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Lower Bound on the Blow-up Rate of the Axisymmetric Navier–Stokes Equations

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Consider axisymmetric strong solutions of the incompressible Navier–Stokes equations in $\mathbb{R}^3$ with nontrivial swirl. Such solutions are not known to be globally defined, but it is shown in ([1], Partial regularity of suitable weak solutions of the Navier–Stokes equations. *Communications on Pure and Applied Mathematics*, 35 (1982), 771–831) that they could only blow up on the axis of symmetry. Let z denote the axis of symmetry and r measure the distance to the z-axis. Suppose the solution satisfies the pointwise scale invariant bound $|v(x,t)| \le C_*(r^2 - t)^{-1/2}$ for $-T_0 \le t < 0$ and $0 < C_* < \infty$ allowed to be large, we then prove that v is regular at time zero.

1 Introduction

The incompressible Navier–Stokes equations in *cartesian coordinates* are given by

$$\partial_t v + (v \cdot \nabla) v + \nabla p = \Delta v, \quad \operatorname{div} v = 0. \tag{N−S}$$

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The velocity field is $v(x,t) = (v_1, v_2, v_3) : \mathbb{R}^3 \times [-T_0, 0) \to \mathbb{R}^3$ and $p(x,t) : \mathbb{R}^3 \times [-T_0, 0) \to \mathbb{R}$ is the pressure. It is a long standing open question to determine if solutions with large smooth initial data of finite energy remain regular for all time.

In this paper, we consider the special class of solutions which are *axisymmetric*. This means, in *cylindrical coordinates* r, θ, z with $(x_1, x_2, x_3) = (r\cos\theta, r\sin\theta, z)$, that the solution is of the form

$$v(x,t) = v_r(r,z,t)e_r + v_\theta(r,z,t)e_\theta + v_z(r,z,t)e_z. \tag{1.1}$$

The components v_r, v_θ, v_z do not depend upon θ and the basis vectors e_r, e_θ, e_z are

$$e_r = \left(\frac{x_1}{r}, \frac{x_2}{r}, 0\right), \quad e_\theta = \left(-\frac{x_2}{r}, \frac{x_1}{r}, 0\right), \quad e_z = (0, 0, 1).$$

The main result of our paper shows that axisymmetric solutions must blow up faster than the scaling invariant rate which appears in equation (1.2).

For $R > 0$, define $B(x_0, R) \subset \mathbb{R}^3$ as the ball of radius R centered at x_0. The parabolic cylinder is $Q(X_0, R) = B(x_0, R) \times (t_0 - R^2, t_0) \subset \mathbb{R}^{3+1}$ centered at $X_0 = (x_0, t_0)$. If the center is the origin, we use the abbreviations $B_R = B(0, R)$ and $Q_R = Q(0, R)$.

Theorem 1.1. Let (v, p) be an axisymmetric solution of the Navier–Stokes equations (N–S) in $D = \mathbb{R}^3 \times (-T_0, 0)$ for which $v(x,t)$ is smooth in x and Hölder continuous in t. Suppose the pressure satisfies $p \in L^{5/3}(D)$ and v is pointwise bounded as

$$|v(x,t)| \le C_*(r^2 - t)^{-1/2}, \quad (x,t) \in D. \tag{1.2}$$

The constant $C_* < \infty$ is allowed to be large. Then $v \in L^\infty(B_R \times [-T_0, 0])$ for any $R > 0$. □

We remark that the exponent 5/3 for the norm of p can be replaced. However, it is the natural exponent occurring in the existence theory for weak solutions, see e.g. [28], [1].

Recall the natural scaling of Navier–Stokes equations: If (v, p) is a solution to (N–S), then for any $\lambda > 0$, the following rescaled pair is also a solution:

$$v^\lambda(x,t) = \lambda v(\lambda x, \lambda^2 t), \quad p^\lambda(x,t) = \lambda^2 p(\lambda x, \lambda^2 t). \tag{1.3}$$

Suppose a solution $v(x,t)$ of the Navier–Stokes equations blows up at $X_0 = (x_0, t_0)$. Leray [18] proved that the blow-up rate in time is at least

$$\|v(\cdot,t)\|_{L^\infty_x} \geq C(t_0 - t)^{-1/2}.$$

Caffarelli, Kohn, and Nirenberg [1] showed that for such a blow-up solution, the average of $|v|$ over $Q(X_0, R)$ satisfies

$$\left(\frac{1}{|Q_R|}\int_{Q(X_0,R)} |v|^3 + |p|^{3/2} dxdt\right)^{1/3} \geq \frac{C}{R}.$$

See also [21, 24, 39]. Thus, the natural rate for blowup is at least

$$|v(x,t)| \sim \frac{O(1)}{[(x_0 - x)^2 + t_0 - t]^{1/2}}. \tag{1.4}$$

Both this and the rate (1.2) are invariant under the natural scaling (1.3).

The Serrin-type criteria [6, 8, 9, 15, 30, 31, 33] states that v is regular if it satisfies

$$\|v\|_{L^s_t L^q_x(Q_1)} < \infty, \quad \frac{3}{q} + \frac{2}{s} \leq 1,\ s, q \in (2,\infty), \quad \text{or} \quad (s,q) = (2,\infty). \tag{1.5}$$

Above, for a domain $\Omega \subset \mathbb{R}^3$, we use the definition

$$\|v\|_{L^s_t L^q_x(\Omega\times(t_1,t_2))} := \| \|v(x,t)\|_{L^q_x(\Omega)} \|_{L^s_t(t_1,t_2)}.$$

For any $X_0 = (x_0, t_0) \in Q_1$, equation (1.5) implies the following local smallness of v:

$$\lim_{R\downarrow 0} \|v\|_{L^s_t L^q_x(Q(X_0,R))} = 0. \tag{1.6}$$

Therefore, equation (1.5) is a so-called ϵ-regularity criterion since it implies that the norm is locally small. For $(q,s) = (3,\infty)$, equation (1.6) does not follow from equation (1.5). Hence, the $(q,s) = (3,\infty)$ end point regularity criterion (1.5) proved in [5, 29] is not an ϵ-regularity-type theory.

However, these criteria do not rule out blowup with the natural scaling rate (1.4). It is a fundamental problem in the study of the incompressible Navier–Stokes equations to determine if solutions to (N–S) with the following scaling invariant bound are regular:

$$|v(x,t)| \le \frac{C}{[(x_0-x)^2+t_0-t]^{1/2}}. \tag{1.7}$$

If a self-similar solution satisfies this bound, then it is known to be zero [37] (the self-similar solution from [24] belongs to $L_t^\infty L_x^3$).

Theorem 1.1 rules out singular axisymmetric solutions satisfying the bound (1.7). In fact, equation (1.2) is considerably weaker than equation (1.7) and is also not a borderline case of the Serrin-type criterion. For example, equation (1.2) implies that $v \in L^q(Q_1)$ for $q < 4$, but not for $q \in [4,5)$. The borderline of the Serrin-type criterion, on the other hand, is $v \in L^5(Q_1)$. Finally, we remark that the assumption (1.2) has been improved in two recent preprints [3] and [13].

We now recall the previous results on the regularity of axisymmetric solutions to the Navier–Stokes equations. Global in-time regularity was first proved under the *no swirl* assumption, $v_\theta = 0$, independently by Ukhovskii–Yudovich [38] and Ladyzhenskaya [16]. See [17] for a refined proof and [12] for similar results in the half-space setting.

When the swirl component v_θ is not assumed to be trivial, global regularity is unknown. But it follows from the partial regularity theory of [1] that singular points can only lie on the axis of symmetry. Any off-axis symmetry would imply a whole circle of singular points, which contradicts [1]. Neustupa–Pokorný [25, 26] proved regularity assuming the zero-dimensional condition $v_r \in L_t^s L_x^q$ with $3/q + 2/s = 1$, $3 < q \le \infty$. Regularity criteria can also be put on the vorticity field $\omega = \operatorname{curl} v$,

$$\omega(x,t) = \omega_r e_r + \omega_\theta e_\theta + \omega_z e_z, \tag{1.8}$$

where

$$\omega_r = -\partial_z v_\theta, \quad \omega_\theta = \partial_z v_r - \partial_r v_z, \quad \omega_z = (\partial_r + r^{-1})v_\theta.$$

Chae–Lee [2] proved regularity assuming finiteness of another zero-dimensional integral: $\omega_\theta \in L_t^s L_x^q$ with $3/q + 2/s = 2$. Jiu–Xin [11] proved regularity if the sum of the zero-dimensional *scaled* norms $\int_{Q_R} (R^{-1}|\omega_\theta|^2 + R^{-3}|v_\theta|^2)dz$ is sufficiently small for all $R > 0$ small enough. Recently, Hou–Li [10] constructed a family of global solutions with large initial data.

The main idea of our proof is as follows. The bound (1.2) ensures that the first blow-up time is no earlier than $t = 0$. For $t \in (-T_0, 0)$, we show that the swirl component

v_θ gains a modicum of regularity: For some small $\alpha = \alpha(C_*) > 0$, equation (1.2) enables us to conclude that

$$|v_\theta(t, r, z)| \le C r^{\alpha - 1}. \tag{1.9}$$

We prove equation (1.9) in Section 3. This estimate breaks the scaling, thereby transforming the problem from order one to ϵ-regularity, which is shown to be sufficient in Section 2.

2 Proof of Main Theorem

In this section, we prove Theorem 1.1. First we show that our solutions are in fact suitable weak solutions. Then we make use of equation (1.9), to establish our main theorem.

2.1 Suitable weak solution

We recall from [1, 21, 28] that a *suitable weak solution* of the Navier–Stokes equations in a domain $\Omega \subset \mathbb{R}^3 \times \mathbb{R}$ is defined to be a pair (v, p) satisfying

$$v \in L^\infty_t L^2_x(\Omega), \quad \nabla v \in L^2(\Omega), \quad p \in L^{3/2}(\Omega). \tag{2.1}$$

Further, (v, p) solve (N–S) in the sense of distributions and satisfy the *local energy inequality*:

$$2\int_\Omega |\nabla v|^2 \varphi \le \int_\Omega \{|v|^2(\partial_t \varphi + \Delta\varphi) + (|v|^2 + 2p) v \cdot \nabla\varphi\}, \qquad \forall \varphi \in C^\infty_c(\Omega), \quad \varphi \ge 0. \tag{2.2}$$

To prove interior regularity, we do not need to specify the initial or boundary data.

We define a solution $v(x, t)$ to be *regular* at a point X_0 if $v \in L^\infty(\Omega(X_0, R))$ for some $R > 0$. Otherwise $v(x, t)$ is *singular* at X_0. We will use the following regularity criterion.

Lemma 2.1. Suppose that (v, p) is a suitable weak solution of (N–S) in $\Omega(X_0, 1)$. Then there exists an $\epsilon_1 > 0$, so that X_0 is a regular point if

$$\limsup_{R \downarrow 0} \frac{1}{R^2} \int_{\Omega(X_0, R)} |v|^3 \le \epsilon_1. \tag{2.3}$$

□

This regularity criterion, which is a variant of the criterion in [1], was proven in [36]; see [9] for more general results. The condition (2.3) does not explicitly involve the pressure, but one does require $p \in L^{3/2}(Q(X_0, 1))$ because the pair (v, p) is assumed to be a suitable weak solution.

2.2 Preliminary estimates

In this section, we show that the solution (v, p) in Theorem 1.1 is sufficiently integrable to be a suitable weak solution, and we derive estimates depending only upon C_* of equation (1.2).

We estimate the pressure with weighted singular integral estimates. We therefore first estimate v in weighted spaces. Fix $\beta \in (1, 5/3)$. For $t \in (-T_0, 0)$ by equation (1.2), we have

$$\int_{\mathbb{R}^3} \frac{|v(x,t)|^4}{|x|^\beta}\, dx \le \int_{\mathbb{R}^3} \frac{1}{|x|^\beta} \frac{C_* r\,dr\,dz}{(r^2-t)^2} = \int_{|z|\ge 1} + \int_{|z|<1, r>1} + \int_{|z|<1, r<1} = I_1 + I_2 + I_3.$$

Each of these integrals can be estimated as follows:

$$\begin{aligned}
|I_1| &\le \int_{|z|>1} \frac{dz}{|z|^\beta} \int_0^\infty \frac{C_* r\,dr}{(r^2-t)^2} \le c|t|^{-1},\\
|I_2| &\le \int_1^\infty r^{-\beta} \frac{C_*}{(r^2-t)^2} r\,dr \le c,\\
|I_3| &\le \int_0^1 (1+r^{1-\beta}) \frac{c}{(r^2-t)^2} r\,dr \le c|t|^{-(1+\beta)/2}.
\end{aligned}$$

Summing the estimates and using $\beta > 1$, we get

$$\int_{\mathbb{R}^3} \frac{1}{|x|^\beta} |v(x,t)|^4\, dx \le c + c|t|^{-(1+\beta)/2}.$$

Define R_i's to be the Riesz transforms: $R_i = \frac{\partial_i}{\sqrt{-\Delta}}$. We consider the singular integral

$$\tilde{p}(x,t) = \int \sum_{i,j} \partial_i \partial_j (v_i v_j)(y) \frac{1}{4\pi|x-y|}\, dy = \sum_{i,j} R_i R_j (v_i v_j).$$

To show that this singular integral is well defined for every t, we use the $L^q(\mathbb{R}^3)$-estimates for singular integrals with A_q weight [32]. Specifically, we use $q = 2$ and the A_2 weight function $|x|^{-\beta}$. We have the estimate

$$\int \frac{1}{|x|^{\beta}} |\tilde{p}(x,t)|^2 \, dx \le c \int \frac{1}{|x|^{\beta}} |v(x,t)|^4 \, dx \le c + c|t|^{-(1+\beta)/2}. \tag{2.4}$$

Choose $\gamma \in (1/2 + 5\beta/6, 3)$. Hölder's inequality gives us the bound

$$\int_{|x|>1} \frac{|\tilde{p}(x,t)|^{5/3}}{|x|^{\gamma}} dx \le \left(\int_{|x|>1} \frac{|\tilde{p}(x,t)|^2}{|x|^{\beta}} dx \right)^{5/6} \left(\int_{|x|>1} |x|^{-(\gamma - \frac{5}{6}\beta)6} dx \right)^{1/6} < \infty.$$

We will use these bounds to show that the pressure p can be identified with $\tilde{p}$.

Let $h(x,t) = p(x,t) - \tilde{p}(x,t)$. Then h is harmonic in x, $\Delta_x h(x,t) = 0$, and by assumption $p(\cdot, t) \in L^{5/3}(\mathbb{R}^3)$ for almost every t. For each such t, we have

$$\int_{|x|>1} \frac{|h(x,t)|^{5/3}}{|x|^{\gamma}} dx \le c \int_{|x|>1} |p(x,t)|^{5/3} dx + c \int_{|x|>1} \frac{|\tilde{p}(x,t)|^{5/3}}{|x|^{\gamma}} dx < \infty.$$

We may thus conclude from using a Liouville theorem that $h(x,t) = 0$ for all x if $\gamma < 3$.

To see the last assertion, fix a radial smooth function $\phi(x) \ge 0$ supported in $2 < |x| < 4$ satisfying $\int \phi = 1$. For any $x \in \mathbb{R}^3$ with $R > |x|$, we have

$$h(x,t) = \int h(y,t) R^{-3} \phi(x + y/R) \, dy.$$

This is the mean-value theorem for harmonic functions. Define $A = B_{5R} - B_R$, then

$$|h(x,t)| \le cR^{-3} \int_A |h(y,t)| dy \le cR^{-3+(6+3\gamma)/5} \left(\int_A |y|^{-\gamma} |h(y,t)|^{5/3} dy \right)^{3/5}.$$

This clearly vanishes as $R \to \infty$. Thus, $p(x,t) = \tilde{p}(x,t)$ for all x and for almost every t.

Next we show that (v, p) form a suitable weak solution. From Hölder's inequality, equation (2.4), and $\beta < 5/3$, we conclude that

$$\int_{Q_1} |p(x,t)|^{3/2} \, dx dt \le c \int_{-1}^{0} \left(\int_{B_1} \frac{1}{|x|^{\beta}} |p(x,t)|^2 \, dx \right)^{3/4} dt \le c. \tag{2.5}$$

The pointwise estimate (1.2) on v implies

$$v \in L^s_t L^q_x(Q_1), \quad \frac{1}{q} + \frac{1}{s} > \frac{1}{2}. \tag{2.6}$$

We will use $(s, q) = (3, 3)$. We also see from equation (1.2) that $v \in L^4(B_1 \times (-T_0, -\epsilon))$ for any small $\epsilon > 0$. Thus, the vector product of (N–S) with $u\varphi$ for any $\varphi \in C_c^\infty(Q_1)$ is integrable in Q_1 and we can integrate by parts to get the local energy inequality (2.2) with $Q = Q_1$. In fact, we have equality.

Now, for any $R \in (0, 1)$ and $t_0 \in (-R^2, 0)$, we can choose a sequence of φ which converges a.e. in Q_R to $H(t_0 - t)$, the Heaviside function that equals 1 for $t < t_0$ and 0 for $t > t_0$. Since the limit of $\partial_t\varphi$ is the negative delta function in t, this gives us the estimate

$$\operatorname*{ess\,sup}_{-R^2<t<0} \int_{B_R} |v(x,t)|^2 dx + \int_{Q_R} |\nabla v|^2 \le C_R \int_{Q_1} (|v|^3 + |p|^{3/2}). \tag{2.7}$$

These estimates show that (v, p) is a suitable weak solution of (N–S) in Q_R. Note that these bounds depend on C_* of equation (1.2) only, not on $\|p\|_{L^{5/3}(\mathbb{R}^3\times(-T_0,0))}$.

2.3 Scaling limit

To show Theorem 1.1, it suffices to show that every point on the z-axis is regular. Suppose now that a point $x_* = (0, 0, x_3)$ on the z-axis is a singular point of v. We will derive a contradiction. Define $X_* = (x_*, 0)$. Let (v^λ, p^λ) be rescaled solutions of (N–S) defined by

$$v^\lambda(x,t) = \lambda v(\lambda(x - x_*), \lambda^2 t), \quad p^\lambda(x,t) = \lambda^2 p(\lambda(x - x_*), \lambda^2 t). \tag{2.8}$$

Fix $R_* > 0$ to be chosen; by Lemma 2.1 there is a sequence λ_k, $k \in \mathbb{N}$, so that $\lambda_k \to 0$ as $k \to \infty$ and

$$\frac{1}{R_*^2} \int_{Q_{R_*}} |v^{\lambda_k}|^3 = \frac{1}{(R_*\lambda_k)^2} \int_{Q(X_*, R_*\lambda_k)} |v|^3 > \epsilon_1. \tag{2.9}$$

We will derive a contradiction to this statement.

For (v^λ, p^λ) with $0 < \lambda < 1$, the pointwise estimate (1.2) is preserved,

$$|v^\lambda(x,t)| \le C_*(r^2 - t)^{-1/2}, \quad (x,t) \in \mathbb{R}^3 \times (-T_0, 0).$$

We also have by rescaling

$$p^\lambda(x,t) = \int \sum_{i,j} \partial_i\partial_j\big(v_i^\lambda v_j^\lambda\big)(y) \frac{1}{4\pi|x-y|}\, dy,$$

The argument in the previous section provides the uniform bounds for $q \in (1,4)$,

$$\int_{Q_1} |v^\lambda|^q + |p^\lambda|^{3/2} \le C, \quad \operatorname*{ess\,sup}_{-R^2<t<0} \int_{B_R} |v^\lambda(x,t)|^2 dx + \int_{Q_R} |\nabla v^\lambda|^2 \le C. \tag{2.10}$$

Above the bound for p_λ follows from equation (2.5), the bound for $|v^\lambda|^q$ follows from equation (1.2), and the energy bound then follows from equation (2.7).

Thus from the sequence λ_k we can extract a subsequence, still denoted by λ_k, so that $(v^{\lambda_k}, p^{\lambda_k})$ weakly converges to some limit function $(\bar{v}, \bar{p})$

$$v^{\lambda_k} \rightharpoonup \bar{v} \quad \text{in } L^q(Q_R), \quad \nabla v^{\lambda_k} \rightharpoonup \nabla \bar{v} \quad \text{in } L^2(Q_R), \quad p^{\lambda_k} \rightharpoonup \bar{p} \quad \text{in } L^{3/2}(Q_R).$$

Moreover, since (v^λ, p^λ) solves (N–S) with bound (2.10), we also have the uniform bound

$$\|\partial_t v^\lambda\|_{L^{3/2}((-R^2,0);H^{-2}(B_R))} < C.$$

We can then apply Theorem 2.1 of [35, Chapter III] to conclude that v^{λ_k} remain in a compact set of $L^{3/2}(Q_R)$. Therefore, (a further subsequence of) $v^{\lambda_k} \to \bar{v}$ strongly in $L^{3/2}(Q_R)$. Since the v^{λ_k} remain bounded in $L^q(Q_R)$ for all $q < 4$, we deduce that $v^{\lambda_k} \to \bar{v}$ strongly in $L^q(Q_R)$ for all $1 \le q < 4$.

2.4 The limit solution

The convergence established at the end of Section 2.3 is sufficient to conclude that the limit function $(\bar{v}, \bar{p})$ is a suitable weak solution of the Navier–Stokes equations in Q_R, as in [1, 21]. Since v satisfies equation (1.2), so does $\bar{v}$. Hence $\bar{v}$ is regular at any interior point of Q_R, and $t = 0$ is the first time when $\bar{v}(x,t)$ could develop a singularity.

To gather more information, we use axisymmetry. We will argue in this section and the next that the estimate (1.9) (proven in the Section 3) is enough to conclude that our solution is regular. In particular, equation (1.9) tells us that

$$\int_{Q_R} \left|v_\theta^\lambda\right| \le C\lambda^\alpha \to 0 \quad \text{as } \lambda \downarrow 0.$$

Thus, the limit $\bar{v}$ has no-swirl, $\bar{v}_\theta = 0$.

Let $\bar\omega = \nabla \times \bar v$ be the vorticity of $\bar v$. The θ component of $\bar\omega$, $\bar\omega_\theta = \partial_z \bar v_r - \partial_r \bar v_z$, solves

$$\left(\partial_t + \bar b \cdot \nabla - \Delta + \frac{1}{r^2}\right)\bar\omega_\theta - \frac{\bar v_r}{r}\bar\omega_\theta = 0.$$

We have used $\bar v_\theta = 0$. Above

$$\bar b = \bar v = \bar v_r e_r + \bar v_z e_z, \quad \bar b \cdot \nabla = \bar v_r \partial_r + \bar v_z \partial_z, \quad \operatorname{div} \bar b = 0.$$

We record the Laplacian for axisymmetric functions

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}.$$

Next define $\Omega = \bar\omega_\theta / r$. Then Ω solves

$$\left(\partial_t + \bar b \cdot \nabla - \Delta - \frac{2}{r}\partial_r\right)\Omega = 0. \tag{2.11}$$

We now derive L^q estimates on Ω using estimates for the Stokes system.

Since $\bar v$ satisfies equation (1.2), it also satisfies equation (2.6). We will use both $(s, q) = (5/2, 5)$ and $(s, q) = (5/4, 5/4)$. We rewrite (N–S) as a Stokes system with force

$$(\partial_t - \Delta)\bar v_i + \nabla_i \bar p = \partial_j f_{ij}, \quad \operatorname{div} \bar v = 0, \quad f_{ij} = -\bar v_i \bar v_j.$$

By the interior estimates of Stokes system (shown in the Appendix), we have

$$\|\nabla \bar v\|_{L_t^{5/4} L_x^{5/2}(Q_{5/8})} \le C \|\bar v\|^2_{L_t^{5/2} L_x^5(Q_{3/4})} + C\|\bar v\|_{L^{5/4}(Q_{3/4})} \le C.$$

Hence, Ω has the bound

$$\|\Omega\|_{L^{20/19}(Q_{5/8})} \le \|\nabla \bar v\|_{L_t^{5/4} L_x^{5/2}(Q_{5/8})} \|1/r\|_{L_t^\infty L_x^{20/11}(Q_{5/8})} \le C. \tag{2.12}$$

In Section 2.6, we obtain $\Omega \in L^\infty$ from equations (2.11), (2.12), and a local maximum estimate. Then in Section 2.7, we show that this is sufficient to conclude Theorem 1.1.

2.5 Energy estimates

We derive parabolic De Giorgi-type energy estimates for equation (2.11). To do this we assume that

$$|\bar{b}(r,z,t)| \leq C_*/r.$$

This assumption on $\bar{b}$ is substantially weaker than the one from Theorem 1.1.

Consider a test function $0 \leq \zeta_1(x,t) \leq 1$ defined on Q_1 for which $\zeta_1 = 0$ on $\partial B_1 \times [-1^2, 0]$ and $\zeta_1 = 1$ on Q_σ for $0 < \sigma < 1$. Suppose that $\zeta_1(x,-1) = 0$. Now consider the rescaled test function $\zeta(x,t) = \zeta_1(x/R, t/R^2)$ on Q_R. Define $(u)_\pm = \max\{\pm u, 0\}$ for a scalar function u. Multiply equation (2.11) by $p(\Omega - k)_\pm^{p-1}\zeta^2$ for $1 < p \leq 2$ and $k \geq 0$ to obtain

$$\begin{aligned}
&\int_{B_R} \zeta^2(\Omega-k)_\pm^p \bigg|_{-R^2}^t + \frac{4(p-1)}{p}\int_{-R^2}^t dt' \int_{B_R} |\nabla((\Omega-k)_\pm^{p/2}\zeta)|^2 \\
&\quad = 2\int_{-R^2}^t dt' \int_{B_R} (\Omega-k)_\pm^p \left(\zeta\frac{\partial\zeta}{\partial t} + |\nabla\zeta|^2 + \frac{2-p}{p}\zeta\Delta\zeta - 2\zeta\frac{\partial_r\zeta}{r} + \bar{b}\cdot\zeta\nabla\zeta\right) \\
&\qquad - 2\int_{-R^2}^t dt' 2\pi \int dz\, \zeta^2(\Omega-k)_\pm^p \bigg|_{r=0}.
\end{aligned}$$

Notice that the last term has a good sign.

Let $v_\pm \equiv (u-k)_\pm^{p/2}$. To estimate the term involving b, we use Young's inequality

$$\int_{\mathbb{R}^3} v_\pm^2 b\zeta \cdot \nabla\zeta \leq \delta\frac{R^{-1+\epsilon}}{1+\epsilon}\int_{\mathbb{R}^3} v_\pm^2\zeta^2|b|^{1+\epsilon} + C_\delta\frac{\epsilon R^{-2+(1+\epsilon)/\epsilon}}{1+\epsilon}\int_{\mathbb{R}^3} v_\pm^2\zeta^2\left[\frac{|\nabla\zeta|}{\zeta}\right]^{(1+\epsilon)/\epsilon}.$$

This holds for small $\delta > 0$ and $\epsilon > 0$ to be chosen. Further choose ζ to decay like $(1 - |x|/R)^n$ near the boundary of B_R. If n is large enough (depending on ϵ), we have

$$C_\delta\frac{\epsilon R^{-2+(1+\epsilon)/\epsilon}}{1+\epsilon}\int_{\mathbb{R}^3} v_\pm^2\zeta^2\left[\frac{|\nabla\zeta|}{\zeta}\right]^{(1+\epsilon)/\epsilon} \leq CR^{-2}\int_{B_R} v_\pm^2.$$

We also use the Hölder and Sobolev inequalities to obtain

$$\begin{aligned}
\delta\frac{R^{-1+\epsilon}}{1+\epsilon}\int_{\mathbb{R}^3} v_\pm^2\zeta^2|b|^{1+\epsilon} &\leq \delta\left(R^{(-1+\epsilon)3/2}\int_{B_R}|b|^{(1+\epsilon)3/2}\right)^{2/3}\int_{\mathbb{R}^3}|\nabla(v_\pm\zeta)|^2 \\
&\leq \delta C\int_{\mathbb{R}^3}|\nabla(v_\pm\zeta)|^2.
\end{aligned}$$

The last inequality is satisfied, for example, if $|b| \le C_*/r$ and $\epsilon < 1/3$. We conclude

$$\int_{\mathbb{R}^3} v_\pm^2 b\zeta \cdot \nabla\zeta \le \delta C \int_{\mathbb{R}^3} |\nabla(v_\pm \zeta)|^2 + CR^{-2} \int_{B_R} v_\pm^2. \tag{2.13}$$

The key point which we used here to control the more singular drift term was to split b from the main part of the term $v_\pm\zeta$, using the Young and Sobolev inequalities instead of standard techniques which utilize the Hardy inequality-type spectral gap estimate to control $|b|v_\pm^2\zeta^2$ in one step. We choose δ sufficiently small in order to absorb this term into the dissipation.

We have $(\partial_r\zeta)/r = (\partial_\rho\zeta)/\rho$, where $\rho = |x|$ since ζ is radial, so that the singularity $1/\rho$ is effectively $1/R$. We thus have

$$\sup_{-\sigma^2R^2<t<0} \int_{B_{\sigma R}\times\{t\}} |(\Omega - k)_\pm|^p + \int_{Q_{\sigma R}} |\nabla(\Omega - k)_\pm^{p/2}|^2 \le \frac{C}{(1-\sigma)^2R^2} \int_{Q_R} |(\Omega - k)_\pm|^p. \tag{2.14}$$

Our goal will be to establish L^p to L^∞ bounds for functions in this energy class.

2.6 Local maximum estimate

The estimates in this section will be proven for a general function $u = \Omega$ satisfying equation (2.14).

Lemma 2.2. Suppose $u = \Omega \in L^p(Q_{\bar{R}})$ satisfies equation (2.14) for $1 < p \le 2$ and $R < \bar{R}$. Then

$$\sup_{Q_{R/2}} u_\pm \le C(p, C_*) \left(R^{-3-2} \int_{Q_R} |u_\pm|^p \right)^{1/p}.$$

□

This estimate can be found in [20] for $p = 2$. The proof is similar and we include it so that the proof of Theorem 3.1, which uses Lemma 2.2, is self-contained. Our choice of p is made merely because those are the ones we need, although others are possible.

Proof. For $K > 0$ to be determined and N a positive integer, we define

$$k_N = k_N^\pm = (1 \mp 2^{-N})K, \quad R_N = (1 + 2^{-N})R/2, \quad \rho_N = \frac{R}{2^{N+3}},$$
$$R_{N+1} < \bar{R}_N = (R_N + R_{N+1})/2 < R_N.$$

Notice that

$$R_N - \bar{R}_N = (R_N - R_{N+1})/2 = (2^{-N} - 2^{-N-1})R/4 = \rho_N.$$

Define $Q_N = Q(R_N)$ and $\bar{Q}_N = Q(\bar{R}_N) \subset Q_N$. Choose a smooth test function ζ_N satisfying $\zeta_N \equiv 1$ on $\bar{Q}_N$, $\zeta \equiv 0$ outside Q_N and vanishing on it's spatial boundary, $0 \le \zeta_N \le 1$ and $|\nabla \zeta_N| \le \rho_N^{-1}$ in Q_N. Further, let

$$A^{\pm}(N) = \{X \in Q_N : \pm(u - k_{N+1})(X) > 0\}.$$

And $A_{N,\pm} = \left|A^{\pm}(N)\right|$. Let $v_\pm = \zeta_N (u - k_{N+1})_\pm^{p/2}$.

Hölder's inequality gives us

$$\begin{aligned}\int_{Q_{N+1}} |(u - k_{N+1})_\pm|^p &\le \int_{\bar{Q}_N} |v_\pm|^2 \\ &\le \left(\int_{\bar{Q}_N} |v_\pm|^{2(n+2)/n}\right)^{n/(n+2)} A_{N,\pm}^{2/(n+2)}.\end{aligned}$$

We will use the following parabolic Sobolev inequality which holds for functions vanishing on ∂B_R:

$$\int_{Q_R} |u|^{2(n+2)/n} \le C(n) \left(\sup_{-R^2<t<0} \int_{B_R \times \{t\}} |u|^2\right)^{2/n} \int_{Q_R} |\nabla u|^2.$$

See [20, Theorem 6.11, p. 112]. We are interested in the form

$$\int_{Q_R} |u^{p/2}|^{2(n+2)/n} \le C(n) \left(\sup_{-R^2<t<0} \int_{B_R \times \{t\}} |u|^p\right)^{2/n} \int_{Q_R} |\nabla u^{p/2}|^2.$$

Above and below, n is the spatial dimension, so that $n = 3$. As in the above followed by Young's inequality then followed by equation (2.14), we obtain

$$\begin{aligned}\left(\int_{\bar{Q}_N} |v_\pm|^{2(n+2)/n}\right)^{n/(n+2)} &\le C \left(\sup_{-R_N^2<t<0} \int_{B(R_N)\times\{t\}} |v_\pm|^2\right)^{2/(n+2)} \left(\int_{Q_N} |\nabla v_\pm|^2\right)^{n/(n+2)} \\ &\le C \left(\sup_{-R_N^2<t<0} \int_{B(R_N)\times\{t\}} |v_\pm|^2 + \int_{Q_N} |\nabla v_\pm|^2\right)\end{aligned}$$

$$
\begin{aligned}
&\le C\left(\sup_{-R_N^2<t<0}\int_{B(R_N)\times\{t\}}|(u-k_{N+1})_\pm|^p+\int_{Q_N}|\nabla(u-k_{N+1})_\pm^{p/2}|^2\right)\\
&\quad+\frac{C_*}{\rho_N^2}\int_{Q_N}|(u-k_{N+1})_\pm|^p\\
&\le\frac{C_*}{\rho_N^2}\int_{Q_N}|(u-k_{N+1})_\pm|^p\le\frac{C_*}{\rho_N^2}\int_{Q_N}|(u-k_N)_\pm|^p.
\end{aligned}
$$

Further assume $K^p\ge R^{-n-2}\int_{Q(R)}|u_\pm|^p$. And define

$$
Y_N\equiv K^{-p}R^{-n-2}\int_{Q_N}|(u-k_N)_\pm|^p.
$$

Since $k_N^\pm$ are increasing for $+$ or decreasing for $-$ and Q_N are decreasing, Y_N is decreasing.

Chebyshev's inequality tells us that

$$
\begin{aligned}
A_{N,\pm}&=\left|\left\{Q_N:\pm\left(u-k_{N+1}^\pm\right)>0\right\}\right|=\left|\left\{Q_N:\pm\left(u-k_N^\pm\right)>\pm\left(k_{N+1}^\pm-k_N^\pm\right)\right\}\right|\\
&=|\{Q_N:\pm(u-k_N)>K/2^{N+1}\}|\le 2^{p(N+1)}R^{n+2}Y_N.
\end{aligned}
$$

Putting all of this together yields

$$
\begin{aligned}
\int_{Q_{N+1}}|(u-k_{N+1})_\pm|^p&\le\left(\int_{\tilde{Q}_N}|v_\pm|^{2(n+2)/n}\right)^{n/(n+2)}A_{N,\pm}^{2/(n+2)},\\
&\le\left(\frac{C_*}{\rho_N^2}\int_{Q_N}|(u-k_N)_\pm|^p\right)(2^{p(N+1)}R^{n+2}Y_N)^{2/(n+2)}\\
&\le\left(\frac{C_*}{\rho_N^2}K^pR^{n+2}Y_N\right)(2^{p(N+1)}R^{n+2}Y_N)^{2/(n+2)}\\
&=C_*K^p2^{2(N+3)}2^{2p(N+1)/(n+2)}R^{n+2}Y_N^{1+\frac{2}{n+2}}.
\end{aligned}
$$

We have thus shown that

$$
Y_{N+1}\le C(N)Y_N^{1+\frac{2}{n+2}}.
$$

Here $C(N)=C_*2^{2(N+3)}2^{2p(N+1)/(n+2)}$. We now choose K as

$$
K^p=\left(1+\frac{1}{C_0}\right)R^{-n-2}\int_{Q_0}|u_\pm|^p.
$$

Above, the constant C_0 is chosen to ensure that $Y_N\to 0$ as $N\to\infty$. ■

2.7 Regularity of the original solution

The limiting solution Ω satisfies equations (2.11), (2.12), and (2.14). We conclude from Lemma 2.2 that

$$\Omega \in L^\infty(Q_{5/16}).$$

We further know that $\operatorname{curl} \bar{v} = \bar{\omega}_\theta e_\theta \in L^\infty(Q_{5/16})$ from the above estimate on Ω, since $\bar{v}_\theta = 0$. Also $\operatorname{div} \bar{v} = 0$ from the equation. Next $\bar{v} \in L^\infty_t L^1_x(Q_{5/16})$ by equation (1.2). We thus conclude $\nabla \bar{v} \in L^\infty_t L^4_x(Q_{1/4})$ by Lemma A.1. Thus, $\bar{v} \in L^\infty(Q_{1/4})$ by embedding.

Now we can deduce regularity of the original solution from the regularity of the limit solution. We have shown that

$$|\bar{v}(x,t)| \le C'_* \quad \text{in } Q_{1/4}.$$

Above, C'_* depends upon C_* but not on the subsequence λ_k. Since the constant can be tracked, we may initially choose R_* sufficiently small to guarantee that

$$\frac{1}{R_*^2}\int_{Q_{R_*}} |\bar{v}|^3 \le \epsilon_1/2,$$

where ϵ_1 is the small constant in Lemma 2.1. Since $v^{\lambda_k} \to \bar{v}$ strongly in L^3 for k sufficiently large, we have

$$\frac{1}{R_*^2}\int_{Q_{R_*}} |v^\lambda|^3 \le \frac{1}{R_*^2}\int_{Q_{R_*}} |\bar{v}|^3 + \frac{1}{R_*^2}\int_{Q_{R_*}} |v^\lambda - \bar{v}|^3 \le \epsilon_1.$$

But this is a contradiction to equation (2.9). Thus, every point x_* on the z-axis is regular; that is, there is a radius $R_{x_*} > 0$ so that $v \in L^\infty(Q(x_*, R_{x_*}))$. Since any finite portion of the z-axis can be covered by a finite subcover of $\{Q(x_*, R_{x_*})\}$, we have proved Theorem 1.1.

The rest of the paper is devoted to proving the key Theorem 3.1.

3 Hölder Estimate for Axisymmetric Solutions

We now move from cartesian to cylindrical coordinates via the standard change of variables $x = (x_1, x_2, x_3) = (r\cos\theta, r\sin\theta, z)$. For axisymmetric solutions (v, p) of the form (1.1), the Navier–Stokes equations (N–S) take the form

$$\begin{aligned}
\frac{\partial v_r}{\partial t} + b \cdot \nabla v_r - \frac{v_\theta^2}{r} + \frac{\partial p}{\partial r} &= \left(\Delta - \frac{1}{r^2}\right) v_r, \\
\frac{\partial v_\theta}{\partial t} + b \cdot \nabla v_\theta + \frac{v_\theta v_r}{r} &= \left(\Delta - \frac{1}{r^2}\right) v_\theta, \\
\frac{\partial v_z}{\partial t} + b \cdot \nabla v_z + \frac{\partial p}{\partial z} &= \Delta v_z, \\
\frac{1}{r}\frac{\partial (r v_r)}{\partial r} + \frac{\partial v_z}{\partial z} &= 0.
\end{aligned}$$

The vector b is given by

$$b = v_r e_r + v_z e_z, \quad \operatorname{div} b = 0.$$

The equations of the vorticity $\omega = \operatorname{curl} v$, decomposed in the form (1.8), are

$$\begin{aligned}
\frac{\partial \omega_r}{\partial t} + b \cdot \nabla \omega_r - \omega_r \partial_r v_r - \omega_z \partial_z v_r &= \left(\Delta - \frac{1}{r^2}\right) \omega_r, \\
\frac{\partial \omega_\theta}{\partial t} + b \cdot \nabla \omega_\theta - 2\frac{v_\theta}{r} \partial_z v_\theta - \frac{v_r}{r} \omega_\theta &= \left(\Delta - \frac{1}{r^2}\right) \omega_\theta, \\
\frac{\partial \omega_z}{\partial t} + b \cdot \nabla \omega_z - \omega_z \partial_z v_z - \omega_r \partial_r v_z &= \Delta \omega_z.
\end{aligned}$$

We are interested in the equation for v_θ, which is independent of the pressure.

Consider the change of variable $\Gamma = r v_\theta$, which is well known (see the references in the introduction). The function Γ is smooth and satisfies

$$\frac{\partial \Gamma}{\partial t} + b \cdot \nabla \Gamma - \Delta \Gamma + \frac{2}{r}\frac{\partial \Gamma}{\partial r} = 0. \tag{3.1}$$

Note that the sign of the term $\frac{2}{r}\frac{\partial \Gamma}{\partial r}$ is opposite to that of equation (2.11). It follows directly from equation (1.2) that $\|\Gamma\|_{L^\infty_{t,x}} \le C_*$; see [2] for related estimates. Since v is smooth, we have $\Gamma(t, 0, z) = 0$ for $t < 0$. The smoothness and axisymmetry assumptions also imply that $v_\theta(t, 0, z) = 0$, but we will not use this fact. The main result of this section is the following.

Theorem 3.1. Suppose that $\Gamma(x,t)$ is a smooth bounded solution of equation (3.1) in Q_2 with smooth $b(x,t)$, both may depend on θ, and

$$\Gamma|_{r=0} = 0, \quad \operatorname{div} b = 0, \quad |b| \le C_*/r \quad \text{in } Q_2.$$

Then there exist constants C and $\alpha > 0$, which depend only upon C_* such that

$$|\Gamma(x,t)| \le C \|\Gamma\|_{L^\infty_{t,x}(Q_2)} r^\alpha \quad \text{in } Q_1.$$

□

We remark that the condition above is substantially weaker than equation (1.2), and we do not need Γ to be axisymmetric. In the rest of this section, we will prove the theorem. Here we are facing two difficulties: First, the condition $\Gamma|_{r=0} = 0$ precludes a direct lower bound on the fundamental solution and a Harnack inequality on Γ (since, when $b = 0$, $\Gamma = r^2$ is a non-negative solution which does not satisfy the usual Harnack inequality.) Second, the condition $b \le C/r$ is weaker than the standard assumption $b \le C/|x|$ (see the discussion). It turns out that one can develop new techniques incorporating the methods introduced by De Giorgi [4] and Moser [22] to overcome these two points. However, we do not know if one can follow the approach of Nash [7, 23] which relies critically on a Gaussian lower bound of the fundamental solution. The proof of Theorem 3.1 is independent of the rest of the paper.

The following related equation has been previously studied by Zhang [41]:

$$\frac{\partial u}{\partial t} + b \cdot \nabla u - \Delta u = 0.$$

He has shown among other things Hölder continuity of solutions to this equation if $b = b(x)$ is independent of time and b satisfies an integral condition, which is fulfilled if say b is controlled by $1/|x|$. His proof makes use of Moser iteration and Gaussian bounds.

3.1 Notation, reformulation, and energy inequalities

Let $X = (x,t)$. Define the modified parabolic cylinder at the origin

$$Q(R,\tau) = \{X : |x| < R, -\tau R^2 < t < 0\}.$$

Here $R > 0$ and $\tau \in (0, 1]$. We sometimes for brevity write $Q_R = Q(R) = Q(R, 1)$. Let

$$m_2 \equiv \inf_{Q(2R)} \Gamma, \quad M_2 \equiv \sup_{Q(2R)} \Gamma, \quad M \equiv M_2 - m_2 > 0.$$

Notice that $m_2 \le 0 \le M_2$ since $\Gamma|_{r=0} = 0$.

Now we reformulate the problem in $Q(2R)$ into a new function, u, which will be zero when $|\Gamma|$ is at its maximum value. Specifically, we define

$$u \equiv \begin{cases} 2(\Gamma - m_2)/M & \text{if } -m_2 > M_2, \\ 2(M_2 - \Gamma)/M & \text{else.} \end{cases} \tag{3.2}$$

In either case, u solves equation (3.1) and $0 \le u \le 2$ in $Q(2R)$. We will further use

$$a \equiv u|_{r=0} = \frac{2}{M}\left(\sup_{Q(2R)} |\Gamma|\right) = \frac{2}{M}\max\{M_2, -m_2\} \ge 1,$$

which follows from our conditions.

We now derive energy estimates for equation (3.1). Define $v_\pm = (u-k)_\pm$ with $k \ge 0$. We have $v_+ \le (2-k)_+$ and $v_- \le k$. Consider a radial test function $0 \le \zeta(x,t) \le 1$ for which $\zeta = 0$ on $\partial B_R \times [-\tau R^2, 0]$ and $\frac{\partial \zeta}{\partial r} \le 0$. We multiply equation (3.1) for $u - k$ with $\zeta^2 v_\pm$ and integrate over $\mathbb{R}^3 \times [t_0, t]$ to obtain

$$\frac{1}{2}\left[\int_{\mathbb{R}^3} |\zeta v_\pm|^2\right]_{t_0}^{t} + \int_{t_0}^{t}\int_{\mathbb{R}^3} |\nabla(\zeta v_\pm)|^2 = \int_{t_0}^{t}\int_{\mathbb{R}^3} v_\pm^2 \left(b\zeta\cdot\nabla\zeta + \zeta\frac{\partial\zeta}{\partial t} + |\nabla\zeta|^2 + \frac{2\zeta}{r}\frac{\partial\zeta}{\partial r}\right) + 2\pi[(a-k)_\pm]^2 \int_{t_0}^{t}\int_{\mathbb{R}} dz\, \zeta^2|_{r=0}. \tag{3.3}$$

We need to estimate all the terms in parenthesis.

Choose $\sigma \in (1/4, 1)$, we require that the test function satisfies $\zeta \equiv 1$ on $Q(\sigma R, \tau)$. If we further choose $\zeta(x, t_0) = 0$ then, using equation (2.13), we estimate equation (3.3) as follows:

$$\sup_{-\tau\sigma^2 R^2 < t < 0} \int_{B(\sigma R)\times\{t\}} v_\pm^2 + \int_{Q(\sigma R, \tau)} |\nabla v_\pm|^2 \le \frac{C_{**}}{\tau(1-\sigma)^2 R^2}\int_{Q(R,\tau)} v_\pm^2 + C\tau R^3[(a-k)_\pm]^2. \tag{3.4}$$

If we alternatively choose $\zeta = \zeta(x)$, then equation (3.3) takes the form

$$\sup_{t_0<s<t} \int_{B(\sigma R)\times\{s\}} v_\pm^2 + \int_{t_0}^{t} \int_{B(\sigma R)} |\nabla v_\pm|^2 - \int_{B_R\times\{t_0\}} v_\pm^2 \leq \frac{C_{**}}{(1-\sigma)^2 R^2} \int_{t_0}^{t} \int_{B_R} v_\pm^2 + C\tau R^3[(a-k)_\pm]^2. \tag{3.5}$$

Notice that there is no τ^{-1} appearing in this energy inequality (3.5) compared to equation (3.4).

The energy estimates (3.4) and (3.5) are the standard parabolic De Giorgi classes except for the last term. Our goal will be to use them to show that the set where Γ is very close to its largest absolute value or, equivalently, the set where u is almost zero is as small as you wish. We establish this fact in the following series of lemmas.

3.2 Initial estimates

Later on, we will use the two standard lemmas in a nonstandard iteration scheme of sorts to show that the set where u is almost zero has very small Lebesgue measure.

Lemma 3.2. Suppose there exists a $t_0 \in [-\tau R^2, 0]$, $K \in (0,1)$, and $\gamma \in (0,1)$ so that

$$|\{x \in B_R : u(x,t_0) \leq K\}| \leq \gamma |B_R|.$$

Further suppose that u satisfies equation (3.5) for v_-. Then for all $\eta \in (0, 1-\sqrt{\gamma})$ and $\mu \in (\gamma/(1-\eta)^2, 1)$, there exists $\theta \in (0,1)$ such that

$$|\{x \in B_R : u(x,t) \leq \eta K\}| \leq \mu |B_R|, \quad \forall t \in [t_0, t_0 + (\tau \wedge \theta)R^2].$$

Here θ depends only on the constants in equation (3.5) and γ. □

We note that, in the proof, it can happen that $\theta(\gamma) \to 0$ as $\gamma \uparrow 1$, but if τ is sufficiently small, then we may take $\theta = \tau$ when γ is close enough to zero. And if γ is small, then μ can be taken almost as small.

Proof. We consider $v_- = (u-K)_-$. The energy inequality (3.5) for this function is

$$\begin{aligned}\int_{B(\sigma R)\times\{t\}} v_-^2 &\leq \int_{B_R\times\{t_0\}} v_-^2 + \frac{C_{**}}{(1-\sigma)^2 R^2} \int_{t_0}^{t} \int_{B_R} v_-^2 + C\tau R^3[(a-K)_-]^2 \\ &\leq K^2 |B_R| \left(\gamma + \frac{C_{**}(\tau \wedge \theta)}{(1-\sigma)^2}\right).\end{aligned}$$

We have used $(a-K)_- = 0$. The Chebyshev inequality tells us that

$$|\{x \in B(\sigma R) : u(x,t) \leq \eta K\}| \cdot (K - \eta K)^2 \leq \int_{B(\sigma R)\times\{t\}} v_-^2.$$

The region $B_R - B_{\sigma R}$ has volume $(1-\sigma^3)|B_R|$. Thus,

$$\begin{aligned}\frac{|\{x \in B_R, u(x,t) \leq \eta K\}|}{|B_R|} &\leq \frac{|\{x \in B(\sigma R), u(x,t) \leq \eta K\}|}{|B_R|} + (1-\sigma^3)\\ &\leq (1-\eta)^{-2}\left(\gamma + \frac{C_{**}(\tau \wedge \theta)}{(1-\sigma)^2}\right) + (1-\sigma^3).\end{aligned}$$

Now let σ be so close to one that $\frac{\gamma}{(1-\eta)^2} + (1-\sigma^3) < \mu$. Then, with τ fixed, choose θ small enough that the whole thing is $\leq \mu$. ■

Lemma 3.2 shows continuity in time of the Lebesgue measure of the set where u is small and Lemma 3.3 below shows that if the set where u is small is less than the whole set, then the set where u is even smaller can be made tiny. This is an extremely weak way to measure diffusion.

Lemma 3.3. Suppose that $u(x,t)$ satisfies equation (3.4) for v_-. In addition,

$$|\{x \in B_R : u(x,t) \leq K\}| \leq \gamma |B_R|, \quad \forall t \in [t_0, t_0 + \theta R^2] = I,$$

where $K, \theta > 0$, $\gamma \in (0,1)$ and $B_R \times I \subset Q(R,\tau)$. Then for all $\epsilon \in (0,1)$, there exists a $\delta \in (0,1)$ such that

$$|\{X \in B_R \times I : u(X) \leq \delta\}| \leq \epsilon |B_R \times I|.$$

□

Proof. We denote, for $n = 0, 1, 2, 3, \ldots$,

$$A_n(t) = \{x \in B_R : u(x,t) \leq 2^{-n} K\}, \quad A_n = \{(x,t) : t \in I, x \in A_n(t)\}.$$

Note that $A_n(t) \subset B_R$ and $A_n \subset B_R \times I$. Clearly, $|A_{n+1}| \leq |A_n| \leq |A_0| \leq \gamma |B_R \times I|$. And

$$\left|A_n^c(t)\right| = |\{x \in B_R : u(x,t) > 2^{-n} K\}| = |B_R| - |A_n(t)| \geq (1-\gamma)|B_R|.$$

Since $\gamma < 1$, we know that $A_n^c(t)$ does not have measure zero.

We invoke the following known inequality, a version of which can already be seen in Lemma II of [4]. For any $v \in W^{1,1}(B_R)$ and for any $\alpha, \beta \in \mathbb{R}$ with $\alpha < \beta$, we have

$$|\{x \in B_R : v(x) \le \alpha\}| \le \frac{C R^{3+1}/(\beta-\alpha)}{|\{x \in B_R : v(x) > \beta\}|} \int_{B_R \cap \{\alpha < v \le \beta\}} |\nabla v|,$$

where $C > 0$ only depends on the dimension. To show it, let $E = \{x \in B_R : v(x) > \beta\}$ and we may assume $|E| > 0$. This inequality can be shown by plugging

$$U(x) = \min((\beta - v(x))_+, \beta - \alpha), \quad f(x) = \frac{|B_R|}{|E|} 1_E(x)$$

into the Poincaré inequality from [20, Proposition 6.14], which implicitly assumes the normalization condition $\frac{1}{|B_R|}\int_{B_R} f = 1$.

Now let $\beta = 2^{-n}K$ and $\alpha = 2^{-n-1}K$. We have

$$|A_{n+1}(t)| \le \frac{C 2^{n+1} R}{K(1-\gamma)} \int_{A_n(t)-A_{n+1}(t)} |\nabla u| = \frac{C 2^{n+1} R}{K(1-\gamma)} \int_{A_n(t)-A_{n+1}(t)} |\nabla (u-a)^-|.$$

We use the Cauchy–Schwartz inequality to bound this integral as

$$\begin{aligned} |A_{n+1}| = \int_I |A_{n+1}(t)| &\le \frac{C 2^{n+1} R}{K(1-\gamma)} \int_{A_n - A_{n+1}} |\nabla (u-a)^-| \\ &\le \frac{C 2^{n+1} R}{K(1-\gamma)} |A_n - A_{n+1}|^{1/2} \left(\int_{A_n - A_{n+1}} |\nabla (u-a)^-|^2 \right)^{1/2}. \end{aligned}$$

The energy inequality (3.4), with σR and R replaced by R and $2R$ results in

$$\begin{aligned} |A_{n+1}| &\le \frac{C 2^{n+1} R}{K(1-\gamma)} |A_n - A_{n+1}|^{1/2} \left(\frac{C}{\tau R^2} \int_{Q(2R,\tau)} |(u-a)^-|^2 \right)^{1/2} \\ &\le \frac{C 2^{n+1} R}{K(1-\gamma)} |A_n - A_{n+1}|^{1/2} |B(2R)|^{1/2} a = \frac{C R^{5/2}}{K(1-\gamma)} |A_n - A_{n+1}|^{1/2}. \end{aligned}$$

Square both sides of this inequality and dividing by $|B_R \times I|^2$ to obtain

$$\frac{|A_{n+1}|^2}{|B_R \times I|^2} \le \frac{C}{\theta K^2 (1-\gamma)^2} \left(\frac{|A_n|}{|B_R \times I|} - \frac{|A_{n+1}|}{|B_R \times I|} \right).$$

Summing in n, we get

$$
\begin{aligned}
n\frac{|A_n|^2}{|B_R\times I|^2} \le \sum_{j=1}^{n}\frac{|A_j|^2}{|B_R\times I|^2} &\le \frac{C}{\theta K^2(1-\gamma)^2}\sum_{j=1}^{n}\left(\frac{|A_{j-1}|}{|B_R\times I|}-\frac{|A_j|}{|B_R\times I|}\right)\\
&= \frac{C}{\theta K^2(1-\gamma)^2}\left(\frac{|A_0|}{|B_R\times I|}-\frac{|A_n|}{|B_R\times I|}\right)\\
&\le \frac{C}{\theta K^2(1-\gamma)^2}\frac{|A_0|}{|B_R\times I|}\le \frac{C\gamma}{\theta K^2(1-\gamma)^2}.
\end{aligned}
$$

■

We complete the proof by choosing n sufficiently large.

3.3 Estimate on the measure of the set where u is small

The next lemma allows us to apply all the machinery above.

Lemma 3.4. There exists a $\kappa\in(0,1)$ such that $0<\lambda<\min\{\kappa\tau, 1/8\}$ implies

$$|\{X\in Q(R,\tau): u(X)\le\lambda^2\}|\le(1-4\lambda)|Q(R,\tau)|.$$

□

Proof. We establish a contradiction using energy estimates. Suppose the opposite

$$|\{X\in Q(R,\tau): u(X)\le\lambda^2\}|>(1-4\lambda)|Q(R,\tau)|.$$

Or equivalently,

$$|\{X\in Q(R,\tau): u(X)>\lambda^2\}|<4\lambda|Q(R,\tau)|. \tag{3.6}$$

This condition will imply a contradiction to the size condition on $a\ge 1$.

We will test equation (3.1) with $pu^{p-1}\zeta^2$ for $0<p<1$ and $\zeta\ge 0$. Since $u=0$ sometimes, in general we should test equation (3.1) for $u+\epsilon$ with $p(u+\epsilon)^{p-1}\zeta^2$ and then send $\epsilon\downarrow 0$ to obtain our estimates. However, since the result is the same, to simplify the presentation we will omit these details. We have

$$
\begin{aligned}
\int_{Q(R,\tau)} pu^{p-1}\zeta^2\frac{\partial u}{\partial t} &= \left[\int_{B_R}\zeta^2u^p\right]_{t_1}^{0}-\int_{Q(R,\tau)}u^p2\zeta\frac{\partial\zeta}{\partial t}\equiv I_1+I_2,\\
\int_{Q(R,\tau)} pu^{p-1}\zeta^2(-\Delta u) &= \frac{4(p-1)}{p}\int_{Q(R,\tau)}|\nabla(u^{p/2}\zeta)|^2\\
&\quad+\int_{Q(R,\tau)}2u^p\left[-|\nabla\zeta|^2+\frac{p-2}{p}\zeta\Delta\zeta\right]\equiv I_3+I_4,
\end{aligned}
$$

$$\int_{Q(R,\tau)} pu^{p-1}\zeta^2 b\cdot\nabla u = -\int_{Q(R,\tau)} 2u^p b\cdot\zeta\nabla\zeta \equiv I_5,$$
$$\int_{Q(R,\tau)} pu^{p-1}\zeta^2\frac{2}{r}\partial_r u = -\int_{Q(R,\tau)} 4u^p\zeta\zeta_\rho/\rho - \int_{-\tau R^2}^0 dt\int_{\mathbb{R}} dz\, 2(\zeta^2 u^p)|_{r=0} \equiv I_6 + I_7.$$

In the computation of I_6, we have used $\zeta_r/r = \zeta_\rho/\rho$, which follows if $\zeta = \zeta(\rho, t)$ where $\rho = |x| = \sqrt{r^2+z^2}$. Notice that $\sum_{j=1}^7 I_j = 0$. For arbitrary $p \in (0,1)$, we see that I_3 and I_7 are both non-positive.

We choose $\zeta = \zeta_1(\rho)\zeta_2(t)$, where $\zeta_1(\rho) = 1$ in $B(R/2)$ and $\zeta_1(\rho)$ has compact support in B_R; also $\zeta_2(t) = 1$ if $t \in [-\frac{7}{8}\tau R^2, -\frac{1}{8}\tau R^2]$ and $\zeta_2(t)$ has compact support in $(-\tau R^2, 0)$. Thus, $I_1 = 0$ and we have

$$\frac{6}{4}\tau R^3 a^p \le -I_7 = \sum_{j=2}^{6} I_j.$$

We estimate each of the terms I_2 through I_6 to obtain a contradiction.

By the argument in equation (2.13), we have

$$|I_5| \le \frac{2(1-p)}{p}\int_{Q(R,\tau)} |\nabla(u^{p/2}\zeta)|^2 + \frac{C}{R^2}\int_{Q(R,\tau)} u^p.$$

Also note $\nabla\zeta = 0$ in $B(R/2)$ and so the singularity $1/\rho$ is effectively $1/R$. Thus,

$$I_2 \le \frac{C}{\tau R^2}\int_{Q(R,\tau)} u^p, \quad \sum_{j=3}^{6} I_j \le \frac{C}{R^2}\int_{Q(R,\tau)} u^p.$$

Assuming equation (3.6) and using $0 \le u \le 2$, we have

$$a^p \le \frac{C}{\tau^2 R^5}\int_{Q(R,\tau)} u^p \le \frac{C}{\tau^2 R^5}\{\lambda^{2p}|Q(R,\tau)| + 2^p(4\lambda|Q(R,\tau)|)\} \le \frac{C_2}{\tau}(\lambda^{2p}+\lambda).$$

Here $C_2 = C_2(C_*)$. Take $p = 1/2$ and $\kappa = \frac{1}{4C_2}$ to get $a^p < 1$, a contradiction. ■

Lemma 3.4 is the starting point of our iteration scheme. From this lemma, we know that there is a $t_1 \in [-\tau R^2, -2\lambda\tau R^2]$ so that

$$|\{x \in B_R : u(x, t_1) \le \lambda^2\}| \le (1-2\lambda)|B_R|. \tag{3.7}$$

Then apply Lemma 3.2 with $K = \lambda^2$ to equation (3.7) to see, for say $\eta = \lambda$ and $\mu = 1 - \lambda$, that

$$|\{x \in B_R : u(x,t) \le \lambda^3\}| \le (1-\lambda)|B_R|, \quad \forall t \in [t_1, t_1 + \theta_* R^2] \equiv I_*.$$

Here $\theta_* = \theta \wedge \tau$ and θ is the constant chosen in Lemma 3.2. From here, Lemma 3.3 allows us to conclude

$$|\{X \in B_R \times I_* : u(X) \le \delta_*\}| \le \frac{\epsilon_*}{2}|B_R \times I_*|,$$

where $\epsilon_* > 0$ is as small as you want and $\delta_* = \delta_*(\epsilon_*)$.

Then, as in equation (3.7), there exists a $t_2 \in I_*$ (so that $t_2 \le -\lambda\tau R^2$) such that

$$|\,\{x \in B_R : u(x,t_2) \le \delta_*\}\,| \le \epsilon_* |B_R|. \tag{3.8}$$

Uptill now, all the small parameters that we have chosen depend upon τ. But above, ϵ_* can be taken arbitrarily small, independent of the size of τ. This is the key point that enables us to proceed. It is the reason why we are required to do this procedure twice.

Now suppose $1 - \sigma^3 = 1/4$ and choose first $\tau < 1/8$, so that $C_{**}\tau/(1-\sigma)^2 \le 1/4$. Then take δ_* from equation (3.8) with $\epsilon_* < 1/16$ playing the role of γ in Lemma 3.2. Also $\eta < 1/2$. With all this, from Lemma 3.2, we can choose $\mu < 1$ so that

$$|\{x \in B_R : u(x,t) \le \eta\delta_*\}| \le \mu|B_R|, \quad \forall t \in [t_2, t_2 + \tau R^2] \equiv I.$$

Further, it is safe to assume that $\theta_* \le \lambda$; we see that $t_2 \le -\lambda\tau R^2$ and so $[-\lambda\tau R^2, 0] \subset I$. Finally, apply Lemma 3.3 again to obtain

$$|\{X \in Q(R,\lambda\tau) : u(X) \le \delta\}| \le \epsilon |Q(R,\lambda\tau)|, \tag{3.9}$$

with $\epsilon > 0$ arbitrarily small. This is a key step in what follows.

Let $U = \delta - u$, where δ is the constant from equation (3.9). U is clearly a solution of equation (3.1) and $U|_{r=0} = \delta - a < 0$. We apply equation (3.4) to U on $Q(2d)$ (with $\tau = 1$) to get

$$\sup_{-\sigma^2 d^2 < t < 0} \int_{B(\sigma d)\times\{t\}} |(U-k)^+|^2 + \int_{Q(\sigma d)} |\nabla (U-k)^+|^2 \le \frac{C_{**}}{(1-\sigma)^2 d^2} \int_{Q(d)} |(U-k)^+|^2.$$

This holds for all $k > 0$ and $\sigma \in (0, 1)$. So we can apply Lemma 2.2 to conclude

$$\sup_{Q(d/2)} (\delta - u) \leq \left(\frac{C}{|Q(d)|} \int_{Q(d)} |(\delta - u)^+|^2 \right)^{1/2}. \tag{3.10}$$

This inequality combined with equation (3.9) will produce a lower bound.

3.4 Regularity from a lower bound

Let $d = \sqrt{\lambda\tau} R$ so that $Q(d) \subset Q(R, \lambda\tau)$. By equations (3.10) and (3.9),

$$\begin{aligned} \delta - \inf_{Q(d/2)} u &\leq \left(\frac{C}{|Q(d)|} \int_{Q(d)} |(\delta - u)^+|^2 \right)^{1/2} \\ &\leq \left(\frac{C\delta^2 \epsilon |Q(R, \lambda\tau)|}{|Q(d)|} \right)^{1/2} = C\delta\epsilon^{1/2} (\lambda\tau)^{-3/4}, \end{aligned}$$

which is less than $\frac{\delta}{2}$ if ϵ is chosen sufficiently small. We conclude

$$\inf_{Q(d/2)} u \geq \frac{\delta}{2}.$$

This is the lower bound we seek. From it we will deduce an oscillation estimate.

This entails a bit of algebra. We define

$$m_d \equiv \inf_{Q(d/2)} \Gamma, \quad M_d \equiv \sup_{Q(d/2)} \Gamma.$$

Then from equation (3.2), we have

$$\inf_{Q(d/2)} u = \begin{cases} 2(m_d - m_2)/M & \text{if } -m_2 > M_2, \\ 2(M_2 - M_d)/M & \text{else.} \end{cases}$$

Notice that both expressions above are non-negative in any case; thus we can add them together to observe that

$$\frac{\delta}{2} \leq \frac{2}{M} \{M - \operatorname{osc}(\Gamma, d/2)\}.$$

Here $\operatorname{osc}(\Gamma, d/2) = M_d - m_d$ and $\operatorname{osc}(\Gamma, 2R) = M_2 - m_2 = M$. We rearrange the above equation as

$$\operatorname{osc}(\Gamma, d/2) \leq \left(1 - \frac{\delta}{4}\right) \operatorname{osc}(\Gamma, 2R).$$

This is enough to produce the desired Hölder continuity via the following.

3.5 Iteration argument

Suppose we have a nondecreasing function ω on an interval $(0, R_0]$ which satisfies

$$\omega(\tau R) \leq \gamma \omega(R),$$

with $0 < \gamma, \tau < 1$. Then for $R \leq R_0$, we have

$$\omega(R) \leq \frac{1}{\gamma} \left(\frac{R}{R_0}\right)^{\alpha} \omega(R_0), \tag{3.11}$$

where $\alpha = \log \gamma / \log \tau > 0$.

Iterating, as in equation (3.11), we get, for $C_\Gamma = \left(1 - \frac{\delta}{4}\right)^{-1} \sup_{Q(1)} \Gamma$, that

$$\operatorname{osc}(\Gamma, R) \leq C_\Gamma R^{\alpha}, \quad \forall R \in (0, 1), \tag{3.12}$$

for $\alpha = 2 \log(1 - \frac{\delta}{4}) / \log(\lambda\tau/16) > 0$. Thus, Γ is Hölder continuous near the origin. We have proved Theorem 3.1.

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Appendix

Here we collect some estimates needed for Section 2.

Lemma A.1. Let $B_{R_2} \subset B_{R_1} \subset \mathbb{R}^3$ be concentric with $0 < R_2 < R_1$. Let v be a vector field defined in B_{R_1}. Let $1 < q < \infty$ and $0 < \alpha < 1$. Then for $k = 0, 1, \ldots$ there is a constant c depending on R_2, R_1, q, α, k, so that

$$\|\nabla^{k+1} v\|_{L^q(B_{R_2})} \leq c\|\nabla^k \operatorname{div} v\|_{L^q(B_{R_1})} + c\|\nabla^k \operatorname{curl} v\|_{L^q(B_{R_1})} + c\|v\|_{L^1(B_{R_1})}$$

and

$$\|\nabla^{k+1} v\|_{C^\alpha(B_{R_2})} \leq c\|\nabla^k \operatorname{div} v\|_{C^\alpha(B_{R_1})} + c\|\nabla^k \operatorname{curl} v\|_{C^\alpha(B_{R_1})} + c\|v\|_{L^1(B_{R_1})}.$$

□

This is well known, see [24].

Lemma A.2 (Interior estimates for Stokes system). Fix $R \in (0, 1)$. Let $1 < s, q < \infty$, and $f = (f_{ij}) \in L^s_t L^q_x(Q_1)$. Assume that $v \in L^s_t L^1_x(Q_1)$ is a weak solution of the Stokes system

$$\partial_t v_i - \Delta v_i + \partial_i p = \partial_j f_{ij}, \quad \operatorname{div} v = 0 \quad \text{in } Q_1.$$

Then v satisfies, for some constant $c = c(q, s, R)$,

$$\|\nabla v\|_{L^s_t L^q_x(Q_R)} \leq c\|f\|_{L^s_t L^q_x(Q_1)} + c\|v\|_{L^s_t L^1_x(Q_1)}. \tag{A.1}$$

If instead v is a weak solution of

$$\partial_t v_i - \Delta v_i + \partial_i p = g_i, \quad \operatorname{div} v = 0 \quad \text{in } Q_1,$$

then

$$\|\nabla^2 v\|_{L^s_t L^q_x(Q_R)} \leq c\|g\|_{L^s_t L^q_x(Q_1)} + c\|v\|_{L^s L^1(Q_1)}. \tag{A.2}$$

□

An important feature of these estimates is that a bound of the pressure p is not needed in the right side. A similar estimate for the time-independent Stokes system appeared in [34]. Note that these estimates improve the spatial regularity only. One cannot improve the temporal regularity, in view of Serrin's example of a solution $v(x,t) = f(t)\nabla h(x)$ where $h(x)$ is harmonic.

Proof. Denote by P the Helmholtz projection in $\mathbb{R}^3$, $(Pg)_i = g_i - R_i R_k g_k$, where R_i is the ith Riesz transform. Let $\tau = R^{1/4} \in (R, 1)$ and choose $\zeta(x,t) \in C^\infty(\mathbb{R}^4)$, $\zeta \ge 0$, $\zeta = 1$ on Q_τ, and $\zeta = 0$ on $\mathbb{R}^3 \times (-\infty, 0] - Q_1$. For a fixed i, define

$$\tilde{v}_i(x,t) = \int_{-1}^{t} \Gamma(x-y, t-s)\, \partial_j(F_{ij})(y,s)\, dy\, ds,$$

where Γ is the heat kernel and $F_{ij} = f_{ij}\zeta - R_i R_k(f_{kj}\zeta)$. The function $\tilde{v}_i$ satisfies

$$(\partial_t - \Delta)\tilde{v}_i = \partial_j F_{ij} = \left[P\partial_j\zeta(f_{kj})_{k=1}^3\right]_i, \quad \operatorname{div}\tilde{v} = 0.$$

The $L_t^s L_x^q$-estimates for the parabolic version of singular integrals and potentials (see [19, 27], also see [14, 40] and their references), and the usual version of L^q-estimates for singular integrals ([32]), give

$$\|\nabla\tilde{v}\|_{L_t^s L_x^q(Q_1)} + \|\tilde{v}\|_{L_t^s L_x^q(Q_1)} \le c\|F\|_{L_t^s L_x^q} \le c\|f\|_{L_t^s L_x^q(Q_1)}. \tag{A.3}$$

Furthermore, for some function $\tilde{p}(x,t)$,

$$(\partial_t - \Delta)\tilde{v} + \nabla\tilde{p} = \partial_j(\zeta f_{ij}), \quad \operatorname{div}\tilde{v} = 0.$$

The differences $u = v - \tilde{v}$ and $\pi = p - \tilde{p}$ satisfy the homogeneous Stokes system

$$\partial_t u - \Delta u + \nabla\pi = 0, \quad \operatorname{div} v = 0 \quad \text{in } Q_\tau.$$

Its vorticity $\omega = \operatorname{curl} u$ satisfies the heat equation $(\partial_t - \Delta)\omega = 0$. Let $W = \zeta_\tau\omega$, where $\zeta_\tau(x,t) = \zeta(x/\tau, t/\tau^2)$. It satisfies

$$(\partial_t - \Delta)W = G := w(\partial_t - \Delta)\zeta_\tau - 2(\partial_m\zeta_\tau)\partial_m\omega.$$

And thus, for $(x,t) \in Q_{\tau^2}$,

$$\omega_i(x,t) = W_i(x,t) = \int_{-1}^{t}\int \Gamma(x-y, t-s)\, G_i(y,s)\, dy ds = \int_{-1}^{t}\int H_{x,t}^{i,j}(y,s)\, u_j(y,s)\, dy ds$$

where, using $\omega_i = -\delta_{ijk}\partial_k u_j$,

$$H^{i,j}_{x,t}(y,s) = \partial_{y_k}\delta_{ijk}\left\{\Gamma(x-y,t-s)(\partial_t - \Delta)\zeta_\tau + 2\,\mathrm{div}[\Gamma(x-y,t-s)\nabla\zeta_\tau]\right\}.$$

The functions $H^{i,j}_{x,t}$ are smooth with uniform L^∞-bound for $(x,t) \in Q_{\tau^3}$. Thus,

$$\|\mathrm{curl}\, u\|_{L^\infty(Q_{\tau^3})} \le C\,\|u\|_{L^1(Q_1)}.$$

Since $\mathrm{div}\, u = 0$, we have for any $q < \infty$, using Lemma A.1,

$$\|\nabla u\|_{L^s_t L^q_x(Q_R)} \le c\|u\|_{L^s_t L^1_x(Q_1)} \le c\|v\|_{L^s_t L^1_x(Q)} + c\|\tilde{v}\|_{L^s_t L^1_x(Q)}. \tag{A.4}$$

The sum of equations (A.3) and (A.4) gives equation (A.1). The proof of (A.2) is similar: one defines

$$\tilde{v}_i(x,t) = \int_{-1}^{t} \Gamma(x-y,t-s)\,F_i(y,s)\,dy\,ds, \quad F_i = g_i\zeta - R_i R_k(g_k\zeta)$$

and obtains $\|\nabla^2\tilde{v}\|_{L^s_t L^q_x(Q_1)} + \|\tilde{v}\|_{L^s_t L^q_x(Q_1)} \le c\|F\|_{L^s_t L^q_x} \le c\|g\|_{L^s L^q(Q_1)}$. One then estimates $\|\nabla^2(v-\tilde{v})\|_{L^s_t L^q_x(Q_R)}$ in the same way. ■

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