LOWER SEMICONTINUITY OF INTEGRAL FUNCTIONALS

BY

LEONARD D. BERKOVITZ(1)

ABSTRACT. It is shown that the integral functional $I(y,z) = \int_G f(t,y(t),z(t))\,d\mu$ is lower semicontinuous on its domain with respect to the joint strong convergence of $y_k \to y$ in $L_p(G)$ and the weak convergence of $z_k \to z$ in $L_q(G)$, where $1 \le p \le \infty$ and $1 \le q \le \infty$, under the following conditions. The function f: $(t,x,w) \to f(t,x,w)$ is measurable in t for fixed (x,w), is continuous in (x,w) for a.e. t, and is convex in w for fixed (t,x).

1. **Introduction.** To show that a functional attains a minimum on a given subset $\mathfrak{S}$ of its domain of definition $\mathfrak{D}$ the following procedure is commonly used. One first shows that for an appropriate topology on $\mathfrak{D}$ the set $\mathfrak{S}$ is compact and then one shows that the functional is lower semicontinuous on $\mathfrak{S}$ with respect to the topology in question. In this paper we shall consider the lower semicontinuity of certain integral functionals that arise in various minimization problems.

In [3] F. Browder studied the weak sequential lower semicontinuity of the functional

$$(1.1) \qquad J(\phi) = \int_G f(t,(M\phi)(t),(L\phi)(t))\,d\mu,$$

where G is a bounded open set in R^n, μ is a finite regular Borel measure on G, ϕ is an element of a Banach space $\mathfrak{F}$, L is a bounded linear mapping of $\mathfrak{F}$ into $L_p(G)$ and M is a compact linear mapping of $\mathfrak{F}$ into $L_q(G)$. Browder obtained two principal results [3, Theorems 4 and 5]. The first result involved certain Hölder type conditions on f and has applications to elliptic boundary value problems. The second result involved certain growth conditions on f and simplified some proofs and strengthened some theorems of Fichera [4] as well as some classical results in the calculus of variations. For further bibliographic details and discussion of applications see [3] and [4].

B. T. Poljak [9] studied the semicontinuity of the functional

$$(1.2) \qquad I(y,z) = \int_G f(t,y(t),z(t))\,d\mu$$

with respect to convergence of sequences $(y_k,z_k) \to (y,z)$, where the convergence of y_k is convergence in $L_p(G)$ and the convergence of z_k is weak

convergence in $L_q(G)$, $1 \leq p \leq \infty$, $1 \leq q \leq \infty$. Here p and q need not be conjugate indices. Poljak's result [9, Theorem 12] was an improvement over Browder's Theorem 4 in that weaker continuity properties on f and less restrictive Hölder type conditions were assumed.

In this paper we shall also study the semicontinuity of (1.2) with respect to the convergence studied in [9]. Our result in Theorem 1 below will be appreciably stronger than that of Poljak (and hence stronger than Theorem 4 of [3]) in that no Hölder type condition on f will be assumed. Theorem 1 below will also include Theorem 5 of [3] and a classical theorem of Morrey [7, Theorem 4.1]. It can also be used to generalize some of the existence theorems of Lions [6] for problems of optimal control. Some of the principal ideas of the proof were used by us in investigations concerned with the existence of optimal controls in nonlinear control problems [1], [2].

A result similar to our Corollary 1, with $p = q = 2$, was given by V. I. Plotnikov [8, Lemma A]. Our proof is completely different from Plotnikov's and is much simpler than his.

2. Notation and definitions. We shall use single letters to denote vectors, we shall use subscripts to distinguish vectors, and we shall use superscripts to denote components of vectors. The letter t will denote a vector $(t^1, \ldots, t^\nu)$ in real euclidean space R^ν, $\nu \geq 1$, the letter x will denote a vector $(x^1, \ldots, x^n)$ in R^n, and the letter w a vector in R^m, $m \geq 1$. The euclidean norm of a vector will be denoted by $|x|$. The inner product of two vectors x_1 and x_2 will be written as $\langle x_1, x_2 \rangle$. Thus $|x| = (\langle x, x \rangle)^{1/2}$.

Let $f: (t, x, w) \rightarrow f(t, x, w)$ be a real valued function defined in $R^\nu \times R^n \times R^m$. Let G be a bounded open region on the t-space R^ν and let $d\mu$ be a finite regular Borel measure on G. Let $\mathcal{Y}$ denote the Banach space of functions $y = (y^1, \ldots, y^n)$ defined on G with range in R^n whose components y^i are in $L_{p_i}(G)$, with respect to $d\mu$, where $1 \leq p_i \leq \infty$. The norm of an element $y = (y^1, \ldots, y^n)$ in $\mathcal{Y}$ is given by

$$\|y\| = \left(\sum_{i=1}^n \|y^i\|_{p_i}^2 \right)^{1/2},$$

where $\|y^i\|_{p_i}$ denotes the $L_{p_i}(G)$ norm of y^i. Let $\mathcal{Z}$ denote the Banach space of functions $z = (z^1, \ldots, z^m)$ defined on G with range in R^m whose components are in $L_{q_i}(G)$ with respect to $d\mu$, $1 \leq q_i \leq \infty$, and where $\|z\|$ is defined in a manner analogous to $\|y\|$.

A sequence $\{(y_k, z_k)\}$ in $\mathcal{Y} \times \mathcal{Z}$ is said to converge to an element (y, z) in $\mathcal{Y} \times \mathcal{Z}$ strongly in y and weakly in z if $\lim(y_k, z_k) = (y, z)$ in the product topology of $\mathcal{Y} \times \mathcal{Z}$ determined by the strong topology of $\mathcal{Y}$ and the weak topology of $\mathcal{Z}$.

Let $\mathcal{D}$ denote the set of elements (y, z) in $\mathcal{Y} \times \mathcal{Z}$ for which the integral in (1.2) exists and is finite. Then $\mathcal{D}$ is the domain of the functional I. We henceforth assume that $\mathcal{D}$ is not empty.

3. Statement and discussion of theorem.

Theorem 1. *Let f be continuous in (x, w) for a.e. t in G and let f be measurable on G for each (x, w) in R^{n+m}. For each (t, x) let f be a convex function of w. Let ψ be a function in $L_1(G)$ such that for all (x, w) in R^{n+m} and a.e. t in G*

$$(3.1) \qquad\qquad f(t, x, w) \geq \psi(t).$$

If $\{(y_k, z_k)\}$ is a sequence in $\mathfrak{D}$ converging to an element (y, z) in $\mathfrak{Y} \times \mathfrak{Z}$ strongly in y and weakly in z and if $\liminf I(y_k, z_k) < \infty$, then (y, z) is in $\mathfrak{D}$ and

$$(3.2) \qquad\qquad \liminf I(y_k, z_k) \geq I(y, z).$$

Note that "(y, z) is in $\mathfrak{D}$" is a conclusion of the theorem and not an assumption.

If we assume (y, z) to be in $\mathfrak{D}$ and if $\liminf I(y_k, z_k) = +\infty$, then the relation (3.2) automatically holds. Thus, we have the following corollary.

Corollary. *The functional I is sequentially lower semicontinuous on $\mathfrak{D}$ with respect to strong convergence in y and weak convergence in z.*

Remark 1. The condition (3.1) is used in the proof to show that $\liminf I(y_k, z_k) > -\infty$ and to justify the use of Fatou's lemma at a certain point. Upon reading the proof it will be clear that the condition (3.1) can be replaced by certain weaker conditions as in Poljak's paper. For example if all $p_i < \infty$ and all $q_i < \infty$ we can replace (3.1) by the condition

$$f(t, x, w) \geq \psi(t) - \beta\left(\sum_{i=1}^{n} |x^i|^{p_i} + \sum_{i=1}^{m} |w^i|^{q_i}\right),$$

where $\beta > 0$. If all $p_i = \infty$ and all $q_i < \infty$, then we may replace (3.1) by the following condition. For each $M > 0$, there exist a function ψ_M in $L_1(G)$ and a constant $\beta_M > 0$ such that for all x in R^n with $|x| \leq M$, all w in R^m and a.e. t in G

$$f(t, x, w) \geq \psi_M(t) - \beta_M \sum_{i=1}^{m} |w^i|^{q_i}.$$

A similar condition can be imposed if all $q_i = \infty$ and all $p_i < \infty$. Finally, if all $p_i = \infty$ and all $q_i = \infty$ then we need merely require that for each $M > 0$ there is a function ψ_M in $L_1(G)$ such that (3.1) holds with ψ replaced by ψ_M for all $|x| \leq M$, $|w| \leq M$, and a.e. t in G.

Remark 2. In [7, Theorem 4.1] Morrey showed that for f continuous, f convex in w and appropriate assumptions on the boundary of G, the functional $\int_G f(t, \phi, \nabla\phi)\, dt$ is sequentially lower semicontinuous on its domain $\mathfrak{D}$ with respect to weak convergence of sequences $\{\phi_k\}$ in $H_1^1(G)$. If $\phi_k \to \phi$ weakly in $H_1^1(G)$ then $\nabla\phi_k \to \nabla\phi$ weakly in $L_1(G)$. Also, for the region G studied by Morrey, $\phi_k \to \phi$ weakly in $H_1^1(G)$ implies $\phi_k \to \phi$ in $L_1(G)$. Thus, Morrey's result follows from our corollary.

Remark 3. In the introduction we discussed the relationship of our results to those of Poljak [9] and hence also to Theorem 4 of [3]. In Theorem 5 of [3] the integrand f in (1.1) is assumed to be continuous in (t, x, w) and convex in w, but is not assumed to satisfy a Hölder condition. Instead, it is assumed that there is a function f_0 that is convex in w such that $f_0(w)/|w| \to \infty$ as $|w| \to \infty$ and such that $f(t, x, w) \geq f_0(w)$. It is not hard to see that this assumption implies that f is bounded from below by an integrable function as in (3.1). Since in (1.1) the mapping L is bounded and the mapping M is compact it follows that if $\phi_k \to \phi$ weakly in $\mathfrak{F}$ then $L\phi_k \to L\phi$ weakly in $L_q(G)$ and $M\phi_k \to M\phi$ strongly in $L_p(G)$. Hence if for ϕ in $\mathfrak{F}$ we set $z = L\phi$ and $y = M\phi$, we get Theorem 5 of [3].

Remark 4. In some problems of optimal control it is required to minimize the functional (1.2) subject to contol constraints involving z (the control) and the so-called "state equation" constraints, which involve z, y (the state) and differential operators applied to y. In problems in which the state equations are linear in w (the control variable) and x (the state variable) and the differential operators are linear, a semicontinuity theorem for the functional is quite often used as one of the cornerstones of the proof.

In his book, Lions [6] usually considers quadratic functionals in which f is essentially of the form

$$f(t, x, w) = \langle x, Q(t)x \rangle + \langle b, x \rangle + \langle w, R(t)w \rangle$$

and uses well-known results about the lower semicontinuity of such functionals. Theorem 1 enables one to extend many of the results of Lions to more general functionals (1.2) in which f satisfies the hypotheses of Theorem 1. Other examples of control problems in which one requires the semicontinuity of the functional (1.2) can be found in [8] and [9].

4. Proof of Theorem 1. The following lemma is crucial to the proof of Theorem 1.

Lemma 1. *Let $g: (t, x) \to g(t, x)$ be a mapping from $G \times R^r \to R^1$ that is continuous in x for a.e. t in G and measurable in t for fixed x in R^r. Let $\{v_k\}$ and $\{u_k\}$ be sequences in $\mathfrak{B} = \prod_{i=1}^r L_{s_i}(G)$ such that $\|u_k\| \leq R$ and $\|v_k\| \leq R$ for some $R > 0$ and such that $(u_k - v_k) \to 0$ in measure on G. Then*

$$g(t, u_k(t)) - g(t, v_k(t)) \to 0$$

in measure on G.

A proof of this lemma can be found in [5, p. 355]. In [5] it is assumed that all of the s_i are equal, say $s_i = s$. To obtain the statement used here take $s = \min s_i$ and recall that G has finite measure.

In the course of the proof of Theorem 1 we shall select subsequences of various sequences. Unless stated otherwise we shall relabel the subsequence with the labeling of the original sequence.

Let

$$(4.1) \qquad \gamma = \liminf I(y_k, z_k).$$

By virtue of (3.1), $\gamma > -\infty$. Since by hypothesis $\gamma < \infty$, we have that γ is finite.
We first select a subsequence $\{(y_k, z_k)\}$ such that

$$(4.2) \qquad \lim I(y_k, z_k) = \gamma.$$

From the definition of $\mathcal{Y}$ in §2 it follows that the sequences of components $\{y_k^i\}$ converge in $L_{p_i}(G)$ to y^i. Hence $y_k \to y$ in measure on G.

Since $y_k \to y$ strongly in $\mathcal{Y}$ and $z_k \to z$ weakly in $\mathcal{Z}$, the sequences $\{(y_k, z_k)\}$ and $\{(y, z_k)\}$ are bounded in norm in the space $\mathcal{Y} \times \mathcal{Z}$. Moreover if $v_k = (y_k, z_k)$ and $u_k = (y, z_k)$, then $v_k - u_k \to 0$ in measure. Let

$$(4.3) \qquad h_k(t) = f(t, y_k(t), z_k(t)) - f(t, y(t), z_k(t));$$

then by Lemma 1, $h_k \to 0$ in measure on G.

Since $h_k \to 0$ in measure, there exists a subsequence $\{(y_k, z_k)\}$ such that for a.e. t in G

$$(4.4) \qquad h_k(t) \to 0.$$

Since $z_k \to z$ weakly in $\mathcal{Z}$, we obtain the following statement from Mazur's theorem. For each integer j there exists an integer n_j, a set of integers $i = 1, \ldots, k$, where $k = k(j)$ depends on j, and a set of numbers $\alpha_{1j}, \ldots, \alpha_{kj}$ satisfying

$$(4.5) \qquad \alpha_{ij} \geq 0, \qquad \sum_{i=1}^{k} \alpha_{ij} = 1,$$

such that $n_{j+1} > n_j + k(j)$ and

$$(4.6) \qquad \left\| z - \sum_{i=1}^{k} \alpha_{ij} z_{n_j+i} \right\| < \frac{1}{j}.$$

Let

$$(4.7) \qquad \psi_j = \sum_{i=1}^{k} \alpha_{ij} z_{n_j+i}.$$

From (4.6) we have that $\psi_j \to z$ in $\mathcal{Z}$. Hence there is a subsequence $\{\psi_j\}$ such that

$$(4.8) \qquad \psi_j(t) \to z(t) \quad \text{a.e.}$$

We now suppose that (4.7) is the subsequence for which (4.8) holds. Corresponding to (4.7) we define a sequence $\{\lambda_j\}$ as follows:

$$(4.9) \qquad \lambda_j(t) = \sum_{i=1}^{k} \alpha_{ij} f(t, y_{n_j+i}(t), z_{n_j+i}(t)),$$

where for each j the numbers α_{ij}, the indices $n_j + i$, and hence the functions z_{n_j+i}, are as in (4.7). Define $\lambda(t) = \liminf \lambda_j(t)$. From (4.9), (4.5), and (3.1) we get that for each j and a.e. t in G

$$(4.10) \qquad \lambda_j(t) \geq \sum_{i=1}^{k} \alpha_{ij} \psi(t) = \psi(t).$$

We may therefore apply Fatou's lemma and get that

$$\int_G \lambda(t)\, d\mu \leq \liminf \int_G \lambda_j(t)\, d\mu$$

$$= \liminf \sum_{i=1}^{k} \alpha_{ij} \int_G f(t, y_{n_j+i}(t), z_{n_j+i}(t))\, d\mu$$

$$= \liminf \sum_{i=1}^{k} \alpha_{ij} I(y_{n_j+i}, z_{n_j+i}).$$

We had originally selected a subsequence $\{(y_k, z_k)\}$ such that $I(y_k, z_k) \to \gamma$. Therefore it follows from (4.5) that

$$(4.11) \qquad \int_G \lambda(t)\, d\mu \leq \gamma.$$

From (4.10) we get that $\lambda(t) \geq \psi(t)$ a.e. Hence λ is finite a.e.

Let G_1 denote the set of points at which $\lambda(t)$ is infinite. Let G_2 denote the set of points at which (4.4) fails and let G_3 denote the set of points at which (4.8) fails. Let G_4 denote the set of points in G at which f is not continuous in (x, w). Let $G_5 = \bigcup_{i=1}^{4} G_i$. The μ-measure of G_5 is zero. Let $G' = G - G_4$.

Let

$$\omega_j(t) = \sum_{i=1}^{k} \alpha_{ij} f(t, y(t), z_{n_j+i}(t)),$$

where the numbers $\alpha_{ij}, i = 1, \ldots, k$, the indices $n_j + i$ and the functions z_{n_j+i} are as in (4.7). From (4.9) and (4.3) we have

$$\lambda_j(t) - \omega_j(t) = \sum_{i=1}^{k} \alpha_{ij} h_{n_j+i}(t).$$

Since $h_k(t) \to 0$ on G', it follows from (4.5) that

$$(4.12) \qquad \lambda_j(t) - \omega_j(t) \to 0 \quad \text{on } G'.$$

Let t be an arbitrary point in G'. There exists a subsequence, which may depend on t, such that $\lambda_j(t) \to \lambda(t)$. For this subsequence we have $\omega_j(t) \to \lambda(t)$ by virtue of (4.12). Since for fixed (t, x), f is a convex function of w, we have

$$\omega_j(t) = \sum_{i=1}^{k} \alpha_{ij} f(t, y(t), z_{n_j+i}(t)) \geq f(t, y(t), \psi_j(t)),$$

where we have used (4.5) and (4.7). If we now let $j \to \infty$ and make use of (4.8), the continuity of f with respect to w, and the fact that $\omega_j(t) \to \lambda(t)$ for the point t and subsequence in question, we get

(4.13) $$\lambda(t) \geq f(t, y(t), z(t)).$$

Since t was an arbitrary point of G', we have that (4.13) holds a.e. in G.

Under our hypothesis on f the mapping $t \to f(t, y(t), z(t))$ is measurable. It then follows from (4.13) and (3.1) that f is in $L_1(G)$ and that

$$\int_G \lambda(t) \, d\mu \geq \int_G f(t, y(t), z(t)) \, d\mu > -\infty.$$

Thus (y, z) is in $\mathfrak{D}$.

From (4.11) and the preceding inequality we get

$$\gamma \geq \int_G f(t, y(t), z(t)) \, d\mu.$$

The relation (3.2) now follows from the definition of γ in (4.1) as $\liminf I(y_k, z_k)$, where $\{(y_k, z_k)\}$ is the original sequence.

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DEPARTMENT OF MATHEMATICS, PURDUE UNIVERSITY, WEST LAFAYETTE, INDIANA 47907