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LOWER BOUNDS FOR THE ZEROS OF BESSEL FUNCTIONS

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ABSTRACT. Let $j_{p,n}$ denote the nth positive zero of J_p, $p \geqslant 0$. Then

$$j_{p,n} \geqslant (j_{0,n}^2 + p^2)^{1/2}.$$

We begin by considering the eigenvalue problem

$$-(xy')' + x^{-1}y = \lambda^2 x^{2p-1}y, \qquad \lambda, p > 0, \tag{1}$$

$$y(a) = y(1) = 0, \qquad 0 < a < 1. \tag{2}$$

For simplicity of notation we will set $q = p^{-1}$. It is easily verified that the general solution of (1) is

$$y(x) = C_1 J_q(\lambda q x^{1/q}) + C_2 Y_q(\lambda q x^{1/q})$$

and that the eigenvalues are given by

$$J_q(\lambda q) Y_q(\lambda q a^{1/q}) - J_q(\lambda q a^{1/q}) Y_q(\lambda q) = 0.$$

If $z_n(a, r)$ denotes the nth positive zero of $J_r(z) Y_r(z a^{1/q}) - J_r(z a^{1/q}) Y_r(z) = 0$, then the nth eigenvalue, $\lambda_n^2(a)$, of (1), (2) is given by

$$\lambda_n^2(a) = (z_n(a, q)/q)^2. \tag{3}$$

Let $j_{r,n}$ denote the nth positive zero of J_r. On p. 38 of [4] it is shown that $z_n(a, r) \to j_{r,n}$ as $a \to 0^+$ whenever r is a positive integer. The restriction on r is extrinsic so that

$$\lim_{a \to 0^+} z_n(a, r) = j_{r,n}, \qquad r \geqslant 0. \tag{4}$$

Let $R[p, y]$ denote the Rayleigh quotient

$$R[p, y] = \int_a^1 (-(xy')' + x^{-1}y) y \, dx \Big/ \int_a^1 x^{2p-1} y^2 \, dx.$$

It is well known that the eigenvalues $\{\lambda_n^2(p)\}$ of (1), (2) can be obtained from the Rayleigh quotient [5]. Let V denote the linear space of all functions in $C^2((a, 1))$ which satisfy the boundary conditions (2). Then

$$\lambda_1^2(p) = \min_{y \in V, y \neq 0} R[p, y].$$

Let $y_1, y_2, \ldots, y_n$ be n functions in V, A denote the subspace of V spanned by $y_1, y_2, \ldots, y_n$ and $A^\perp$ denote the orthogonal complement of A relative to V. Then

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$$\lambda_{n+1}^2(p) = \max_{A} \min_{y \in A^{\perp}, y \neq 0} R[p, y]$$

where the maximum is taken over all sets of n functions in V.

Whenever $p > 0$ we have that $x^{2p-1} < x^{-1}$ for all $x \in (0, 1)$. Then

$$R[p,y] = \frac{\int_a^1 -(xy')'y\,dx}{\int_a^1 x^{2p-1}y^2\,dx} + \frac{\int_a^1 x^{-1}y^2\,dx}{\int_a^1 x^{2p-1}y^2\,dx} \geqslant Q[p,y] + 1, \tag{5}$$

where

$$Q[p,y] = \int_a^1 -(xy')'y\,dx \Big/ \int_a^1 x^{2p-1}y^2\,dx$$

is the Rayleigh quotient for the eigenvalue problem

$$-(xy')' = \mu^2 x^{2p-1} y, \tag{6}$$

$$y(a) = y(1) = 0, \tag{7}$$

Equation (6) is equivalent to

$$x^2y'' + yx' + \mu^2 x^{2p} y = 0. \tag{8}$$

It is easily checked that the general solution of (8) and, hence, of (6) is (recall that $q = p^{-1}$)

$$y(x) = C_1 J_0(\mu q x^{1/q}) + C_2 Y_0(\mu q x^{1/q})$$

and that the eigenvalues are given by

$$J_0(\mu q) Y_0(\mu q a^{1/q}) - J_0(\mu q a^{1/q}) Y_0(\mu q) = 0.$$

In particular the nth eigenvalue, $\mu_n^2(a)$, of (6), (7) is given by

$$\mu_n^2(a) = (z_n(a, 0)/q)^2. \tag{9}$$

From (3), (5), and (9) we obtain

$$(z_n(a, q)/q)^2 \geqslant (z_n(a, 0)/q)^2 + 1. \tag{10}$$

If we now replace q by p, let $a \to 0^+$ in (10), and using (4) we find that $(j_{p,n}/p)^2 \geqslant (j_{0,n}/q)^2 + 1$.

Theorem. *$j_{p,n} \geqslant ((j_{0,n})^2 + p^2)^{1/2}$ whenever $p \geqslant 0$.*

Corollary. *$j_{p,n} > ((n - \frac{1}{4})^2\pi^2 + p^2)^{1/2}$ whenever $p \geqslant 0$.*

Proof. It is known (see [9, p. 489]) that the positive zeros of J_0 lie in the intervals $(m\pi + \frac{3}{4}\pi, m\pi + \frac{7}{8}\pi)$ for $m = 0, 1, 2, \ldots$. Hence, $j_{0,n} > (n-1)\pi + \frac{3}{4}\pi = (n - \frac{1}{4})\pi$. The desired result follows.

In [8] it is shown that

$$j_{p,n} = p + a_n p^{1/3} + b_n p^{-1/3} + O(p^{-1}) \qquad (n = 1, 2, \ldots),$$

where a_n and b_n are independent of p. Hence,

$$j_{p,n}^2 = p^2 + c_n p^{4/3} + O(p^{2/3}) \qquad (n = 1, 2, \ldots),$$

where c_n is independent of p. This shows that the second term of the lower bound for $j_{p,n}$ given in the Theorem is of the wrong order. Other asymptotic expansions for $j_{p,n}$ may be found in [1], [2], and [6].

In [3] it is shown that for $0 \leqslant p \leqslant \frac{1}{2}$

$$p\pi/2 + \left(n - \tfrac{1}{4}\right)\pi \leqslant j_{p,n}. \tag{11}$$

For $p = 0$ the result of the Theorem is exact, while the expression in (11) has a strict inequality. Hence, our result is stronger than (11) whenever p is sufficiently small. However, when $p = \frac{1}{2}$, the result in (11) is exact. Hence, for $0 \leqslant p \leqslant \frac{1}{2}$ neither result implies the other. It should be emphasized that the Theorem is valid for all $p \geqslant 0$, while (11) is valid only for $0 \leqslant p \leqslant \frac{1}{2}$.

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