

LOWER BOUNDS OF RATES OF DECAY FOR SOLUTIONS TO THE NAVIER-STOKES EQUATIONS

MARIA E. SCHONBEK

To Ron

INTRODUCTION

We study the asymptotic behavior of solutions to the Navier-Stokes equations in two and three space dimensions

$$(1.1) \quad \begin{aligned} u_t + u \nabla u + \nabla p &= \Delta u, \\ \operatorname{div} u &= 0. \end{aligned}$$

Our earlier work [4, 5] dealt with the upper bounds on the L^2 decay rates of solutions to the Navier-Stokes equations in three space dimensions with large data. It was established that if $u_0 \in L^2 \cap L^p$, $1 \leq p < 2$, then

$$|u(\cdot, t)|_{L^2}^2 \leq C(1+t)^{-\alpha(p)},$$

where $\alpha(p) = \frac{3}{4}(2/p - 1)$ and the constant C depends on the L^2 and L^p norms of the initial data u_0 .

This paper deals with the more subtle problem of deriving lower bounds on the energy decay rates. We show that for a certain class of initial data the solutions $u(x, t)$ to the 2D and 3D Navier-Stokes equations admit an algebraic lower bound on the energy decay. Specifically, there are two cases to consider. In the first case, the average of the initial data $\int u_0 dx$ is nonzero. This case was treated in the earlier paper [5] where it was established that

$$(1.2) \quad |u(\cdot, t)|_{L^2}^2 \geq C(1+t)^{-n/2}$$

for $n = 2, 3$. In the second case the average is zero, i.e., the Fourier transform at the origin is zero. Here the lower bound on the velocity of decay rate depends on the order of the zero of the initial data. More precisely, if the zero is of order one and the data $u_0 \in L^1 \cap H^1$ and certain weighted L^p spaces described below then there is a lower bound of the form

$$(1.3) \quad |u(\cdot, t)|_{L^2}^2 \geq C(1+t)^{-\alpha(n)},$$

Received by the editors July 1, 1989 and, in revised form, July 24, 1990.

1991 *Mathematics Subject Classification*. Primary 35K55.

The work of the author was partially supported by National Science Foundation grants DMS-8614887 and DMS 8610730.

where $\alpha(n) = n/2 + 1$ and C depends on a few parameters of the data. If the zero is of order greater than one and the data is taken outside a set M of radially equidistributed energy, then in two dimensions the lower bound will be described by (1.3). The lower bound obtained in 2D is uniform, the one in 3D is not. The estimate in two dimensions is sharp. An example suggested by A. Majda shows that there are solutions to the Navier-Stokes equations with data in M that decay exponentially.

The algebraic lower bound is a consequence of the nonlinear structure of the equations. In contrast, solutions to the heat equation decay at an exponential rate if the initial data is highly oscillatory. The inertial term $\operatorname{div}(u \otimes u)$ in the Navier-Stokes equations appears to convert short waves into long waves, reducing the decay rate. Even for most cases of highly oscillatory initial data (i.e., containing just short waves), energy will be transmitted to the lower end of the scale, thereby producing long waves that reduce the decay rate of the solutions.

Our approach in the case of zero average data is first to find conditions for the data such that the corresponding solution to the heat equation decays at a very slow rate. These conditions will be met by the solution $u(x, t)$ of the Navier-Stokes at some time $t_0 \geq 0$. That is, short waves are transformed into long waves. Hence the solution to the heat equation, which takes on as initial data $u(x, t_0)$ for some appropriate $t_0 \geq 0$, has a lower bound on their rate of decay. Specifically, if v is a solution of $v_t = \Delta v$, $v(x, 0) = u(x, t_0)$ then $|v(\cdot, t)|_{L^2}^2 \geq (1+t)^{-\alpha(n)}$.

This information is used to insure that the solutions to the Navier-Stokes equations cannot decay any faster. The result follows using Fourier analysis of the Navier-Stokes equations. The argument relies on a technique that involves splitting the frequency space into two time-dependent sets. This technique was developed in [4] to study the upper bound.

For solutions in two spatial dimensions, the results are valid for classical solutions. For three dimensions, the results are valid for suitable Leray-Hopf solutions in the sense of Caffarelli, Kohn, and Nirenberg [1]. We expect that the results can be extended to n dimensions, $n > 3$, using the results of Wiegner [6], Kayikiya and Miyakawa [2].

2. ESTIMATES ON SOLUTIONS TO THE HEAT EQUATIONS

In this section we describe a class of initial data D , for which the solutions to the heat equations admit a lower bound on the L^2 decay rate. In §3, it is shown that if $u(x, t)$ is a solution to the Navier-Stokes in two spatial dimensions with data outside a set M of radially equidistributed energy, then there is $t_0 \geq 0$ for which $u(x, t_0) \in D$. Hence if $u(x, t_0)$ is taken as initial data, the corresponding solution to the heat equation will have a lower bound on the L^2 decay. This information will be used to obtain a lower bound on the L^2 rate of decay for solutions to the Navier-Stokes equations. For data in M ,

an example is given in §5 of a solution to the 2D Navier-Stokes equation that decays exponentially fast.

The section concludes with a lemma establishing an upper bound on the L^∞ rate of decay for the gradient of the solutions to the heat equation for solutions that decay in L^2 at a given rate.

Theorem 2.1. *Let $v_0 \in L^2(\mathbf{R}^n)$. Let v be a solution to the heat equation with data v_0 . Suppose that there exist functions l and h such that the Fourier transform of v_0 for $|\xi| \leq \delta$, $\delta > 0$, admits the representation*

$$\hat{v}_0(\xi) = \xi \cdot l(\xi) + h(\xi), \quad l = (l_1, \dots, l_n),$$

where l and h satisfy the following conditions:

- (i) $|h(\xi)| \leq M_0 |\xi|^2$ for some $M_0 > 0$;
- (ii) l is homogeneous of degree zero; and
- (iii) $\alpha_1 = \int_{|\omega|=1} |\omega \cdot l(\omega)|^2 d\omega > 0$.

Let $M_1 = \sup_{|y|=1} |l(y)|$, $M_2 = \sup_{\delta/2 \leq |y| \leq 1} |\nabla l(y)|$, $K = \max(M_0, M_1, M_2)$. Then there exist constants C_0 and C_1 such that

$$C_0(t+1)^{-(n/2+1)} \leq |v(\cdot, t)|_{L^2}^2 \leq C_1(t+1)^{-(n/2+1)},$$

where C_0 and C_1 both depend on n, M_0, M_1, δ , and $|v_0|_{L^2}$ and C_0 also depends on K and α .

Proof. Note first that condition (iii) is not necessary for the upper bound $|v(\cdot, t)|_{L^2}^2$.

Upper bound. By Plancharel’s theorem

$$\int_{\mathbf{R}^n} |v|^2 dx = \int_{\mathbf{R}^n} |\hat{v}|^2 d\xi = \int_{\mathbf{R}^n} |\hat{v}_0|^2 e^{-2|\xi|^2 t} d\xi.$$

Let $A = \{\xi : |\xi| \leq \delta\}$. Then

$$\begin{aligned} \int_{\mathbf{R}^n} |v|^2 dx &= \int_A |\hat{v}_0|^2 e^{-2|\xi|^2 t} d\xi + \int_{A^c} |\hat{v}_0|^2 e^{-2|\xi|^2 t} d\xi \\ (2.1) \qquad &\leq \int_A |\hat{v}_0|^2 e^{-2|\xi|^2 t} d\xi + e^{-\delta^2 t} |v_0|_{L^2}^2. \end{aligned}$$

To estimate the integral on the right-hand side, use the representation of v_0 in terms of l and h .

$$\begin{aligned} \int_A |\hat{v}_0|^2 e^{-2|\xi|^2 t} d\xi &= \int_A |\xi \cdot l(\xi)|^2 e^{-2|\xi|^2 t} d\xi + 2 \operatorname{Re} \left(\int_A \xi \cdot l(\xi) \bar{h}(\xi) e^{-2|\xi|^2 t} d\xi \right) \\ &\quad + \int_A |h(\xi)|^2 e^{-2|\xi|^2 t} d\xi \\ (2.2) \qquad &\leq 2 \int_A |\xi \cdot l(\xi)|^2 e^{-2|\xi|^2 t} d\xi + M_0^2 \int_A |\xi|^4 e^{-2|\xi|^2 t} d\xi. \end{aligned}$$

Recall that $l(\xi)$ is homogeneous of degree zero. Combine (2.1) and (2.2) and make the change of variables $y = \sqrt{2t}\xi$. Then

$$(2.3) \quad \int_{\mathbf{R}^n} |v|^2 dx \leq 2(2t)^{-n/2+1} \int_A |y \cdot l(y)|^2 e^{-y^2} dy + M_0(2t)^{-n/2-2} \int_{A(t)} |y|^4 e^{-y^2} dy,$$

where $A(t) = \{y : |y| \leq \delta\sqrt{2t}\}$. Since $M_1 = \sup_{|y|=1} l(y)$ and l is homogeneous of degree zero,

$$|y \cdot l(y)|^2 \leq |y|^2 M_1^2.$$

Hence from (2.3)

$$\int_{\mathbf{R}^n} |v|^2 dx \leq C/(2t)^{n/2+1},$$

where C depends on M_0, M_1, δ , and n . Recall that

$$\int_{\mathbf{R}^n} |v|^2 dx \leq |v_0|_{L^2}^2;$$

hence

$$\int_{\mathbf{R}^n} |v|^2 dx \leq C_1(t+1)^{-(n/2+1)},$$

with $C_1 = C_1(M_0, M_1, \delta, n, |v_0|_{L^2})$.

Lower bound. Choose $\delta_1 < \delta$ such that $4M_0M_1\delta_1 \leq \alpha_1$, and let $A_1 = \{\xi : |\xi| \leq \delta_1\}$. Then

$$\begin{aligned} \int_{\mathbf{R}^n} |v|^2 dx &\geq \int_{A_1} |\hat{v}_0|^2 e^{-2|\xi|^2 t} d\xi \geq \int_{A_1} (|\xi \cdot l(\xi)|^2 - 2M_0M_1|\xi|^3) e^{-2|\xi|^2 t} d\xi \\ &= \omega_n \int_0^{\delta_1} r^{n+1} (\alpha_1 - 2M_0M_1r) e^{-2r^2 t} dr \\ &\geq \frac{1}{2} \omega_n \alpha_1 \int_0^{\delta_1} r^{n+1} e^{-2r^2 t} dr \\ &\geq \frac{\omega_n \alpha_1 e^{-2}}{2(n+2)} t^{-(n/2+1)} \quad \text{for } t \geq \delta_1^{-1}. \end{aligned}$$

For $t < \delta_1^{-2}$ we have

$$\begin{aligned} \int_{\mathbf{R}^n} |v|^2 dx &= \int_{\mathbf{R}^n} |\hat{v}_0|^2 e^{-2|\xi|^2 t} d\xi \geq \int_{\mathbf{R}^n} |\hat{v}_0|^2 e^{-2|\xi|^2 \delta_1^{-2}} d\xi \\ &\geq (1+t)^{-(n/2+1)} \int_{\mathbf{R}^n} |\hat{v}_0|^2 e^{-2|\xi|^2 \delta_1^{-2}} d\xi. \end{aligned}$$

The lower bound follows for

$$C_0 = \min \left\{ \frac{\omega_n \alpha_1 e^{-1}}{2(n+2)}, \int_{\mathbf{R}^n} |\hat{v}_0|^2 e^{-2|\xi|^2 \delta_1^{-2}} d\xi \right\}.$$

Corollary 2.2. *Let v be a solution to the heat equation with data $v_0 \in L^2(\mathbf{R}^n)$, where v_0 has the Fourier representation described in Theorem 2.1 and l and h*

satisfy (i), (ii). If in addition l satisfies

- (1) $\omega_0 \cdot l(\omega_0) = \alpha \neq 0$ for some $\omega_0 \in S^{n-1}$ and
- (2) $\xi \cdot l(\xi) \in C^1(\mathbf{R}^n \setminus 0)$,

then the conclusion of Theorem 2.1 holds.

Proof. It is necessary to show that (1) and (2) imply conditions of Theorem 2.1. Note that if $\omega_0 \cdot l(\omega_0) = \alpha \neq 0$ for some $\omega_0 \in S^{n-1}$ and $\xi \cdot l(\xi) \in C^1(\mathbf{R}^n \setminus 0)$, it follows that there exists an open ball centered at ω_0 , $B_r(\omega_0)$ of radius $r > 0$, such that for $\omega \in B_r(\omega_0)$, $\omega \cdot l(\omega) \geq \alpha/2$. Hence

$$\int_{|\omega|=1} |\omega \cdot l(\omega)|^2 d\omega > \int_{S \cap B_r(\omega_0)} |\omega \cdot l(\omega)|^2 d\omega \geq \alpha/2L > 0,$$

where $L = \int_{S \cap B_r(\omega_0)} d\omega$ and $S = \{\omega : |\omega| = 1\}$.

The next lemma establishes an upper bound on L^∞ rate of decay for gradient of solutions to the heat equation which decay at a given rate in L^2 .

Lemma 2.3. *Let $v(x, t)$ be a solution to the heat equation. Suppose $|v(\cdot, t)|_{L^2}^2 \leq C(t + 1)^{-(n/2+1)}$. Then*

$$|\nabla v(\cdot, t)|_\infty < C(t)^{-(n/2+1)}.$$

Proof.

$$\begin{aligned} |\nabla v(\cdot, t)|_\infty &\leq |\widehat{\nabla} v(\cdot, t)|_{L^1} \leq \int |\xi| |\widehat{v}(t/2)| e^{-2|\xi|^2 t/2} d\xi \\ &\leq |v(\cdot, t/2)|_{L^2} \cdot \left(\int |\xi|^2 e^{-2|\xi|^2 t} d\xi \right)^{1/2} \\ &= C |v(\cdot, t/2)|_{L^2} t^{-(n/2+1)/2} \leq C t^{-(n/2+1)}. \end{aligned}$$

3. THE INITIAL DATA

In this section a class of initial data is found for which the solutions to the Navier-Stokes equations admit the Fourier representation

$$\hat{u}_k(\xi, t_0) = \xi \cdot l_k(\xi, t_0) + h_k(\xi, t_0)$$

for some $t_0 \geq 0$. Here l_k and h_k satisfy the conditions given in Theorem 2.1. Hence the solution to the heat equation started at $u(x, t_0)$ has a lower bound for L^2 decay.

The initial data will belong to the intersection of L^1, H^1 , some weighted spaces, and the complement of a set of radially equidistributed energy. The condition of not having equidistributed energy is essential in two dimensions. Specifically, an example will be given where the data has radially equidistributed energy and solution decays exponentially.

The data to be considered has Fourier transform vanishing at the origin. For nonvanishing data, the reader is referred to [4]. There are two cases.

Case 1. The zero of the origin is of order one.

Case 2. The zero is of order greater than one.

In case one, the data can have equidistributed energy and t_0 is 0. For case two, the data has to lie outside a set of equidistributed energy.

We define the following weighted spaces and norms.

$$\begin{aligned} W_1 &= \left\{ u : \int_{\mathbf{R}^n} |x|^2 |u| dx < \infty \right\}; \\ W_2 &= \left\{ u : \int_{\mathbf{R}^n} |u|^2 |x| dx < \infty \right\}; \\ |u|_{W_1} &= \int_{\mathbf{R}^n} |x|^2 |u| dx; \\ |u|_{W_2} &= \left(\int_{\mathbf{R}^n} |u|^2 |x| dx \right)^{1/2}. \end{aligned}$$

Note that if $u \in W_1 \cap W_2 \cap L^2$ then $\int |x| |u| dx < \infty$, since

$$\begin{aligned} \int_{\mathbf{R}^n} |x| |u| dx &= \int_{|x| \leq 1} |x| |u| dx + \int_{|x| \geq 1} |x| |u| dx \\ &\leq \int_{|x| \leq 1} (|x|^2 + |u|^2) dx + \int_{|x| \geq 1} |x|^2 |u| dx < \infty. \end{aligned}$$

$$\begin{aligned} V(\mathbf{R}^n) &= C_0^\infty(\mathbf{R}^n) \cap \{u : \nabla \cdot u = 0\}, \\ H &= H(\mathbf{R}^n) = \text{closure of } V \text{ in } L^2. \end{aligned}$$

Note that the choice of the weighted spaces insures that the data has at least two Fourier derivatives in L^2 .

Theorem 3.1. *Let $g \in H \cap W_1 \cap W_2(\mathbf{R}^n)$, $n = 2, 3$. If g has a zero of order one at the origin, then there exists $\delta > 0$ such that for $|\xi| \leq \delta$*

$$\hat{g}(\xi) = \xi \cdot l(\xi) + h(\xi),$$

since l and h satisfy the hypothesis of Theorem 2.1, with $M_0 = \sup_{|x| \leq \delta} |\nabla^2 \hat{g}(\xi)|$ and α depending only on $\widehat{\nabla g}(0)$.

Proof. Since $g \in W_1 \cap W_2$,

$$\begin{aligned} \left| \frac{\partial}{\partial \xi_i} \hat{g}(\xi) \right| &\leq \int |x| |g| dx \leq C, \\ \left| \frac{\partial^2}{\partial \xi_i \partial \xi_j} \hat{g}(\xi) \right| &\leq \int_{\mathbf{R}^n} |x_i| |x_j| |g| dx \leq \int_{\mathbf{R}^n} |x|^2 |g| dx \leq C. \end{aligned}$$

Since $\hat{g}(0) = 0$ for $|\xi| \leq \delta$, $\delta > 0$,

$$\hat{g}(\xi) = \widehat{\nabla g}(0) \cdot \xi + \widehat{\nabla^2 g}(\bar{\xi}) \xi^2.$$

Since $\widehat{\nabla g}(0) \neq 0$ by hypothesis, to finish the proof let

$$l(\xi) = \widehat{\nabla g}(0), \quad h(\xi) = \widehat{\nabla g}(\bar{\xi}) \xi^2.$$

Here $M_0 = \sup_{|x| \leq \delta} |\widehat{\nabla^2 g}(\xi)|$ and $\omega_0 \in S^{n-1}$ is chosen so that $\omega_0 \cdot \widehat{\nabla g}_k \neq 0$.

The following notation will be used.

$$\begin{aligned} \alpha_i^j(t_0, u) &= \int_0^{t_0} \int_{\mathbf{R}^n} |u_i|^2 - |u_j|^2 dx dt, & \beta_i^j(t_0, u) &= \int_0^{t_0} \int_{\mathbf{R}^n} u_i u_j dx dt, \\ A_i^j(\mathbf{R}^n) &= \left\{ u : \int_{\mathbf{R}^n} |u_i|^2 dx = \int_{\mathbf{R}^n} |u_j|^2 dx \right\}, \\ B_i^j &= B_i^j(\mathbf{R}^n) = \left\{ u : \int_{\mathbf{R}^n} u_i u_j dx = 0 \right\}. \end{aligned}$$

The two-dimensional case is considered first. For sake of notation, let $M = A_1^2 \cap B_1^2$.

Theorem 3.2. *Let $g \in H^1 \cap H \cap W_2 \cap M^c(\mathbf{R}^2)$. Let $u(x, t)$ be a solution to the Navier-Stokes equations with data g . If g has zero of order greater than one, then there exist t_0 and $\delta > 0$ such that for $|\xi| \leq \delta$*

$$\hat{u}_k(\xi, t_0) = \xi \cdot l_k(\xi, t_0) + h_k(\xi, t_0),$$

where $t_0 = t_0(|g|_{H^1}, |g|_{W_2})$ and $l_k(\cdot, t_0)$ and $h_k(\cdot, t_0)$ satisfy

- (i) $|h_k(\xi)| \leq M_0 |\xi|^2$;
- (ii) l_k is homogeneous of degree zero;
- (iii) $\omega_0 \cdot l_k(\omega_0) = \alpha \neq 0$ for some $\omega_0 \in S^{n-1}$ and at least one component l_k ; and
- (iv) $\xi \cdot l_k(\xi) \in C^1(\mathbf{R}^n \setminus \{0\})$.

The constant M_0 depends only on $|g|_{L^2}$, $|g|_{W_2}$, and δ . The constant α is a multiple of $\alpha_1^2(t_0, g)$.

Proof. Take the Fourier transform of the Navier-Stokes equations

$$\begin{aligned} \hat{u}_t + |\xi|^2 \hat{u} &= -\widehat{u \nabla u} - \widehat{\nabla p} = -\widehat{H}, \\ \hat{u}(\xi, 0) &= \hat{g}(\xi). \end{aligned}$$

Hence

$$(3.1) \quad \hat{u}_k(\xi, t) = \hat{g}_k(\xi) e^{-|\xi|^2 t} - \int_0^t \widehat{H}_k(\xi, s) e^{-|\xi|^2(t-s)} ds.$$

W.l.o.g. choose $k = 1$. Recall that

$$\Delta p = - \sum \frac{\partial^2}{\partial x_i \partial x_j} u_i u_j$$

so that

$$\widehat{\partial_{x_1} p} = -i \xi_1 \sum_{\xi, \xi_j} \frac{\xi_i \xi_j}{|\xi|^2} \widehat{u_i u_j}.$$

Let $a_{ij} = \widehat{u_i u_j}$. Then

$$(3.2) \quad \widehat{H}_1(\xi, s) = -i \left(\sum \xi_i a_{i1} - \xi_1 / |\xi|^2 \sum \xi_i \xi_j a_{ij} \right) \\ = -i \left(\xi_1 (1 - \xi_1^2 / |\xi|^2) a_{11} - \xi_2 (1 - 2\xi_1^2 / |\xi|^2) a_{12} - \frac{\xi_1 \xi_2^2}{|\xi|^2} a_{22} \right).$$

Note that $1 - \xi_1^2 / |\xi|^2 = \xi_2^2 / |\xi|^2$; hence

$$(3.3) \quad \widehat{H}_1(\xi, s) = -i\xi \cdot (\xi_2^2 / |\xi|^2 (a_{11} - a_{22}), (1 - 2\xi_1^2 / |\xi|^2) a_{12}).$$

Proposition A.1 (see Appendix) establishes that for $|\xi| \leq \delta$

$$|\nabla_\xi a_{ij}(\xi, t)| \leq C(t),$$

where $C(t)$ denotes constant in ξ that depends on $|g|_{L^2}$, $|g|_{W_2}$, δ , and t .

Let $a_{ij}^0 = a_{ij}(0, t)$. Thus a_{ij} can be represented as

$$(3.4) \quad a_{ij} = a_{ij}^0 + \xi \cdot \nabla_\xi a_{ij}(\bar{\xi}), \quad 0 \leq \bar{\xi} \leq \xi.$$

By (3.3)

$$(3.5) \quad \widehat{H}_1(\xi, s) = -i\xi \cdot [(\xi_2^2 / |\xi|^2)(a_{11}^0 - a_{22}^0), (1 - 2\xi_1^2 / |\xi|^2)a_{12}^0 + \tilde{h}_1(\xi, s)] \\ = i\xi(H_1^0(\xi, s) + \tilde{h}_1(\xi, s)),$$

and by (3.3) and (3.4)

$$|\tilde{h}_1(\xi, s)| \leq C(s)|\xi|.$$

From (3.1) and (3.5)

$$\hat{u}(\xi, t) = -i\xi \cdot \left(\int_0^t H_1^0(\xi, s) e^{-|\xi|^2(t-s)} ds + \int_0^t \tilde{h}_1(\xi, s) e^{-|\xi|^2(t-s)} ds + e^{|\xi|^2 t} \hat{g}(\xi) \right) \\ = \xi \cdot \int_0^t (-i)H_1^0 ds + h_1(\xi, s),$$

where

$$h_1(\xi, s) = -i\xi \cdot \left(\int_0^t H_1^0(\xi, s) (e^{-|\xi|^2(t-s)} - 1) ds + \int_0^t \tilde{h}_1(\xi, s) e^{-|\xi|^2(t-s)} ds + e^{-|\xi|^2 t} \hat{g}(\xi) \right).$$

Since

$$e^{-|\xi|^2(t-s)} - 1 = O(|\xi|^2); \\ |\xi \cdot \tilde{h}_1(s)| \leq M|\xi|^2 \text{ form } |\xi| \leq \delta, \quad M \text{ depending on } C(t); \\ \hat{g}(\xi) = O(|\xi|^2),$$

it follows that $|h_1(\xi, t)| \leq M_0|\xi|^2$, M_0 depending only $|g|_{L^2}$, δ , and t . Let

$$l_1(\xi, t) = (l_1^1, l_1^2) = \left(-i \int_0^t (\xi_2^2 / |\xi|^2)(a_{11}^0 - a_{22}^0) ds, \int_0^t (1 - 2\xi_1^2 / |\xi|^2) a_{12}^0 ds \right).$$

We first show that for some $t_0 > 0$ and some $\omega_0 \in S^2$, $\omega_0 \cdot l_1(\omega_0, t_0) \neq 0$. If $u_0 \notin A_1^2$ choose $\omega_0 = (1, 1)/\sqrt{2}$; then

$$(3.6) \quad \omega_0 \cdot l_1(\omega_0, t_0) = \frac{-i}{2\sqrt{2}} \int_0^{t_0} \int_{\mathbf{R}^2} u_1^2 - u_2^2 dx dy ds.$$

If $u_0 \notin B_1^2$ choose $\omega_0 = (1, 0)$; then

$$(3.7) \quad \omega_0 \cdot l_1(\omega_0, t_0) = \int_0^{t_0} \int_{\mathbf{R}^2} u_1 u_2 dx dy ds.$$

By Lemmas A.2 and A.3 of the appendix since $g \in M^c$ there exists $t_0 = t_0(|g|_{L^2}, |g|_{W_2}, \alpha)$ such that RHS of (3.6) and (3.7) are not zero. Hence

$$\omega_0 \cdot l_1(\omega_0, t_0) \neq 0.$$

Conditions (i), (ii), and (iv) are trivially satisfied for $l_1(\xi, t_0)$ and $h_1(\xi, t_0)$.

Theorem 3.3. *Let $g \in H \cap W_2(\mathbf{R}^3)$. Let $u(x, t)$ be a suitable Leray-Hopf solution in the sense of Caffarelli, Kohn, and Nirenberg with data g . Let g have a zero at the origin of order greater than one. If there exists t_0 such that*

$$\alpha_i^j(t_0, 3) = \int_0^{t_0} \int_{\mathbf{R}^3} |u_i|^2 - |u_j|^2 dx dt \neq 0, \quad i, j = 1, 2, 3$$

or

$$\beta_i^j(t_0, 3) = \int_0^{t_0} \int_{\mathbf{R}^3} u_i u_j dx dt \neq 0, \quad i \neq j,$$

then there exists $\delta > 0$ such that for $|\xi| \leq \delta$,

$$\hat{u}_k(\xi, t_0) = \xi \cdot l_k(\xi, t_0) + h_k(\xi, t_0),$$

where $l_k(\xi, t_0)$ and $h_k(\xi, t_0)$ satisfy the same conditions as in Theorem 3.2.

Proof. Recall that a weak solution with data g satisfies

$$(3.8) \quad \langle u(t), \phi(t) \rangle - \int_0^t \left\{ \left\langle u(s) \frac{\partial}{\partial s} \phi(s) \right\rangle + \langle \nabla u(s), \nabla \phi(s) \rangle + \langle (u(s) \cdot \nabla) u(s), \phi(s) \rangle \right\} ds - \langle g, \phi(0) \rangle = 0$$

for all smooth vectors ϕ with compact support and $\text{div } \phi = 0$. Following Wiegner's argument [6], we choose ϕ to be the solution to the heat equation with data $\phi_0 \in C_0^\infty(\mathbf{R}^3)$ and $\text{div } \phi_0 = 0$. This ϕ is smooth and bounded in L^∞ and (3.8) holds for ϕ by approximation. Let $t_0 > 0$ be fixed and $t^* > t_0$. For $0 \leq s \leq t$ let

$$\phi(s) = F^{-1}(F(\phi_0) \exp(-|\xi|^2, t^* - s)),$$

which is the solution to the homogeneous heat system with data ϕ_0 at time $t^* - s$. It is easy to show that for such a choice of ϕ , (3.8) yields

$$(3.9) \quad \hat{u}_k(\xi, t_0) = \sum_{j=1}^3 (\delta_{jk} - \xi_k \xi_j) |\xi|^{-2} \left(\hat{g}_j e^{-|\xi|^2 t_0} - \int_0^{t_0} \widehat{(u(s) \cdot \nabla) u_j(s)} e^{-\xi^2(t_0-s)} ds \right).$$

For more details we refer the reader to [6]. By hypothesis

$$\hat{g}_j(\xi) = \hat{g}_j(0) + \widehat{\nabla}g_j(0) \cdot \xi + \widehat{\nabla^2}g_j(\bar{\xi})(\xi, \xi) = \widehat{\nabla^2}g_j(\bar{\xi})(\xi, \xi).$$

Hence we only have to consider the terms in

$$\sum_{j=1}^3 (\delta_{kj} - \xi_k \xi_j) |\xi|^{-2} \int_0^{t_0} \widehat{u} \widehat{\nabla} u_j(s) e^{-|\xi|^2(t_0-s)} ds.$$

By Note A.4 (see Appendix) the last expression can be rewritten as

$$(3.10) \quad -i \sum_{j=1}^3 (\delta_{kj} - \xi_k \xi_j) |\xi|^{-2} \int_0^{t_0} \sum_{i,j=1}^3 \xi_i a_{ij}^0(s) ds + h(\xi),$$

where $|h(\xi)| \leq K|\xi|^2$ with K depending only on the L^2, W_2 norms of the data, and t_0 . The vector function $l_k = (l_k^1, l_k^2, l_k^3)$ can be chosen so that the components have the form

$$l_k^i = -i \sum_{j=1}^3 (\delta_{kj} - (\xi_k \xi_j)) |\xi|^{-2} \int_0^{t_0} \sum_{i,j=1}^3 a_{ij}^0(s) ds.$$

From (3.9) and (3.10) it follows that

$$\hat{u}_k(\xi, t_0) = \xi \cdot l_k(\xi, t_0) + h_k(\xi, t_0)$$

with $|h_k(\xi, t_0)| \leq K_0|\xi|^2$, and K_0 depends only on $\sup_{|\xi| \leq \delta} |\widehat{\nabla^2}g(\xi)|$, the L^2, L_{W_2} norms of g and t_0 . Conditions (i), (ii), and (iv) of Theorem 3.2 follow trivially. To establish (iii) choose $\omega_0 = (1/\sqrt{2})(e_i + e_j)$ if $\alpha_i^j \neq 0$ with e_i the i th element of the canonical basis of \mathbf{R}^3 . If $\beta_i^j \neq 0$

$$\text{let } \omega_0 = e_j \text{ if } j \neq k \text{ or } \omega_0 = e_i \text{ if } i \neq k.$$

4. THE KEY THEOREM

The decay rates for L^2 norms of solutions to the heat equation are compared to the decay rates of L^2 norms of solutions to the Navier-Stokes equations. It is shown that if the upper bounds and lower bounds of decay rates of solutions to the heat equation are of order $(t + 1)^{-(n/4+1/2)}$, then the same rates are valid for solutions to the Navier-Stokes equations starting with the same data.

A formal argument is given for solutions in n dimensions, $n \geq 2$. The proof is rigorous for $n = 2$. For $n = 3$, the rigorous results are less strong, since they are obtained only under the supposition of the existence of a sequence of approximate solutions (such as those constructed by Caffarelli, Kohn, and Nirenberg [1]) which converge strongly in $L^2([0, 1] \times \mathbf{R}^3)$. Passing to the limit a lower bound will be obtained for almost all t . We expect that the result for $n = 3$ can be improved. For $n > 3$ it is expected that the proof can be applied

to the approximate solutions constructed by Kayikiya and Miyakawa [2] and passing to the limit. The upper limits of the rates of decay are included for completeness. The bounds for $n = 2$ were established by Wiegner in [6]. The bounds for $n = 3$ were established in [5, 6].

The proof for the lower bound is based on an analysis of the Fourier transform of the difference between the solutions to the heat equation and the Navier-Stokes equations starting with the same data. The argument is by contradiction.

Theorem 4.1. *Let $u_0 \in L^1 \cap W_2 \cap H(\mathbf{R}^n)$, $n = 2, 3$. Let v be a solution to the heat equation with data u_0 . Suppose*

$$(4.1) \quad C_0(1+t)^{-(n/2+1)} \leq |v(\cdot, t)|_{L^2}^2 \leq C_1(1+t)^{-(n/2+1)}.$$

For $n = 2$ let $u(\cdot, t)$ be a solution to the Navier-Stokes equations with data u_0 . Then there exist constants M_0 and M_1 such that

$$(4.2) \quad M_0(1+t)^{-(n/2+1)} \leq |u(\cdot, t)|_{L^2}^2 \leq M_1(1+t)^{-(n/2+1)},$$

where M_0 and M_1 depend on C_1, n , and the L^1 and L^2 norms of u_0 , and M_0 depends also on C_0 and the W_2 norm of u_0 .

For $n = 3$, the upper bound in (4.2) holds for all t and the lower bound holds for a.e. t , where $u(x, t)$ is a Leray-Hopf solution in the sense of Caffarelli, Kohn, and Nirenberg.

Proof.

Upper bound. See [5, 6].

Lower bound. We first present a proof that is rigorous for $n = 2$ and formal for $n \geq 3$. We will indicate the modifications necessary to make the proof rigorous for $n = 3$.

Outline of the proof. There are two cases to consider. Let β be a fixed constant, which will be defined below. For $n = 2$ the cases are the following.

Case 1. Given t there exists $T > t$ such that

$$\left| \int_0^T \int_{\mathbf{R}^n} |u_1|^2 - |u_2|^2 dx ds \right| < \beta \sqrt{C_0} \quad \text{and} \quad \left| \int_0^T \int_{\mathbf{R}^n} u_1 u_2 dx ds \right| < \beta \sqrt{C_0}.$$

Case 2. There exists T_0 such that for all $t \geq T_0$

$$\left| \int_0^t \int_{\mathbf{R}^n} |u_1|^2 - |u_2|^2 dx ds \right| \geq \beta \sqrt{C_0} \quad \text{or} \quad \left| \int_0^t \int_{\mathbf{R}^n} u_1 u_2 dx ds \right| \geq \beta \sqrt{C_0}.$$

If $n > 2$ the two cases are slightly different.

Case 1*. Given t there exists $T > t$ such that

$$\left| \int_0^T \int_{\mathbf{R}^n} |u_i|^2 - |u_j|^2 dx ds \right| < \beta \sqrt{C_0}, \quad 1 \leq i, j \leq n,$$

and

$$\left| \int_0^T \int_{\mathbf{R}^n} u_i u_j dx ds \right| < \beta \sqrt{C_0}, \quad 1 \leq i, j \leq n, \quad i \neq j.$$

Case 2*. There exists T_0 such that for all $t \geq T_0$

$$\left| \int_0^t \int_{\mathbf{R}^n} |u_i|^2 - |u_j|^2 dx ds \right| \geq \beta \sqrt{C_0} \quad \text{for at least one pair } i, j$$

or

$$\left| \int_0^t \int_{\mathbf{R}^n} u_i u_j dx ds \right| \geq \beta \sqrt{C_0} \quad \text{for at least one pair } i, j \text{ with } i \neq j.$$

Outlines of the proofs of Cases 1 and 2 are given first followed by a detailed proof of each one.

Outline of Case 1. In this case an increasing sequence $\{r_m\}$, $r_m = r_m(\beta)$, $r_m \rightarrow \infty$ as $m \rightarrow \infty$, can be constructed such that

(4.3)

$$\left| \int_0^{r_m} \int_{\mathbf{R}^n} |u_1|^2 - |u_2|^2 dx ds \right| < \beta \sqrt{C_0} \quad \text{and} \quad \left| \int_0^{r_m} \int_{\mathbf{R}^n} u_1 u_2 dx ds \right| < \beta \sqrt{C_0}.$$

Let $\omega = v - u$ be the difference of the solution to the heat equation and the solution to the Navier-Stokes equations with the same data u_0 . It will be shown first that for t sufficiently large

(4.4)

$$|\omega(\cdot, t)|_{L^2}^2 < C(t+1)^{-2\alpha} + O((t+1)^{-\gamma}),$$

where $\alpha = n/4 + 1/2$, $\gamma > 2\alpha$, and $C \leq C_0/4$. Briefly, inequality (4.4) is obtained as follows. The difference ω can be considered as a solution to an inhomogeneous heat equation

$$\omega_t = \Delta \omega + A,$$

where $A = -(u \nabla u + \nabla p)$. Using the Fourier splitting method [4, 5], it follows that the L^2 norm of ω is bounded by three terms. Two of these terms come from the inhomogeneous part and decrease at a faster rate than α . This faster decay is a consequence of the terms either being cubic in (ω, u) or involving the gradient v , which improves the decay rate. The third term is the critical one and has the form

$$\int_{S(t)} |\hat{\omega}(\xi, t)|^2 d\xi,$$

where $S(t)$ is a ball of radius $(t+1)^{-n/2}$ and as such decays like $(t+1)^{-n/2}$. The hypothesis of Case 1 is used (or Case 1* if $n > 2$) to show that

$$|\hat{\omega}(\xi, t)| \leq \left[2\beta \sqrt{C_0} + C_2(1+r_m)^{-n} \right] |\xi| + O(|\xi|^2),$$

where $s \geq 2$ and C_2 is an appropriate constant. Hence

$$\int_{S(t)} |\hat{\omega}(\xi, t)|^2 d\xi \leq [8\beta^2 C_0 + 2C_2(1+r_m)^{-2n}] (t+1)^{-(n/2+1)} + O(t^{-\alpha}),$$

where $\alpha > n/2 + 1$. Here C_2 and C_3 depend on the L^2 and L^1 norms of u_0 . Choosing β sufficiently small and r_m sufficiently large, (4.4) will yield for $t \geq T_0$, $T_0 = T_0(|u_0|_{L^2}, |u_0|_{L^1}, n, C_0, C_1)$

$$|\omega(\cdot, t)|_{L^2}^2 \leq C_0/4(t+1)^{-(n/2+1)}.$$

Hence the lower bound of $|u(\cdot, t)|_{L^2}$ follows for $t \geq T_0$. For $t \leq T_0$ the bound is a consequence of the decay of energy of u .

Outline of Case 2. Here we study the difference $W = U - V$ where $U = u(x, t + T)$ with $T > T_0$ for some appropriate T and V is the solution to the heat equation

$$\begin{aligned} V_t &= \Delta V, \\ V(x, 0) &= u(x, T). \end{aligned}$$

By Theorems 3.2 and 2.1 and the hypothesis of Case 2 (or Case 2* if $n > 2$) the L^2 norm of V has the following upper and lower bounds:

$$C_0\beta_1(t + 1)^{-(n/2+1)} \leq |V(\cdot, t)|_{L^2}^2 \leq K_1(t + 1)^{-(n/2+1)}$$

for some appropriate constant β_1 . As in Case 1, W satisfies an inhomogeneous heat equation. Again using the Fourier splitting method, the critical term to bound is

$$\int_{S(t)} |\widehat{W}(\xi, t)|^2 d\xi.$$

Hypothesis of Case 2 yields

$$|\widehat{W}(\xi, t)| \leq C(T + 1)^{-n/2}|\xi|,$$

and as in Case 1, if T sufficiently large and $t > T$, it will follow that

$$\int_{\mathbf{R}^n} |\widehat{W}(\xi, t)|^2 dx \leq C(T + 1)^{-n/2}(t + 1)^{-(n/2+1)} \leq \frac{C_0\beta_1}{4}(t + 1)^{-(n/2+1)}.$$

The details of the proof are given next. We give the proof for $n = 2$; if $n > 2$ the hypotheses are given by Cases 1* and 2* and (3.5) needs to be modified appropriately.

Detailed proofs of Cases 1 and 2 follow.

Proof of Case 1. Let $\{r_k\}$ be an increasing sequence such that (4.3) holds. Let $\omega = v - u$ be the difference between the solution to the heat equation and the solution to the Navier-Stokes. Then

$$(4.5) \quad \omega_t = \Delta\omega - (u\nabla u + \nabla p).$$

Multiplying (4.5) by ω and integrating in space yields, after some integration by parts,

$$\frac{d}{dt} \int_{\mathbf{R}^n} \frac{|\omega|^2}{2} dx = - \int_{\mathbf{R}^n} |\nabla\omega|^2 dx - \int_{\mathbf{R}^n} (u - v)u\nabla u dx - \int_{\mathbf{R}^n} \omega\nabla p dx.$$

Since $\operatorname{div} \omega = 0$, the last integral vanishes. Moreover, since $\operatorname{div} u = 0$,

$\int uu\nabla u \, dx = 0$. Hence

$$\begin{aligned} \frac{d}{dt} \int_{\mathbf{R}^n} |\omega|^2 \, dx &= 2 \int_{\mathbf{R}^n} |\nabla \omega|^2 \, dx - 2 \sum_{j=1}^n \int_{\mathbf{R}^n} v_j \sum_{i=1}^n \frac{\partial}{\partial x_i} (u_i u_j) \, dx \\ &= -2 \int_{\mathbf{R}^n} |\nabla \omega|^2 \, dx + 2 \sum_{j=1}^n \int_{\mathbf{R}^n} (u_i u_j) \frac{\partial}{\partial x_i} v_j \, dx \\ &\leq -2 \int_{\mathbf{R}^n} |\nabla \omega|^2 \, dx + K |\nabla v|_\infty \int_{\mathbf{R}^n} |u|^2 \, dx, \end{aligned}$$

where $K = 2n^2$. Using Plancherel’s theorem the last equality reads

$$\frac{d}{dt} \int_{\mathbf{R}^n} |\hat{\omega}|^2 \, d\xi \leq -2 \int_{\mathbf{R}^n} |\xi|^2 |\hat{\omega}|^2 \, d\xi + K |\nabla v|_\infty \int_{\mathbf{R}^n} |u|^2 \, dx.$$

Let

$$S(t) = \left\{ \xi : |\xi| \leq \left(\frac{2n}{t+1} \right)^{1/2} \right\}.$$

Splitting the frequency domain to sets $S(t)$ and $S(t)^c$, arguments of previous work done in [5], yields

$$\begin{aligned} (4.6) \quad \frac{d}{dt} \left((t+1)^{4n} \int_{\mathbf{R}^n} |\omega|^2 \, dx \right) &\leq (t+1)^{4n-1} \left[4 \int_{S(t)} |\hat{\omega}|^2 \, d\xi + K |\nabla v|_\infty \int_{\mathbf{R}^n} |u|^2 \, dx \right] \\ &= I_1 + I_2. \end{aligned}$$

Consider first I_2 . By Lemma 2.1

$$(4.7) \quad I_2 \leq KC_1 t^{-n/2-1} (t+1)^{4n-1} \int_{\mathbf{R}^n} |u|^2 \, dx.$$

Recall that by the results of Wiegner [6] there is a constant C depending only on the norms of the initial data such that

$$|\omega|_{L^2}^2 \leq C(t+1)^{-(n/2+1)}.$$

Since

$$|v|_{L^2}^2 \leq C(t+1)^{-(n/2+1)},$$

it follows that if $C_2 = C + C_1$ then

$$(4.8) \quad |u(\cdot, t)|_{L^2}^2 \leq C_2(t+1)^{-(n/2+1)}.$$

Hence by (4.8) and Lemma 2.1 there is a constant M such that for $t \geq 1$

$$(4.9) \quad I_2 \leq M(t+1)^{3n-3},$$

where $M_1 = KC_1C_2$. To bound integral I_1 the following estimates on $\hat{\omega}(\xi, t)$ are necessary. Taking the Fourier transform on the equations for the difference yields

$$\begin{aligned} \hat{\omega}_{k,t}(\xi, t) + |\xi|^2 \hat{\omega}_k &= -\widehat{u\nabla u}_k - \widehat{\nabla_k p} = -\widehat{H}_k, \quad k = 1, 2; \\ \hat{\omega}_k(\xi, 0) &= 0. \end{aligned}$$

For sake of notation, let $\omega_k = \omega$, $H_k = H$. Since

$$\widehat{H} = \widehat{u} \nabla u - \widehat{\nabla} p = -i \left[\sum_{j=1}^n \xi_j \widehat{u}_j \widehat{u}_k - \xi_k \sum_{i,j} \frac{\xi_r \xi_j}{|\xi|^2} \widehat{u}_r \widehat{u}_j \right],$$

it follows that

$$|\widehat{H}(\xi, t)| \leq K_1 |\xi| |u(\cdot, t)|_{L^2}^2$$

with $K_1 = n(n + 1)$. Hence for $t \geq r_m$, where r_m will be chosen below,

$$|\widehat{\omega}(\xi, t)| \leq |\widehat{\omega}(\xi, r_m)| e^{-|\xi|^2(t-r_m)} + K_1 |\xi| \int_{r_m}^t e^{-|\xi|^2(t-s)} |u(\cdot, s)|_{L^2}^2 ds = \text{I} + \text{II}.$$

Bound for I. Hypothesis for Case 1 (Case 1* if $n > 2$) is used to show that $\widehat{\omega}(\xi, r_m)$ is sufficiently small. Then r_m is chosen so large that the decay of the energy will imply that the coefficient of II is sufficiently small. Since r_m was chosen so that

$$\left| \int_0^{r_n} \int_{\mathbf{R}^2} |u_1|^2 - |u_2|^2 dx \right| \leq \beta \sqrt{C_0}$$

and

$$\left| \int_0^{r_m} \int_{\mathbf{R}^2} u_1 u_2 dx \right| \leq \sqrt{C_0} \beta,$$

and by (3.5)

$$\widehat{H}(\xi, s) = i\xi \cdot \left(\frac{\xi_2^2}{|\xi|^2} (a_{11}^0 - a_{22}^0), 1 - \frac{2\xi_1^2}{|\xi|^2} a_{12}^0 \right) + O(|\xi|^2)$$

(Recall that if $n > 2$, \widehat{H} will be modified appropriately.) It follows that

$$|\widehat{\omega}(\xi, r_m)| \leq \int_0^{r_m} |\widehat{H}(\xi, s)| ds \leq 2\beta \sqrt{C_0} |\xi| + O(|\xi|^2) r_m.$$

Bound for II. From (4.8)

$$\begin{aligned} \text{II} &\leq K_1 |\xi| C_2 \int_{r_m}^t \frac{1}{(1+s)^{n/2+1}} ds \\ &\leq \frac{1}{n/2+2} K_1 |\xi| C_2 (1+r_m)^{-n/2} \leq 2n(1+r_m)^{-n/2} C_2 |\xi|. \end{aligned}$$

Hence

$$\begin{aligned} I_1 &\leq (t+1)^{4n-1} \int_{S(t)} [\text{I} + \text{II}]^2 d\xi \\ &\leq 4n(t+1)^{4n-1} \left[4\beta^2 C_0^2 \int_{S(t)} |\xi|^2 d\xi + 4n^2 (1+r_m)^{-n} C_2^2 \int_{S(t)} |\xi|^2 d\xi \right. \\ &\quad \left. + \int_{S(t)} O(|\xi|^4) r_m^2 d\xi \right] \\ &\leq (t+1)^{4n-1} \frac{C_0}{8} (t+1)^{-n/2-1} + O((t+1)^{(7/2)n-3}), \end{aligned}$$

where we let β be so that $16\beta^2 A_n = 1/16$ with $A_n = 2\pi(2n)^{n/2+1}/(n+2)$ and then choose r_m so that $A_n 16n^2(1+r_m)^{-n} C_2^2 \leq C_0/16$. Integrate (4.6) over $[1, t]$ and use the bounds for I_1 and I_2 to obtain

$$(4.10) \quad \int_{\mathbf{R}^n} |\omega(x, t)|^2 dx \leq (t+1)^{-4n} \int_{\mathbf{R}^n} |\omega(x, 1)|^2 dx + M(t+1)^{-(n+2)} + \frac{C_0}{8}(t+1)^{-(n/2+1)} + \text{H.O.T.}$$

Recall that

$$|\omega(x, 1)|_{L^2}^2 \leq |u(\cdot, 1)|_{L^2}^2 + |v(\cdot, 1)|_{L^2}^2 \leq 2|u_0|_{L^2}^2.$$

Hence (4.10) implies

$$\int_{\mathbf{R}^n} |\omega(x, t)|^2 dx \leq \frac{C_0}{8}(t+1)^{-(n/2+1)} + \text{H.O.T.}$$

By (3.5)

$$\begin{aligned} \text{H.O.T.} &= \left[\int_0^t (s+1)^{4n-1} \int_{S(s)} O(|\xi|^4) d\xi ds r_m \right] (t+1)^{-4n} \\ &\quad + M(t+1)^{-(n+2)} + 2|u_0|_{L^2}^2 (t+1)^{-4n} \\ &\leq C(r_m, |u_0|_{W_2}, |u_0|_{L^1}, |u_0|_{L^2})(t+1)^{-\sigma}, \end{aligned}$$

where $\sigma \geq n/2 + 2$. Hence for $t \geq T_0$ with T_0 depending only on $r_m, |u_0|_{L^1}, |u_0|_{L^2}, |u_0|_{W_2}$, and C_0 ,

$$\int_{\mathbf{R}^n} |\omega(x, t)|^2 dx \leq \frac{C_0}{4}(t+1)^{-(n/2+1)}.$$

Thus for $t \geq T_0$

$$|u(\cdot, t)|_{L^2} = |v|_{L^2} - |\omega|_{L^2} \geq \sqrt{C_0}/2(t+1)^{-n/2-1}.$$

If $t < T_0$ then

$$\begin{aligned} |u(\cdot, t)|_{L^2}^2 &\geq |u(\cdot, T_0)|_{L^2}^2 \geq C_0/4 \left[\frac{t+1}{T_0+1} \right]^{n/2+1} \left[\frac{1}{t+1} \right]^{n/2+1} \\ &\geq \frac{C_0}{4}(T_0+1)^{-(n/2+1)}(t+1)^{-(n/2+1)} \end{aligned}$$

and this proves Case 1 since T_0 depends only on $|u_0|_{L^1}, |u_0|_{L^2}, |u_0|_{W_2}, n$, and C_0 .

The following auxiliary computations will be needed for Case 2.

1. Lower bound for α_1 . Recall from Theorems 2.1 and 3.2 that

$$\alpha_1 = \int_{|\omega|=1} |\omega \cdot l_1(\omega)|^2 d\sigma,$$

where

$$l_1(\omega, t) = \left(-i \int_0^t (\omega_2^2/|\omega|^2)(a_{11}^0 - a_{22}^0) ds, \int_0^t (1 - 2\omega_1^2/|\omega|^2)a_{12}^0 ds \right),$$

$$a_{ij}^0 = \int_{\mathbf{R}^n} u_i u_j dx.$$

Hence

$$\alpha_1 = \int_{|\omega|=1} \omega_1^2 \omega_2^4 \left(\int_0^t a_{11}^0 - a_{22}^0 ds \right) + (1 - 2\omega_1^2)\omega_2^2 \left(\int_0^t a_{12}^0 ds \right)^2 d\sigma.$$

Since by hypothesis of Case 2 for $t \geq T_0$,

$$\int_0^t a_{11}^0 - a_{22}^0 ds = \int_0^t \int_{\mathbf{R}^n} u_1^2 - u_2^2 dx ds \geq \beta \sqrt{C_0},$$

$$\int_0^t a_{12}^0 ds = \int_0^t \int_{\mathbf{R}^n} u_1 u_2 dx ds \geq \beta \sqrt{C_0},$$

it follows that

$$\alpha_1 \geq \beta^2 C_0 \int_{|\omega|=1} \omega_1^2 \omega^4 + (1 - 2\omega_1^2)^2 \omega_2^2 d\sigma = \beta^2 C_0 \gamma.$$

2. Lower bound for the solution to heat equation $V(x, t)$ with data $u(x, T_1)$, $T_1 \geq T_0$, T_0 given by hypothesis of Case 2. By Theorems 2.1 and 3.2

$$\hat{u}_1(\xi, T_1) = \xi \cdot l_1(\xi, T_1) + h_1(\xi, T_1)$$

and $|V(x, t)|_{L^2}$ satisfies for $t \geq \delta_1$, $\delta_1 = (\delta(T_1))^{-1}$ (where $4\delta M_0 M_1 \leq \alpha_1$),

$$|V(\cdot, t)|_{L^2}^2 \geq \chi_0 (t + 1)^{-(n/2+1)},$$

where $\chi_0 = \alpha_1 \omega_n e^{-1}/2(n + 2)$. By the computations for α_1 it follows that

$$\chi_0 \geq \beta^2 C_0 \gamma \omega_n e^{-1}/2(n + 2) = \chi_1.$$

Hence for $t \geq \delta = \delta_1(T_1)$

$$(4.11) \quad |V(\cdot, t)|_{L^2}^2 \geq \chi_1 (t + 1)^{-(n/2+1)},$$

and χ_1 is independent of T_1 .

Proof of Case 2. Let V be the solution to the heat equation with data $V(x, 0) = u(x, T_1)$, for T_1 satisfying

- (i) $T_1 > T_0$, T_0 given by hypothesis of Case 2;
- (ii) $4n^2 C^2 (1 + T_1)^{-n} \leq \chi_1 A_n^{-1}/8$,

where χ_1 is given by the auxiliary computations 1. The constant C is such that $|u(\cdot, t)|_{L^2}^2 \leq C(t + 1)^{-(n/2+1)}$ and $A_n = 2\pi(2n)^{n/2+1}/(n + 2)$. Thus by Theorems 2.1 and 3.2 it follows that for $t \geq \delta_1 = \delta(T_1)^{-1}$

$$(4.12) \quad \chi_1 (t + 1)^{-(n/2+1)} \leq |V(\cdot, t)|_{L^2}^2 \leq K_0 (t + 1)^{-(n/2+1)},$$

where K_0 depends on the L^2 norm of u_0 and χ_1 depends on β and C_0 . Note that χ_1 is independent of T_1 , but if T_1 tends to infinity then the lower bound of (4.12) is only valid for $t \geq \delta_1 = \delta(T)^{-1}$ and $\delta(T_1)^{-1}$ tends to infinity as T_1 tends to infinity. Let

$$U(x, t) = u(x, t + T_1).$$

We study the difference $W = V - U$. Here the hypothesis of Case 2 (Case 2^* if $n > 2$) together with the decay of energy of the solutions to Navier-Stokes will imply that $|W(\cdot, t)|_{L^2}^2 \leq C(t+1)^{-(n/2+1)}$ with C sufficiently small. As in Case 1, W satisfies an inhomogeneous heat equation. The Fourier splitting method will yield

$$(4.13) \quad \frac{d}{dt} \left[(t+1)^{4n} \int_{\mathbf{R}^n} |W|^2 dx \right] \leq (t+1)^{4n-1} \int_{S(t)} |\widehat{W}|^2 d\xi + K|\nabla V|_\infty \int_{\mathbf{R}^n} |u|^2 dx.$$

The second term can be bounded as in Case 1 for $t \geq 1$

$$(4.14) \quad K|\nabla V|_\infty \int_{\mathbf{R}^n} |U|^2 dx \leq M(t+1)^{3(n-1)}.$$

To bound the first term note that the Fourier transform of the equation for W yields

$$\widehat{W} + |\xi|^2 \widehat{W} = -U\nabla U - \nabla P = \widehat{H},$$

where $P(x, t) = p(x, t + T_1)$. As Before $|\widehat{H}(\xi, t)| \leq K_1 |\xi| |U(\cdot, t)|_{L^2}^2$; hence

$$\begin{aligned} |\widehat{W}(\xi, t)| &\leq \int_0^t |\widehat{H}(\xi, s)| ds \leq K_1 |\xi| \int_0^t |U(x, s)|_{L^2}^2 ds \\ &= K_1 |\xi| \int_0^t |u(x, s + T_1)|_{L^2}^2 ds = K_1 |\xi| \int_{T_1}^{t+T_1} |u(x, s)|_{L^2}^2 ds \\ &\leq CK_1 |\xi| \int_{T_1}^{t+T_1} \left[\frac{1}{1+s} \right]^{n/2+1} ds \leq 2nC |\xi| \left[\frac{1}{1+T_1} \right]^{n/2} \\ &= 2nC |\xi| \left(\frac{1}{1+T_1} \right)^{n/2}. \end{aligned}$$

Hence the first term in (4.13) can be bounded as follows:

$$\begin{aligned} (t+1)^{4n-1} \int_{S(t)} |\widehat{W}|^2 d\xi &\leq 4n^2 C^2 (1+T_1)^{-n} (t+1)^{4n-1} \int_{S(t)} |\xi|^2 d\xi \\ &\leq \frac{\chi_1}{8} (t+1)^{7/2n-2}. \end{aligned}$$

The last inequality follows by the choice of T_1 made at the beginning of Case 2. Combining this last bound with (4.13) and (4.14) yields

$$\frac{d}{dt} \left[(t+1)^{4n} \int_{\mathbf{R}^n} |W|^2 dx \right] \leq \frac{\chi_1}{8} (t+1)^{7/2n-2} + M(t+1)^{3(n-1)}.$$

Integrating over $[\delta_1, t]$ gives

$$\int_{\mathbf{R}^n} |W|^2 dx \leq \frac{\chi_1}{8}(t+1)^{-n/2-1} + M(t+1)^{-n-2} + \int_{\mathbf{R}^n} |W(x, \delta_1)|^2 dx(t+1)^{-2n}.$$

Note that

$$\begin{aligned} \int_{\mathbf{R}^n} |W(x, \delta_1)|^2 dx &\leq \int_{\mathbf{R}^n} |V(x, 0)|^2 dx + \int_{\mathbf{R}^n} |U(x, 0)|^2 dx \\ &\leq 2 \int |u(x, T_1)|^2 dx \leq 2 \int |u_0(x)|^2 dx. \end{aligned}$$

Hence

$$(4.15) \quad \int_{\mathbf{R}^n} |W|^2 dx \leq \frac{\chi_1}{8}(t+1)^{-(n/2+1)} + \text{H.O.T.}$$

for t large enough. That is for $t \geq \max(T_0, T_2)$, where

$$T_2 = T_2(|u_0|_{L^1}, |u_0|_{L^2}, n, \alpha_1, \delta_1, \beta_1, C_0).$$

The last inequality combined with (4.11) yields

$$|U(\cdot, t)|_{L^2} \geq |V(\cdot, t)|_{L^2} - |W(\cdot, t)|_{L^2} \geq (1 - 1/2)\sqrt{\chi_1}(t+1)^{-(n/2+1)/2};$$

hence for $t \geq T_3 = T_1 + T_2$

$$|u(\cdot, t)|_{L^2} \geq \frac{\sqrt{\chi_1}}{2}(t+1)^{-(n/2+1)/2}.$$

For $t < T_3$ the decay of energy of u yields

$$|u(\cdot, t)|_{L^2}^3 \geq |u(\cdot, T_3)|_{L^2}^2 \geq \frac{\chi_1}{4} \left[\frac{1+t}{1+T_3} \right]^{n/2+1} (1+t)^{-(n/2+1)}.$$

Thus for Case 2 let $M_0 = (\chi_1/4)(T_3 + 1)^{-(n/2+1)}$ and the lower bound follows. In order to give the formal argument when $n = 3$, the hypotheses needed are given by Case 2* and (3.5) has to be modified adequately.

In order to make the proof rigorous for $n = 3$, apply the formal proof with appropriate modifications to the subsequence u_δ of appropriate solutions that converges strongly in $L^2_{loc}(R_+ \times R^n)$. For $n > 3$, a similar argument should work if applied to the approximate solutions constructed by Kayikiya and Miyakawa. Let $n = 3$. Let us recall that the approximation solutions u_δ constructed by Caffarelli, Kohn, and Nirenberg satisfy

$$\begin{aligned} u_t + \psi_\delta \nabla u + \nabla p &= \Delta u, \\ \operatorname{div} u &= 0, \\ u(x, 0) &= g, \end{aligned}$$

where $\psi_\delta = \delta^{-4} \psi * \bar{u}$ and

$$\begin{aligned} \psi \in C^\infty, \quad \psi \geq 0, \quad \iint \psi dx dt &= 1, \\ \operatorname{supp} \psi \subset \{(x, t) : |x|^2 < t, 1 < t < 2\}, \end{aligned}$$

and let $D = \mathbf{R}^3 \times (0, T)$,

$$\bar{u} = \begin{cases} u(x, t) & (x, t) \in D, \\ 0 & \text{otherwise.} \end{cases}$$

Suppose that u_δ is a subsequence that converges strongly to u in $L^2_{\text{loc}}(D)$ where u is a Leray-Hopf solution of Navier-Stokes equations. The steps to show that the approximate solutions satisfy

$$|u_\delta(\cdot, t)|_{L^2} \leq M_0(t + 1)^{-(n/2+1)}$$

are obtained combining the arguments of [5] with the formal proof. The lower bound for the limiting Leray-Hopf solution $u(x, t)$ follows, a.e. in t , taking the limit as $\delta \rightarrow 0$. The details are omitted.

5. THE LOWER BOUNDS

The results obtained in the previous sections are combined to establish the lower bounds for the rates of decay for solutions to the Navier-Stokes equations in two and three spatial dimensions.

Theorem 5.1. *Let $u_0 \in L^1 \cap H(\mathbf{R}^n)$, $n = 2$. Then*

(i) *If $\hat{u}(0) = \int u(x, t) dx = \int u(x) dx \neq 0$, then there exist constants C_0 and C_1 depending only on L^1 and*

$$C_0(t + 1)^{-n/2} \leq |u(\cdot, t)|_{L^2}^2 \leq C_1(t + 1)^{-n/2}.$$

If the average $\int u(x, t) dx = \int u(x) dx = 0$ then there are three cases to consider.

(ii) *Let $u_0 \in W_1 \cap W_2$. Suppose that $\hat{\omega}_0(\xi)$ has a zero of order one at the origin; then there exist constants C_2 and C_3 such that*

$$C_1(t + 1)^{-(n/2+1)} \leq |u(\cdot, t)|_{L^2}^2 \leq C_2(t + 1)^{-(n/2+1)}.$$

The constants depend only on the L^2 , W_1 , and W_2 norms of the data.

(iii) *Suppose that $u_0 \in H^1 \cap M^c \cap W_2$. If $\hat{u}(\xi)$ has a zero at the origin of order greater than one, then there exist constants C_4, C_5 such that*

$$C_4(t + 1)^{-2} \leq |u(\cdot, t)|_{L^2}^2 \leq C_5(t + 1)^{-2},$$

where C_4 depends only on the L^1 and L^2 norms of u_0 and C_5 depends on the L^1, L^2, H^1, W_2 norms of the data and

$$\alpha = \int_{\mathbf{R}^2} |u_1^0|^2 - |u_2^0|^2 dx \quad \text{or} \quad \beta = \int_{\mathbf{R}^2} u_1^0 u_2^0 dx.$$

Proof.

Upper bound. Case (i). See [5]. Cases (ii), (iii) follow from Theorems 2.1, 3.2, 3.3 and the decay rates for the difference ω between the solution to the heat equation and the solution to the Navier-Stokes equation obtained by Wiegner [6].

Lower bound. Case (i). The proof can be found in [5, Theorem (4.2)]. There are a few simple modifications.

1. The more restrictive hypothesis of Theorem (4.2) [5], $|\hat{u}(\xi)| \neq 0$ for $|\xi| \leq \delta$, can be deduced from $\hat{u}(0) \neq 0$, since $u_0 \in L^1$ implies $\hat{u} \in C^1$.

2. The necessary upper bounds for ω when $n = 2$ can be found in [6].

Case (ii). Follows from Theorems 2.1, 3.1, and 4.1.

Case (iii). Follows from Theorems 2.1, 3.2, and 4.1.

Theorem 5.2. *Let $u_0 \in L^1 \cap H(\mathbf{R}^n)$, $n = 3$. Let $u(x, t)$ be a suitable Leray-Hopf solution in the sense of Caffarelli, Kohn, and Nirenberg and the lower bound holds for almost all t . Then (i) and (ii) of Theorem 5.1 hold, where the upper bound is for all t and the lower bound is for a.e. t .*

(iii) *If there exists t_0 such that $u \in M_{t_0}^c$ and also $u_0 \in W_2$, then*

$$C_6(t + 1)^{-5/2} \leq |u(\cdot, t)|_{L^2}^2 \leq C_7(t + 1)^{-5/2},$$

where C_7 depends only on the L^1 and L^2 norms of the data and C_6 depends on the L^1, L^2, W_2 norms of the data and $\alpha_j^i = \alpha_j^i(t_0, 3)$ and $\beta_j^i = \beta_j^i(t_0, 3)$, where α_j^i and β_j^i were defined in Theorem 3.

Proof.

Upper bound. See Theorem 5.1, apply proof to approximation solutions, and pass to the limit.

Lower bound. Cases (i), (ii). Same steps as in Theorem 5.1 applied to special subsequence of approximate solutions, which is supposed to exist by hypothesis and pass to limit. Bound will be valid for a.e. t .

Case (iii). Follows from Theorems 2.1, 3.3, and 4.1 applied to special sequence of approximate solutions existing by hypothesis and pass to the limit. The bound will hold for almost all t .

We expect that Theorems 5.1 and 5.2 can be extended to n dimensions for all $n \geq 4$ using the approximate solutions constructed by Kayikiya and Miyakawa. The hypothesis on the data $u_0 \in M^c$ for $n = 2$ is optimal as the following example shows. In three dimensions we expect that the condition $u \in M_{t_0}^c(\mathbf{R}^3)$ is necessary but we have not found an example where the lower bound fails if $u \in M_{t_0}(\mathbf{R}^3)$. The example in two dimensions we will present was suggested by A. Majda.

Example 5.3. Exponentially decaying vorticity in 2 spatial dimensions with data $u_0 \in M$.

Let $u(x, t)$ be a solution to the 2D Navier-Stokes equation with radial vorticity. Suppose also that $u_0 \in M \cap L^1 \cap L^2$ is such that $\text{curl } u_0 = \omega_0$ satisfies

- (i) $\omega_0 \in L^1$,
- (ii) $\nabla u_0 \in L^1$,
- (iii) $\hat{\omega}(\xi) = 0$ for $|\xi| \leq \delta$, some $\delta > 0$.

Then

$$(5.1) \quad u(x, t) = \begin{pmatrix} -x_2/r^2 \\ x_1/r^2 \end{pmatrix} \int_0^r s\omega(s) ds, \quad r^2 = x_1^2 + x_2^2,$$

where ω is the vorticity. Here $u_0(x) = u(x, 0) \in M(\mathbf{R}^2)$, since

$$\int_{\mathbf{R}^2} u_1^2(x, 0) dx = \int_{\mathbf{R}^2} u_2^2(x, 0) dx$$

and

$$\int_{\mathbf{R}^2} u_1^0(x, 0)u_2^0(x, 0) dx = 0.$$

The vorticity equation is reduced to

$$\begin{aligned} \omega_t &= \Delta\omega, \\ \omega(|x|, 0) &= \omega(x, 0) = \omega_0(r). \end{aligned}$$

The term $u \cdot \nabla\omega$ is zero, since the vorticity is radial. By (iii) follows

$$\begin{aligned} \int_{\mathbf{R}^2} |\omega(x, t)|^2 dx &= \int_{\mathbf{R}^2} |\hat{\omega}(\xi, t)|^2 d\xi = \int_{\mathbf{R}^2} |\hat{\omega}_0(\xi)|^2 e^{-|\xi|^2 t} d\xi \\ &\leq e^{-\delta^2 t} \int_{\mathbf{R}^2} |\hat{\omega}_0(\xi)|^2 d\xi. \end{aligned}$$

Hence $|\omega|_{L^2}^2 \leq ke^{-|\delta|^2 t}$. We only give an outline for the estimate

$$\int_{\mathbf{R}^2} |u|^2 dx \leq C_0 \exp(-C_1 t).$$

We use the explicit form of u and the decay of $|\omega|_{L^2}$. This estimate will follow from the explicit form of u, v and the decay of $|\omega|_{L^2}$.

By the Jensen inequality and (5.1) it follows that

$$\begin{aligned} |u|^2 + |v|^2 &\leq \frac{1}{r^2} \left(\int_0^r s\omega ds \right)^2 \leq \int_0^r s^2 \omega^2 \frac{ds}{r} \leq \frac{1}{2\pi} \int_0^{2\pi} \int_0^r s\omega^2 ds \\ &= \frac{1}{2\pi} \int_{\mathbf{R}^2} |\omega|^2 dx \leq C_0 e^{-\delta^2 t}; \end{aligned}$$

hence

$$|u|_\infty + |v|_\infty \leq C \exp(-\delta^2 t/2).$$

The L^2 decay of u will follow from a time dependent L^1 estimate, more precisely from

$$|u(\cdot, t)|_{L^1} \leq Ct,$$

where C depends on L^1, L^2 norms of u and ω and also on the L^1 norm of $\nabla\omega$.

APPENDIX

Proposition A.1. *Let $u_0 \in H(\mathbf{R}^2) \cap W_2$. Then if u is a solution of Navier-Stokes with data u_0*

$$|\nabla_\xi a_{ij}(\xi, t)| \leq C(t),$$

where $a_{ij} = \widehat{u_i u_j}$ and $C(t)$ depends on $|u_0|_{L^2}$.

Proof.

$$(A.1) \quad |\partial_{\xi_k} a_{ij}(\xi, t)| \leq \int_{\mathbf{R}^2} |x_k| |u_i u_j| dx \leq \int_{\mathbf{R}^2} |x_k| |u|^2 dx.$$

Hence we need to bound $\int_{\mathbf{R}^2} |x| |u|^2 dx$ in terms of the initial data. Multiply the Navier-Stokes equation by $|x_k| u_j$ and integrate in space. We do it componentwise and sum

$$(A.2) \quad \begin{aligned} \frac{d}{dt} \int_{\mathbf{R}^2} |x_k| u_j u_j dx &= - \int_{\mathbf{R}^2} |x_k| u_j \sum_i u_i \partial_i u_j dx \\ &\quad - \int_{\mathbf{R}^2} |x_k| u_j \partial_j p dx + \sum_s \int_{\mathbf{R}^2} |x_k| u_j \partial_{ss} u_j dx \\ &= \text{I} + \text{II} + \text{III}; \end{aligned}$$

$$(A.3) \quad \begin{aligned} \text{I} &= \frac{1}{2} \sum_i \int_{\mathbf{R}^2} \text{sign } x_k u_i u_j^2 dx \leq \frac{1}{2} \int_{\mathbf{R}^2} |u|^2 + \frac{1}{2} \int_{\mathbf{R}^2} |u|^4 \\ &\leq \frac{1}{2} \int_{\mathbf{R}^2} |u|^2 + \frac{1}{2} \int_{\mathbf{R}^2} |\nabla u|^2, \end{aligned}$$

where $\int |u|^4$ follows by Ladyzenkaya [3, Lemma 7].

For the second term it is convenient to bound the sum of all the terms and use that u is divergent free:

$$(A.4) \quad \begin{aligned} \sum_j \text{II} &\leq \int_{\mathbf{R}^2} |\text{sign}(x_k)| |u_j| |p| dx \\ &\leq \int_{\mathbf{R}^2} |u|^2 + \int_{\mathbf{R}^2} |p|^2 \leq \int_{\mathbf{R}^2} |u|^2 + 4 \int_{\mathbf{R}^2} |u|^4 \leq \int_{\mathbf{R}^2} |u|^2 + \int_{\mathbf{R}^2} |\nabla u|^2. \end{aligned}$$

The bound on p above follows by recalling that the pressure satisfies an elliptic equation, which is obtained by taking the divergence of the Navier-Stokes equation,

$$\Delta p = - \sum_{i,j} \frac{\partial^2}{\partial x_i \partial x_j} u_i u_j;$$

hence

$$\hat{p} = - \sum_{i,j} \frac{\xi_i \xi_j}{|\xi|^2} \widehat{u_i u_j}$$

and

$$(A.5) \quad \begin{aligned} \int_{\mathbf{R}^2} |p|^2 dx &= \int_{\mathbf{R}^2} |\hat{p}|^2 d\xi \leq \int_{\mathbf{R}^2} \sum_{i,j} \frac{\xi_i \xi_j}{|\xi|^2} \widehat{u_i u_j} d\xi \\ &\leq 4 \int_{\mathbf{R}^2} |u|^4 dx \leq 8 \int_{\mathbf{R}^2} |\nabla u|^2 dx. \end{aligned}$$

Finally III is estimated as follows:

$$\begin{aligned}
 \text{III} &= \sum_s \int_{\mathbf{R}^2} \text{sign}(x_k) u_j \partial_s u_j \, dx - \int_{\mathbf{R}^2} |x_k| |\nabla u_j|^2 \, dx \\
 \text{(A.6)} \quad &\leq 3 \int_{\mathbf{R}^2} |u|^2 \, dx + \int_{\mathbf{R}^2} |\nabla u|^2 \, dx.
 \end{aligned}$$

Hence integrating over $[0, t]$ (A.2) and combining (A.1), (A.2), (A.3), (A.4), and (A.6)

$$\begin{aligned}
 |\partial_{\xi_k} a_{ij}(\xi, t)| &\leq \int_{\mathbf{R}^2} |x_k| |u|^2 \, dx \\
 &\leq C \left(\int_0^t \int_{\mathbf{R}^2} |u|^2 \, dx \, ds + \int_0^t \int_{\mathbf{R}^2} |\nabla u|^2 \, dx \, ds \right) + \int_{\mathbf{R}^2} |x_k| |u_0|^2 \\
 &\leq C(t+1) \int_{\mathbf{R}^2} |u_0|^2 \, dx + \int_{\mathbf{R}^2} |x_k| |u_0|^2 \, dx,
 \end{aligned}$$

where $C \leq 40$.

Lemma A.2. Let $u_0 = (u_0^1, u_0^2) \in M^c \cap H^1(\mathbf{R}^2)$. Let $\alpha = \int_{\mathbf{R}^2} |u_0^1|^2 - |u_0^2|^2 \, dx \neq 0$. There then exists t_0 such that if $u(x, t)$ is a solution to the Navier-Stokes equation with data u_0

$$\left| \int_0^T \int_{\mathbf{R}^2} |u_1|^2 - |u_2|^2 \, dx \, dt \right| \geq (\alpha/2)T$$

for all $T \leq t_0$, t_0 depending only on the H^1 norm of u_0 .

Proof. W.l.o.g. suppose that $\alpha > 0$. If not take

$$\alpha = \int_{\mathbf{R}^2} |u_0^2|^2 - |u_0^1|^2 \, dx.$$

Let

$$F(t) = \int_{\mathbf{R}^2} |u_1|^2 - |u_2|^2 \, dx.$$

The following estimate is needed.

$$\left| \frac{d}{dt} F(t) \right| \leq C \int_{\mathbf{R}^2} |\nabla u|^2 \, dx.$$

The last estimate follows from multiplying the equation for the first component of Navier-Stokes by u_1 , the second by u_2 , and integrating in space. Hence

$$\begin{aligned}
 \left| \frac{d}{dt} F(t) \right| &\leq \left| \int_{\mathbf{R}^2} u_1 \sum_i u_i \partial_i u_1 - u_1 \partial_1 p + u_1 \Delta u_1 \, dx \right. \\
 &\quad \left. - \int_{\mathbf{R}^2} u_2 \sum_i u_i \partial_i u_2 - u_2 \partial_2 p + u_2 \Delta u_2 \, dx \right|.
 \end{aligned}$$

Thus from (A.5)

$$\left| \frac{d}{dt} F(t) \right| \leq 2n \int_{\mathbf{R}^2} |\nabla u|^2 \, dx + \int_{\mathbf{R}^2} |u|^4 \, dx.$$

By Lemma (7) in [3] it follows that if $|u(x, t)| \rightarrow 0$ as $|x| \rightarrow \infty$,

$$\int_{\mathbf{R}^2} |u|^4 dx \leq 2 \int_{\mathbf{R}^2} |\nabla u|^2 dx \int_{\mathbf{R}^2} |u|^2 dx \leq C \int_{\mathbf{R}^2} |\nabla u|^2 dx$$

with $C = 4nC_0$, $C_0 = |u_0|_{L^2}^2$. Recall that solutions to the Navier-Stokes equations in two spatial dimensions satisfy

$$\int_{\mathbf{R}^2} |\nabla u|^2 dx \leq \int_{\mathbf{R}^2} |\nabla u_0|^2 dx.$$

The last inequality follows from the special relation between the partial differentials of the components of the gradient. This relation is a consequence of the solution being divergence free. By the mean value theorem it follows that

$$|F(t) - F(0)| \leq |F'(\bar{s})|t \leq Ct \int_{\mathbf{R}^2} |\nabla u|^2(\bar{s}) dx \leq Ct \int_{\mathbf{R}^2} |\nabla u_0|^2 dx.$$

Thus $F(t) \geq F(0) - C_1t$, where $C_1 = C \int_{\mathbf{R}^2} |\nabla u_0|^2 dx$. Integrating over $[0, T]$ yields

$$\int_0^T F(t) dt \geq F(0)T - C_1T^2/2$$

and for any $T \leq F(0)/2C_1$ it follows that

$$\int_0^T F(t) dt \geq F(0)T/2.$$

Let $t_0 = F(0)/2C_1$, and the lemma follows.

Lemma A.3. Let $u_0 = (u_0^1, u_0^2) \in N^c \cap H^1$. Let $\alpha = \int_{\mathbf{R}^2} u_0^1 u_0^2 dx \neq 0$. Then there exists $t_0 > 0$ such that

$$\left| \int_0^T \int_{\mathbf{R}^2} u_0^1 u_0^2 dx \right| \geq \alpha/2T$$

for all $T \leq t_0$, t_0 depending only on the H^1 norms of u_0 .

Proof. W.l.o.g. suppose $\alpha > 0$. Following the lines of the proof for Lemma A.2 we only need to show that

$$\left| \frac{d}{dt} G(t) \right| \leq C \int_{\mathbf{R}^2} |\nabla u_0|^2 dx,$$

where

$$G(t) = \int_{\mathbf{R}^2} u_0^1 u_0^2 dx.$$

Note that

$$\begin{aligned} \left| \frac{d}{dt} G(t) \right| &\leq \left| \int_{\mathbf{R}^2} u_1 u_{2t} + u_{1t} u_2 dx \right| \\ &= \left| - \int_{\mathbf{R}^2} u_1 \sum_i u_i \partial_i u_2 dx - \int_{\mathbf{R}^2} u_1 \partial_2 p dx + \int_{\mathbf{R}^2} u_1 \Delta u_2 dx \right. \\ &\quad \left. - \int_{\mathbf{R}^2} u_2 \sum_i u_i \partial_i u_1 dx - \int_{\mathbf{R}^2} u_2 \partial_1 p dx + \int_{\mathbf{R}^2} u_2 \Delta u_1 dx \right| \\ &\leq C \left(\int_{\mathbf{R}^2} |u|^4 dx + \int_{\mathbf{R}^2} |\Delta u|^2 dx + \int_{\mathbf{R}^2} |p|^4 dx \right) \end{aligned}$$

and from the observation in the previous lemma it follows that last term is bounded by the L^2 norm of the gradient of u and hence

$$\left| \frac{d}{dt} G(t) \right| \leq C \int_{\mathbf{R}^2} |\nabla u_0|^2 dx.$$

And the proof now is the same as for Lemma A.2.

Note A.4. We recall that in Lemma (8.2) [1], Caffarelli, Kohn, and Nirenberg showed that for a suitable Leray-Hopf solution for almost all t

$$\int_{\mathbf{R}^3} |u(x, t)|^2 |x| dx \leq A(t)$$

with $A(t)$ depending only on the L^2 and W_2 norms of the data. Let

$$A = \left\{ t : \left| \frac{\partial}{\partial \xi} \widehat{u_i u_j}(\xi, t) \right| \leq A(t) \right\},$$

where $u = (u_1, u_2, u_3)$ is a suitable Leray-Hopf solution to Navier-Stokes. By Lemma 8.2 [1] such a set is nonempty. Moreover, A can be chosen so that $m(A^c) = 0$. With the notation $a_{ij} = \widehat{u_i u_j}$, $a_{ij}^0(t) = \widehat{u_i u_j}(0, t)$ iff $t \in A$

$$a_{ij}(\xi, t) = a_{ij}^0(t) + \xi \cdot \nabla_{\xi} a_{ij}(\bar{\xi}, t).$$

ACKNOWLEDGMENTS

The suggestion to look at the lower bounds of rates of solutions decay the Navier-Stokes equation was made by Ron DiPerna. I would like to thank A. Majda for his helpful comments. I would also like to thank the referee for his careful reading of the work and his many helpful suggestions.

REFERENCES

1. L. Caffarelli, R. Kohn, and L. Nirenberg, *Partial regularity of suitable weak solutions of the Navier-Stokes equations*, Comm. Pure Appl. Math. **35** (1982), 771–831.
2. R. Kayikiya and T. Miyakawa, *On the L^2 decay of weak solutions of the Navier-Stokes equations in \mathbf{R}^n* , Math. Z. **192** (1986), 135–148.
3. O. A. Ladyzenkaya, *The mathematical theory of viscous incompressible flows*, 2nd ed., Gordon and Breach, New York, 1969.

4. M. E. Schonbek, L^2 decay for weak solutions of the Navier-Stokes equations, Arch. Rational Mech. Anal. **88** (1985), 209–222.
5. ———, Large time behavior of solutions to the Navier-Stokes equations, Comm. Partial Differential Equations **11** (1986), 753–763.
6. M. Wiegner, Decay results for weak solutions of the Navier-Stokes equations in \mathbf{R}^n , J. London Math. Soc. (2) **35** (1987), 303–313.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA, SANTA CRUZ, CALIFORNIA 95064