{K_{k'}k'}{\pi(p_0-1)} \operatorname{sn}\left(\frac{K_{k'}j}{p_0-1}, k'\right), \\ \alpha_j''(Q) &= \frac{K_{k'}^3k'}{\pi(p_0-1)^3} \operatorname{sn}\left(\frac{K_{k'}j}{p_0-1}, k'\right) \left(1 + k'^2 - 2k'^2 \operatorname{sn}^2\left(\frac{K_{k'}j}{p_0-1}, k'\right)\right), \\ \beta(Q) &= \frac{K_{k'}k'}{2\pi(p_0-1)}, \quad \beta''(Q) = \frac{K_{k'}^3k'^2k'}{2\pi(p_0-1)^3}. \end{aligned} \tag{5.7}$$

From (5.5c) and (5.5b) we have

$$\eta_{\nu_1+1} = 3 + O(\exp(A\beta(Q)/T)). \tag{5.8}$$

Substituting this into (5.5a) and (5.5b), we have

$$\begin{aligned} \epsilon_j(x) &= -A\alpha_j(x) + O(T \exp(A\beta(Q))) + O(T \exp(A\alpha_1(Q))) \\ &\qquad\qquad\qquad \text{for } j=1, 2, \dots, \nu_1-1, \\ \epsilon_{\nu_1}(x) &= A\beta(x) + T \ln 2 + O(T \exp(A\beta(Q))) + O(T \exp(A\alpha_1(Q))). \end{aligned} \tag{5.9}$$

From (5.5e) one has

$$f(T) = f(0) - 2T^{3/2} \beta(Q) \sqrt{\frac{2\pi}{-A\beta''(Q)}} \exp(A\beta(Q)/T) \{1 + O(T)\}$$

$$-T^{3/2} \alpha_1(Q) \sqrt{\frac{2\pi}{-A\alpha_1''(Q)}} \exp(A\alpha_1(Q)/T) \{1 + O(T)\}. \tag{5.10}$$

At $\text{sn}(K_k/(p_0-1), k') > \frac{1}{2}$ the second term of (5.10) is dominant because of $\beta(Q) < \alpha_1(Q)$. It is interesting that the energy gap appearing in specific heat cannot be analytically continued as a function of $p_0 = K_l/\zeta$ at $p_0 = 1 + K_k/(\text{sn}^{-1}(\frac{1}{2}, k'))$.

It is well known that $f(T)$ is invariant under the transformation $(J_x, J_y, J_z) \rightarrow (J_x, -J_y, -J_z)$. Then we restrict ourselves to the case $J_z > J_y \geq |J_x|$. The energy gap appearing in specific heat is

$$\frac{J_z \text{sn}(2\zeta, l) K_k k'}{2\zeta} \quad \text{for } \frac{K_l - \zeta}{\zeta} \geq \frac{\text{sn}^{-1}(\frac{1}{2}, k')}{K_k}, \tag{5.11a}$$

$$\frac{J_z \text{sn}(2\zeta, l) K_k k'}{\zeta} \text{sn}\left(\frac{K_k(K_l - \zeta)}{\zeta}, k'\right) \quad \text{for } \frac{K_l - \zeta}{\zeta} \leq \frac{\text{sn}^{-1}(\frac{1}{2}, k')}{K_k}, \tag{5.11b}$$

where k, l and ζ are defined by (5.4c) in the condition $0 \leq \zeta \leq K_l$.

The low-temperature free energy, at $J_z = J_y > 0$ and $J_y \geq J_x > -J_y$, is obtained if we put $\mu_0 H = 0$ in Eqs. (4.1) and (4.2). Then the specific heat is proportional to T at low-temperature. But we cannot obtain low-temperature free energy at $J_z \geq J_y = -J_x > 0$.

§ 6. Summary and discussion

For the Heisenberg-Ising ring, $f(0, 0)$ and $\lim_{H \rightarrow 0} \lim_{T \rightarrow 0} \chi$ were obtained by Yang and Yang,⁶⁾ and des Cloizeaux and Gaudin.⁶⁾ In Table II these quantities are given as functions of $\theta (\Delta = \cos \theta)$ at $J > 0$. An interesting feature is that these quantities are analytic as functions of Δ at $-1 < \Delta < 1$.

Table II. Analytic expression of physical quantities at $J > 0$ and $0 \leq \theta < \pi$.

$\lim_{H \rightarrow 0} \lim_{T \rightarrow 0} \chi$	$\lim_{H \rightarrow 0} \lim_{T \rightarrow 0} C_H/T$	The ground state energy per site
$\frac{4\theta \mu_0^2}{J(\pi - \theta)\pi \sin \theta}$	$\frac{2\theta}{3J \sin \theta}$	$-J \sin \theta \int_{-\infty}^{\infty} \frac{\text{sh}(\pi - \theta)\omega \, d\omega}{2 \text{ch} \theta \omega \text{sh} \pi \omega}$

We can discuss the low-temperature properties by considering a few kinds of excitations when $\mu_0 H \gg T$, because other excitations have a finite energy gap. On the other hand in the case $H = 0$, we must consider all kinds of excitations, because the energy gap vanishes. Therefore, integral equations for low-temperature thermodynamics have many unknown functions.

a) *Relations between the results for $|\Delta| \geq 1$ and those for $|\Delta| < 1$*

Gaudin⁷⁾ gave a set of integral equations for H-I ring at $\Delta \geq 1$ which included author's set for $\Delta = 1$ ⁸⁾ as a special case. His set of equations is equivalent to the following set:

$$\varepsilon_1(x) = -As_1(x) + s_1^* T \ln(1 + \exp(\varepsilon_2(x)/T)), \quad (6.1a)$$

$$\varepsilon_j(x) = s_1^* T \ln(1 + \exp(\varepsilon_{j-1}/T)) (1 + \exp(\varepsilon_{j+1}/T)), \quad (6.1b)$$

$$\lim_{j \rightarrow \infty} \frac{\varepsilon_j(x)}{j} = 2\mu_0 H, \quad (6.1c)$$

$$f(T, H) = f(0, 0) - T \int_{-Q}^Q s_1(x) \ln(1 + \exp(\varepsilon_1(x)/T)) dx, \quad (6.1d)$$

where

$$s_1(x) = \sum_{j=-\infty}^{\infty} \frac{1}{4} \operatorname{sech} \frac{\pi}{2} (x - 2jQ) = \frac{K_{k'}}{2\pi} \operatorname{dn}(K_{k'} x; k), \quad (6.1e)$$

$$Q = \frac{\pi}{\emptyset}, \quad \frac{K_k}{K_{k'}} = Q, \quad (6.1f)$$

and \emptyset is determined by $\operatorname{ch} \emptyset = \Delta$. Johnson and McCoy³⁾ calculated the low-temperature free energy using these equations.

At $J > 0$, $0 < 2\mu_0 H < As_1(Q)$, $\Delta > 1$ one obtains

$$\begin{aligned} \varepsilon_1(x) &= -As_1(x) + T \ln(2 \operatorname{ch}(\mu_0 H/T)) + O(T \exp(-\varepsilon_1(Q)/T)), \\ \varepsilon_j(x) &= T \ln \left(\left(\frac{\operatorname{sh}(j\mu_0 H/T)}{\operatorname{sh}(\mu_0 H/T)} \right)^2 - 1 \right) + O(T \exp(-\varepsilon_1(Q)/T)) \quad \text{for } j > 1. \end{aligned} \quad (6.2a)$$

Then one has

$$f(T, H) = f(0, 0) - T^{3/2} s_1(Q) \sqrt{\frac{2\pi}{As_1''(Q)}} \exp(-As_1(Q)/T) 2 \operatorname{ch} \frac{\mu_0 H}{T} (1 + O(T)). \quad (6.2b)$$

At $J > 0$, $J(1 + \Delta) > 2\mu_0 H > As_1(Q)$, $\Delta \geq 1$ we can treat the problem in the same way as § 3, and obtain

$$\varepsilon_1(x) = -\frac{2\pi J \operatorname{sh} \emptyset}{\emptyset} a_1(x) + 2\mu_0 H + \int_{-Q}^Q a_2(x-x') T \ln(1 + \exp(-\varepsilon_1(x')/T)) dx', \quad (6.3a)$$

where

$$a_j(x) = \frac{\emptyset}{2\pi} \frac{\operatorname{sh} j\emptyset}{\operatorname{ch} j\emptyset - \cos \emptyset x}, \quad j=1, 2. \quad (6.3b)$$

These correspond to Eqs. (3.4a) and (3.4b). The free energy is given by

$$f(T, H) - f(0, H) = -\frac{\pi \emptyset T^2}{6J \operatorname{sh} \emptyset} \frac{V(K)}{W(K)} + O(T^3) \quad (6.4a)$$

and K is determined by

$$2\mu_0 H = \frac{2\pi J \operatorname{sh} \emptyset}{\emptyset} \frac{V(K)}{Z(K)}, \quad (6.4b)$$

where

$$W(x) + \int_{-K}^K a_2(x-x') W(x') dx' = a_1(x), \tag{6.4c}$$

$$V(x) + \int_{-K}^K a_2(x-x') V(x') dx' = -\frac{d}{dx} a_1(x), \tag{6.4d}$$

$$Z(x) + \int_{-K}^K a_2(x-x') Z(x') dx' = 1. \tag{6.4e}$$

The second term of (3.11) and (6.4a) are analytic functions of θ^2 and Φ^2 , respectively. Putting $\Phi^2 \rightarrow -\theta^2$ one finds that (6.4a), (6.4b) and $\Delta = \text{ch } \Phi$ analytically continue to (3.11), (3.6a) and $\Delta = \cos \theta$ as functions of θ^2 , respectively. Then one finds that $\lim_{T \rightarrow 0} C_H/T$ is analytic as a function of Δ at $\Delta = 1, H \neq 0$.

At $J > 0, \Delta \geq 1, 2\mu_0 H \geq J(1 + \Delta)$ one has

$$\varepsilon_j(x) = -\frac{J \text{sh } \Phi \text{ sh } j\Phi}{\text{ch } j\Phi - \cos \Phi x} + 2j\mu_0 H + O(T^{-3/2}\lambda), \tag{6.5a}$$

$$\lambda = \exp\left(\frac{-2\mu_0 H + J(1 + \Delta)}{T}\right), \tag{6.5b}$$

therefore

$$f(T, H) = -\mu_0 H - \frac{T^{3/2}}{\pi} \sqrt{\frac{2}{J}} \int_0^\infty \ln(1 + \lambda \exp(-x^2)) dx + O(\lambda T^2). \tag{6.5c}$$

In the case $J < 0, \Delta > 1$ and $\mu_0 H \gg T$ one has

$$\varepsilon_j(x) = \frac{J \text{sh } \Phi \text{ sh } j\Phi}{\text{ch } j\Phi - \cos \Phi x} + 2j\mu_0 H + O(T^{-3/2}\lambda), \tag{6.6a}$$

$$\lambda = \exp\left(\frac{-2\mu_0 H - J(1 - \Delta)}{T}\right), \tag{6.6b}$$

$$f(T, H) = -\mu_0 H - T^{3/2} \frac{1}{\sqrt{2\pi|J|}} \exp\left(\frac{-2\mu_0 H - J(1 - \Delta)}{T}\right). \tag{6.6c}$$

As $f(T, H)$ is invariant under the transformation $(J, \Delta) \rightarrow (-J, -\Delta)$, one has

$$f(T, H) = -\mu_0 H - T^{3/2} \frac{1}{\sqrt{2\pi J}} \exp\left(\frac{-2\mu_0 H + J(1 + \Delta)}{T}\right), \tag{6.7}$$

for $J > 0, \Delta \leq -1$ and $\mu_0 H \gg T$. The results, (6.5c) and (6.7), analytically continue as functions of Δ to those of § 3 b) at $\Delta = \pm 1$.

The results for the Heisenberg-Ising ring are summarized in Fig. 2. In the shaded region and on the segment d , the system shows T -linear low-temperature specific heat. On the line a , C_H is proportional to $T^{1/2}$. In the region $2\mu_0 H > (1 + \Delta)$, specific heat is proportional to $T^{-3/2} \exp(-\alpha/T)$.

Putting $l=1$ in Eq. (5.2a), one can derive the low-temperature free energy

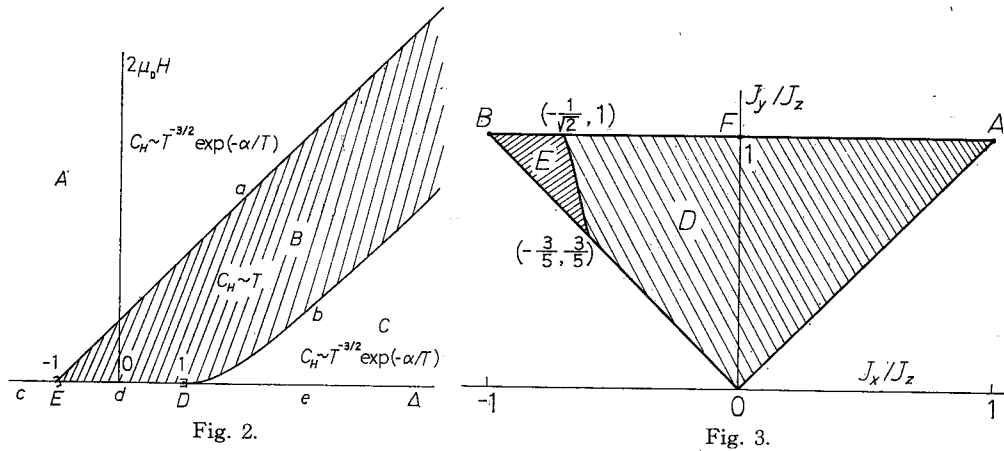


Fig. 2. Properties of the low-temperature specific heat of the Heisenberg-Ising model at $J=+1$. Regions A, B, C contain neither lines a, b, c, d, e nor end points D, E . Point D is contained in line d and point E is contained in line c .
 Fig. 3. Properties of the low-temperature specific heat of the X - Y - Z model in zero field.

of Heisenberg-Ising model in zero field. This coincides with (6.2b) at $H=0$. We summarize the results for the X - Y - Z model in zero field in Fig. 3. On the line AFB except B , the specific heat (C) is proportional to T . In the shaded region and on lines AO except points A and O , C to $T^{-3/2} \exp(-\alpha/T)$. On the point O (Ising case), C to $T^{-2} \exp(-\alpha/T)$. In the regions D and E , the energy gap α is given by (5.11a) and (5.11b) respectively. One should note that the gap cannot be analytically continued from region D to E .

b) *Unsolved problems*

We can obtain neither low-temperature specific heat of H-I ring at $H=0, \Delta \leq -1, J > 0$ nor that of X - Y - Z ring at $J_z \geq J_y = -J_x > 0$.

If the magnetic susceptibility is invariant in reversing the order of the limits $T \rightarrow 0, H \rightarrow 0$, i.e.

$$\lim_{T \rightarrow 0} \lim_{H \rightarrow 0} \chi = \lim_{H \rightarrow 0} \lim_{T \rightarrow 0} \chi,$$

it is required

$$C''(0) = \frac{1}{2(\pi - \theta)} \quad \text{for (4.1),}$$

and

$$C''(0) = \frac{1}{2\theta} \quad \text{for (4.2).}$$

But we can prove neither its validity nor $C(0) = \pi/6$ from Eqs. (4.1) or Eqs. (4.2).

Acknowledgment

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Appendix A

—Calculation of the third term of high-temperature expansion—

Here we continue the calculation of Appendix C of Ref. 1). From Eqs. (C. 3) of Ref. 1) we have

$$\sum_{i=1}^{\infty} [\delta_{ji}(1 + (\eta_j^{(0)})^{-1}) - D_{ji}^*] c_i^{(2)} = \frac{1}{2} ((\eta_j^{(0)})^{-1} + (\eta_j^{(0)})^{-2}) (c_j^{(1)})^2. \quad (A.1)$$

Then we have

$$\widehat{c_j^{(2)}}(\omega) = \sum_{i=1}^{\infty} \widehat{B_{ji}}(\omega) \frac{(\eta_i^{(0)})^{-1} + (\eta_i^{(0)})^{-2}}{2} \widehat{\{c_i^{(1)}\}^2}(\omega),$$

where $\widehat{B_{jl}}(\omega)$ is the jl element of inverse matrix of $\{\delta_{jl}(1 + (\eta_j^{(0)})^{-1}) - \widehat{D_{jl}}(\omega)\}$. One can show

$$\widehat{B_{ii}}(\omega) = -2 \operatorname{ch} \omega c_i^{(1)}(\omega) (-1)^{r(i)+1},$$

where $r(j)$ is defined by $m_{r(j)} \leq j < m_{r(j)+1}$, therefore

$$\widehat{c_i^{(2)}} = \frac{1}{2} \sum_{l=1}^{\infty} (-1)^{r(l)} \{(\eta_l^{(0)})^{-1} + (\eta_l^{(0)})^{-2}\} 2 \operatorname{ch} \omega c_l^{(1)}(\omega) \widehat{\{c_l^{(1)}\}^2}(\omega). \quad (A.2)$$

Then the third term of $1/T$ expansion of f is

$$\begin{aligned} -A^2 \int_{-\infty}^{\infty} s_1(x) c_1^{(2)}(x) dx &= -\frac{A^2}{2} \sum_{i=1}^{\infty} (-1)^{r(i)} \{(\eta_i^{(0)})^{-1} + (\eta_i^{(0)})^{-2}\} \int_{-\infty}^{\infty} \{c_i^{(1)}(x)\}^2 dx \\ &= -\frac{A^2}{2} \left[\sum_{i=0}^{\infty} (-1)^i \left(\sum_{l=m_i+1}^{m_{i+1}-1} U_l \right) + \sum_{i=1}^{\infty} (-1)^i U_{m_i} \right], \end{aligned} \quad (A.3)$$

where

$$U_i = [(\eta_i^{(0)})^{-1} + (\eta_i^{(0)})^{-2}] \int_{-\infty}^{\infty} \{c_i^{(1)}(x)\}^2 dx.$$

Substituting (C. 2) and (C. 6) of Ref. 1) into the above equation and carrying out integration, we obtain

$$\begin{aligned} U_i &= \frac{1}{8\pi p_0^2 f_{m_{i+1}} \{f(1)\}^3} \left[\frac{1}{f_i f_{i+1}} \{h_{m_{i+1}}^{(3)} - h_{i+1}^{(3)} - 6 \cot \psi_{m_{i+1}} (h_i^{(2)} - h_{i+1}^{(2)}) + 2(\psi_i + \psi_{i+1})\} \right. \\ &+ \frac{1}{f_{i+1} f_{i+2}} \{h_{m_{i+1}}^{(3)} + h_{i+1}^{(3)} + 6 \cot \psi_{m_{i+1}} (h_{i+1}^{(2)} - h_{i+2}^{(2)}) + 2(\psi_{i+1} + \psi_{i+2})\} \\ &\left. + \frac{f_{i+2}}{f_i^2 f_{i+1}} h_i^{(3)} - \frac{f_i}{f_{i+1} f_{i+2}^2} h_{i+2}^{(3)} \right] \quad \text{for } m_i \leq l < m_{i+1}, \end{aligned}$$

where

$$\begin{aligned}\psi_i &= q_i \pi / p_0, & \psi_{i+1} &= \pi (q_i + (-1)^{i+1} p_{i+1}) / p_0, \\ \psi_{i+2} &= (q_i + 2(-1)^{i+1} p_{i+1}) / p_0, & \psi_{m_{i+1}} &= q_{m_{i+1}} \pi / p_0, \\ h_j^{(3)} &= -3 \cot \psi_j + (1 + 3 \cot^2 \psi_j) \psi_j, \\ h_j^{(2)} &= 1 - \cot \psi_j \cdot \psi_j, \\ f_i &= f(n_i - 1), & f_{i+1} &= f(n_i + y_i - 1), \\ f_{i+2} &= f(n_i + 2y_i - 1), & f_{m_{i+1}} &= f(m_{i+1} - 1).\end{aligned}$$

Using

$$\begin{aligned}\frac{f_{i+2}}{f_i^2 f_{i+1}} &= -\frac{1}{f_i f_{i+1}} + \frac{2 \operatorname{ch} y_i \mu_0 H}{f_i^2}, \\ \frac{f_i}{f_{i+1} f_{i+2}} &= -\frac{1}{f_{i+1} f_{i+2}} + \frac{2 \operatorname{ch} y_i \mu_0 H}{f_{i+2}^2}\end{aligned}$$

and

$$\frac{1}{f_i f_{i+1}} = \frac{1}{f_{m_{i+1}}} \left(\frac{e^{-n_i \mu_0 H}}{f_i} - \frac{e^{-n_{i+1} \mu_0 H}}{f_{i+1}} \right),$$

one finds that almost all the terms in (A.3) cancel out except a few terms, and obtains finally

$$(A.3) = -\frac{J^2}{4} \left\{ \frac{1 + 2\Delta^2}{(2 \operatorname{ch} \mu_0 H)^2} - \frac{6\Delta^2}{(2 \operatorname{ch} \mu_0 H)^4} \right\}. \quad (A.4)$$

This coincides with the third term of (1.2).

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