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LOWER BOUNDS FOR MATRICES ON BLOCK WEIGHTED SEQUENCE SPACES I

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Abstract. In this paper we consider some matrix operators on block weighted sequence spaces $l_p(w,F)$. The problem is to find the lower bound of some matrix operators such as Hausdorff and Hilbert matrices on $l_p(w,F)$. This study is an extension of papers by G. Bennett, G.J.O. Jameson and R. Lashkaripour.

Keywords: block weighted sequence spaces, lower bound, inequality, Hausdorff matrix, Hilbert matrix

MSC 2010: 47B37, 46B45, 26D15

1. Introduction

Suppose $p \geqslant 1$ and $w = (w_n)$ is a decreasing non-negative sequence. We define the weighted sequence space $l_p(w)$ as

$$l_p(w) := \left\{ x = (x_n) \colon \sum_{n=1}^{\infty} w_n |x_n|^p \text{ is finite} \right\},$$

with a norm $\|\cdot\|_{p,w}$ which is defined in the following way:

$$\|x\|_{p,w} = \left(\sum_{n=1}^{\infty} w_n |x_n|^p \right)^{1/p}.$$

Assume that F is a partition of positive integers. If $F = (F_n)$, where each F_n is a finite interval of positive integers and

$$\max F_n < \min F_{n+1} \quad (n = 1, 2, \ldots),$$

we define the block weighted sequence space $l_p(w, F)$ as

$$l_p(w,F) := \left\{ x = (x_n) \colon \sum_{n=1}^{\infty} w_n |\langle x, F_n\rangle|^p \text{ is finite} \right\},$$

where $\langle x, F_n\rangle = \sum_{j\in F_n} x_j$. The norm on $l_p(w, F)$, denoted by $\|\cdot\|_{p,w,F}$, is defined as follows:

$$\|x\|_{p,w,F} = \left(\sum_{n=1}^{\infty} w_n |\langle x, F_n\rangle|^p \right)^{1/p}.$$

For a certain I_n such as $I_n = \{n\}$, $I = (I_n)$ is a partition of positive integers, $l_p(w, I) = l_p(w)$ and also $\|x\|_{p,w,I} = \|x\|_{p,w}$.

We write $\|A\|_{p,w,F}$ for the norm of A as an operator from $l_p(w, I)$ into $l_p(w, F)$. The problem of the norm of matrix operators on $l_p(w)$ and $l_p(w, F)$ is considered in [5], [6], [7] and [8].

We consider lower bounds L of the form

$$\|Ax\|_{p,w,F} \geqslant L\|x\|_{p,w,I},$$

for all decreasing non-negative sequences x. The constant L is independent of x. We seek the largest possible value of L, and denote the best lower bound by $L_{p,w,F}(A)$ for matrix operators from $l_p(w, I)$ into $l_p(w, F)$. Also, if A is an operator from $l_p(w, I)$ into itself, we denote the best lower bound by $L_{p,w,I}(A)$. We shall use all the above notation when $p < 1$.

In Section 2, we generalize two techniques obtained by Bennett in Section 7 of [1] and deduce the lower bound for the Hausdorff matrix. In Section 3, we also generalize Theorem 1 of [4] for matrix operators from $l_p(w, I)$ into $l_p(w, F)$ and study the lower bound problem for the Hilbert and Copson matrices.

Throughout this paper, we denote the transpose matrix of A by A^t, and the conjugate exponent of p by p^*, so that $p^* = p/(p-1)$.

2. Hausdorff matrix operator

In this part we consider the Hausdorff matrix operator $H(\mu) = (h_{j,k})$ with entries of the form

$$h_{j,k} = \begin{cases} \binom{j-1}{k-1} \Delta^{j-k} a_k & \text{if } 1 \leqslant k \leqslant j, \\ 0 & \text{if } k > j, \end{cases}$$

where Δ is the difference operator, that is

$$\Delta a_k = a_k - a_{k+1},$$

and (a_k) is a sequence of real numbers, normalized so that $a_1 = 1$. If

$$a_k = \int_0^1 \theta^{k-1}\,\mathrm{d}\mu(\theta) \quad (k = 1, 2, \ldots),$$

where μ is a probability measure on $[0,1]$, then for all $j, k = 1, 2, \ldots$,

$$h_{j,k} = \begin{cases} \binom{j-1}{k-1} \int_0^1 \theta^{k-1}(1-\theta)^{j-k}\,\mathrm{d}\mu(\theta) & \text{if } 1 \leqslant k \leqslant j, \\ 0 & \text{if } k > j. \end{cases}$$

The Hausdorff matrix contains some famous classes of matrices. These classes are as follows:

i) Choice $\mathrm{d}\mu(\theta) = \alpha(1-\theta)^{\alpha-1}\,\mathrm{d}\theta$ gives the Cesàro matrix of order α;
ii) Choice $\mathrm{d}\mu(\theta) =$ *point evaluation at* $\theta = \alpha$ gives the Euler matrix of order α;
iii) Choice $\mathrm{d}\mu(\theta) = |\log\theta|^{\alpha-1}/\Gamma(\alpha)\,\mathrm{d}\theta$ gives the Hölder matrix of order α;
iv) Choice $\mathrm{d}\mu(\theta) = \alpha\theta^{\alpha-1}\,\mathrm{d}\theta$ gives the Gamma matrix of order α.

The Cesàro, Hölder and Gamma matrices have non-negative entries whenever $\alpha > 0$, and also the Euler matrix is non-negative when $0 \leqslant \alpha \leqslant 1$.

In this section, we are considering the lower bound problem for the Hausdorff matrix (general form), and also for the Cesàro, Hölder and Gamma matrices.

Proposition 2.1. *Let $A = (a_{n,k})$ be an upper-triangle matrix with non-negative entries and let $0 < p \leqslant 1$. If*

$$\sup_n \sum_{k=n}^{\infty} a_{n,k} = k_1 > 0, \quad \inf_k \sum_{n=1}^{k} a_{n,k} = k_2,$$

then

$$L_{p,w,I}(A) \geqslant k_1^{(p-1)/p} k_2^{1/p}.$$

Proof. Suppose x is a non-negative sequence. Applying Hölder's inequality we have

$$\begin{aligned} \sum_{k=n}^{\infty} a_{n,k} w_k x_k^p &= \sum_{k=n}^{\infty} a_{n,k}^{1-p} (a_{n,k} w_k^{1/p} x_k)^p \\ &\leqslant \Big(\sum_{k=n}^{\infty} a_{n,k}\Big)^{1-p} \Big(\sum_{k=n}^{\infty} a_{n,k} w_k^{1/p} x_k\Big)^p \\ &\leqslant k_1^{1-p} \Big(\sum_{k=n}^{\infty} a_{n,k} w_k^{1/p} x_k\Big)^p. \end{aligned}$$

Since A is an upper-triangle matrix with non-negative entries and w is decreasing, we have

$$\begin{aligned}k_1^{1-p}\sum_{n=1}^{\infty}w_n\Big(\sum_{k=1}^{\infty}a_{n,k}x_k\Big)^p &= k_1^{1-p}\sum_{n=1}^{\infty}w_n\Big(\sum_{k=n}^{\infty}a_{n,k}x_k\Big)^p\\ &\geqslant k_1^{1-p}\sum_{n=1}^{\infty}\Big(\sum_{k=n}^{\infty}a_{n,k}w_k^{1/p}x_k\Big)^p \geqslant \sum_{n=1}^{\infty}\Big(\sum_{k=n}^{\infty}a_{n,k}w_kx_k^p\Big)\\ &= \sum_{k=1}^{\infty}w_kx_k^p\Big(\sum_{n=1}^{k}a_{n,k}\Big) \geqslant k_2\sum_{k=1}^{\infty}w_kx_k^p.\end{aligned}$$

Hence $\|Ax\|_{p,w,I}^p \geqslant k_1^{p-1}k_2\|x\|_{p,w,I}^p$ and so we have the desired conclusion. □

In the next statement, we seek a lower bound for the quasi-Hausdorff matrix when the sequences are non-negative. We recall the transpose of the Hausdorff matrix which is called the quasi-Hausdorff matrix.

Theorem 2.1. *Suppose that $H(\mu)$ is the Hausdorff matrix and $0 < p \leqslant 1$. Then*

$$\|H^t(\mu)x\|_{p,w,I} \geqslant \Big(\int_0^1 \theta^{(1-p)/p}\,\mathrm{d}\mu(\theta)\Big)\|x\|_{p,w,I}$$

for every non-negative sequence x. The constant is the best possible.

Proof. Let $E(\alpha)$ be the Euler matrix of order α. Since all row sums of $E^t(\alpha)$ are $1/\alpha$ and all column sums are 1, applying Proposition 2.1 we obtain

$$L_{p,w,I}(E^t(\alpha)) \geqslant \alpha^{(1-p)/p}.$$

Let $F = (F_n)$ be defined as above. We now apply Minkowski's inequality to show

$$\begin{aligned}\|H^t(\mu)x\|_{p,w,I} &= \Big(\sum_{n=1}^{\infty}w_n\Big(\sum_{j\in F_n}(H^t(\mu)x)_j\Big)^p\Big)^{1/p} = \Big(\sum_{n=1}^{\infty}w_n\Big(\sum_{k=1}^{\infty}h_{k,n}x_k\Big)^p\Big)^{1/p}\\ &= \Big(\sum_{n=1}^{\infty}w_n\Big(\int_0^1\sum_{k=1}^{\infty}e_{k,n}x_k\,\mathrm{d}\mu(\alpha)\Big)^p\Big)^{1/p}\\ &\geqslant \int_0^1\Big(\sum_{n=1}^{\infty}w_n\Big(\sum_{k=1}^{\infty}e_{k,n}x_k\Big)^p\Big)^{1/p}\,\mathrm{d}\mu(\alpha)\\ &= \int_0^1\|E^t(\alpha)x\|_{p,w}\,\mathrm{d}\mu(\alpha) \geqslant \Big(\int_0^1\alpha^{(1-p)/p}\,\mathrm{d}\mu(\alpha)\Big)\|x\|_{p,w,I}.\end{aligned}$$

This completes the proof of the above inequality. Therefore for any real number $\alpha > 0$ we have

$$\text{(I)} \qquad \|H^t(\mu)x\|_{p,w+\alpha,I} \geqslant \Big(\int_0^1 \theta^{(1-p)/p}\, \mathrm{d}\mu(\theta)\Big)\|x\|_{p,w+\alpha,I}$$

for all non-negative sequences x in $l_p(w, I)$. We will show that the above constant is the best possible.

Let $\varrho > 1/p$ and let n be a fixed integer such that $n \geqslant \varrho$. We define x by

$$x_k = \begin{cases} 0 & \text{if } k < n, \\ \binom{k-\varrho}{k-n} \Big/ \binom{k}{n} & \text{if } k \geqslant n. \end{cases}$$

Since

$$x_k = \frac{(k-\varrho)\ldots(n+1-\varrho)}{k\ldots(n+1)} \sim k^{-\varrho}$$

when $k \to \infty$, it follows that $\|x\|_p < \infty$ and $\|x\|_p \to \infty$ when $\varrho \to 1/p$. Since w is decreasing and also $w_k + \alpha \geqslant \alpha$ for all k, we have

$$\alpha^{1/p}\|x\|_p \leqslant \|x\|_{p,w+\alpha,I} \leqslant (w_1+\alpha)^{1/p}\|x\|_p,$$

so $\|x\|_{p,w+\alpha,I} < \infty$ and $\|x\|_{p,w+\alpha,I} \to \infty$ when $\varrho \to 1/p$. Moreover, for all $m > n$ we have

$$(H^t(\mu)x)_m = x_m \int_0^1 \theta^{\varrho-1}\, \mathrm{d}\mu(\theta),$$

hence

$$\begin{aligned} \|H^t(\mu)x\|_{p,w+\alpha,I}^p &= \sum_{m=1}^{n}(w_m+\alpha)\Big(\sum_{k=m}^{\infty} h_{k,m}x_k\Big)^p + \sum_{m=n+1}^{\infty}(w_m+\alpha)(H^t(\mu)x)_m^p \\ &\leqslant n(w_1+\alpha)\sup_{k,m}|h_{k,m}|^p\|x\|_1^p + \Big(\int_0^1 \theta^{\varrho-1}\,\mathrm{d}\mu(\theta)\Big)^p\|x\|_{p,w+\alpha,I}^p \end{aligned}$$

and also

$$(L_{p,w+\alpha,I}(H^t(\mu)))^p \leqslant \frac{n(w_1+\alpha)\sup_{k,m}|h_{k,m}|^p\|x\|_1^p}{\|x\|_{p,w+\alpha,I}^p} + \Big(\int_0^1 \theta^{\varrho-1}\,\mathrm{d}\mu(\theta)\Big)^p.$$

If $\varrho \to 1/p$, then

$$L_{p,w+\alpha,I}(H^t(\mu)) \leqslant \int_0^1 \theta^{(1-p)/p}\,\mathrm{d}\mu(\theta).$$

Therefore

$$L_{p,w+\alpha,I}(H^t(\mu)) = \int_0^1 \theta^{(1-p)/p}\,\mathrm{d}\mu(\theta),$$

and the constant in (I) is the best possible. Hence for all m there is a non-negative sequence $y_m \in l_p(w,I)$ such that

$$\frac{\|H^t(\mu)y_m\|_{p,w+\alpha,I}}{\|y_m\|_{p,w+\alpha,I}} < \int_0^1 \theta^{(1-p)/p}\,\mathrm{d}\mu(\theta) + \frac{1}{m}.$$

Since $\|H^t(\mu)y_m\|_{p,w,I} \leqslant \|H^t(\mu)y_m\|_{p,w+\alpha,I}$, we have

$$\begin{aligned}\frac{\|H^t(\mu)y_m\|_{p,w+\alpha,I}}{\|y_m\|_{p,w+\alpha,I}} &\geqslant \frac{\|H^t(\mu)y_m\|_{p,w,I}}{\|y_m\|_{p,w+\alpha,I}}\\ &= \frac{\|y_m\|_{p,w,I}}{\|y_m\|_{p,w+\alpha,I}} \cdot \frac{\|H^t(\mu)y_m\|_{p,w,I}}{\|y_m\|_{p,w,I}}\\ &\geqslant \frac{\|y_m\|_{p,w,I}}{\|y_m\|_{p,w+\alpha,I}} L_{p,w,I}(H^t(\mu)),\end{aligned}$$

and so

$$\frac{\|y_m\|_{p,w,I}}{\|y_m\|_{p,w+\alpha,I}} L_{p,w,I}(H^t(\mu)) \leqslant \int_0^1 \theta^{(1-p)/p}\,\mathrm{d}\mu(\theta) + \frac{1}{m}.$$

If $\alpha \to 0$, since $\|x\|_{p,w+\alpha,I} < \infty$, we have $\|x\|_{p,w+\alpha,I} \to \|x\|_{p,w,I}$ and so

$$L_{p,w,I}(H^t(\mu)) \leqslant \int_0^1 \theta^{(1-p)/p}\,\mathrm{d}\mu(\theta) + \frac{1}{m}.$$

Now, if $m \to \infty$, we have

$$L_{p,w,I}(H^t(\mu)) \leqslant \int_0^1 \theta^{(1-p)/p}\,\mathrm{d}\mu(\theta).$$

Therefore

$$L_{p,w,I}(H^t(\mu)) = \int_0^1 \theta^{(1-p)/p}\,\mathrm{d}\mu(\theta).$$

This completes the proof of the theorem. □

Example. We denote the Gamma matrix of order 2 by $\Gamma(2)$. If $\Gamma^t(2) = (b_{i,j})$ is the transpose of the Gamma matrix, we have

$$b_{i,j} = \begin{cases} \dfrac{i}{\frac{1}{2}j(j+1)} & \text{if } j \geqslant i,\\ 0 & \text{if } j < i \end{cases}$$

and

$$L_{p,w,I}(\Gamma^t(2)) = \frac{2p}{p+1}.$$

We now give a lower bound for the quasi-Hausdorff matrix when the sequences are non-negative. We recall the transpose of the Hausdorff matrix which is called the quasi-Hausdorff matrix.

Proposition 2.2. *Let $0 < p, q < 1$, and let A be a matrix with non-negative entries. Then*

$$\|Ax\|_{q,w,I} \geqslant L\|x\|_{p,w,I}$$

for all non-negative x if and only if

$$\|A^t y\|_{p^*,w,I} \geqslant L\|y\|_{q^*,w,I}$$

for all non-negative y, where p^*, q^* are the conjugate exponents of p and q, respectively.

Proof. Suppose u is a sequence with non-negative entries. First we show that

(I) $\quad \|u\|_{t,w,I} = \inf\{\langle u,v\rangle :\ v \text{ is a non-negative sequence and } \|v\|_{t^*,w,I} \geqslant 1\}$

for $0 < t < 1$ or $t < 0$, where $\langle u,v\rangle = \sum\limits_{k=1}^{\infty} w_k u_k v_k$.

Let v be a non-negative sequence such that $\|v\|_{w,t^*,I} \geqslant 1$. Then applying Hölder's inequality, we deduce that

$$\langle u,v\rangle = \sum_{k=1}^{\infty} w_k u_k v_k = \sum_{k=1}^{\infty} w_k^{1/t+1/t^*} u_k v_k \geqslant \Big(\sum_{k=1}^{\infty} w_k u_k^t\Big)^{1/t} \Big(\sum_{k=1}^{\infty} w_k v_k^{t^*}\Big)^{1/t^*}$$
$$= \|u\|_{t,w,I}\|v\|_{t^*,w,I} \geqslant \|u\|_{t,w,I}.$$

Hence $\inf\langle u,v\rangle \geqslant \|u\|_{t,w,I}$.

We divide the proof of the converse inequality in two cases as follows:

Case 1. If $u > 0$, we take

$$\tilde{v}_k = u_k^{t-1}, \quad v_k = \frac{\tilde{v}_k}{\|\tilde{v}\|_{t^*,w,I}}.$$

Hence $\|\tilde{v}\|_{t^*,w,I} = \|u\|_{t,w,I}^{t-1}$ and $\langle u,v\rangle = \|u\|_{t,w,I}$, so that

$$\inf\langle u,v\rangle \leqslant \|u\|_{t,w,I}.$$

Case 2. If some $u_k = 0$, we consider (i), (ii):

(i) For $t < 0$, $\|u\|_{w,t,I} = 0$ and set

$$v_n = \begin{cases} 0 & \text{for } n \neq k, \\ \dfrac{1}{w_k^{1/t^*}} & \text{for } n = k. \end{cases}$$

(ii) For $0 < t < 1$, we set

$$\tilde{v}_k = \begin{cases} u_k^{t-1} & \text{for } u_k > 0, \\ \Big(\dfrac{\varepsilon}{w_k 2^k}\Big)^{1/t^*} & \text{for } u_k = 0 \end{cases}$$

and $v_k = \tilde{v}_k / \|\tilde{v}\|_{t^*,w,I}$, where ε is positive.

Hence $\|v\|_{t^*,w,I} = 1$, $\|\tilde{v}\|_{t^*,w,I} \geqslant 1/(\varepsilon + \|u\|_{t,w,I}^t)^{-1/t^*}$ and also

$$\langle u, v\rangle \leqslant \|u\|_{t,w,I}^t (\varepsilon + \|u\|_{t,w,I}^t)^{-1/t^*},$$

so that

$$\inf \langle u, v\rangle \leqslant \|u\|_{t,w,I}^t (\varepsilon + \|u\|_{t,w,I}^t)^{-1/t^*}.$$

If ε tends to zero, we have

$$\inf \langle u, v\rangle \leqslant \|u\|_{t,w,I}.$$

This completes the proof of (I).

Applying (I) twice, we deduce that

$$\begin{aligned} \inf_{\|x\|_{p,w,I} \geqslant 1} \|Ax\|_{q,w,I} &= \inf_{\|x\|_{p,w,I} \geqslant 1} \inf_{\|y\|_{q^*,w,I} \geqslant 1} \langle Ax, y\rangle \\ &= \inf_{\|x\|_{p,w,I} \geqslant 1} \inf_{\|y\|_{q^*,w,I} \geqslant 1} \big\langle x, A^t y\big\rangle \\ &= \inf_{\|y\|_{q^*,w,I} \geqslant 1} \inf_{\|x\|_{p,w,I} \geqslant 1} \big\langle x, A^t y\big\rangle \\ &= \inf_{\|y\|_{q^*,w,I} \geqslant 1} \|A^t y\|_{p^*,w,I} \end{aligned}$$

and so we have the statement. □

In the next statement, we are seeking a lower bound of the Hausdorff matrix when the sequences are non-negative.

Corollary 2.2. *Suppose that $p < 0$, and let $H(\mu)$ be the Hausdorff matrix. Then*

$$\|H^t(\mu)x\|_{p,w,I} \geqslant \Big(\int_0^1 \theta^{-1/p}\,\mathrm{d}\mu(\theta)\Big) \|x\|_{p,w,I}$$

for every non-negative sequence x. The constant is the best possible.

P r o o f. Since $0 < p^* < 1$, applying Theorem 2.1 and Proposition 2.1 we obtain the statement. □

Corollary 2.3. *Assume $0 < p \leqslant 1$, and let $H(\mu)$ be the Hausdorff matrix. Then*

$$\|H^t(\mu)x\|_p \geqslant \left(\int_0^1 \theta^{(1-p)/p}\,\mathrm{d}\mu(\theta)\right)\|x\|_p$$

for every non-negative sequence x. The constant is the best possible.

P r o o f. By taking $w_n = 1$ for all n in the previous corollary, we have the above inequality. □

Corollary 2.4. *If $p > 0$ and $H(\mu)$ is the Hausdorff matrix, then*

$$\sum_{n=1}^{\infty} w_n \left(\sum_{k=1}^{n} \frac{h_{n,k}}{|x_k|}\right)^{-p} \leqslant \left(\int_0^1 \theta^{1/p}\,\mathrm{d}\mu(\theta)\right)^{-p} \sum_{k=1}^{\infty} w_k|x_k|^p$$

for every non-negative sequence, and the constant is the best possible.

P r o o f. Suppose that y is a sequence with non-negative entries. Since $-p < 0$, applying Corollary 2.2, we arrive at

$$\|H^t(\mu)y\|_{-p,w,I} \geqslant \left(\int_0^1 \theta^{1/p}\,\mathrm{d}\mu(\theta)\right)\|y\|_{-p,w,I}.$$

Hence

$$\sum_{n=1}^{\infty} w_n \left(\sum_{k=1}^{n} h_{n,k} y_k\right)^{-p} \leqslant \left(\int_0^1 \theta^{1/p}\,\mathrm{d}\mu(\theta)\right)^{-p} \sum_{k=1}^{\infty} w_k|y_k|^p.$$

By replacing y_k with $1/|x_k|$ for $k = 1, 2, \ldots$, we have the result. □

3. Lower bound of matrix operators

In this section, we deal with the problem of finding a lower bound of certain matrix operators from $l_p(w, I)$ into $l_p(w, F)$, which is considered for some matrix operators such as Cesàro, Copson and Hilbert operators in [2], [4] and [9] on $l_p(w)$ and on Lorentz sequence spaces $d(w, p)$. We generalize Theorem 1 from [4] to certain matrix operators from $l_p(w, I)$ into $l_p(w, F)$.

Lemma 3.1 ([4], Lemma 2)**.** *Let $p \geqslant 1$. Suppose that (a_j), (x_j) are non-negative sequences and (x_j) is decreasing and tends to 0. Write $A_n = \sum_{j=1}^{n} a_j$ (with $A_0 = 0$), and $B_n = \sum_{j=1}^{n} a_j x_j$. Then*

(i) $B_n^p - B_{n-1}^p \geqslant (A_n^p - A_{n-1}^p) x_n^p$ *for all n;*

(ii) *if $\sum_{j=1}^{\infty} a_j x_j$ is convergent, then*

$$\left(\sum_{j=1}^{\infty} a_j x_j\right)^p \geqslant \sum_{n=1}^{\infty} A_n^p (x_n^p - x_{n+1}^p).$$

Corollary 3.1. *If (x_j) is a non-negative decreasing sequence and $X_n = x_1 + \ldots + x_n$, then for each n*

$$X_n^p - X_{n-1}^p \geqslant [n^p - (n-1)^p] x_n^p.$$

Proof is elementary. □

Theorem 3.1. *Suppose $p \geqslant 1$ and let $A = (a_{i,j})$ be a matrix operator from $l_p(w, I)$ into $l_p(w, F)$ with non-negative entries. Write $r_{j,i} = \sum_{k=1}^{i} a_{j,k}$ and*

$$S_i = \sum_{n=1}^{\infty} w_n \left(\sum_{j \in F_n} r_{j,i}\right)^p.$$

Then

$$L_{p,w,F}^p(A) = \inf_n \frac{S_n}{V_n}.$$

Proof. Let x be in $l_p(v, I)$ such that $x_1 \geqslant x_2 \geqslant \ldots \geqslant 0$ and $m = \inf(S_n/V_n)$. Since $\sum_{i \in F_n} y_i = \sum_{j=1}^{\infty} c_{n,j} x_j$, where $c_{n,j} = \sum_{i \in F_n} a_{i,j}$, hence $\sum_{j=1}^{k} c_{n,j} = \sum_{i \in F_n} r_{i,k}$. Applying Lemma 3.1, we obtain

$$\begin{aligned}\|Ax\|_{p,w,F}^p = \sum_{n=1}^{\infty} w_n \left(\sum_{j \in F_n} y_j\right)^p &\geqslant \sum_{n=1}^{\infty} w_n \sum_{i=1}^{\infty} \left(\sum_{j \in F_n} r_{j,i}\right)^p (x_i^p - x_{i+1}^p) \\ &= \sum_{i=1}^{\infty} S_i (x_i^p - x_{i+1}^p).\end{aligned}$$

Since

$$\|x\|_{p,w,I}^p = \sum_{n=1}^{\infty} W_n (x_n^p - x_{n+1}^p),$$

we deduce that

$$\|Ax\|_{p,w,F}^p \geqslant m\|x\|_{p,w,I}^p.$$

Therefore

$$L_{p,w,F}^p(A) \geqslant m.$$

Further, if we take $x_1 = x_2 = \ldots = x_n = 1$ and $x_k = 0$ for all $k \geqslant n+1$, then

$$\|x\|_{p,w,I}^p = W_n, \quad \|Ax\|_{p,w,F}^p = S_n.$$

Hence

$$L_{p,w,F}^p(A) \leqslant m.$$

This completes the proof of the theorem. □

Corollary 3.2. *Suppose $A = (a_{i,j})$ is a matrix operator from $l_p(w, I)$ into itself with non-negative entries. We write $r_{i,n} = \sum_{j=1}^{n} a_{i,j}$ and $S_n = \sum_{i=1}^{\infty} w_i r_{i,n}^p$ and $W_n = w_1 + \ldots + w_n$. Then*

$$L_{p,w,I}(A)^p = \inf_n \frac{S_n}{W_n}.$$

Note 3.1. For $p > 1$, the last part of Theorem 3.1 shows that $\|A\|_{p,w,F}^p \geqslant \sup_n(S_n/W_n)$, but $l_p(w,F) = l_p(w)$ when $F_i = \{i\}$ and equality does not hold (see [5]).

Write

$$t_n = \sum_{i=1}^{\infty} w_i \Big(\sum_{j \in F_i} a_{j,n} \Big)^p$$

and

$$s_n = S_n - S_{n-1} = \sum_{i=1}^{\infty} w_i \Big[\Big(\sum_{j \in F_i} r_{j,n} \Big)^p - \Big(\sum_{j \in F_i} r_{j,n-1} \Big)^p \Big],$$

where $S_n = s_1 + \ldots + s_n$. For $p = 1$ we have $t_n = s_n$. It is elementary that $\inf_n(S_n/W_n) \geqslant \inf_n(s_n/w_n)$. We now apply Lemma 3.1 to deduce the following result.

Proposition 3.1. *If A satisfies all conditions mentioned in Theorem* 3.1 *and $(a_{i,j})$ decreases with j for each i, then*

$$L_{p,w,F}(A)^p \geqslant \inf_n [n^p - (n-1)^p]\frac{t_n}{w_n}.$$

Proof. Corollary 3.2 yields that

$$\Big(\sum_{j\in F_i} r_{j,n}\Big)^p - \Big(\sum_{j\in F_i} r_{j,n-1}\Big)^p \geqslant [n^p - (n-1)^p]\Big(\sum_{j\in F_i} a_{j,n}\Big)^p.$$

Thus

$$s_n \geqslant [n^p - (n-1)^p]\sum_{i=1}^{\infty} v_i\Big(\sum_{j\in F_i} a_{j,n}\Big)^p = [n^p - (n-1)^p]t_n$$

and so we have the statement. □

Let $p \geqslant 1$ and let A be a matrix operator with non-negative entries. If $y = Ax$ and w is a decreasing sequence, then for each non-negative sequence x we have

$$\|Ax\|_{p,w,F}^p = \sum_{i=1}^{\infty} w_i\Big(\sum_{j\in F_i} y_j\Big)^p \geqslant \sum_{i=1}^{\infty} w_i \sum_{j\in F_i} y_j^p \geqslant \sum_{i=1}^{\infty} w_i y_i^p = \|Ax\|_{p,w}^p.$$

It follows that

$$L_{p,w,F}(A) \geqslant L_{p,w}(A).$$

Corollary 3.3. *Suppose that A is the Cesàro operator and $p \geqslant 1$. If $w_n = 1/n$, then*

$$L_{p,w,F}(A) \geqslant 1.$$

Proof. If we apply Theorem 4 from [4], we deduce that $L_{p,w}(A) = 1$ and so we have the statement. □

The Copson matrix is an upper triangular matrix. We will solve the lower bound problem for this operator by the next statement. In fact, we characterize a class of operators for which the lower bound constant is equal to one.

Theorem 3.2. *Assume that A is an upper triangular matrix, i.e. $a_{n,k}=0$ for $n>k$, and $\sum\limits_{n=1}^{k} a_{n,k}=1$ for all k (in other words, A is a quasi-summability matrix). Let $p\geqslant 1$ and let $w=(w_n)$ be a non-negative decreasing sequence. Then*

$$L_{p,w,F}(A)=1.$$

P r o o f. If we apply Proposition 2 from [4], we have $L_{p,w}(A)=1$. Hence $L_{p,w,F}(A)\geqslant L_{p,v}(A)=1$. Since $1\in F_1$ and $Ae_1=e_1$, we deduce that

$$\|Ae_1\|_{p,w,F}=\|e_1\|_{p,w,I}=w_1.$$

This completes the proof of the theorem. □

In the next statement, we consider the lower bound constant for the Hilbert matrix operator H.

Theorem 3.3. *Suppose that H is the Hilbert matrix operator and $p\geqslant 1$. Let $F_i=\{2i-1,2i\}$ and $w_n=1/n^{\alpha}$, where $0<\alpha<1$. Then*

$$L_{p,w,F}(H)^p\geqslant\sum_{k=1}^{\infty}\frac{1}{k^{\alpha}(k+1/2)^p}.$$

P r o o f. Let $E_k=\{i\in Z\colon\ (k-1)n<i\leqslant kn\}$, where $k\geqslant 1$. If $i\in E_k$, then

$$\Big(\frac{i}{n}\Big)^{\alpha}(2i+n)^p\leqslant k^{\alpha}(2kn+n)^p.$$

Since E_k has n members,

$$n^{p+\alpha-1}\sum_{i\in E_k}\frac{1}{i^{\alpha}(2i+n)^p}\geqslant\frac{n^p}{k^{\alpha}(2kn+n)^p}=\frac{1}{k^{\alpha}(2k+1)^p}.$$

Hence

$$n^{p+\alpha-1}\sum_{k=1}^{\infty}\frac{1}{k^{\alpha}(2k+n)^p}\geqslant\sum_{k=1}^{\infty}\frac{1}{k^{\alpha}(2k+1)^p},$$

and also

$$\inf_n n^{p+\alpha-1}\sum_{k=1}^{\infty}\frac{1}{k^{\alpha}(2k+n)^p}=\sum_{k=1}^{\infty}\frac{1}{k^{\alpha}(2k+1)^p}.$$

We now apply Proposition 3.1, and with the above notation we have

$$t_n=\sum_{i=1}^{\infty}\frac{1}{i^{\alpha}}\Big(\frac{1}{2i-1+n}+\frac{1}{2i+n}\Big)^p,$$

and so

$$L_{p,w,F}(H)^p \geqslant \inf_n [n^p - (n-1)^p] n^\alpha t_n.$$

Since $n^p - (n-1)^p \geqslant n^{p-1}$, we deduce that

$$\begin{aligned} L_{p,w,F}(H)^p &\geqslant \inf_n n^{p+\alpha-1} \sum_{i=1}^{\infty} \frac{1}{i^\alpha} \Big(\frac{1}{2i-1+n} + \frac{1}{2i+n} \Big)^p \\ &\geqslant 2^p \inf_n n^{p+\alpha-1} \sum_{k=1}^{\infty} \frac{1}{k^\alpha (2k+n)^p} \\ &= 2^p \sum_{k=1}^{\infty} \frac{1}{k^\alpha (2k+1)^p}. \end{aligned}$$

This completes the proof of the theorem. □

References

[1] *G. Bennett*: Factorizing the Classical Inequalities. Mem. Amer. Math. Soc. 576, 1996.

[2] *D. Foroutannia, R. Lashkaripour*: Lower bounds for summability matrices on weighted sequence spaces. Lobachevskii J. Math *27* (2007), 15–29.

[3] *G. H. Hardy, J. E. Littlewood, G. Pólya*: Inequalities, 2nd edition. Cambridge University Press, Cambridge, 1988.

[4] *G. J. O. Jameson, R. Lashkaripour*: Lower bounds of operators on weighted l_p spaces and Lorentz sequence spaces. Glasg. Math. J. *42* (2000), 211–223.

[5] *G. J. O. Jameson, R. Lashkaripour*: Norms of certain operators on weighted l_p spaces and Lorentz sequence spaces. JIPAM, J. Inequal. Pure Appl. Math. *3* (2002). Electronic only.

[6] *R. Lashkaripour, D. Foroutannia*: Inequalities involving upper bounds for certain matrix operators. Proc. Indian Acad. Sci., Math. Sci. *116* (2006), 325–336.

[7] *R. Lashkaripour, D. Foroutannia*: Norm and lower bounds of operators on weighted sequence spaces. Math. Vesn. *59* (2007), 47–56.

[8] *R. Lashkaripour, D. Foroutannia*: Some inequalities involving upper bounds for some matrix operators I. Czechoslovak Math. J *57* (2007), 553–572.

[9] *J. Pečarić, I. Perić, R. Roki*: On bounds for weighted norms for matrices and integral operators. Linear Algebra Appl. *326* (2001), 121–135.

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