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LOWER BOUNDS OF THE DISCRETIZATION ERROR FOR PIECEWISE POLYNOMIALS

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ABSTRACT. Assume that V_h is a space of piecewise polynomials of a degree less than $r \geq 1$ on a family of quasi-uniform triangulation of size h. There exists the well-known upper bound of the approximation error by V_h for a sufficiently smooth function. In this paper, we prove that, roughly speaking, if the function does not belong to V_h, the upper-bound error estimate is also sharp.

This result is further extended to various situations including general shape regular grids and many different types of finite element spaces. As an application, the sharpness of finite element approximation of elliptic problems and the corresponding eigenvalue problems is established.

1. Introduction

Error analyses for many numerical methods are mostly presented for upper-bound estimates of the approximation error. This paper focuses on the lower-bound error estimate and its applications for piecewise polynomial approximation in Sobolev spaces. Our work was inspired by some recent studies of lower-bound approximations of eigenvalues by finite element discretization for some elliptic partial differential operators [10, 15]. One crucial technial ingredient that is needed in the analysis in [10, 15] is some lower bound of the eigenfunction discretization error by the finite element method.

Lower-bound error estimates have been studied in the literature for some special cases. In Babuška and Miller [4, 5], lower bounds of the discretization error were obtained for a second-order elliptic problem by a bilinear element discretization using the Taylor expansion method under the assumption that the solution is smooth enough on the prescribed domain. More recently in Křížek, Roos, and Chen [13] (one of the studies that inspired this paper), two-sided bounds where the constants are almost 1 were obtained for the discretization error of linear and bilinear elements on the uniform meshes by superconvergence theory and interpolation error

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estimate. In Widlund [20, 21], a type of inverse theorem was given to indicate the function conditions necessary for obtaining the desired level of accuracy. Subsequently, Babuška, Kellogg, and Pitkäranta [3] proved the optimality of the error estimate for a second-order elliptic problem by the linear finite element method on the polygonal domain with graded mesh. In Liu and Xu [16], a lower-bound result by Hermite spline space for special functions is provided on one-dimensional structured meshes.

The aim of this paper is to derive lower-bound results of the error by piecewise polynomial approximation for much more general classes of problems under much weaker and more natural assumptions on grids and smoothness of functions to be approximated. As a special application, the lower bounds of the discretization error by a variety of finite element spaces can easily be obtained. For example, the following lower error bounds (see Sections 3 and 4) are valid for finite element (consisting of piecewise polynomials of a degree less than r) approximation to $2m$-th order elliptic boundary value problems:

$$\|u-u_h\|_{j,p,h} \geq Ch^{r-j}, \quad 0 \leq j \leq r,$$

where the positive constant C is independent of the mesh size h. This kind of result plays a very important role in the analysis of lower-bound eigenvalue approximations in [10, 15].

The outline of the rest of the paper is as follows. Section 2 focuses on the general derivation of the lower bounds of the error arising from piecewise polynomial approximation. Section 3 considers the lower bounds of the discretization error of the second-order elliptic problem and the corresponding eigenpair problem by finite element method. Section 4 concerns a generalization of the results from Section 3 to the $2m$-th order elliptic problem and the corresponding eigenpair problem. Section 5 presents some brief concluding remarks.

2. Notation and basic results

In this section, we introduce the used notation and then state some lower-bound results associated with the piecewise polynomial approximation error, which is a basic tool in this paper.

Here we assume that $\Omega \subset \mathcal{R}^n$ $(n \geq 1)$ is a bounded polytopic domain with a Lipschitz continuous boundary of $\partial\Omega$. Throughout this paper, we use the standard notation for the usual Sobolev spaces and the corresponding norms, semi-norms, and inner products as in [6, 8]. Let us introduce the multi-index notation. A multi-index α is an n-tuple of nonnegative integers α_i. The length of α is given by

$$|\alpha| = \sum_{i=1}^{n} \alpha_i.$$

The derivative $D^\alpha v$ is then defined by

$$D^\alpha v = \Big(\frac{\partial}{\partial x_1}\Big)^{\alpha_1} \cdots \Big(\frac{\partial}{\partial x_n}\Big)^{\alpha_n} v.$$

For a subdomain G of Ω, the usual Sobolev spaces $W^{m,p}(G)$ with the norm $\|\cdot\|_{m,p,G}$ and the semi-norm $|\cdot|_{m,p,G}$ are used. In the case of $p=2$, we have $H^m(G) = W^{m,2}(G)$ and the index p is omitted. The L^2-inner product on G is denoted by $(\cdot,\cdot)_G$. For $G \subset \Omega$, we write $G \subset\subset \Omega$ to indicate that $\mathrm{dist}(\partial\Omega, G) > 0$ and $\mathrm{meas}(G) > 0$.

We introduce a face-to-face partition $\mathcal{T}_h$ of the computational domain Ω into elements K (triangles, rectangles, tetrahedrons, bricks, etc.) such that

$$\bar{\Omega} = \bigcup_{K\in\mathcal{T}_h} K$$

and where $\mathcal{E}_h$ denotes the set of all the $(n-1)$-dimensional facets of all elements $K \in \mathcal{T}_h$. Here $h := \max_{K\in\mathcal{T}_h} h_K$ and $h_K = \operatorname{diam} K$ denote the global and local mesh size, respectively [6, 8]. We also define $\mathcal{T}_h^G = \{K \in \mathcal{T}_h \text{ and } K \subset G\}$ and $h_G = \max_{K\in\mathcal{T}_h^G} h_K$. A family of partitions $\mathcal{T}_h$ is said to be *regular*, if it satisfies the following condition [6]:

$$\exists\sigma > 0 \text{ such that } \; h_K/\tau_K > \sigma \quad \forall K \in \mathcal{T}_h,$$

where τ_K is the maximum diameter of the inscribed ball in $K \in \mathcal{T}_h$. A regular family of partitions $\mathcal{T}_h$ is called *quasi-uniform*, if it satisfies the following condition [6, 8]:

$$\exists\beta > 0 \quad \text{such that} \quad \max\{h/h_K,\ K \in \mathcal{T}_h\} \le \beta.$$

Based on the partition $\mathcal{T}_h$, we build the finite element space V_h of piecewise polynomial functions (see [6, 8]). In order to perform the error analysis, we define the following piecewise-type semi-norm for $v \in W^{j,p}(G) \cup V_h$ with $G \subseteq \Omega$:

$$|v|_{j,p,G,h} \;\; := \;\; \left(\sum_{K\in\mathcal{T}_h^G} \int_K \sum_{|\alpha|=j} |D^\alpha v|^p dK \right)^{\frac{1}{p}}, \qquad 1 \le p < \infty$$

and

$$|v|_{j,\infty,G,h} \;\; := \;\; \max_{K\in\mathcal{T}_h^G} |v|_{j,\infty,K}.$$

Then the corresponding norm can be defined by

$$\|v\|_{j,p,G,h} \;\; := \;\; \left(\sum_{i=0}^{j} |v|^p_{i,p,G,h} \right)^{\frac{1}{p}}$$

and

$$\|v\|_{j,\infty,G,h} \;\; := \;\; \max_{0\le i\le j} |v|_{i,\infty,\Omega,h}.$$

We drop G when $G = \Omega$. Throughout this paper, the symbol C (with or without subscript) stands for a positive generic constant which may attain different values at its different occurrences and which is independent of the mesh size h, but may depend on the exact solution u.

Theorem 2.1. *Assume $u \in W^{r+\delta,p}(G)$ $(\delta > 0)$ and that a multi-index γ with $|\gamma| = r$ exists such that $\|D^\gamma u\|_{0,p,G} > 0$ and $D^\gamma v_h = 0$ for all $K \in \mathcal{T}_h^G$ and for all $v_h \in V_h$. Then the following lower bound of the approximation error holds when the family of partitions $\{\mathcal{T}_h\}$ is quasi-uniform:*

$$\inf_{v_h\in V_h} \|u - v_h\|_{j,p,G,h} \ge C_1 h^{r-j}, \qquad 0 \le j \le r, \tag{2.1}$$

where $1 \le p \le \infty$ and C_1 is dependent on u.

Proof. We prove the result (2.1) by a process of reduction. The assumption that (2.1) is not correct means that for an arbitrarily small $\varepsilon > 0$ there exist a small enough h and $v_h \in V_h$ such that

$$\frac{\|u - v_h\|_{j,p,G,h}}{h^{r-j}} < \varepsilon. \tag{2.2}$$

Furthermore, let h be small enough to satisfy $h^\delta \|u\|_{r+\delta,p,G} < \varepsilon$. Now we can show that (2.2) leads to a contradiction.

Combining $u \in W^{r+\delta,p}(G)$, (2.2), the quasi-uniform property of $\mathcal{T}_h$, and the inverse inequality for finite element functions, we have

$$\begin{aligned}
|u - v_h|_{r,p,G,h} &\leq \|u - \Pi_h^r u\|_{r,p,G,h} + \|\Pi_h^r u - v_h\|_{r,p,G,h} \\
&\leq C_2 h^\delta \|u\|_{r+\delta,p,G} + C_3 h^{j-r} \|\Pi_h^r u - v_h\|_{j,p,G,h} \\
&\leq C_2 h^\delta \|u\|_{r+\delta,p,G} + C_3 h^{j-r} \|\Pi_h^r u - u\|_{j,p,G,h} \\
&\quad + C_3 h^{j-r} \|u - v_h\|_{j,p,G,h} \\
&\leq (C_2 + C_3 C_4) h^\delta \|u\|_{r+\delta,p,G} + C_2 \varepsilon \\
&\leq (C_2 + C_3 C_4 + C_3)\varepsilon, \qquad (2.3)
\end{aligned}$$

where $\Pi_h^r u$ denotes a piecewise r degree polynomial interpolant of u (discontinuous or continuous) such that

$$\|u - \Pi_h^r u\|_{\ell,p,G,h} \leq C_4 h^{r+\delta-\ell} \|u\|_{r+\delta,p,G}, \quad 0 \leq \ell \leq r.$$

Then the condition $D^\gamma v_h = 0$ leads to

$$\|D^\gamma u\|_{0,p,G} = \|D^\gamma (u - v_h)\|_{0,p,G,h} \leq |u - v_h|_{r,p,G,h} \leq C\varepsilon,$$

where $C = C_2 + C_3 C_4 + C_3$. This contradicts the inequality $\|D^\gamma u\|_{0,p,G} > 0$. Thus, the assumption (2.2) is not true. Therefore, the lower-bound result (2.1) holds, and the proof is complete. □

Remark 2.2. For $j = r$, we can improve the result (2.1) for $\delta = 0$, as we have

$$\|D^\gamma u\|_{0,p,G} = \|D^\gamma (u - v_h)\|_{0,p,G,h} \leq |u - v_h|_{r,p,G,h} \leq C\varepsilon$$

under the condition (2.2) for $j = r$. This contradicts the inequality $\|D^\gamma u\|_{0,p,G} > 0$.

The result in Theorem 2.1 can be extended to a regular family partitions and to more general Sobolev space norms.

Theorem 2.3. *Assume $u \in W^{r+\delta,p}(G)$ $(\delta > 0)$ and that a multi-index γ with $|\gamma| = r$ exists such that $\|D^\gamma u\|_{0,p,G} > 0$ and $D^\gamma v_h = 0$ for all $K \in \mathcal{T}_h^G$ and for all $v_h \in V_h$. Then we have the following lower bound of the approximation error when the family of partitions $\{\mathcal{T}_h\}$ is regular:*

$$\inf_{v_h \in V_h} \left(\sum_{K \in \mathcal{T}_h^G} h_K^{p(j-r)} \|u - v_h\|_{j,p,K}^p \right)^{\frac{1}{p}} \geq C_5, \quad 0 \leq j \leq r \tag{2.4}$$

and

$$\inf_{v_h \in V_h} \left(\sum_{K \in \mathcal{T}_h^G} h_K^{p\left((j-r)+n(\frac{1}{p}-\frac{1}{q})\right)} \|u - v_h\|_{j,q,K}^p \right)^{\frac{1}{p}} \geq C_6, \quad 0 \leq j \leq r, \tag{2.5}$$

where $1 \le p < \infty$, $1 \le q \le \infty$ ($W^{r+\delta,p}(G)$ can be embedded in $W^{j,q}(G)$) and where C_5 and C_6 are positive constants independent of mesh size h_G, but dependent on u.

Proof. We only prove the result (2.5), as (2.4) can be deduced directly from (2.5).

To get (2.5), we use a similar reduction as in the proof of Theorem 2.1. The assumption that (2.5) is not correct means that for an arbitrarily small $\varepsilon > 0$, there exist a small enough h_G and $v_h \in V_h$ such that

$$\left(\sum_{K \in \mathcal{T}_h^G} h_K^{p\left((j-r)+n(\frac{1}{p}-\frac{1}{q})\right)} \left\| u - v_h \right\|_{j,q,K}^p \right)^{\frac{1}{p}} \quad < \quad \varepsilon. \tag{2.6}$$

Furthermore, let h_G be small enough to satisfy $h_G^\delta \|u\|_{r+\delta,p,G} < \varepsilon$.

We will show that this assumption leads to a contradiction. Combining $u \in W^{r+\delta,p}(G)$, (2.6), and the inverse inequality for piecewise polynomial functions, we have

$$\begin{aligned}
|u - v_h|_{r,p,G,h} &\le \|u - \Pi_h^r u\|_{r,p,G,h} + \|\Pi_h^r u - v_h\|_{r,p,G,h} \\
\le \quad & C_7 h_G^\delta \|u\|_{r+\delta,p,G} + \left(\sum_{K \in \mathcal{T}_h^G} C_{8,K} h_K^{p\left((j-r)+n(\frac{1}{p}-\frac{1}{q})\right)} \left\| \Pi_h^r u - v_h \right\|_{j,q,K}^p \right)^{\frac{1}{p}} \\
\le \quad & C_7 h_G^\delta \|u\|_{r+\delta,p,G} + C_8 \left(\sum_{K \in \mathcal{T}_h^G} h_K^{p\left((j-r)+n(\frac{1}{p}-\frac{1}{q})\right)} \left\| \Pi_h^r u - v_h \right\|_{j,q,K}^p \right)^{\frac{1}{p}} \\
\le \quad & C_7 h_G^\delta \|u\|_{r+\delta,p,G} + C_8 \left(\sum_{K \in \mathcal{T}_h^G} h_K^{p\left((j-r)+n(\frac{1}{p}-\frac{1}{q})\right)} \left\| u - \Pi_h^r u \right\|_{j,q,K}^p \right)^{\frac{1}{p}} \\
& + C_8 \left(\sum_{K \in \mathcal{T}_h^G} h_K^{p\left((j-r)+n(\frac{1}{p}-\frac{1}{q})\right)} \left\| u - v_h \right\|_{j,q,K}^p \right)^{\frac{1}{p}} \\
\le \quad & (C_7 + C_8 C_9) h_G^\delta \|u\|_{r+\delta,p,G} + C_8 \varepsilon \\
\le \quad & (C_7 + C_8 C_9 + C_8)\varepsilon, \qquad (2.7)
\end{aligned}$$

where $\Pi_h^r u$ denotes a piecewise r degree polynomial interpolant of u (discontinuous or continuous) for which we have the following error estimate [6, 8]:

$$\|u - \Pi_h^r u\|_{\ell,q,K} \quad \le \quad C_{9,K} h_K^{r+\delta-\ell+n\left(\frac{1}{q}-\frac{1}{p}\right)} \|u\|_{r+\delta,p,K}, \qquad 0 \le \ell \le r, \quad \forall K \in \mathcal{T}_h.$$

In (2.7), the constants C_8 and C_9 defined, respectively, by

$$C_8 := \max_{K \in \mathcal{T}_h^G} C_{8,K} \quad \text{and} \quad C_9 := \max_{K \in \mathcal{T}_h^G} C_{9,K},$$

are dependent on the shape of the elements in $\mathcal{T}_h^G$. However, they are independent in terms of size; i.e., C_8 and C_9 are independent of h.

Then combining (2.7) and the condition $D^\gamma v_h = 0$ ($|\gamma| = r$) leads to

$$\|D^\gamma u\|_{0,p,G} = \|D^\gamma (u - v_h)\|_{0,p,G,h} \le |u - v_h|_{r,p,G,h} \le C\varepsilon,$$

where $C = C_7 + C_8 C_9 + C_8$. This contradicts the condition $|D^\gamma u|_{0,p,G} > 0$. Thus, the assumption (2.6) is not true. Hence, the lower-bound result (2.5) holds, and the proof is complete. □

Remark 2.4. Similar to Remark 2.2, when $j = r$, we can prove the results (2.4) and (2.5) for $\delta = 0$.

3. Lower bounds for a second-order elliptic problem

In this section, as an application of Theorems 2.1 and 2.3, we derive the lower bounds of the discretization error for a second-order elliptic problem and the corresponding eigenpair problem by the finite element method.

Here we are concerned with the Poisson problem

$$\left\{ \begin{array}{rcll} -\Delta u & = & f & \text{in } \Omega, \\ u & = & 0 & \text{on } \partial\Omega, \end{array} \right. \tag{3.1}$$

and the corresponding eigenpair problem:

Find (λ, u) such that $\|u\|_0 = 1$ and

$$\left\{ \begin{array}{rcll} -\Delta u & = & \lambda u & \text{in } \Omega, \\ u & = & 0 & \text{on } \partial\Omega. \end{array} \right. \tag{3.2}$$

Based on the partition $\mathcal{T}_h$ on $\bar{\Omega}$, we define a suitable finite element space V_h (conforming or nonconforming for the second-order elliptic problem) with piecewise polynomials of a degree less than r.

Then the finite element approximation of (3.1) consists of finding $u_h \in V_h$ such that

$$a_h(u_h, v_h) \quad = \quad (f, v_h) \quad \forall v_h \in V_h, \tag{3.3}$$

where

$$a_h(u_h, v_h) = \sum_{K \in \mathcal{T}_h} \int_K \nabla u_h \nabla v_h dK.$$

From the standard error estimate theory of the finite element method, it is known that the upper bound of the discretization error (see [6, 8]) given as

$$\|u - u_h\|_{\ell,p,h} \quad \leq \quad Ch^{s-\ell}\|u\|_{s,p}, \qquad 0 \leq \ell \leq 1, \quad 0 < s \leq r, \quad 1 < p < \infty, \tag{3.4}$$

holds.

From Theorems 2.1 and 2.3, we state the following lower-bound results of the discretization error.

Corollary 3.1. *Assume a subdomain* $G \subset\subset \Omega$ *exists such that* $u \in W^{r+\delta,p}(G)$ $(\delta > 0)$ *and assume a multi-index* γ *with* $|\gamma| = r$ *exists such that* $\|D^\gamma u\|_{0,p,G} > 0$ *and for all* $K \in \mathcal{T}_h^G$ *and for all* $v_h \in V_h$, $D^\gamma v_h = 0$. *If the family of partitions* $\{\mathcal{T}_h\}$ *is quasi-uniform, the finite element solution* $u_h \in V_h$ *in* (3.3) *has the following lower bound of the discretization error,*

$$\|u - u_h\|_{j,p,h} \geq C_{10} h^{r-j}, \qquad 0 \leq j \leq r, \tag{3.5}$$

where $1 \leq p \leq \infty$, C_{10} *is a positive constant dependent on* u *and the error estimate* (3.4) *is optimal for* $s = r$ *and* $0 \leq \ell \leq 1$.

Proof. First we have the following property:

$$\frac{\|u - u_h\|_{j,p,h}}{h^{r-j}} \quad \geq \quad \frac{\|u - u_h\|_{j,p,G,h}}{h^{r-j}} \geq \inf_{v_h \in V_h} \frac{\|u - v_h\|_{j,p,G,h}}{h^{r-j}}.$$

Therefore the desired result (3.5) can be directly deduced by (2.1). $\square$

Remark 3.1. The interior regularity result $u \in W^{r+\delta,p}(G)$ for a subdomain $G \subset\subset \Omega$ and $\delta > 0$ for the elliptic problem (3.1) can be obtained if the right-hand side f is smooth enough (see [11, Theorem 8.10]).

Corollary 3.2. *Assume a subdomain $G \subset\subset \Omega$ exists such that $u \in W^{r+\delta,p}(G)$ ($\delta > 0$) and assume a multi-index γ with $|\gamma| = r$ exists such that $\|D^\gamma u\|_{0,p,G} > 0$ and for all $K \in \mathcal{T}_h^G$ and for all $v_h \in V_h$, $D^\gamma v_h = 0$. If the family of partitions $\{\mathcal{T}_h\}$ is regular, the finite element solution $u_h \in V_h$ in (3.3) has the following lower bound of the discretization error:*

$$\left(\sum_{K\in\mathcal{T}_h^G} h_K^{p(j-r)} \|u-u_h\|_{j,p,K}^p\right)^{\frac{1}{p}} \geq C_{11}, \qquad 0 \leq j \leq r \tag{3.6}$$

and

$$\left(\sum_{K\in\mathcal{T}_h^G} h_K^{p\left((j-r)+n(\frac{1}{p}-\frac{1}{q})\right)} \|u-u_h\|_{j,q,K}^p\right)^{\frac{1}{p}} \geq C_{12}, \qquad 0 \leq j \leq r, \tag{3.7}$$

where $1 \leq p < \infty$, $1 \leq q \leq \infty$ ($W^{r+\delta,p}(G)$ can be embedded in $W^{j,q}(G)$) and where C_{11} and C_{12} are positive constants dependent on u.

Proof. The proof can be given using the property

$$\sum_{K\in\mathcal{T}_h^G} h_K^{p\left((j-r)+n(\frac{1}{p}-\frac{1}{q})\right)} \|u-u_h\|_{j,q,K}^p \geq \inf_{v_h\in V_h} \sum_{K\in\mathcal{T}_h^G} h_K^{p\left((j-r)+n(\frac{1}{p}-\frac{1}{q})\right)} \|u-v_h\|_{j,q,K}^p$$

and Theorem 2.3. ☐

Remark 3.2. In [7], the lower bound of the discretization error by the Wilson element is analyzed under the conditions of the rectangular partition and the regularity $u \in W^{3,\infty}(\Omega)$.

Now let us consider the lower-bound analysis of the eigenpair problem (3.2) by the finite element method. The finite element approximation $(\lambda_h, u_h) \in \mathcal{R} \times V_h$ of (3.2) satisfies $\|u_h\|_0 = 1$ and

$$a_h(u_h, v_h) \quad = \quad \lambda_h(u_h, v_h) \quad \forall v_h \in V_h. \tag{3.8}$$

For the eigenfunction approximation u_h in (3.8), the following lower-bound results hold.

Corollary 3.3. *Assume a multi-index γ with $|\gamma| = r$ exists such that $D^\gamma v_h = 0$ for all $K \in \mathcal{T}_h$ and for all $v_h \in V_h$. If the family of partitions $\{\mathcal{T}_h\}$ is quasi-uniform, the eigenpair approximation $(\lambda_h, u_h) \in \mathcal{R} \times V_h$ in (3.8) satisfies this lower bound of the discretization error*

$$\|u-u_h\|_{j,p,h} \geq C_{13} h^{r-j}, \qquad 0 \leq j \leq r, \tag{3.9}$$

where $1 \leq p \leq \infty$ and C_{13} is a positive constant dependent on u.

Furthermore, if the family of partitions $\{\mathcal{T}_h\}$ is regular, (λ_h, u_h) has the following lower bounds of the discretization error

$$\left(\sum_{K\in\mathcal{T}_h^G} h_K^{p(j-r)} \|u-u_h\|_{j,p,K}^p\right)^{\frac{1}{p}} \geq C_{14}, \qquad 0 \leq j \leq r \tag{3.10}$$

and

$$\left(\sum_{K\in\mathcal{T}_h^G} h_K^{p\left((j-r)+n(\frac{1}{p}-\frac{1}{q})\right)}\left\|u-u_h\right\|_{j,q,K}^p\right)^{\frac{1}{p}} \geq C_{15}, \qquad 0\leq j\leq r, \tag{3.11}$$

where $1\leq p<\infty$, $1\leq q\leq\infty$, C_{14} *and* C_{15} *are positive constants dependent on* u.

Proof. First, it is easy to show that the eigenfunctions of problem (3.2) cannot be polynomial of the bounded degree on any subdomain $G\subset\subset\Omega$. We prove this by a process of reduction. Assume that the exact eigenfunction is a polynomial function and that $u\in\mathcal{P}_\ell(G)$ for some integer $\ell>0$. Directly from the definition of problem (3.2), we have

$$-\Delta^{\lceil\frac{\ell}{2}\rceil}u \;=\; (-1)^{\lceil\frac{\ell}{2}\rceil-1}\lambda^{\lceil\frac{\ell}{2}\rceil}u, \tag{3.12}$$

where $\lceil\frac{\ell}{2}\rceil$ denotes the smallest integer not smaller than $\frac{\ell}{2}$. As $-\Delta^{\lceil\frac{\ell}{2}\rceil}u=0$, we have $u=0$ on G. This means the exact eigenfunction cannot be polynomial of the bounded degree and has the property of

$$|u|_{r,p,G}>0.$$

The proof of this corollary can be obtained with the same argument as in the proof of Corollaries 3.1 and 3.2. □

Now, we present some conforming and nonconforming elements that yield the lower bound of the discretization error with the help of Corollaries 3.1, 3.2, and 3.3.

In order to describe the results, we introduce the index set

$$Ind_r := \left\{\text{multi index } \alpha \text{ with } |\alpha|=r\right\}. \tag{3.13}$$

First we obtain the lower-bound results for the standard Lagrange-type elements

$$V_h \;=\; \left\{v_h|_K\in\mathcal{P}_\ell(K) \text{ or } \mathcal{Q}_\ell(K)\ \ \forall K\in\mathcal{T}_h\right\}, \tag{3.14}$$

where $\mathcal{P}_\ell(K)$ denotes the space of polynomials with a degree not greater than ℓ and $\mathcal{Q}_\ell(K)$ denotes the space of polynomials with a degree not greater than ℓ for each variable. From Corollaries 3.1, 3.2, and 3.3, the lower-bound results in this section hold with $r=\ell+1$ and $\gamma\in Ind_r$ for the $\mathcal{P}_\ell(K)$ case and they hold with $r=\ell+1$ and $\gamma\in Ind_r\backslash Ind_{Q,\ell}$ for the $\mathcal{Q}_\ell(K)$ case with

$$Ind_{Q,\ell} := \left\{\text{multi index } \alpha \text{ with } \alpha_i\leq\ell\right\}.$$

Then it is also easy to check the lower-bound results for four types of nonconforming elements: the Crouzeix–Raviart (CR), the Extension of Crouzeix–Raviart (ECR), the Q_1 rotation ($Q_1^{\rm rot}$), and the Extension of the Q_1 rotation ($EQ_1^{\rm rot}$):

- The CR element space proposed by Crouzeix and Raviart [9] is defined on simplicial partitions by

$$V_h \;=\; \Big\{v\in L^2(\Omega):\ v|_K\in\mathcal{P}_1(K),\\ \int_F v|_{K_1}ds=\int_F v|_{K_2}ds \text{ if } K_1\cap K_2=F\in\mathcal{E}_h\Big\}.$$

  The lower-bound result holds with $r=2$ and $\gamma\in Ind_2$.

- The ECR element space, proposed by Hu, Huang, and Lin [10] and by Lin, Xie, Luo, and Li [15], is defined on simplicial partitions by

$$V_h = \Big\{ v \in L^2(\Omega): \ v|_K \in \mathcal{P}_{ECR}(K), \\ \int_F v|_{K_1} ds = \int_F v|_{K_2} ds \text{ if } K_1 \cap K_2 = F \in \mathcal{E}_h \Big\},$$

where $\mathcal{P}_{ECR}(K) = \mathcal{P}_1(K) + \text{span}\big\{ \sum_{i=1}^n x_i^2 \big\}$. The lower-bound result holds with $r = 2$ and γ with $\gamma_i = 1, \gamma_j = 1$, $1 \le i < j \le n$.
- The Q_1^{rot} element space, proposed by Rannacher and Turek [17] and by Arbogast and Chen [2], is defined on n-dimensional block partitions by

$$V_h = \Big\{ v \in L^2(\Omega) : v|_K \in Q_{\text{Rot}}(K), \\ \int_F v|_{K_1} ds = \int_F v|_{K_2} ds \text{ if } K_1 \cap K_2 = F \in \mathcal{E}_h \Big\},$$

where $Q_{\text{Rot}}(K) = \mathcal{P}_1(K) + \text{span}\big\{ x_i^2 - x_{i+1}^2 \mid 1 \le i \le n-1 \big\}$. The lower-bound result holds with $r = 2$ and γ with $\gamma_i = 1, \gamma_j = 1$, $1 \le i < j \le n$.
- The EQ_1^{rot} element space, proposed by Lin, Tobiska, and Zhou [14], is defined on n-dimensional block partitions by

$$V_h = \Big\{ v \in L^2(\Omega) : v|_K \in Q_{\text{ERot}}(K), \\ \int_F v|_{K_1} ds = \int_F v|_{K_2} ds \text{ if } K_1 \cap K_2 = F \in \mathcal{E}_h \Big\},$$

where $Q_{\text{Rot}}(K) = \mathcal{P}_1(K) + \text{span}\big\{ x_i^2 \mid 1 \le i \le n \big\}$. The lower-bound result holds with $r = 2$ and γ with $\gamma_i = 1, \gamma_j = 1$, $1 \le i < j \le n$.

All lower bounds of the above four examples are sharp, if the solution is smooth enough. For other types of finite elements, we could also obtain the lower-bound results with the corresponding r and γ as in this section.

4. Lower bounds for the $2m$-th order elliptic problem

We consider similar lower bounds of the discretization error for the $2m$-th order elliptic problem and the corresponding eigenpair problem by the finite element method. This is a natural generalization of the results in Section 3.

The $2m$-th order Dirichlet elliptic problem for a given integer $m \ge 1$ is defined as

$$\begin{cases} (-1)^m \Delta^m u = f & \text{in } \Omega, \\ \frac{\partial^j u}{\partial^j \nu} = 0 & \text{on } \partial\Omega \text{ and } 0 \le j \le m-1, \end{cases} \tag{4.1}$$

where ν denotes the unit outer normal. The corresponding weak form of problem (4.1) is to seek $u \in H_0^m(\Omega)$ such that

$$a(u, v) = (f, v) \quad \forall v \in H_0^m(\Omega), \tag{4.2}$$

where

$$a(u, v) = \int_\Omega \sum_{|\alpha| = m} D^\alpha u D^\alpha v \, d\Omega.$$

Based on the partition $\mathcal{T}_h$ of $\bar{\Omega}$, we build a suitable finite element space V_h (conforming or nonconforming for the $2m$-th order elliptic problem) with a piecewise polynomial of a degree less than r. The finite element approximation of (4.1) is to seek $u_h \in V_h$, thus satisfying

$$a_h(u_h, v_h) \quad = \quad (f, v_h) \quad \forall v_h \in V_h, \tag{4.3}$$

where

$$a_h(u_h, v_h) = \sum_{K\in\mathcal{T}_h} \int_K \sum_{|\alpha|=m} D^\alpha u_h D^\alpha v_h dK.$$

We also consider the corresponding $2m$-th order elliptic eigenpair problem:

Find $(\lambda, u) \in \mathcal{R} \times H_0^m(\Omega)$ such that $\|u\|_0 = 1$ and

$$a(u, v) \quad = \quad \lambda(u, v) \quad \forall v \in H_0^m(\Omega). \tag{4.4}$$

In this section, we assume that the following upper bound of the discretization error holds:

$$\|u - u_h\|_{m,h} \quad \leq \quad Ch^{s-m}\|u\|_s, \quad 0 < s \leq r. \tag{4.5}$$

Similar to Corollaries 3.1 and 3.2, the finite element approximation u_h yields the following lower-bound results.

Corollary 4.1. *Assume a subdomain $G \subset\subset \Omega$ exists such that $u \in W^{r+\delta,p}(G)$ ($\delta > 0$) and a multi-index γ with $|\gamma| = r$ exists such that $\|D^\gamma u\|_{0,p,G} > 0$ and for all $K \in \mathcal{T}_h^G$ and for all $v_h \in V_h$, $D^\gamma v_h = 0$. If the family of partitions $\{\mathcal{T}_h\}$ is quasi-uniform, the finite element solution $u_h \in V_h$ in (4.3) has the lower bound of the discretization error*

$$\|u - u_h\|_{j,p,h} \geq C_{16}h^{r-j}, \quad 0 \leq j \leq r, \tag{4.6}$$

where $1 \leq p \leq \infty$, C_{16} is a positive constant dependent on u, and the error estimate (4.5) is optimal for $s = r$.

Proof. First we have the following property:

$$\frac{\|u - u_h\|_{j,p,h}}{h^{r-j}} \quad \geq \quad \frac{\|u - u_h\|_{j,p,G,h}}{h^{r-j}} \geq \inf_{v_h\in V_h} \frac{\|u - v_h\|_{j,p,G,h}}{h^{r-j}}.$$

Therefore the desired result (4.6) can be directly deduced by (2.1). □

Remark 4.1. The interior regularity result $u \in W^{r+\delta,p}(G)$ for a subdomain $G \subset\subset \Omega$ and $\delta > 0$ for problem (4.1) can be obtained if the right-hand side f is smooth enough [12, Theorem 7.1.2].

Corollary 4.2. *Assume a subdomain $G \subset\subset \Omega$ exists such that $u \in W^{r+\delta,p}(G)$ ($\delta > 0$) and a multi-index γ with $|\gamma| = r$ exists such that $\|D^\gamma u\|_{0,p,G} > 0$ and for all $K \in \mathcal{T}_h^G$ and for all $v_h \in V_h$, $D^\gamma v_h = 0$. If the family of partitions $\{\mathcal{T}_h\}$ is regular, the finite element solution $u_h \in V_h$ in (3.3) has the following lower bound of the discretization error*

$$\left(\sum_{K\in\mathcal{T}_h^G} h_K^{p(j-r)}\|u - u_h\|_{j,p,K}^p\right)^{\frac{1}{p}} \geq C_{17}, \quad 0 \leq j \leq r \tag{4.7}$$

and

$$\left(\sum_{K\in\mathcal{T}_h^G} h_K^{p\left((j-r)+n(\frac{1}{p}-\frac{1}{q})\right)}\|u - u_h\|_{j,q,K}^p\right)^{\frac{1}{p}} \geq C_{18}, \quad 0 \leq j \leq r, \tag{4.8}$$

where $1 \le p < \infty$, $1 \le q \le \infty$ ($W^{r+\delta,p}(G)$ *can be embedded in* $W^{j,q}(G)$) *and where* C_{17} *and* C_{18} *are positive constants independent of mesh size* h_G, *but dependent on* u.

Now we introduce the corresponding lower-bound analysis of the eigenpair problem (4.4). We define the corresponding discrete eigenpair problem in the finite element space:

Find $(\lambda_h, u_h) \in \mathcal{R} \times V_h$ such that $\|u_h\|_0 = 1$ and

$$a_h(u_h, v_h) = \lambda_h(u_h, v_h) \quad \forall v_h \in V_h. \tag{4.9}$$

The eigenfunction approximation u_h in (4.9) also gives the lower-bound results as follows.

Corollary 4.3. *Assume a multi-index* γ *with* $|\gamma| = r$ *exists such that* $D^\gamma v_h = 0$ *for all* $K \in \mathcal{T}_h$ *and for all* $v_h \in V_h$. *If the family of partitions* $\{\mathcal{T}_h\}$ *is quasi-uniform, the eigenpair approximation* $(\lambda_h, u_h) \in \mathcal{R} \times V_h$ *in* (4.9) *yields the following lower bound of the discretization error:*

$$\|u - u_h\|_{j,p,h} \ge C_{19} h^{r-j}, \quad 0 \le j \le r, \tag{4.10}$$

where $1 \le p \le \infty$ *and* C_{19} *is a positive constant independent of mesh size.*

Furthermore, if the family of partitions $\{\mathcal{T}_h\}$ *is only regular,* (λ_h, u_h) *has the following lower bounds of the discretization error*

$$\left(\sum_{K \in \mathcal{T}_h^G} h_K^{p(j-r)} \|u - u_h\|_{j,p,K}^p \right)^{\frac{1}{p}} \ge C_{20}, \quad 0 \le j \le r \tag{4.11}$$

and

$$\left(\sum_{K \in \mathcal{T}_h^G} h_K^{p\left((j-r)+n(\frac{1}{p}-\frac{1}{q})\right)} \|u - u_h\|_{j,q,K}^p \right)^{\frac{1}{p}} \ge C_{21}, \quad 0 \le j \le r, \tag{4.12}$$

where $1 \le p < \infty$, $1 \le q \le \infty$ *and where* C_{20} *and* C_{21} *are positive constants dependent on* u.

Proof. Similar to Corollary 3.3, it is easy to obtain that the eigenfunction of problem (4.4) cannot be a polynomial function of the bounded degree on any subdomain $G \subset\subset \Omega$. This means the eigenfunction has the following property:

$$|u|_{r,p,G} > 0.$$

Then the proof can be obtained with the same argument used for the proof of Corollary 4.1 and for Corollary 4.2. □

Now, we give some types of conforming and nonconforming elements that can produce the lower bound of the discretization error with the help of Corollaries 4.1, 4.2, and 4.3.

First we would like to restate that for the two-dimensional case ($n = 2$) there exist elements such as the Argyris and Hsieh–Clough–Tocher elements [8] that yield lower-bound results from Corollaries 4.1, 4.2, and 4.3 for the biharmonic problem. The lower-bound results in this section hold for the Argyris element with $m = 2$, $r = 6$, $\gamma \in Ind_6$ and for the Hsieh–Clough–Tocher element with $m = 2$, $r = 4$, $\gamma \in Ind_4$.

Next, we consider a family of nonconforming elements referred to as MWX (Morley-Wang-Xu) as proposed by Wang and Xu [18] and apply it to the $2m$-th order elliptic problem and the corresponding eigenpair problem under consideration. The MWX element with $n \geq m \geq 1$ is the triple $(K, \mathcal{P}_K, D_K)$, where K is an n-simplex and $\mathcal{P}_K = \mathcal{P}_m(K)$. For a description of the set D_K of degrees of freedom, see [18].

In order to understand this element, we list some special cases as in [18] for $1 \leq m \leq 3$. If $m = 1$ and $n = 1$, we obtain the well-known conforming linear elements. This is the only conforming element in this family of elements. For $m = 1$ and $n \geq 2$, we obtain the well-known nonconforming linear element (CR). If $m = 2$, we recover the well-known nonconforming Morley element for $n = 2$ and its generalization to $n \geq 2$ (see Wang and Xu [19]). For $m = 3$ and $n = 3$, we obtain a cubic element on a simplex that has 20 degrees of freedom.

Based on the above description of the MWX element, we know that

$$|v_h|_{1+m,p,h} \equiv 0 \qquad \forall v_h \in V_h.$$

Then with the help of Corollaries 4.1, 4.2, and 4.3, we get the lower-bound results in this section with $r = m + 1$ and $\gamma \in Ind_r$.

We can also obtain the lower bound results in this section for other types of elements with suitable r and γ for the $2m$-th order elliptic problem (4.1) and the corresponding eigenvalue problem (4.4).

5. Concluding remarks

In this paper, a type of lower-bound results of the error by piecewise polynomial approximation is presented. As applications, we give the lower bounds of the discretization error for the second-order elliptic and $2m$-th order elliptic problem by finite element methods. From the analysis, the idea and methods here can be extended to other problems and numerical methods that are based on the piecewise polynomial approximation.

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