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LOWER BOUNDS FOR DISCRIMINANTS OF NUMBER FIELDS. II

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Abstract. If K is an algebraic number field with r_1 real and $2r_2$ complex conjugate fields, $n = r_1 + 2r_2$ is the degree of the field, and D is absolute value of the discriminant of K, then we show

$$D^{1/n} \geqq (60)^{r_1/n}(22)^{2r_2/n} + o(1) , \qquad n \to \infty .$$

If the zeta function of K has no zeros $\beta + i\gamma$ with $\beta > 1/2$ and $0 < |\gamma| < 3$, then we show

$$D^{1/n} \geqq (188)^{r_1/n}(41)^{2r_2/n} + o(1) , \qquad n \to \infty .$$

Let K be an algebraic number field of degree $n = n_K$, with r_1 real and $2r_2$ complex conjugate fields, and let $D = D_K$ be the absolute value of the discriminant of K. Until recently the best bound for D (at least for large n) was the one due to Rogers [4] and Mulholland [1]:

$$(1) \qquad D^{1/n} \geqq (32.561\cdots)^{r_1/n}(15.775\cdots)^{2r_2/n} + o(1) , \qquad n \longrightarrow \infty .$$

Their proofs depended on geometry of numbers methods.

The above bound was improved recently in [2], where some applications of lower bounds for discriminants were also discussed. It was shown there that

$$(2) \qquad D^{1/n} \geqq (55)^{r_1/n}(21)^{2r_2/n} + o(1) , \qquad n \longrightarrow \infty ,$$

and that if $\zeta_K(s)$, the zeta function of K, has no zeros $\beta + i\gamma$ with $\beta > 1/2$ and $(1-\beta)/2 < |\gamma| < 10$, then

$$(3) \qquad D^{1/n} \geqq (136)^{r_1/n}(34.5)^{2r_2/n} + o(1) , \qquad n \longrightarrow \infty .$$

These bounds were proved by a new analytical method, based on an identity of Stark [5] expressing D in terms of the zeros of $\zeta_K(s)$.

In this paper we will obtain even better bounds for D by employing a slight variant of the method of [2]. Let

$$Z(s) = -\frac{\zeta_K'}{\zeta_K}(s) ,$$

$$Z_j(s) = \frac{(-1)^j}{j!}\frac{d^j}{ds^j}Z(s) , \qquad j = 1, 2, \cdots .$$

Since for $\sigma = \operatorname{Re}(s) > 1$ we have

$$Z(s) = \sum_P \frac{\log NP}{(NP)^s - 1},$$

where P runs through the prime ideals of K, we see that $Z(\sigma) > 0$, $Z_j(\sigma) > 0$ for $\sigma > 1$. With this notation we can now state our main results.

THEOREM 1. *If*

$$\begin{aligned} f_1 = f_1(K) = {} & 2Z(1.01) + 1.015Z_1(2) + 0.0564Z_2(1.2) \\ & + 1.974Z_2(2.3) + 1.246Z_3(1.7) + 0.2776Z_3(2) \\ & + 2.588Z_3(2.5) + 0.007532Z_4(1.2) + 0.0865Z_4(1.4) \\ & + 0.2Z_4(1.6) + 2.946Z_4(2.25) + 0.234Z_4(2.5) \\ & + 1.464Z_4(2.75) \end{aligned}$$

and

$$\begin{aligned} f_2 = f_2(K) = {} & 2Z(1.1) + 1.184Z_1(2) + 0.298Z_2(1.2) \\ & + 2.04Z_2(2.3) + 1.076Z_3(1.7) + 2.946Z_3(2.5) \\ & + 0.0884Z_4(1.6) + 3.57Z_4(2.25) + 0.1834Z_4(2.5) \\ & + 1.516Z_4(2.75), \end{aligned}$$

then

$$D \geqq (60.1)^{r_1}(22.2)^{2r_2}e^{f_1-254} \tag{4}$$

and

$$D \geqq (58.6)^{r_1}(21.8)^{2r_2}e^{f_2-70}. \tag{5}$$

THEOREM 2. *If $\zeta_K(s)$ has no zeros $\rho = \beta + i\gamma$ with $\beta > 1/2$ and $0 < |\gamma| < 3$, then*

$$D \geqq (188.3)^{r_1}(41.6)^{2r_2}e^{f_3-3.7\times 10^8}, \tag{6}$$

where

$$\begin{aligned} f_3 = f_3(K) = {} & 2Z(1.001) + 0.995Z_1(2) + 2.448Z_2(2.5) \\ & + 8.662Z_3(2.5) + 1.902Z_4(1.75) + 16.276Z_4(2) \\ & + 300Z_6(4.25) + 0.0228Z_8(1.1) + 0.0342Z_9(1.1) \\ & + 473Z_9(2.1) + 12{,}660Z_9(3.1) + 3{,}340Z_9(4.1) \\ & + 85{,}000Z_9(4.6) + 9.16Z_{10}(1.55) + 7{,}090Z_{14}(2.3) \\ & + 3{,}460{,}000Z_{14}(3.5). \end{aligned}$$

Before proceeding to the proof we should make a few comments about these results. In general we cannot say anything about f_1, f_2, and f_3 except that they are positive. In those cases in which K is known to have many prime ideals with small norms, the contributions of the f_i can be used to obtain significantly better estimates. For example, if 2 splits completely in K, then by bounding the Z_i from below by the sums over the n prime ideals P which divide 2 we obtain

$$D > (495)^{r_1}(183)^{2r_2}e^{-254}, \tag{4'}$$

$$D > (472)^{r_1}(175)^{2r_2}e^{-70}, \tag{5'}$$

and if ζ_K has no zeros $\beta + i\gamma$ with $\beta > 1/2$ and $0 < |\gamma| < 3$, then

$$D > (4{,}696)^{r_1}(1{,}037)^{2r_2}e^{-3.7\times 10^8}. \tag{6'}$$

If 3 splits complely, we obtain

$$D > (351)^{r_1}(129)^{2r_2}e^{-254}, \tag{4''}$$

$$D > (335)^{r_1}(124)^{2r_2}e^{-70}, \tag{5''}$$

$$D > (3{,}326)^{r_1}(734)^{2r_2}e^{-3.7\times 10^8}, \tag{6''}$$

and if both 2 and 3 split completely, the corresponding bounds are

$$D > (2{,}890)^{r_1}(1{,}070)^{2r_2}e^{-254}, \tag{4'''}$$

$$D > (2{,}700)^{r_1}(1{,}000)^{2r_2}e^{-70}, \tag{5'''}$$

and

$$D > (82{,}900)^{r_1}(18{,}300)^{2r_2}e^{-3.7\times 10^8}. \tag{6'''}$$

These results are interesting in that they relate the arithmetic properties of a field to the size of the discriminant. We should note that, as is shown in [2], there exist infinitely many fields with discriminants smaller than the bound (5‴), say.

The estimate (6) can be obtained on the assumption of a smaller zero-free region than the one of Theorem 2. Furthermore, it can be seen from the proof of Theorem 2 that there exists a positive constant c such that if D_K does not satisfy (6), then $\zeta_K(s)$ has $\geqq cn_K$ zeros $\rho = \beta + i\gamma$ with $\beta > 1/2$ and $0 < |\gamma| < 3$.

The very large size of the error term 3.7×10^8 in (6) is due to the fact that since Theorem 2 assumes an unproved hypothesis, no attempt was made to obtain a small error term. However, by choosing the variables in the proof somewhat differently one could obtain an estimate which would be non-trivial even for small n (although the main term would probably have to be decreased). Similarly, (4) and (5) are but two of an infinite family of estimates that can be derived by our method. Unfortunately, in most cases each estimate of this type requires a separate proof. However, in [3] a simple version of the method of this paper is used to derive a very flexible result, which although not as good as (4) or (5) for large n, is better than they are for moderately large n (approximately for $n \leqq 130$ if K is totally real and $n \leqq 210$ of K is totally complex) and gives non-trivial bounds even for very small n. Furthermore, that estimate is very easy to prove and does not require the computations needed to establish the theorems of this paper.

PROOF OF THEOREMS 1 AND 2. Let

$$A = D^{1/2} 2^{-r_2} \pi^{-n/2} \tag{7}$$

and

$$\psi(s) = \frac{\Gamma'}{\Gamma}(s) .$$

As in [2] and [3], our starting point is an identity of H. M. Stark [5; Lemma 1], [2; Lemma 1]:

$$\log A = -\frac{r_1}{2}\psi\left(\frac{s}{2}\right) - r_2\psi(s) - \frac{1}{s-1} - \frac{1}{s} + Z(s) + \sum_{\rho}{}' \frac{1}{s-\rho} , \tag{8}$$

valid identically in the complex variable s, where ρ runs through the non-trivial zeros of $\zeta_K(s)$ (i.e., those zeros $\rho = \beta + i\gamma$ for which $0 < \beta < 1$), and $\sum'$ indicates that the ρ and $\bar{\rho}$ terms are to be summed together. Our main goal will be to obtain a lower bound for the sum over the zeros that will be independent of D. For this purpose we will use derivatives of (8), which give relations between the zeros ρ, the $Z_i(s)$, and the derivatives of the digamma function, but do not involve D.

First of all, if ρ is a non-trivial zero of ζ_K, then so are $\bar{\rho}$, $1-\rho$, and $1-\bar{\rho}$. Thus for real σ we can write

$$\sum_{\rho}{}' \frac{1}{\sigma-\rho} = \frac{1}{2}\sum_{\rho} \operatorname{Re}\left\{\frac{1}{\sigma-\rho} + \frac{1}{\sigma-1+\bar{\rho}}\right\} .$$

Hence if

$$E(\sigma, z) = \operatorname{Re}\left\{\frac{1}{\sigma-z} + \frac{1}{\sigma-1+\bar{z}}\right\}$$

and

$$G(\sigma) = -\frac{r_1}{2}\psi\left(\frac{\sigma}{2}\right) - r_2\psi(\sigma) ,$$

then for real σ we can write (8) as

$$\log A = G(\sigma) - \frac{1}{\sigma - 1} - \frac{1}{\sigma} + Z(\sigma) + \frac{1}{2}\sum_{\rho} E(\sigma, \rho) . \tag{9}$$

If for $j = 1, 2, \cdots,$ we let

$$G_j(\sigma) = \frac{(-1)^j}{j!}\frac{d^j}{d\sigma^j}G(\sigma)$$

and

$$E_j(\sigma, z) = -\mathrm{Re}\left\{\frac{1}{(\sigma - z)^j} + \frac{1}{(\sigma - 1 + \bar{z})^j}\right\} ,$$

then by differentiating (9) $j - 1$ times we obtain (for $j \geqq 2$)

$$\frac{1}{2}\sum_{\rho} E_j(\sigma, \rho) = G_{j-1}(\sigma) - \frac{1}{(\sigma - 1)^j} - \frac{1}{\sigma^j} + Z_{j-1}(\sigma) . \tag{10}$$

Suppose now that we can find $\sigma > 1$ and a positive integer N together with $\sigma_1, \cdots, \sigma_N$, each $\sigma_i > 1$, as well as positive real numbers $a_1, \cdots, a_N$, and integers $k_1, \cdots, k_N$, each $k_i \geqq 2$, such that

$$E(\sigma, \rho) \geqq \sum_{l=1}^{N} a_l E_{k_l}(\sigma_l, \rho) \tag{11}$$

holds for all non-trivial zeros ρ of ζ_K. Then, combining (9), (11), and (10), we see that

$$\begin{aligned} \log A \geqq G(\sigma) &+ \sum_{l=1}^{N} a_l G_{k_l - 1}(\sigma_l) - \frac{1}{\sigma - 1} - \frac{1}{\sigma} \\ &- \sum_{l=1}^{N} a_l\left\{\frac{1}{(\sigma_l - 1)^{k_l}} + \frac{1}{\sigma_l^{k_l}}\right\} + Z(\sigma) \\ &+ \sum_{l=1}^{N} a_l Z_{k_l - 1}(\sigma_l) , \end{aligned} \tag{12}$$

which, together with the definition (7) of A, gives us a bound for D of the desired form.

The main difficulty in applying this method is the proper choice of σ, N, σ_l, a_l, and k_l. The values of these variables which give us Theorems 1 and 2 are presented in separate tables. These values were found with the help of linear programming, but as far as the proofs of (4)-(6) are concerned, we only need to show that these values satisfy (11). In the case of Theorem 1, to ensure that (11) holds for all non-trivial zeros ρ, we will prove that

$$E(\sigma, z) \geqq \sum_{l=1}^{N} a_l E_{k_l}(\sigma_l, z) \tag{13}$$

holds for all complex $z = x + iy$ with $0 < x < 1$. Since E and the E_j are even functions of y, it suffices to prove (13) for $y \geqq 0$. However, it was pointed out by H. M. Stark that in fact it suffices to prove (13) just for $x = 1$, $y \geqq 0$. To see this, note that E and the E_j are harmonic functions of z and in the strip $0 \leqq x \leqq 1$ they tend uniformly to 0 as $y \to \infty$. Hence by the maximum principle for harmonic functions (13) must hold throughout the strip $0 \leqq x \leqq 1$, $y \geqq 0$ if it holds on the boundary of that strip. Now for $y = 0$ and $0 \leqq x \leqq 1$ we have $E(\sigma, z) > 0$ and $E_j(\sigma', z) < 0$ for $j \geqq 2$, so (13) is trivially valid. Also, E and the E_j are invariant under the transformation $x \to 1 - x$, so if (13) holds for $x = 1$, $y \geqq 0$, then it also holds for $x = 0$, $y \geqq 0$. Thus we only need to check (13) for $x = 1$, $y \geqq 0$.

To prove Theorem 1 it thus remains to prove (13) for $x = 1$, $y \geqq 0$ for the choices of variables given in the first two tables. For $y \geqq 40$ this can be done by means of very crude estimates, since the terms with $k_l > 2$ are negligible. For $0 \leqq y < 40$, the inequality (13) was proved numerically. If $g(u) \in C^2[a, b]$, say, and $|g''(u)| \leqq Q$, $u \in [a, b]$, then for $x \in [a, b]$, $u_0 \in [a, b]$ we have

$$|g(x) - g(u_0) - g'(u_0)(x - u_0)| \leqq \frac{1}{2} Q |x - u_0|^2$$

by Taylor's formula. In particular, if for some $u_0 \in [a, b]$ we are given $g(u_0) > 0$, $g'(u_0)$, and Q, then we can determine an interval around u_0 on which $g(u)$ is positive. In our case we have to do this with

$$g(y) = E(\sigma, 1 + iy) - \sum_{l=1}^{N} a_l E_{k_l}(\sigma_l, 1 + iy) .$$

Evaluation of $g(y)$ and $g'(y)$ is straightforward. To bound the second derivative, we use the fact that

$$|g''(y)| \leqq \left| \frac{\partial^2}{\partial y^2} E(\sigma, 1 + iy) \right| + \sum_{l=1}^{N} a_l \left| \frac{\partial^2}{\partial y^2} E_{k_l}(\sigma_l, 1 + iy) \right|$$

and

$$\begin{aligned} \left| \frac{\partial^2}{\partial y^2} E_r(\tilde{\sigma}, 1 + iy) \right| &\leqq r(r + 1) \left\{ \frac{1}{|\tilde{\sigma} - 1 - iy|^{r+2}} + \frac{1}{|\tilde{\sigma} - iy|^{r+2}} \right\} \\ &\leqq r(r + 1) \left\{ \frac{1}{|\tilde{\sigma} - 1 - ia|^{r+2}} + \frac{1}{|\tilde{\sigma} - ia|^{r+2}} \right\} \end{aligned}$$

for $y \in [a, b]$, $a > 0$. These bounds, when applied to several hundred subintervals of $[0, 40]$, show that the choices of variables corresponding to (4) and (5) satisfy (13) for $x = 1$, $0 \leqq y \leqq 40$, and this proves Theorem 1.

To prove Theorem 2, it is necessary to verify that the choices of variables corresponding to (6) satisfy (13) for $0 < x < 1$, $y = 0$; $x = 1/2$, $0 \leqq y \leqq 3$; and $0 < x < 1$, $y \geqq 3$. The case $y = 0$, $0 < x < 1$ is again trivial, while the case $x = 1/2$, $0 \leqq y \leqq 3$ can be checked numerically, as is the case of Theorem 1. In the case $0 < x < 1$, $y \geqq 3$, the harmonic functions argument shows that we only need to check when $1/2 \leqq x < 1$, $y = 3$ and $x = 1$, $y \geqq 3$. For $x = 1$ and $y \geqq 40$ it is easy to check that (13) is satisfied, since the terms with $k_l > 2$ are negligible. On the other hand, the two intervals $1/2 \leqq x < 1$, $y = 3$ and $x = 1$, $3 \leqq y \leqq 40$ were again checked numerically, which completed the proof of Theorem 2.

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Estimate (4): $\sigma = 1.01$, $N = 12$

l	k_l	σ_l	a_l
1	2	2	0.5075
2	3	1.2	0.0282
3	3	2.3	0.987
4	4	1.7	0.623
5	4	2	0.1388
6	4	2.5	1.294
7	5	1.2	0.003766
8	5	1.4	0.04325
9	5	1.6	0.1
10	5	2.25	1.473
11	5	2.5	0.117
12	5	2.75	0.732

Estimate (5): $\sigma = 1.1$, $N = 9$

l	k_l	σ_l	a_l
1	2	2	0.592
2	3	1.2	0.149
3	3	2.3	1.02
4	4	1.7	0.538
5	4	2.5	1.473
6	5	1.6	0.0442
7	5	2.25	1.785
8	5	2.5	0.0917
9	5	2.75	0.758

Estimate (6): $\sigma = 1.001$, $N = 15$

l	k_l	σ_l	a_l
1	2	2	0.4975
2	3	2.5	1.224
3	4	2.5	4.331
4	5	1.75	0.951
5	5	2	8.138
6	7	4.25	150
7	9	1.1	0.0114
8	10	1.1	0.0171
9	10	2.1	236.5
10	10	3.1	6,330
11	10	4.1	1,670
12	10	4.6	42,500
13	11	1.55	4.58
14	15	2.3	3,545
15	15	3.2	1.73×10^6

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