Lowering and Raising Operators for the Orthogonal Group in the Chain $O(n) \supset O(n-1) \supset \cdots$, and their Graphs*

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Normalized lowering and raising operators are constructed for the orthogonal group in the canonical group chain $O(n) \supset O(n-1) \supset \cdots \supset O(2)$ with the aid of graphs which simplify their construction. By successive application of such lowering operators for $O(n)$, $O(n-1)$, $\cdots$ on the highest weight states for each step of the chain, an explicit construction is given for the normalized basis vectors. To illustrate the usefulness of the construction, a derivation is given of the Gel'fand–Zetlin matrix elements of the infinitesimal generators of $O(n)$.

1. INTRODUCTION

THE semisimple Lie groups have recently found many new applications in physics. The unitary groups in particular have received wide attention as a result of this renewed importance, and the irreducible representations of $U(n)$ (arbitrary n), have been studied in considerable detail.[1,2] Although the orthogonal group $O(n)$ has received less attention, it recently also found some new applications to physical problems. In particular, the groups $O(5)$ and $O(8)$ have become of interest in nuclear spectroscopy in connection with the quasi-spin formalism for neutron and proton configurations.[3,4] The group chain $O(n) \supset O(n-1) \supset \cdots$ has also been found of interest in general many-body theory in the construction of n-body states of definite permutational symmetry.[5]

The basis vectors of an arbitrary irreducible representation of $O(n)$ are completely characterized by the chain of canonical subgroups $O(n-1) \supset O(n-2) \supset \cdots O(2)$. This canonical group chain has been studied many years ago by Gel'fand and Zetlin,[6] who give the matrix elements of the infinitesimal operators of $O(n)$, for arbitrary n, in this basis.[7] Since the mathematically natural chain of subgroups, such as $O(n) \supset O(n-1) \supset \cdots$, often does not include the subgroups of actual physical interest,[3,4] the application to physical problems, in general, involves a transformation from the mathematically natural to a physically relevant scheme. To effect such a transformation, it becomes important to have an explicit construction of the basis vectors of an arbitrary irreducible representation of the group.

It is the purpose of this paper to give an explicit construction of the basis vectors of the irreducible representations of $O(n)$ in the Gel'fand scheme through the successive application of lowering operators acting on the highest weight state. The concept of lowering (or raising) operators was employed by Nagel and Moshinsky[1] to construct the full set of basis vectors of $U(n)$ in the canonical group chain $U(n) \supset U(n-1) \supset \cdots$. Although the present work has set itself the analogous task for the group chain $O(n) \supset O(n-1) \cdots$ and thus forms a parallel to the work of Nagel and Moshinsky, the techniques employed are somewhat different. In particular, since the lowering (or raising) operators for $O(n)$ are complicated polynomial functions of the infinitesimal generators of the group, a graphical technique has been found useful in the construction of these operators.

In Sec. 2 a review is given of some of the properties of the group $O(n)$ and the canonical chain of subgroups employed in the Gel'fand basis. In Sec. 3 the raising and lowering operators are constructed with the aid of graphs. Section 4 presents the calculation of the normalization coefficients of the lowering operators. These are the fundamental numbers of the construction since the successive application of lowering operators must yield a normalized basis vector for easy application in actual problems. Finally, in Sec. 5, a brief derivation is given of the Gel'fand and Zetlin results for the matrix elements of the infinitesimal operators to illustrate the usefulness of the present construction.

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[4] B. H. Flowers and S. Szpikowski, Proc. Phys. Soc. (London) 84, 673 (1964).

[5] P. Kramer and M. Moshinsky, Nucl. Phys. 82, 241 (1966).

[6] I. M. Gel'fand and M. L. Zetlin, Dokl. Akad. Nauk. USSR 71,, 1017 (1950). I. M. Gel'fand, R. A. Minlos, and Z. Ya. Shapiro *Representations of the Rotation and Lorentz Groups and Their Application* (The Macmillan Company, New York, 1963), p. 353.

[7] The Gel'fand–Zetlin result has also been derived by algebraic techniques by J. D. Louck, Los Alamos Scientific Laboratory Reports LA 2451 (1960).

2. SOME PROPERTIES OF $O(n)$

A. Generators of $O(n)$

The natural infinitesimal generators of $O(n)$ are the set of skew-symmetric, Hermitian operators J_{ij} with the commutation relations

$$[J_{mj}, J_{kl}] = i(\delta_{mk}J_{jl} + \delta_{jl}J_{mk} - \delta_{jk}J_{ml} - \delta_{ml}J_{jk}), \quad (2.1)$$

where m, j, k, and l run from 1 to n. The number of independent generators of $O(n)$ is therefore $\frac{1}{2}n(n-1)$.

The infinitesimal generators of a Lie group are best expressed in standard form[8] in which they are organized into one set of k commuting operators (H type), where k is the rank of the group, and a set of raising and lowering generators[9] (E type). In $O(3)$, for example, H, E_1, E_{-1} correspond to J_{12}, $J_{13} + iJ_{23}$, $J_{13} - iJ_{23}$, respectively. For both $O(2k+1)$ and $O(2k)$ it is convenient to choose the k commuting operators as $J_{12}, J_{34}, J_{56}, \cdots, J_{2k-1,2k}$. It is useful to further classify the raising and lowering generators into two types, those which connect the group $O(n)$ to its subgroups, to be denoted by Q, and those which operate within the space of the subgroups only, to be denoted by ρ, so that there are three types of operators in all. In $O(7)$, for example, operators of type Q are linear combinations of the J_{i7}, while operators of type ρ involve only J_{ij} with *both* $i, j < 7$.

The three types of operators are defined as follows:

(a) $O(2k+1)$

Type (1) $H_\alpha = J_{2\alpha-1,2\alpha}, \quad \alpha = 1, 2, \cdots, k,$

(2) $Q_{2k+1,\pm\alpha} = J_{2\alpha-1,2k+1} \pm iJ_{2\alpha,2k+1}, \quad \alpha = 1, 2, \cdots, k, \quad (2.2)$

(3) $\rho_{\alpha\beta} = [Q_{2k+1,\alpha}, Q_{2k+1,\beta}], \quad \alpha, \beta = \pm 1, \cdots, \pm k, \quad \beta \neq -\alpha;$

(b) $O(2k)$

Type (1) $H_\alpha = J_{2\alpha-1,2\alpha}, \quad \alpha = 1, 2, \cdots, k-1,$

(2) $Q_{2k,k} = J_{2k-1,2k} \quad (=H_k),$

$Q_{2k,\pm\alpha} = J_{2\alpha-1,2k} \pm iJ_{2\alpha,2k}, \quad \alpha = 1, 2, \cdots, k-1, \quad (2.3)$

(3) $\rho_{\alpha\beta} = [Q_{2k,\alpha}, Q_{2k,\beta}], \quad \alpha, \beta = \pm 1, \pm 2, \cdots, \pm(k-1), k, \quad \beta \neq -\alpha,$

(Note that H_k is now included among the type 2 operators, and that $\rho_{\alpha,-\alpha}$ is not of type 3 but, from its definition, is merely equal to $2J_{2\alpha-1,2\alpha}$.) The basic commutators of these operators are then

$$[J_{2\alpha-1,2\alpha}, J_{2\beta-1,2\beta}] = 0, \quad (2.4)$$

$$[J_{2\alpha-1,2\alpha}, \rho_{\beta\gamma}] = (\delta_{\alpha\beta} + \delta_{\alpha\gamma} - \delta_{\alpha,-\beta} - \delta_{\alpha,-\gamma})\rho_{\beta\gamma}, \quad (2.5)$$

$$[J_{2\alpha-1,2\alpha}, Q_{n\beta}] = (\delta_{\alpha\beta} - \delta_{\alpha,-\beta})Q_{n\beta}, \quad (2.6)$$

$$[\rho_{\alpha\beta}, Q_{n\gamma}] = 2(\delta_{\beta,-\gamma}Q_{n\alpha} - \delta_{\alpha,-\gamma}Q_{n\beta}), \quad (2.7)$$

$$[\rho_{\alpha\beta}, \rho_{\gamma\delta}] = 2(\delta_{\alpha,-\delta}\rho_{\beta\gamma} + \delta_{\beta,-\gamma}\rho_{\alpha\delta} - \delta_{\alpha,-\gamma}\rho_{\beta\delta} - \delta_{\beta,-\delta}\rho_{\alpha\gamma}). \quad (2.8)$$

The $\rho_{\alpha\beta}$ can also be represented as Q-type operators of the subgroups of $O(n)$

$$\begin{aligned}
\rho_{\alpha\beta} &= i[Q_{2\beta-1,\alpha} + iQ_{2\beta,\alpha}], \\
\rho_{\alpha,-\beta} &= i[Q_{2\beta-1,\alpha} - iQ_{2\beta,\alpha}], \quad 0 < \alpha < \beta, \\
\rho_{-\alpha,\beta} &= i[Q_{2\beta-1,-\alpha} - iQ_{2\beta,-\alpha}], \quad 2\beta < n, \\
\rho_{-\alpha,-\beta} &= i[Q_{2\beta-1,-\alpha} + iQ_{2\beta,-\alpha}].
\end{aligned} \quad (2.9)$$

B. The Gel'fand Basis

Gel'fand and Zetlin[6] have provided a way to completely specify the basis vectors of the irreducible representations of $O(n)$ according to the canonical chain of subgroups $O(n) \supset O(n-1) \supset \cdots \supset O(2)$. For the case $n = 2k+1$

$$|\mathcal{M}_{n\mu}\rangle = \begin{bmatrix}
m_{2k+1,1} & m_{2k+1,2} & \cdots & m_{2k+1,k-1} & m_{2k+1,k} \\
m_{2k,1} & m_{2k,2} & \cdots & m_{2k,k-1} & m_{2k,k} \\
m_{2k-1,1} & m_{2k-1,2} & \cdots & m_{2k-1,k-1} & \\
\cdot & \cdot & \cdots & m_{2k-2,k-1} & \\
\cdot & \cdot & \cdots & & \\
\cdot & \cdot & \cdots & & \\
m_{51} & m_{52} & & & \\
m_{41} & m_{42} & & & \\
m_{31} & & & & \\
m_{21} & & & &
\end{bmatrix}. \quad (2.10)$$

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[8] G. Racah, CERN reprint 61-8 (1961).

[9] The raising and lowering *generators* are not to be confused with the raising and lowering *operators* which are the subject of this paper. Except for $O(3)$ the lowering and raising *operators* are complicated polynomial functions of the lowering and raising *generators*.

For the case $n = 2k$

$$|\mathcal{M}_{n\mu}\rangle = \begin{bmatrix} m_{2k,1} & m_{2k,2} & \cdots & m_{2k,k-1} & m_{2k,2} \\ m_{2k-1,1} & m_{2k-1,2} & \cdots & m_{2k-1,k-1} & \\ m_{2k-2,1} & m_{2k-2,2} & \cdots & & \\ \cdot & \cdot & & & \\ \cdot & \cdot & & & \\ \cdot & \cdot & & & \\ m_{41} & m_{42} & & & \\ m_{31} & & & & \\ m_{21} & & & & \end{bmatrix}. \tag{2.11}$$

The k numbers in the top row characterize the irreducible representations of $O(n)$. The numbers in the next row characterize one of the possible irreducible representations of $O(n-1)$ contained in the specific irreducible representation of $O(n)$, and so forth for successive subgroups of the chain. The numbers in each row thus characterize one of the possible irreducible representations of a specific subgroup. The numbers m_{61}, m_{62}, m_{63}, for example, characterize one of the irreducible representations of $O(6)$.[10]

The Gel'fand basis vectors are not eigenvectors of the k commuting operators $J_{2\alpha-1,2\alpha}$. The basis differs in this respect from the corresponding Gel'fand–Zetlin basis for the unitary groups.[1] Although the full set of m_{ij} are thus not simply related to the components of the weights, they are nevertheless related to the highest weights of the irreducible representations, since the highest weight state of $O(n)$ is an eigenvector of the set of $J_{2\alpha-1,2\alpha}$. The significance of the $m_{n,i}$ is therefore the following:

(a) For $n = 2k + 1$,

$m_{2k+1,1}$ is the maximum possible eigenvalue of J_{12} in $O(2k+1)$,
$m_{2k+1,2}$ is the maximum possible eigenvalue of J_{34} when the eigenvalue of J_{12} is $m_{2k+1,1}$ in $O(2k+1)$,

⋮

$m_{2k+1,i}$ is the maximum possible eigenvalue of $J_{2i-1,2i}$ when the eigenvalues of $J_{2\alpha-1,2\alpha}$ are equal to $m_{2k+1,\alpha}$ for all $\alpha < i$ in $O(2k+1)$,

⋮

$m_{2k+1,k}$ is the maximum possible eigenvalue of $J_{2k-1,2k}$ when the eigenvalues of $J_{2\alpha-1,2\alpha}$ are equal to $m_{2k+1,\alpha}$ for all $\alpha < k$ in $O(2k+1)$;

(b) For $n = 2k$,

$m_{2k,1}$ is the maximum possible eigenvalue of J_{12} in $O(2k)$,

⋮

$m_{2k,i}$ is the maximum possible eigenvalue of $J_{2i-1,2i}$ when the eigenvalues of $J_{2\alpha-1,2\alpha}$ are equal to $m_{2k,\alpha}$ for all $\alpha < i$ in $O(2k)$,

⋮

$m_{2k,k-1}$ is the maximum possible eigenvalue of $J_{2k-3,2k-2}$ when the eigenvalues of $J_{2\alpha-1,2\alpha}$ are equal to $m_{2k,\alpha}$ for all $\alpha < k-1$ in $O(2k)$,

$m_{2k,k}$ is the eigenvalue of $J_{2k-1,2k}$ when the eigenvalues of $J_{2\alpha-1,2\alpha}$ are equal to $m_{2k,\alpha}$ for all $\alpha < k-1$ in $O(2k)$.

The irreducible representations of the subgroups in the chain are characterized in the same way.

The numbers m_{ij} are simultaneously either integral or half integral with restrictions which have been given by Gel'fand and Zetlin[6]:

$$m_{2p+1,i} \geq m_{2p,i} \geq m_{2p+1,i+1} \quad (i = 1, 2, \cdots, p),$$
$$m_{2p,i} \geq m_{2p-1,i} \geq m_{2p,i+1} \quad (i = 1, 2, \cdots, p-1),$$
$$m_{2p+1,p} \geq |m_{2pp}|. \tag{2.12}$$

These properties are clear once the lowering and raising operators are derived in this paper.

Since the type-1 operators $J_{2\alpha-1,2\alpha}$ are not diagonal in the general Gel'fand basis, it is convenient to define a whole hierarchy of subbases of decreasing

[10] A slight change has been made in the Gel'fand–Zetlin notation. The first index has been shifted up by one unit so that $m_{\nu1}$, $m_{\nu2}$, $\cdots$ characterize the irreducible representation of $O(\nu)$. The chain of numbers thus ends with m_{21} [irreducible representation of $O(2)$], rather than with m_{11}.

complexity:

$$
\begin{aligned}
[\mathcal{M}_{n\mu}^{(2)}] &\equiv [\mathcal{M}_{n\mu}],\\
[\mathcal{M}_{n\mu}^{(3)}] &= [\mathcal{M}_{n\mu};\, m_{31} = m_{21}],\\
[\mathcal{M}_{n\mu}^{(4)}] &= [\mathcal{M}_{n\mu};\, m_{41} = m_{31} = m_{21}],\\
[\mathcal{M}_{n\mu}^{(5)}] &= \left[\mathcal{M}_{n\mu};\, \begin{matrix} m_{51} = m_{41} = m_{31} = m_{21} \\ m_{52} = m_{42} \end{matrix}\right],\\
&\;\;\vdots\\
[\mathcal{M}_{n\mu}^{(2i)}] &= \left[\mathcal{M}_{n\mu};\, m_{2i,\alpha} = m_{\beta\alpha} \quad \begin{matrix} \alpha = 1, 2, \cdots, i \\ \beta = 2\alpha, 2\alpha + 1, \cdots, 2i \end{matrix}\right],\\
[\mathcal{M}_{n\mu}^{(2i+1)}] &= \left[\mathcal{M}_{n\mu};\, m_{2i+1,\alpha} = m_{\beta\alpha} \quad \begin{matrix} \alpha = 1, 2, \cdots, i \\ \beta = 2\alpha, 2\alpha + 1, \cdots, 2i \end{matrix}\right],
\end{aligned}
\tag{2.13}
$$

The base vectors of $[\mathcal{M}_{n\mu}^{(q)}]$ with $q = 2i$ or $q = 2i + 1$ have the special property that they are eigenvectors of the set of commuting operators $J_{2\alpha-1,2\alpha}$ with $\alpha = 1, \cdots, i$. Any vector of $[\mathcal{M}_{n\mu}^{(q)}]$ is specified by $(n - q + 1)$ rows of numbers.

The particular subbasis $[\mathcal{M}_{n\mu}^{(n-1)}]$, made up of the base vectors of highest weight in the immediate subgroup $O(n - 1)$ of $O(n)$, is of greatest importance in the present discussion. Its states are specified by only two rows of numbers and it has the following special properties.

(1) *All* of the type-1 operators, $J_{2\alpha-1,2\alpha}$ (with $\alpha = 1, \cdots, k$ for $n = 2k + 1$, and $\alpha = 1, \cdots, k - 1$ for $n = 2k$), are diagonal in this basis.

(2) All of the type-3 raising generators $\rho_{\alpha\beta}$, $\rho_{\alpha,-\beta}$ $(0 < \alpha < \beta)$ give zero when operating on any vector of the basis $[\mathcal{M}_{n\mu}^{(n-1)}]$. This condition is necessary and sufficient to define the basis $[\mathcal{M}_{n\mu}^{(n-1)}]$. (Note that the generators ρ_{12}, $\rho_{1,-2}$, ρ_{13}, $\rho_{1,-3}$, ρ_{23}, $\rho_{2,-3}, \cdots$ are naturally considered as raising generators, whereas $\rho_{-1,2}$, $\rho_{-1,-2}$, $\rho_{-1,3}$, $\rho_{-1,-3}$, $\rho_{-2,3}$ $\rho_{-2,-3}, \cdots$ are lowering generators.)

The raising and lowering operators which are the subject of this paper are best defined in terms of the subbasis $[\mathcal{M}_{n\mu}^{(n-1)}]$. They are the operators which raise or lower by one integer one of the quantum numbers $m_{n-1,i}$ of the second row without leaving the subbasis $[\mathcal{M}_{n\mu}^{(n-1)}]$, that is the space of base vectors of highest weight in the immediate subgroup. In particular, the full set of states of $[\mathcal{M}_{n\mu}^{(n-1)}]$ can be constructed by repeated operation with the various lowering operators of $O(n)$ on the highest weight state of a specific irreducible representation, namely $[\mathcal{M}_{n\mu}^{(n)}]$. The set of states of $[\mathcal{M}_{n\mu}^{(n-2)}]$ can then be constructed by successive operation with lowering operators of $O(n - 1)$ on the states of $[\mathcal{M}_{n\mu}^{(n-1)}]$ which are highest weight states of irreducible representations of $O(n - 1)$, and so forth, until the full set of Gel'fand states has been reached by successive stepdown operations with the lowering operators of $O(n)$, $O(n - 1), \cdots, O(3)$.

3. THE RAISING AND LOWERING OPERATORS AND THEIR GRAPHS

In $O(3)$ the raising (and lowering) *generators* $J_{13} \pm iJ_{23} \equiv Q_{3,\pm 1}$ are themselves raising (and lowering) *operators*; that is, $Q_{3,+1}$ operating on a state $|l, m\rangle$ converts it into a state $|l, m + 1\rangle$. In $O(n)$, with $n > 3$, the raising and lowering generators $Q_{n,i}$ have matrix elements connecting very many different states of the general Gel'fand basis, and when operating on a state of $[\mathcal{M}_{n\mu}^{(n-1)}]$ do not give states belonging solely to $[\mathcal{M}_{n\mu}^{(n-1)}]$.

By forming polynomial functions of the raising and lowering generators, it is possible to construct raising and lowering operators, to be denoted by $O_{n,\pm i}$, which have the simple property that they raise (or lower) by one integer one of the quantum numbers $m_{n-1,i}$ of the subbasis $[\mathcal{M}_{n\mu}^{(n-1)}]$ without leaving this subbasis, that is, the space of base vectors of highest weight in the immediate subgroup $O(n - 1)$.[11] Specifically $O_{n\pm i}$ is defined by

$$
O_{n,\pm i}\left|\begin{matrix} m_{n1} & m_{n2} & \cdots & m_{ni} & \cdots & m_{nk} \\ m_{n-1,1} & m_{n-1,2} & \cdots & m_{n-1,i} & \cdots & \cdot \end{matrix}\right\rangle
= N'\left|\begin{matrix} m_{n1} & m_{n2} & \cdots & m_{ni} & \cdots & m_{nk} \\ m_{n-1,1} & m_{n-1,2} & \cdots & m_{n-1,i} \pm 1 & \cdots & \cdot \end{matrix}\right\rangle,
\tag{3.1}
$$

where N' is a normalization factor and $|\ \rangle$ denotes a normalized state. To save writing, only the column that suffers change is indicated:

$$
O_{n,\pm i}\left|\begin{matrix} m_{ni} \\ m_{n-1,i} \end{matrix}\right\rangle = N'\left|\begin{matrix} m_{ni} \\ m_{n-1,i} \pm 1 \end{matrix}\right\rangle,
\quad \begin{cases} i = 1, 2, \cdots, k, & n = 2k + 1, \\ i = 1, 2, \cdots, k - 1, & n = 2k. \end{cases}
\tag{3.2}
$$

For $n = 2k$ it is also convenient to introduce the zero-step operator, $O_{2k,k}$

$$
O_{2k,k}\left|\begin{matrix} m_{2k,1} & \cdots & m_{2k,i} & \cdots & m_{2k,k-1} & m_{2k,k} \\ m_{2k-1,1} & \cdots & m_{2k-1,i} & \cdots & m_{2k-1,k-1} & \end{matrix}\right\rangle
= N'\left|\begin{matrix} m_{2k,1} & \cdots & m_{2k,i} & \cdots & m_{2k,k-1} & m_{2k,k} \\ m_{2k-1,1} & \cdots & m_{2k-1,l} & \cdots & m_{2k-1,k-1} & \end{matrix}\right\rangle
\tag{3.3}
$$

[11] For the specific cases $n = 5$ and 6 explicit expressions for raising and lowering operators have been given previously. J. Flores, E. Chacon, P. A. Mello, and M. de Llano, Nucl. Phys. **72**, 352 (1965), and ($n = 5$) K. T. Hecht, *ibid.* **63**, 177 (1965).

or

$$O_{2k,k}\left|\begin{matrix} m_{2k,i} \\ m_{2k-1,i}\end{matrix}\right\rangle = N'\left|\begin{matrix} m_{2k,i} \\ m_{2k-1,i}\end{matrix}\right\rangle. \tag{3.4}$$

Since

$$\left|\begin{matrix} m_{ni} \\ m_{n-1,i}\end{matrix}\right\rangle \text{ and } \left|\begin{matrix} m_{ni} \\ m_{n-1,i} \pm 1\end{matrix}\right\rangle \in [\mathcal{M}_{n\mu}^{(n-1)}],$$

$O_{n,\pm i}$ and $O_{2k,k}$ must satisfy

(1) $$[J_{2\alpha-1,2\alpha}, O_{ni}] = \delta_{\alpha i}O_{ni},$$
$$[J_{2\alpha-1,2\alpha}, O_{n-i}] = -\delta_{\alpha i}O_{n,-i},$$
$$[J_{2\alpha-1,2\alpha}, O_{2k,k}] = 0, \quad 0 < \alpha < k, \tag{3.5}$$

(2) $$[\rho_{\alpha\beta}, O_{n,\pm i}]\left|\begin{matrix} m_{ni} \\ m_{n-1,i}\end{matrix}\right\rangle = 0,$$

$$[\rho_{\alpha\beta}, O_{2k,k}]\left|\begin{matrix} m_{ni} \\ m_{n-1,i}\end{matrix}\right\rangle = 0,$$

$$[\rho_{\alpha,-\beta}, O_{n,\pm i}]\left|\begin{matrix} m_{ni} \\ m_{n-1,i}\end{matrix}\right\rangle = 0, \tag{3.6}$$

$$[\rho_{\alpha,-\beta}, O_{2k,k}]\left|\begin{matrix} m_{ni} \\ m_{n-1,i}\end{matrix}\right\rangle = 0, \quad 0 < \alpha < |\beta|.$$

Equations (3.5) and (3.6) are necessary and sufficient conditions that $O_{n,\pm i}$ be raising (lowering) operators. Equations (3.6) apply to all of the raising generators of the subgroup $O(n-1)$ and ensure that the state $O_{n,\pm i}\,|m_{ni}, m_{n-1,i}\rangle$ is a highest weight state of the subgroup $O(n-1)$ since the state $|m_{ni}, m_{n-1,i}\rangle$ has this property. Since the raising and lowering operators are complicated functions of the generators they are best described in terms of graphs, and manipulations involving these operators are also best performed with the aid of these graphs.

A. Raising Operators and Their Associated Graphs

Contents of R_i graphs

Graphs associated with the raising operator $O_{n,i}$ are to be denoted by R_i; these graphs consist of the following (see Table I).

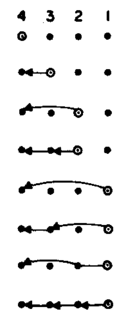
(1) A single row of i ordered points numbered from 1 to i with order increasing from right to left.

(2) A connected chain of arrows always pointing from right to left, with (a) any point $1 \le j \le i$ as starting point, to be indicated by a circle, (b) end point always at i, (c) the arrows which form the links of the connected chain may connect some (possibly all) of the points between the starting point j and the end point i but may skip around others (possibly none).

Operator Representations of the R_i Graphs

Each of the many possible graphs of type R_i represents one of the terms of the raising operator $O_{n,i}$.

TABLE I. The graphs of R_{ni} for any $n > 8$.



GRAPHS	OPERATOR REPRESENTATION OF THE GRAPHS
	$Q_{n4}\ a_{3-4}\ a_{2-4}\ a_{1-4}$
	$(-\rho_{4-3})\ Q_{n3}\ a_{2-4}\ a_{1-4}$
	$(-\rho_{4-2})\ Q_{n2}\ a_{3-4}\ a_{1-4}$
	$(-\rho_{4-3})(-\rho_{3-2})\ Q_{n2}\ a_{1-4}$
	$(-\rho_{4-1})\ Q_{n1}\ a_{3-4}\ a_{2-4}$
	$(-\rho_{4-3})(-\rho_{3-1})\ Q_{n1}\ a_{2-4}$
	$(-\rho_{4-2})(-\rho_{2-1})\ Q_{n1}\ a_{3-4}$
	$(-\rho_{4-3})(-\rho_{3-2})(-\rho_{2-1})\ Q_{n1}$
$a_{i-j} = a_i - a_j$	$a_\alpha = 2(J_{2\alpha-1,2\alpha} + k - \alpha)$, $n = 2k$ or $2k+1$

(1) The circle around the starting point j represents the operator $Q_{n,j}$.

(2) An arrow link of the chain connecting points α and β, with $\alpha < \beta$, represents the operator $(-\rho_{\beta,-\alpha})$. [Note that the operator $(-\rho_{\beta,-\alpha}) = \rho_{-\alpha,\beta}$ with $\alpha < \beta$ is a lowering generator of one of the subgroups of $O(n)$.]

(3) A free point, not connected by one of the arrow links of the chain, is associated in the operator representation of the graph by $a_{j,-i} = a_j - a_i$, $a_\alpha = 2(J_{2\alpha-1,2\alpha} + k - \alpha)$ for $n = 2k$ or $n = 2k + 1$. (Note that $a_{j,-i} = a_{-i,j}$, and the vectors of $[\mathcal{M}_{n\mu}^{(n-1)}]$ are eigenvectors of a_α.)

(4) The full operator represented by one of the R_i graphs is the product of all the factors of type $(-\rho_{\beta,-\alpha})$ and $Q_{n,j}$ implied by the various links of the graph. The order of the Q and ρ operators in the product reading from right to left is the same as the order of the links of the chain again reading from right to left, with $-\rho_{i,-\alpha}$ on the extreme left and $Q_{n,j}$ on the right followed on the right by all the commuting operator functions $a_{-i,j}$.

The Raising Operators

Theorem: O_{ni} is equal to the sum of the operators represented by all possible graphs R_{ni}.

Proof: Since all raising generators $\rho_{\alpha,\beta}$, $\rho_{\alpha,-\beta}$ $(0 < \alpha < |\beta|)$ can be expressed in terms of commutators of generators of the type $\rho_{j,j+1}$ and $\rho_{j,-(j+1)}$, Eqs. (3.5) and (3.6) follow from

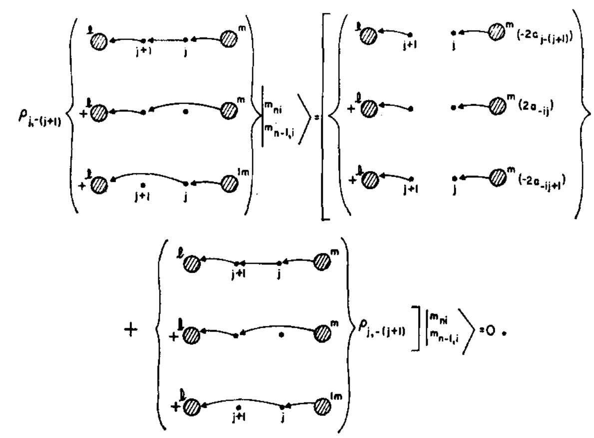
$$[J_{2\alpha-1,2\alpha}, O_{ni}] = \delta_{\alpha i}O_{ni}, \tag{3.7}$$

$$\rho_{j,j+1}O_{ni}\left|\begin{matrix} m_{ni} \\ m_{n-1,i}\end{matrix}\right\rangle = 0, \tag{3.8}$$

$$j = 1, 2, \cdots, (k-1),$$

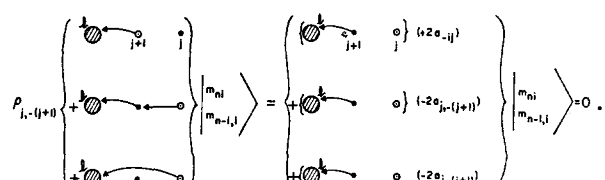
$$\rho_{j,-(j+1)}O_{ni}\left|\begin{matrix} m_{ni} \\ m_{n-1,i}\end{matrix}\right\rangle = 0. \tag{3.9}$$

Equation (3.7) follows at once from the commutators of $J_{2\alpha-1,2\alpha}$ with the ρ's and Q's [Eqs (2.4)–(2.6)]. Equation (3.8) follows from the fact that a ρ with two positive indices, when commuted through to the right of all factors $\rho_{\beta,-\alpha}$ and Q_j of O_{ni}, leaves one ρ with two positive indices (which in turn has one such surviving term when commuted to the right), and a ρ with two positive indices gives zero when operating on a state of $[\mathcal{M}_{n,i}^{(n-1)}]$. Equation (3.9) follows since all the $\rho_{j,-(j+1)}$ commute through to the right side of all factors ρ and Q of $O_{n,i}$ except for the types which involve the indices j and $j+1$, and $\rho_{j,-(j+1)}$ operating on terms including these satisfy the relations written in terms of graphs:



(3.10)

And also



(3.11)

Special examples

$O(6)$

$$O_{61} = Q_{6,1}$$

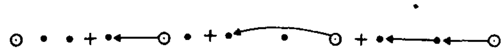
$$O_{62} = Q_{6,2}a_{1-2} + (-\rho_{2-1})Q_{6,1}$$

$O(7)$

$$O_{7,1} = Q_{7,1}$$

$$O_{7,2} = Q_{7,2}a_{1-2} + (-\rho_{2-1})Q_{7,1}$$

$$O_{7,3} = Q_{7,3}a_{1-3}a_{2-3} + (-\rho_{3-2})Q_{7,2}a_{1-3} + (-\rho_{3-1})Q_{7,1}a_{2-3} + (-\rho_{3-2})(-\rho_{2-1})Q_{7,1}$$



B. Lowering Operators and Their Associated Graphs

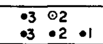
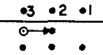
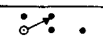
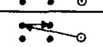
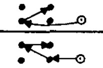
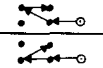
The graphs for the raising operators $O_{n,1}$ are identical for all $k > i$. The graphs for the lowering operators $O_{n,-1}$, however, are not only dependent on the specific value of n but have a slightly different character for the odd- and even-dimensional rotation groups, $n = 2k + 1$ and $n = 2k$, so that the two cases must be discussed separately. Graphs associated with the lowering operators $O_{2k+1,-i}$ are to be denoted by $\mathfrak{L}_{2k+1,i}$

(1) $\mathfrak{L}_{2k+1,i}$ *Graphs*

The $\mathfrak{L}_{2k+1,i}$ graphs consist of the following (see Table II).

(a) Two rows of ordered points, k points in the bottom row numbered from 1 to k with order increasing from right to left, and $k - i + 1$ points in the top row with order decreasing from left to right starting with k at the left and ending with i so that the point, j $(i \leq j \leq k)$, in the top row sits above the point j of the bottom row.

TABLE II. The graphs of $\mathfrak{L}_{7,2}$.

GRAPHS	OPERATOR REPRESENTATION OF THE GRAPHS
	$Q_{7\text{-}2}\ a_{2\text{-}3}\ a_{23}\ a_{22}\ a_{21}$
	$(-\rho_{-23})\ Q_{7\text{-}3}\ a_{23}\ a_{22}\ a_{21}$
	$(-\rho_{-2-3})\ Q_{73}\ a_{2-3}\ a_{22}\ a_{21}$
	$(-\rho_{-23})(-\rho_{-3-2})\ Q_{72}\ a_{23}\ a_{21}$
	$(-\rho_{-2-3})(-\rho_{3-2})\ Q_{72}\ a_{2-3}\ a_{21}$
	$(-\rho_{-2-1})\ Q_{71}\ a_{2-3}\ a_{23}\ a_{22}$
	$(-\rho_{-23})(-\rho_{-3-1})\ Q_{71}\ a_{23}\ a_{22}$
	$(-\rho_{-2-3})(-\rho_{3-1})\ Q_{71}\ a_{2-3}\ a_{22}$
	$(-\rho_{-23})(-\rho_{-3-2})(-\rho_{2-1})\ Q_{71}\ a_{23}$
	$(-\rho_{-2-3})(-\rho_{3-2})(-\rho_{2-1})\ Q_{71}\ a_{2-3}$
$a_{ij} = a_i + a_j$ $\quad a_{i-j} = a_i - a_j$ $\quad a_a = 2(J_{2a-1,2a} + 3 - a)$	

(b) A connected chain of arrows forming a clockwise path, the arrows always pointing from right to left in the bottom row and from left to right in the top row, with (i) any point of either the top or bottom row as starting point, to be indicated by a circle, (ii) end point always at i of the top row, (iii) no vertical arrows (that is, no connections from point l in the bottom row to point l in the top row), (iv) no arrows pointing downward [that is, no arrows with starting points (tails) in the top row and end points (arrowheads) in the bottom row], (v) the arrows which form the links of the connected chain may then be directed from point α in the botton row to point β in the bottom row with ($\beta > \alpha$), from point μ in the top row to point ν in the top row with ($\nu < \mu$), or from point α in the bottom row to any point σ in the top row $\sigma \lesseqgtr \alpha$, but $\sigma \neq \alpha$.

(2) *Operator Representations of the* $\mathfrak{L}_{2k+1,i}$ *Graphs*

(a) The circle around the starting point, say j, represents the operator $Q_{2k+1,j}$ when it is in the bottom row and $Q_{2k+1,-j}$ when in the top row.

(b) An arrow link of the chain connecting point α to point β represents the operator

(i) $(-\rho_{\beta,-\alpha})$ when $\alpha < \beta$, both points in bottom row,

(ii) $(-\rho_{-\beta,\alpha})$ when $\alpha > \beta$, both points in top row,

(iii) $(-\rho_{-\beta,-\alpha})$ with α in the bottom row, β in top row.

(c) A free point not connected by one of the arrow links of the chain is represented by the operator function $a_{i,-\alpha} = a_i - a_\alpha$ when α is in the top row and $a_{i\alpha} = a_i + a_\alpha$ when α is in the bottom row, where $a_\alpha = 2(J_{2\alpha-1,2\alpha} + k - \alpha)$, as before.

(d) The full operator, represented by one of the graphs $\mathfrak{L}_{2k+1,i}$, is again the product of all factors of type ρ and Q implied by the links of the graph. The order of the Q and ρ operators in the product reading from right to left is the same as the order of the links of the chain starting with the encircled point and ending at point i of the top row, with Q followed on the right by all the factors $a_{i\pm\alpha}$ implied by the free points of the graph.

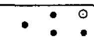
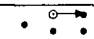
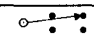
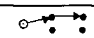
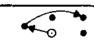
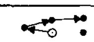
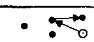
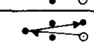
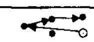
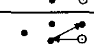
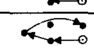
(3) $\mathfrak{L}_{2k,i}$ *Graphs*

Graphs associated with the lowering operators $O_{2k,-i}$ of the orthogonal group in an even number of dimensions are to be denoted by $\mathfrak{L}_{2k,i}$ (see Table III). The graphs $\mathfrak{L}_{2k,i}$ have the same structure as the graphs $\mathfrak{L}_{2k+1,i}$ with the exception that the two points k are replaced by a single point to be placed halfway between the top and bottom rows but to the left of the two points ($k - 1$). The rules for the construction of the operators represented by the graphs $\mathfrak{L}_{2k,i}$ are the same as those for the graphs $\mathfrak{L}_{2k+1,i}$ except for the following.

(a) A free point, not connected by one of the arrow links of the chains and if placed in the αth position of the bottom row, is to be denoted by $b_{i\alpha} = a_{i\alpha} - 2$. If the kth point is a free point it is to be denoted by $c_i = \frac{1}{2}(a_i - 2)$. (Free points of the top row are associated with $a_{i,-\alpha}$, as for $\mathfrak{L}_{2k+1,i}$.)

(b) For the special case $i = k$, required for the zero-step operator, the free points of the bottom row (say in position α) are now to be denoted merely by a_α. (The points of the top row play no role whatsoever in this special case.)

TABLE III. The graphs of $\mathfrak{L}_{6,1}$.

GRAPHS	OPERATOR REPRESENTATION OF THE GRAPHS	
	$Q_{6-1}\ a_{1-2}\ c_1\ b_{12}\ b_{11}$	
	$(-\rho_{-12})\ Q_{6-2}\ c_1\ b_{12}\ b_{11}$	
	$(-\rho_{-13})\ Q_{6-3}\ a_{1-2}\ b_{12}\ b_{11}$	
	$(-\rho_{-12})(-\rho_{-23})\ Q_{6-3}\ b_{12}\ b_{11}$	
	$(-\rho_{-1-2})\ Q_{62}\ a_{1-2}\ c_1\ b_{11}$	
	$(-\rho_{-13})(-\rho_{-3-2})\ Q_{62}\ a_{1-2}\ b_{11}$	
	$(-\rho_{-12})(-\rho_{-23})(-\rho_{-3-2})\ Q_{62}\ b_{11}$	
	$(-\rho_{-12})(-\rho_{-2-1})\ Q_{61}\ c_1\ b_{12}$	
	$(-\rho_{-13})(-\rho_{-3-1})\ Q_{61}\ a_{1-2}\ b_{12}$	
	$(-\rho_{-12})(-\rho_{-23})(-\rho_{-3-1})\ Q_{61}\ b_{12}$	
	$(-\rho_{-1-2})(-\rho_{2-1})\ Q_{61}\ a_{1-2}\ c_1$	
	$(-\rho_{-13})(-\rho_{-3-2})(-\rho_{2-1})\ Q_{61}\ a_{1-2}$	
	$(-\rho_{-12})(-\rho_{-23})(-\rho_{-3-2})(-\rho_{2-1})\ Q_{61}$	
$a_{i\pm j} = a_i \pm a_j$	$b_{ij} = a_{ij} - 2$	$c_i = 1/2(a_i - 2)$
$Q_{6,3} = Q_{6,-3}$	$\rho_{\alpha,3} = \rho_{\alpha,-3}$	$a_\alpha = 2(J_{2\alpha-1,2\alpha} + 3 - \alpha)$

Theorem: $O_{2k+1,-i}$, $O_{2k,-i}$ is equal to the sum of the operators represented by all the possible graphs of $\mathfrak{L}_{2k+1,i}$ and $\mathfrak{L}_{2k,i}$ respectively.

Proof: (a) $[J_{2\alpha-1,2\alpha}, O_{n,-i}] = -O_{n,-i}\delta_{\alpha i}$
This again follows at once from the commutators of Eqs. (2.4)–(2.6).

(b) For the relation

$$\rho_{j,-(j+1)} O_{n,-i} \left| \begin{matrix} m_{ni} \\ m_{n-1,i} \end{matrix} \right\rangle = 0,$$

the proof is essentially the same as that for the raising operator except that there are two sets of terms like those of Eqs. (3.10)–(3.11). One set arises when j and $j + 1$ are both in the bottom row, the other when j and $j + 1$ are both in the top row. Both sets of terms sum to zero independently of each other. [The points m and l of Eqs. (3.10)–(3.11) can now be in either top or bottom row.]

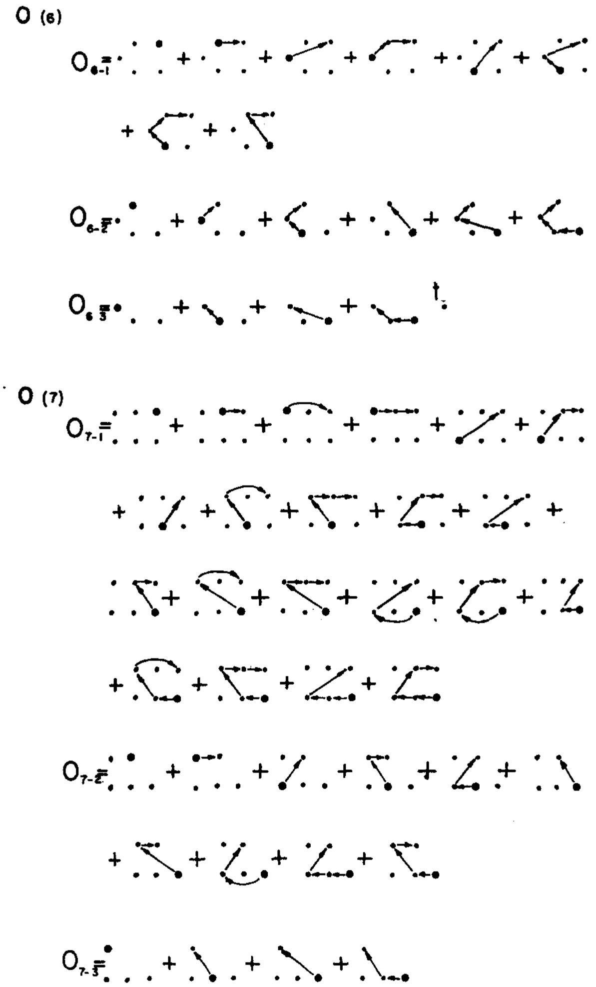
The proof of the relation

$$\rho_{j,j+1} O_{n,-i} \left| \begin{matrix} m_{ni} \\ m_{n-1,i} \end{matrix} \right\rangle = 0$$

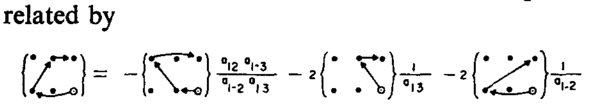
is much more complicated since more summations of graphs are involved. However, the method is identical[12] to that illustrated by Eqs. (3.10)–(3.11).

[12] S. C. Pang, University of Michigan dissertation (to be published).

(4) *Special Examples of Lowering Operators*[13]



Not all the graphs give independent operators. In $O_{7,-1}$, for example, only 15 out of the 21 graphs are independent. The remaining six give operators which can be written as linear combinations of the 15 independent ones. Terms 16 and 18, for example, are related by



It is most convenient, however, to treat all graphs on an equal footing to preserve both the over-all symmetry of the expression for $O_{n,-i}$ and the uniform and simple factoring of the operators associated with an individual graph. The operator representations of the various graphs all have the same structure, differing only in the number of factors of type ρ and a. The total number of operator factors for each graph of $\mathfrak{L}_{ni}$ is equal to $n - i$.

[13]† Note that O_{63} is an example of a neutral or zero-step operator of type $O_{2k,k}$.

C. Some Properties of the Raising and Lowering Operators

The raising and lowering operators which have been constructed have meaning only when they operate on the basis $[\mathcal{M}_{n\mu}^{(n-1)}]$. It is interesting to note that the operators $O_{n,\pm\alpha}$ together with the $J_{2\alpha-1,2\alpha}$ form a Lie algebra with respect to $[\mathcal{M}_{n\mu}^{(n-1)}]$. The raising and lowering operators have not yet been normalized. However, the unnormalized operators $O_{n,\alpha}$ have the simple property

$$[O_{ni}, O_{nj} \,|\mathcal{M}_{n\mu}^{(n-1)}\rangle = 0, \quad i \neq -j. \qquad (3.12)$$

With respect to the basis $[\mathcal{M}_{n\mu}^{(n-1)}]$, the set of operators $O_{n\alpha}$, $O_{n,-\alpha}$, $J_{2\alpha-1,2\alpha}$ thus commutes with any other set $O_{n\beta}$, $O_{n,-\beta}$, $J_{2\beta-1,2\beta}$ $(\beta \neq \alpha)$, so that the Lie algebras mentioned above breaks up into a set of k, $(k-1)$, commuting algebras of order three for dimension $n = 2k+1$ $(n = 2k)$, respectively. Equation (3.12) can be verified by direct computation or obtained from the following considerations.

From the uniqueness of the base vector

$$\begin{vmatrix} m_{ni} & m_{nj} \\ m_{n-1,i} \pm 1 & m_{n-1,j} \pm 1 \end{matrix}\right\rangle$$

the states $O_{ni}O_{nj}\,|\mathcal{M}_{n\mu}^{(n-1)}\rangle$, $O_{nj}O_{ni}\,|\mathcal{M}_{n\mu}^{(n-1)}\rangle$ can differ by at most a constant:

$$O_{ni}O_{nj}\,|\mathcal{M}_{n\mu}^{(n-1)}\rangle = c_{ij}O_{nj}O_{ni}\,|\mathcal{M}_{n\mu}^{(n-1)}\rangle. \qquad (3.13)$$

The constant c_{ij} can be shown to be unity by comparing the coefficients of the terms with the largest number of factors of type ρ on each side of Eq. (3.13). The term with the largest number of factors ρ for a single operator O_{ni} arises from a single graph and has the coefficient unity in all cases except $i < 0$, $n = 2k+1$. In the latter case it arises from two graphs (e.g., graphs Nos. 9 and 10 of Table II) whose summed coefficient (on the right) is equal to a_{ii}. This has the same eigenvalue when operating on the state $|\mathcal{M}_{n\mu}^{(n-1)}\rangle$ or on $O_{nj}\,|\mathcal{M}_{n\mu}^{(n-1)}\rangle$. Thus $c_{ij} = 1$.

4. THE NORMALIZATION

The raising and lowering operators $O_{n,i}$ do not yield normalized basis vectors. It is therefore important to define normalized raising and lowering operators, to be denoted by $U_{n,i}$, which differ from the O_{ni} merely by a normalization factor. The calculation of these normalization factors is presented in this section.

The results for the even- and odd-dimensional orthogonal group are somewhat different. For $n = 2k+1$

the normalized raising and lowering operators are[14]

$$U_{2k+1,i}=\left|\left(\prod_{\alpha=1}^{i-1}\frac{a_{i\alpha}}{(a_{-i,\alpha}+2)(a_{-i\alpha^0}^{(2k+1)}+2)a_{i\alpha^0}}\right)\left(\prod_{\beta=i+1}^{k}\frac{a_{i-\beta}a_{i\beta}}{a_{i\beta^0}^{(2k+1)}(a_{i,-\beta^0}^{(2k+1)}-2)}\right)\frac{2}{a_{ii^0}^{(2k+1)}}\frac{2}{(a_{i^0,-i}^{(2k+1)}+2)}\right|^{\frac{1}{2}}O_{2k+1,i}, \tag{4.1}$$

$$U_{2k+1,-i}=O_{2k+1,-i}\frac{1}{a_{ii}}\left|\frac{2}{a_{ii^0}^{(2k+1)}}\frac{2}{(a_{i^0,-i}^{(2k+1)}+2)}\prod_{\alpha=1}^{i-1}\frac{(a_{-i\alpha}+2)}{a_{i\alpha}(a_{-i,\alpha^0}^{(2k+1)}+2)a_{i\alpha^0}^{(2k+1)}}\prod_{\beta=i+1}^{k}\frac{1}{a_{i,-\beta}a_{i\beta}a_{i\beta^0}^{(2k+1)}(a_{i,-\beta^0}^{(2k+1)}-2)}\right|^{\frac{1}{2}}, \tag{4.2}$$

where

$$a_{i\pm\alpha}=2(J_{2i-1,2i}+k-i)\pm2(J_{2\alpha-1,2\alpha}+k-\alpha) \tag{4.3}$$

with eigenvalue $2(m_{2ki}+k-i)\pm2(m_{2k\alpha}+k-\alpha)$ in the restricted basis $[\mathcal{M}_{2k+1,\mu}^{(2k)}]$. The superscript zero on a subscript of $a_{i\alpha}$ has the following meaning: the eigenvalue of the corresponding $J_{2\alpha-1,2\alpha}$ is to take its highest possible value in $O(2k+1)$. For example, the eigenvalue of $a_{i\alpha^0}^{(2k+1)}$ is

$$2(m_{2ki}+k-i)+2(m_{2k+1,\alpha}+k-\alpha).$$

For $n=2k$ the normalized raising and lowering operators are

$$U_{2ki}=\frac{|b_{ii}c_i|^{\frac{1}{2}}}{2}\left|\left(\prod_{\alpha=1}^{i-1}\frac{b_{i\alpha}}{(a_{-i,\alpha^0}^{(2k)}+2)(a_{-i,\alpha}+2)b_{i\alpha^0}^{(2k)}}\right)\left(\prod_{\beta=i+1}^{k-1}\frac{a_{i-\beta}b_{i\beta}}{(a_{i,-\beta^0}^{(2k)}-2)b_{i\beta^0}^{(2k)}}\right)\frac{2}{b_{ii^0}^{(2k)}}\frac{2}{(a_{i^0,-i}^{(2k)}+2)}\frac{2}{(a_{i,-k^0}^{(2k)}-2)}\frac{2}{b_{ik^0}}\right|^{\frac{1}{2}}O_{2ki}, \tag{4.4}$$

$$U_{2k,-i}=O_{2k,-i}\frac{1}{2}\left|\frac{1}{b_{ii}c_i}\frac{2}{b_{ii^0}^{(2k)}}\frac{2}{(a_{i,-k^0}^{(2k)}-2)}\frac{2}{(a_{i^0,-i}^{(2k)}+2)}\frac{2}{b_{ik^0}^{(2k)}}\prod_{\alpha=1}^{i-1}\frac{(a_{-i\alpha}+2)}{(a_{-i\alpha^0}^{(2k)}+2)b_{i\alpha}b_{i\alpha^0}^{(2k)}}\prod_{\beta=i+1}^{k-1}\frac{1}{a_{i,-\beta}b_{i\beta}(a_{i,-\beta^0}^{(2k)}-2)b_{i\beta^0}^{(2k)}}\right|^{\frac{1}{2}}, \tag{4.5}$$

where $a_{i\alpha}$, $a_{i\alpha^0}^{(2k)}$ are defined as before, and $b_{i\alpha}=a_{i\alpha}-2$, $c_i=\frac{1}{2}(a_i-2)$.

The general basis vector for the orthogonal group can then be generated by successive applications of these operators. Taking the case $n=2k+1$ as an example,

$$|\mathcal{M}_{2k+1\mu}^{(2k)}\rangle=U_{2k+1-1}^{m_{2k+1,1}-m_{2k,1}}U_{2k+1,-2}^{m_{2k+1,2}-m_{2k,2}}\cdots U_{2k+1,-k}^{m_{2k+1,k}-m_{2k,k}}|\mathcal{M}_{2k+1,\mu}^{(2k+1)}\rangle=\overrightarrow{\prod_{\alpha=1}^{k}}U_{2k+1,-\alpha}^{m_{2k+1,\alpha}-m_{2k\alpha}}|\mathcal{M}_{2k+1,\mu}^{(2k+1)}\rangle,$$

$$|\mathcal{M}_{2k+1,\mu}^{(2k-1)}\rangle=U_{2k,-1}^{m_{2k,1}-m_{2k-1,1}}U_{2k,-2}^{m_{2k,2}-m_{2k-1,2}}\cdots U_{2k,-(k-1)}^{m_{2k,k-1}-m_{2k-1,k-1}}|\mathcal{M}_{2k+1,\mu}^{(2k)}\rangle=\overrightarrow{\prod_{\beta=1}^{k-1}}U_{2k,-\beta}^{m_{2k,\beta}-m_{2k-1,\beta}}|\mathcal{M}_{2k+1,\mu}^{(2k)}\rangle, \tag{4.6}$$

$$\vdots$$

$$|\mathcal{M}_{2k+1,\mu}^{(3)}\rangle=U_{3-1}^{m_{31}-m_{21}}|\mathcal{M}_{2k+1,\mu}^{(3)}\rangle.$$

Therefore

$$|\mathcal{M}_{2k+1,\mu}\rangle=\overrightarrow{\prod_{i=3}^{2k+1}}\overrightarrow{\prod_{j=1}^{[i-1]}}U_{i-j}^{m_{i\alpha}-m_{i-1,\alpha}}|\mathcal{M}_{2k+1,\mu}^{(2k+1)}\rangle, \tag{4.7}$$

where

$$[n]=\begin{cases}\frac{1}{2}(n-1), & n\text{ odd},\\ \frac{1}{2}n, & n\text{ even}.\end{cases}$$

The symbol $\overrightarrow{\prod}$ with an arrow means that terms are to be arranged in increasing order from left to right. Note that the eigenvalues of the $a_{i\alpha}$ depend upon the exact position of these factors in the ordered product.

A. Normalization Factor for the Case $n=2k+1$

Since the lowering operators $O_{2k+1,-\alpha}$ form a commuting set of operators in the restricted basis $[\mathcal{M}_{2k+1,\mu}^{(2k)}]$, it is sufficient to consider the special vector $|i\rangle$ in the calculation of the normalization factor associated with $O_{2k+1,-i}$, where $|i\rangle$ is defined by

$$|i\rangle=|[\mathcal{M}_{2k+1,\mu}^{(2k)}];\ m_{2k,\alpha}=m_{2k+1,\alpha}\quad\text{for}\quad\alpha\neq i\rangle. \tag{4.8}$$

Before calculating the normalization factors, a number of preparatory steps are taken.

(1) *The Quadratic Casimir Invariant*

It is well known that $\sum_{i<j}^{2k+1}J_{ij}^2$ is a quadratic invariant of $O(2k+1)$,

$$\sum_{i<j}^{2k+1}J_{ij}^2\,|\mathcal{M}_{2k+1,\mu}^{(\alpha)}\rangle=C_{2k+1}\,|\mathcal{M}_{2k+1,\mu}^{(\alpha)}\rangle\quad\text{for all }\alpha. \tag{4.9}$$

Expressing J_{ij} in terms of the Q operators of both

[14] The superscript $(2k+1)$ will be omitted whenever it is obvious.

Eqs. (2.2) and (2.3) the invariant takes the form

$$\sum_{i<j}^{2k+1} J_{ij}^2 = \sum_{i=1}^{k} Q_{2k+1,-i}Q_{2k+1,i} + \sum_{0<i<j}^{k} Q_{2j,-i}Q_{2j,i} + \sum_{0<i<j}^{k} Q_{2j-1,-i}Q_{2j-1,i} + \sum_{i=1}^{k} J_{2i-1,2i}^2 + \sum_{i=1}^{k}(2k-2i+1)J_{2i-1,2i}. \quad (4.10)$$

By applying (4.10) to

$$|\mathcal{M}_{2k+1,\mu}^{(2k+1)}\rangle$$

and using the fact that the raising generators Q_i give zero when operating on the highest-weight state, the invariant can be evaluated:

$$C_{2k+1} = \sum_{\alpha=1}^{k} m_{2k+1,\alpha}^2 + \sum_{\alpha=1}^{k}(2k-2\alpha+1)m_{2k+1,\alpha}. \quad (4.11)$$

(2) *Some Preparatory Lemmas*

Lemma 1:

$$Q_{n,j}\,|i\rangle = 0 \quad \text{for} \quad 0<j<i, \quad (4.12)$$

where the vector $|i\rangle$ is defined by Eq. (4.8).

Proof:

$$O_{n\alpha} = \sum_{\beta=1}^{\alpha} g_{\alpha\beta}(\rho)Q_{n\beta},$$

$$Q_{n\beta} = \sum_{\beta=1}^{\alpha} h_{\alpha\beta}(\rho)O_{n\beta},$$

since

$$O_{n\beta}\,|i\rangle = 0 \quad \beta \neq i \quad (m_{2k\beta} = m_{2k+1\beta} \quad \beta \neq i),$$

it follows that

$$Q_{n,j}\,|i\rangle = 0, \quad 0<j<i.$$

Lemma 2:

$$\langle i|\sum_{j\geq i}^{k} Q_{2k+1,-j}Q_{2k+1,j}\,|i\rangle = \langle i|\,(m_{2k+1,i} - J_{2i-1,2i}) \times (m_{2k+1,i} + J_{2i-1,2i} + 2k - 2i + 1)\,|i\rangle. \quad (4.13)$$

This is a consequence of Lemma 1 and Eqs. (4.10) and (4.11).

Lemma 3:

$$\langle i|\,Q_{n,-l}Q_{nl}\,|i\rangle = \langle i|\,\frac{2\sum_{m\geq i}^{l-1} Q_{n,-l}Q_{nl}}{a_{i-l}}\,|i\rangle = \langle i|\,\frac{2}{a_{i-l}}\prod_{\alpha=i+1}^{l-1}\frac{a_{i-\alpha}+2}{a_{i-\alpha}}\,Q_{n,-i}Q_{ni}\,|i\rangle, \quad l>i. \quad (4.14)$$

This follows from the relations

(i) $\langle i|\,Q_{n,-\beta}O_{n\beta}\,|i\rangle = 0, \quad i<\beta,$

(ii) $\langle i|\,\rho_{-\epsilon\lambda} = 0, \quad 0<\epsilon<|\lambda|,$

and a process of mathematical induction. Note that (ii) follows from $\langle i|\,\rho_{-\epsilon\lambda} = (-\rho_{\epsilon,-\lambda}\,|i\rangle)^\dagger$ and the fact that $-\rho_{\epsilon,-\lambda}$ with $0<\epsilon<|\lambda|$ is a raising generator of a subgroup of $O(n)$. Set $\beta = i+1$ in relation (i). As a consequence of Lemma 1 only two terms of $O_{n\beta}$ (corresponding to the first two graphs of Table I) survive. Commuting $Q_{n,-(i+1)}$ through the factor $\rho_{i+1,-i}$ and using relation (ii), the term arising through the second graph reduces to

$$-2\langle i|\,Q_{n,-i}Q_{ni}\prod_{\alpha=1}^{i-1} a_{\alpha,-(i+1)}\,|i\rangle.$$

Together with the first term this leads to the special case of Eq. (4.14), with $l = i+1$. By similar techniques the case with arbitrary l can be related to that with $l-1$.

Lemma 4:

$$\langle i|\,Q_{2k+1,-i}Q_{2k+1,i}\,|i\rangle = \langle i|\prod_{l=i+1}^{k}\frac{a_{i,-l}}{(a_{i-l}+2)}(m_{2k+1,i} - J_{2i-1,2i}) \times (m_{2k+1,i} + J_{2i-1,2i} + 2k - 2i + 1)\,|i\rangle. \quad (4.15)$$

This is a direct consequence of Lemma 3 and Lemma 2.

(3) *Evaluation of* $\langle i|\,O_{2k+1,i}O_{2k+1,-i}\,|i\rangle$

All terms in the raising operator $O_{2k+1,i}$, except the one term containing $Q_{2k+1,i}$, have at least one factor $\rho_{-\epsilon\lambda}$ $(0<\epsilon<|\lambda|)$ on the left-hand side. Since $\langle i|\,\rho_{-\epsilon\lambda} = 0$, the basic matrix element reduces to

$$\langle i|\,O_{2k+1,i}O_{2k+1,-i}\,|i\rangle = \langle i|\,Q_{2k+1,i}O_{2k+1,-i}\,|i\rangle\prod_{\alpha=1}^{i-1}\langle i|\,(a_{\alpha-i}+2)\,|i\rangle. \quad (4.16)$$

The matrix element $\langle i|\,Q_{2k+1,i}O_{2k+1,-i}\,|i\rangle$ is evaluated by commuting all of the factors $\rho_{-\epsilon\lambda}$ of $O_{2k+1,-i}$ through to the left-hand side where they give zero when operating on $\langle i|$. After this process only matrix elements of the type $\langle i|\,Q_{2k+1,-j}Q_{2k+1,j}\,|i\rangle$, $(j \geq i)$, survive. Their coefficients are evaluated in Appendix A by a process of summing of graphs. The matrix elements themselves are given by Lemmas 3 and 4. Combining these results (Appendix A), the basic matrix element is

$$\langle i|\,O_{2k+1,i}O_{2k+1,-i}\,|i\rangle = \langle i|\left(\prod_{\alpha=1}^{k} a_{i\alpha}\prod_{\gamma=i+1}^{k}(a_{i,-\gamma}-2)\prod_{\beta=1}^{i-1}(a_{-i\beta}+2)\right) \times (m_{2k+1,i} + J_{2i-1,2i} + 2k - 2i)\,|i\rangle. \; (m_{2k+1,i} - J_{2i-1,2i} + 1) \quad (4.17)$$

In the state $|i\rangle$ all $J_{2\alpha-1,2\alpha}$ except that with $\alpha = i$ yield their highest-weight value

$$\begin{aligned}\langle i| J_{2\alpha-1,2\alpha} |i\rangle &= m_{2k+1,\alpha}, \\ \langle i| J_{2i-1,2i} |i\rangle &= m_{2k,i}.\end{aligned} \quad (\alpha \neq i) \tag{4.18}$$

Thus

$$\begin{aligned}&\langle i| O_{2k+1,i}O_{2k+1,-i} |i\rangle \\ &= \langle \mathcal{M}^{(2k)}_{2k+1,\mu}| (m_{2k+1,i} - J_{2i-1,2i} + 2k - 2i) |\mathcal{M}^{(2k)}_{2k+1,\mu}\rangle \\ &\quad \times \tfrac{1}{2}\langle \mathcal{M}^{(2k)}_{2k+1,\mu}| \prod_{\alpha=1}^{k} a_{i\alpha^0}^{(2k+1)} \prod_{\gamma=i+1}^{k} \\ &\quad \times (a_{i,-\gamma^0}^{(2k+1)} - 2) \prod_{\beta=1}^{i} (a_{-i\beta^0}^{(2k+1)} + 2) |\mathcal{M}^{(2k)}_{2k+1,\mu}\rangle.\end{aligned} \tag{4.19}$$

The superscript zero on a subscript of $a_{i\alpha}$ has been defined in connection with Eq. (4.19). For example,

$$\langle \mathcal{M}^{(2k)}_{2k+1,\mu}| a_{i,-\gamma^0}^{(2k+1)} |\mathcal{M}^{(2k)}_{2k+1,\mu}\rangle = 2(m_{2k,i} - m_{2k+1,\gamma} + \gamma - i). \tag{4.20}$$

B. Normalization Factor for the Case $n = 2k$

(1) *The Quadratic Casimir Invariant*

$$\sum_{i<j}^{2k} J_{ij}^2 |\mathcal{M}^{(\alpha)}_{2k,\mu}\rangle = C_{2k} |\mathcal{M}^{(\alpha)}_{2k,\mu}\rangle \quad \text{for all } \alpha. \tag{4.21}$$

Expressing the J_{ij} in terms of Q operators as before

$$\begin{aligned}\sum_{i<j}^{2k} J_{ij}^2 &= \sum_{i=1}^{k-1} Q_{2k,-i}Q_{2ki} + \sum_{0<i<j}^{k-1} Q_{2j,-i}Q_{2ji} \\ &\quad + \sum_{0<i<j}^{k} Q_{2j-1,-i}Q_{2j-1,i} + \sum_{i=1}^{k} J_{2i-1,2i}^2 \\ &\quad + \sum_{i=1}^{k-1}(2k-2i)J_{2i-1,2i}.\end{aligned} \tag{4.22}$$

$$C_{2k} = \sum_{i=1}^{k} m_{2k,-i}^2 + \sum_{i=1}^{k-1}(2k-2i)m_{2k,i}. \tag{4.23}$$

As before, it is convenient to define the special vector

$$|i\rangle = |\mathcal{M}^{(2k-1)}_{2k,\mu}; m_{2k,\alpha} = m_{2k-1,\alpha} \quad \alpha \neq i\rangle. \tag{4.24}$$

Since the raising operators for $O(2k)$ and $O(2k+1)$ have the same form, Eqs. (4.12) and (4.14) hold, and

$$\langle i| \sum_{j\geq i}^{k-1} Q_{2k,-j}Q_{2kj} |i\rangle = \langle i| \prod_{\alpha=i+1}^{k-1} \frac{(a_{i-\alpha}+2)}{a_{i\alpha}} Q_{2k,-i}Q_{2k,i} |i\rangle. \tag{4.25}$$

Putting this relation back into the expression for the quadratic Casimir invariant gives

$$\begin{aligned}&\langle i| \left[\prod_{\alpha=i+1}^{k-1} \frac{(a_{i-\alpha}+2)}{a_{i-\alpha}}\right] Q_{2k,-i}Q_{2k,i} + J_{2k-1,2k}^2 \\ &\quad + J_{2i-1,2i}^2 + (2k-2i)J_{2i-1,2i} |i\rangle \\ &= m_{2ki}^2 + (2k-2i)m_{2k,i} + m_{2k,k}^2.\end{aligned} \tag{4.26}$$

Unlike the corresponding equation for the case of the odd-dimensional orthogonal group, this relation is not sufficient to evaluate the matrix element $\langle i| Q_{2k,-i}Q_{2k,i} |i\rangle$, since the matrix element

$$\langle i| J_{2k-1,2k}J_{2k-1,2k} |i\rangle$$

is not known. However, there is now one more invariant at our disposal.

(2) *The Quadratic Invariant in the Restricted Basis* $[\mathcal{M}^{(2k-1)}_{2k\mu}]$

Since the (zero-step) neutral operator $O_{2k,k}$ commutes with all raising and lowering operators when applied to the basis $[\mathcal{M}^{(2k-1)}_{2k,\mu}]$, it is an invariant in this restricted basis. To get a relation between the matrix elements of the quadratic factors $J_{2k-1,2k}^2$ and $Q_{2k,-i}Q_{2k,i}$ consider $\langle i| O_{2k,k}O_{2k,k} |i\rangle$, where

$$\begin{aligned}\langle i| O_{2k,k}O_{2k,k} |i\rangle &= \langle i| \prod_{\alpha=1}^{k-1} a_\alpha J_{2k-1,2k}O_{2k,k} |i\rangle \\ &= \langle i| \prod_{\alpha=1}^{k-1} a_\alpha |i\rangle\langle i| J_{2k-1,2k}O_{2k,k} |i\rangle\end{aligned} \tag{4.27}$$

through the relation $\langle i| \rho_{-j\epsilon} = 0,\ 0 < j < |\epsilon|$. Summing up of the matrix elements from all the possible graphs in $O_{2k,k}$ with techniques similar to those illustrated in Appendix A leads to

$$\begin{aligned}&\langle i| O_{2k,k}O_{2k,k} |i\rangle \\ &= \langle i| \prod_{\alpha=1}^{k-1} a_\alpha^2 |i\rangle \\ &\quad \times \langle i| J_{2k-1,2k}^2 - \prod_{\alpha=i+1}^{k-1} \frac{(a_{i-\alpha}+2)}{a_{i-\alpha}a_i} Q_{2k,-i}Q_{2k,i} |i\rangle.\end{aligned} \tag{4.28}$$

On the other hand, since $O_{2k,k}$ is an invariant

$$\langle i| O_{2k,k}O_{2k,k} |i\rangle = \langle \mathcal{M}^{(2k)}_{2k,\mu}| O_{2k,k}O_{2k,k} |\mathcal{M}^{(2k)}_{2k,\mu}\rangle. \tag{4.29}$$

Also

$$\langle i| a_\alpha^2 |i\rangle = \langle \mathcal{M}^{(2k)}_{2k,\mu}| a_\alpha^2 |\mathcal{M}^{(2k)}_{2k,\mu}\rangle \quad \text{for } \alpha \neq i. \tag{4.30}$$

By applying (4.28), (4.29), and (4.30), the quartic invariant leads to the relation

$$\begin{aligned}&\langle i| J_{2k-1,2k}^2 a_i^2 - a_i \prod_{\alpha=i+1}^{k-1} \frac{(a_{i-\alpha}+2)}{a_{i-\alpha}} Q_{2k,-i}Q_{2k,i} |i\rangle \\ &= 4m_{2k,k}^2(m_{2k,i} + k - i)^2.\end{aligned} \tag{4.31}$$

(3) *Evaluation of* $\langle i| O_iO_{-i} |i\rangle$

Since we have two equations and two unknowns we can determine both $\langle i| Q_{2k,-i}Q_{2k,i} |i\rangle$ and $\langle i| J_{2k-1,2k}^2 |i\rangle$. The technique for the summing up of the graphs is similar to the case of $O(2k+1)$ illustrated in Appendix

A (and Sec. 4A3.) and leads to

$$\langle i|\, O_{2k,i}O_{2k,-i}\,|i\rangle$$
$$= \tfrac{1}{4}\langle \mathcal{M}^{(2k-1)}_{2k,\mu}|\prod_{\alpha=i+1}^{k}(a^{(2k)}_{i-\alpha^0}-2)\prod_{\beta=1}^{i}(a^{(2k)}_{-i,\beta^0}+2)$$
$$\times\prod_{\gamma=1}^{k} b^{(2k)}_{i\gamma^0}\,|\mathcal{M}^{(2k-1)}_{2k,\mu}\rangle\langle\mathcal{M}^{(2k-1)}_{2k,\mu}|\,\frac{b^{(2k)}_{ii^0}}{b_{ii}}\,|\mathcal{M}^{(2k-1)}_{2k,\mu}\rangle. \quad (4.32)$$

The superscript zero on a subscript of $a_{i\alpha}$ has the same meaning as before. For example,

$$\langle\mathcal{M}^{(2k-1)}_{2k,\mu}|\, b^{(2k)}_{i\gamma^0}\,|\mathcal{M}^{(2k-1)}_{2k,\mu}\rangle$$
$$= 2(m_{2k-1,i}+m_{2k,\gamma}+2k-i-\gamma-1). \quad (4.33)$$

C. The Normalization Coefficients

Let the normalized lowering (raising) operators be denoted by $U_{n,\pm i}$. If the state $|\mathcal{M}^{(n-1)}_{n\mu}\rangle$ is normalized

$$\langle\mathcal{M}^{(n-1)}_{n\mu};\, m_{n-1,i}|\, U_{ni}U_{n,-i}\,|\mathcal{M}^{(n-1)}_{n\mu};\, m_{n-1,i}\rangle$$
$$= \langle\mathcal{M}^{(n-1)}_{n\mu};\, m_{n-1,i}-1\mid\mathcal{M}^{(n-1)}_{n\mu};\, m_{n-1,i}-1\rangle = 1. \quad (4.34)$$

But

$$\langle\mathcal{M}^{(n-1)}_{n\mu};\, m_{n-1,i}-1| = \langle\mathcal{M}^{(n-1)}_{n\mu};\, m_{n-1,i}|\,(U_{n,-i})^{\dagger}. \quad (4.35)$$

The normalized lowering (raising) operators should thus have the property

$$U_{ni} = (U_{n,-i})^{\dagger}. \quad (4.36)$$

The lowering (raising) operators of type $O_{n,\pm i}$ do not satisfy this relation. However, if O_{ni} is a lowering (raising) operator of $[\mathcal{M}^{(n-1)}_{n\mu}]$, so is

$$f_{ni}(J_{12}, J_{34}, \cdots)O_{ni},$$

where f_{ni} is a function of $J_{2\alpha-1,2\alpha}$ only ($\alpha = 1, \cdots, k$ for $n = 2k+1$, $\alpha = 1, \cdots, k-1$ for $n = 2k$), and where $f_{n,\pm i}$ can be chosen such that

$$f_{ni}O_{ni} = (O_{n,-i}f_{n,-i})^{\dagger}. \quad (4.37)$$

Since any arbitrary function $g(Q,\rho,J)\rho_{\alpha\beta}$, with $0<\alpha<1\,|\beta|$, is a null operator when acting on $[\mathcal{M}^{(n-1)}_{n\mu}]$ and can be added to a raising or lowering operator without changing its raising or lowering property, the functions $f_{ni}, f_{n,-i}$ must be evaluated by comparing the ρ independent terms on each side of Eq. (4.37). This leads to

$$f_{2k+1,i} = \prod_{\alpha=i+1}^{k} a_{i,-\alpha}\prod_{\beta=1}^{k} a_{i\beta}, \quad (4.38)$$

$$f_{2k+1,-i} = \prod_{\gamma=1}^{i-1}(a_{-i,\gamma}+2), \quad (4.39)$$

$$f_{2k,-i} = \prod_{\alpha=1}^{i-1}(a_{-i,\alpha}+2), \quad (4.40)$$

$$f_{2k,i} = c_i b_{ii}\prod_{\beta=1}^{i-1} b_{i\beta}\prod_{\alpha=i+1}^{k-1} a_{i,-\alpha}b_{i\alpha}. \quad (4.41)$$

Thus

$$U_{ni} = (f_{ni}/N_{ni})O_{ni}, \quad (4.42)$$
$$U_{n,-i} = O_{n,-1}(f_{n,-i}/N_{ni}), \quad (4.43)$$

where N_{ni} is a factor which is defined to be real. With

$$\langle\mathcal{M}^{(n-1)}_{n\mu}|\, U_{ni}U_{n,-i}\,|\mathcal{M}^{(n-1)}_{n\mu}\rangle = 1,$$
$$\langle\mathcal{M}^{(n-1)}_{n\mu}|\, f_{ni}O_{ni}O_{n,-i}f_{n,-i}\,|\mathcal{M}^{(n-1)}_{n\mu}\rangle = N^2_{ni}. \quad (4.44)$$

Note that the U_{ni}, unlike the O_{ni}, do not form a commuting set of lowering (raising) operators, $[U_{n\alpha}, U_{n\beta}] \neq 0$, since $[f_{n\alpha}, O_{n\beta}] \neq 0$. However,

$$[O_{ni}O_{n,-i}, f_{n\alpha}] = 0, \quad \text{for any } \alpha. \quad (4.45)$$

Therefore,

$$N^2_{ni} = \langle\mathcal{M}^{(n-1)}_{n\mu}|\, f_{ni}f_{n,-i}\,|\mathcal{M}^{(n-1)}_{n\mu}\rangle\langle i|\, O_{ni}O_{n,-i}\,|i\rangle, \quad (4.46)$$

With Eqs. (4.19), (4.32), and (4.38)–(4.41)

$$N_{2k+1,i} = a^{(2k+1)}_{ii}\left|\prod_{\alpha=i+1}^{k} a_{i,-\alpha}a_{i\alpha}a^{(2k+1)}_{i\alpha^0}(a^{(2k+1)}_{i,-\alpha^0}-2)\right.$$
$$\left.\times\prod_{\beta=1}^{i-1} a_{i\beta}(a_{-i,\beta}+2)(a^{(2k+1)}_{-i,\beta^0}+2)a^{(2k+1)}_{i\beta^0}\right|^{\frac{1}{2}}, \quad (4.47)$$

$$N_{2k,i} = \left|\frac{b_{ii}c_i}{4}\, b^{(2k)}_{ii^0}(a^{(2k)}_{i^0,-i}+2)(a^{(2k)}_{i,-k^0}-2)b^{(2k)}_{ik^0}\right|^{\frac{1}{2}}$$
$$\times\left|\prod_{\alpha=1}^{i-1}(a_{-i,\alpha}+2)b_{i\alpha}b^{(2k)}_{i\alpha^0}(a^{(2k)}_{-i,\alpha^0}+2)\right.$$
$$\left.\times\prod_{\beta=i+1}^{n-1} a_{i,-\beta}b_{i\beta}(a^{(2k)}_{i,-\beta^0}-2)b^{(2k)}_{i\beta^0}\right|^{\frac{1}{2}}. \quad (4.48)$$

5. MATRIX ELEMENTS OF $J_{n-1,n}$

In the evaluation of the matrix elements of the infinitesimal generators, the matrix elements of $J_{n-1,n}$ play the fundamental role since the matrix elements of all other J_{ij} can be simply related to these. Matrix elements of $J_{n-1,n}$ have been given by Gel'fand and Zetlin.[6,7] A derivation of the Gel'fand–Zetlin result is given here to illustrate the usefulness of the lowering (raising) operators.

Since $J_{n-1,n}$ commutes with all J_{ij} with both $i, j < n-1$, $J_{n-1,n}$ is a scalar operator with respect to $O(n-2)$. The matrix elements of $J_{n-1,n}$ are thus diagonal in $m_{n-2,\alpha}$ and independent of $m_{\nu,\alpha}$, $\nu \leq n-3$. With respect to $O(n)$, $J_{n-1,n}$ transforms according to the regular representation $[11000\cdots]$. With respect to $O(n-1)$ its irreducible tensor character is that of the vector representation $[1000\cdots]$. It thus connects states in which any one of the $m_{n-1,\alpha}$ differ by ± 1 only. (For $n-1$ odd, it also has a diagonal matrix element.)

$$\langle\mathcal{M}'_{n\mu}|\, J_{n-1,n}\,|\mathcal{M}_{n\mu}\rangle = \langle\mathcal{M}'^{(n-2)}_{n\mu}|\, J_{n-1,n}\,|\mathcal{M}^{(n-2)}_{n\mu}\rangle$$
$$= \left\langle\begin{matrix} m_{ni} \\ m'_{n-1,i} \\ m_{n-2,i}\end{matrix}\right|\, J_{n-1n}\,\left|\begin{matrix} m_{ni} \\ m_{n-1,i} \\ m_{n-2,i}\end{matrix}\right\rangle. \quad (5.1)$$

For convenience, only the relevant $m_{\nu i}$ in the one column subject to change are written out. The matrix elements in the $[\mathcal{M}_{n\mu}^{(n-2)}]$ basis could be evaluated through a construction involving successive application of lowering operators of type $U_{n,i}$ followed by $U_{n-1,i}$. It is more convenient to factor the matrix element of $J_{n-1,n}$ into two parts by using the Wigner–Eckart theorem. The reduced matrix element, independent of the $m_{n-2,\alpha}$, can be chosen as the matrix element of $J_{n-1,n}$ in the restricted basis $[\mathcal{M}_{n\mu}^{(n-1)}]$, while the $m_{n-2,\alpha}$-dependent factor can be expressed as the matrix element of a vector operator in $(n-1)$-dimensional space

$$\langle \mathcal{M}'_{n\mu}|\, J_{n-1,n}\, |\mathcal{M}_{n\mu}\rangle = \left\langle \begin{matrix} m_{ni} \\ m'_{n-1,i} \\ m_{n-1,i}\end{matrix} \right| J_{n-1,n} \left| \begin{matrix} m_{ni} \\ m_{n-1,i} \\ m_{n-1,i}\end{matrix} \right\rangle \times \left\langle \begin{matrix} m'_{n-1,i} \\ m_{n-2,i}\end{matrix} \right| V \left| \begin{matrix} m_{n-1,i} \\ m_{n-2,i}\end{matrix} \right\rangle, \quad (5.2)$$

where V has irreducible tensor character $[1000\cdots]$ with respect to $O(n-1)$ and $[000\cdots]$ with respect to $O(n-2)$, and its matrix element is normalized to unity when $m_{n-2,i} = m_{n-1,i}$ (all i). The first factor imposes the restriction $m'_{n-1,i} \geq m_{n-1,i}$. However, the matrix element with $m'_{n-1,i} = m_{n-1,i} - 1$ can be obtained from that with $m'_{n-1,i} = m_{n-1,i} + 1$ through the Hermiticity of $J_{n-1,n}$.

A. Evaluation of $\left\langle \begin{matrix} m_{ni} \\ m'_{n-1,i} \\ m_{n-1,i}\end{matrix} \right| J_{n-1i,n} \left| \begin{matrix} m_{ni} \\ m_{n-1,i} \\ m_{n-1,i}\end{matrix} \right\rangle$

(1) $n = 2k$

$O_{2k,k}$ is a linear combination of $J_{2k-1,2k}$ and $Q_{2k,\alpha}$. Re-expressing $O_{2k,k}$ instead as a linear combination of $J_{2k-1,2k}$ and $O_{2k,i}$

$$\left\{ O_{2k,k} = \left[J_{2k-1,2k} \prod_{\alpha=1}^{k-1} a_\alpha^{(2k)} + \sum_{\alpha=1}^{k-1} O_{2k-1,-\alpha} O_{2k,\alpha} h_i \right] \right\} \left| \begin{matrix} m_{2k,i} \\ m_{2k-1,i}\end{matrix} \right\rangle = 0, \quad (5.3)$$

where h_i are functions of $J_{2i-1,2i}$, $(i < k)$, which are to be determined from the conditions required for $O_{2k,k}$

(a) $$[\rho_{jl}, O_{2k,k}] \left| \begin{matrix} m_{n,j} \\ m_{n-1,j}\end{matrix} \right\rangle = 0, \quad 0 < j < |l| \leq k-1, \quad (5.4)$$

(b) $$[Q_{2k-1,l}, O_{2k,k}] \left| \begin{matrix} m_{n,j} \\ m_{n-1,j}\end{matrix} \right\rangle = 0. \quad (5.5)$$

Condition (a) is automatically satisfied. In order to satisfy condition (b):

$$\left\{ [Q_{2k-1,j}, J_{2k-1,2k}] \prod_{\alpha=1}^{k-1} a_\alpha^{(2k)} + \sum_{\alpha=1}^{k-1} [Q_{2k-1,j}, O_{2k-1,-\alpha}] O_{2k\alpha} h_\alpha \right\} \left| \begin{matrix} m_{n,j} \\ m_{n-1,j}\end{matrix} \right\rangle = 0. \quad (5.6)$$

From the coefficients of $Q_{2k,i}$, however, the h_j follow directly

$$h_j = i \left(\prod_{\alpha=1}^{j-1} \frac{a_\alpha^{(2k)}}{(a_{j\alpha}^{(2k)} - 2) a_{-j,\alpha}^{(2k)}} \prod_{\beta=j+1}^{k-1} \frac{a_\beta^{(2k)}}{a_{j,-\beta}^{(2k)} (a_{j\beta}^{(2k)} - 2)} \right) \frac{1}{a_{jj}^{(2k)}}. \quad (5.7)$$

Also

$$O_{2k,k} \left| \begin{matrix} m_{nj} \\ m_{n-1,j}\end{matrix} \right\rangle = \frac{1}{2} \prod_{i=1}^{k} a_{j^0}^{(2k)} \left| \begin{matrix} m_{2k,j} \\ m_{2k-1,j}\end{matrix} \right\rangle. \quad (5.8)$$

Re-expressing the $O_{2k-1,-\alpha}$ and $O_{2k,\alpha}$ of Eq. (5.3) in terms of $U_{2k-1,-\alpha}$ and $U_{2k,\alpha}$, the matrix element of $J_{2k-1,2k}$ can be read off from Eq. (5.3):

$$\left\langle \begin{matrix} m_{2k,j} \\ m_{2k-1,j} \\ m_{2k-1,j}\end{matrix} \right| J_{2k-1,2k} \left| \begin{matrix} m_{2k,j} \\ m_{2k-1,j} \\ m_{2k-1,j}\end{matrix} \right\rangle = \left\langle \begin{matrix} m_{2k,j} \\ m_{2k-1,j}\end{matrix} \right| \frac{a_{k^0}^{(2k)}}{2} \prod_{\alpha=1}^{k} \frac{a_{\alpha^0}^{(2k)}}{a_\alpha^{(2k)}} \left| \begin{matrix} m_{2k,j} \\ m_{2k-1j}\end{matrix} \right\rangle, \quad (5.9)$$

$$\left\langle \begin{matrix} m_{2k,j} \\ m_{2k-1,j}+1 \\ m_{2k-1,j}\end{matrix} \right| J_{2k-1,2k} \left| \begin{matrix} m_{2k,j} \\ m_{2k-1,j} \\ m_{2k-1,j}\end{matrix} \right\rangle = \left\langle \begin{matrix} m_{2k,j} \\ m_{2k-1,j}\end{matrix} \right| \frac{-2i}{a_j^{(2k)}(a_j+1)^{\frac{1}{2}}} \left| \left(\prod_{\alpha=1}^{j-1} \frac{a_{-j\alpha^0}^{(2k)} a_{j\alpha^0}^{(2k)}}{a_{-j\alpha}^{(2k)} a_{j\alpha}^{(2k)}} \prod_{\beta=j+1}^{k-1} \frac{a_{j-\beta^0}^{(2k)} a_{j\beta^0}^{(2k)}}{a_{j-\beta}^{(2k)} a_{j\beta}^{(2k)}} \right) \frac{a_{jj^0}^{(2k)}}{2} \frac{a_{j^0-j}^{(2k)}}{2} \frac{a_{j-k^0}^{(2k)}}{2} \frac{a_{jk^0}^{(2k)}}{2} \right|^{\frac{1}{2}} \left| \begin{matrix} m_{2k,j} \\ m_{2k-1,j}\end{matrix} \right\rangle$$

$$(j = 1, 2, \cdots, k-1) \quad (5.10)$$

(2) *The* $n = 2k + 1$ *Case*

The procedure is similar to that for $n = 2k$. First, since $J_{2k,2k+1}$ has no diagonal matrix elements, in place of the neutral operator there is now the relation

$$\left\{ J_{2k,2k+1} - \left[\sum_{\alpha=1}^{k-1} O_{2k,-\alpha} O_{2k+1,\alpha} h_\alpha + O_{2k+1,k} h_k + O_{2k+1,-k} h_{-k} \right] \right\} \left| \begin{matrix} m_{2k+1,i} \\ m_{2k,i}\end{matrix} \right\rangle = 0, \quad (5.11)$$

where the h_α are evaluated as before through

$$\Big[Q_{2k,j}, J_{2k,2k+1} - \sum_{\alpha=1}^{k-1} O_{2k,-\alpha}O_{2k+1,\alpha}h_\alpha - O_{2k+1,k}h_k - O_{2k+1,-k}h_{-k}\Big]\left|\begin{matrix} m_{2k+1,i} \\ m_{2k,i}\end{matrix}\right\rangle = 0. \quad (5.12)$$

From the coefficients of $Q_{2k+1,j}$,

$$h_j = i\Big[2(b_{jj}^{(2k)} + 4)\Big(\frac{a_{j-k}^{(2k)}}{2}\frac{a_{jk}^{(2k)}}{2}\Big) \times \prod_{\alpha=1}^{j-1} a_{j\alpha}^{(2k)}a_{-j,\alpha}^{(2k)} \prod_{\beta=j+1}^{k-1} a_{j,-\beta}^{(2k)}a_{j\beta}^{(2k)}\Big)^{-1}. \quad (5.13)$$

Similarly, re-expressing the O operators in terms of U operators, the matrix element of $J_{2k,2k+1}$ can be read off from Eq. (5.11).

$$\left\langle\begin{matrix} m_{2k+1,j} \\ m_{2k,j}+1 \\ m_{2k,j}\end{matrix}\right| J_{2k,2k+1} \left|\begin{matrix} m_{2k+1,j} \\ m_{2k,j} \\ m_{2k,j}\end{matrix}\right\rangle = \left\langle\begin{matrix} m_{2k+1,j} \\ m_{2k,j}\end{matrix}\right| \frac{-i}{2} \left|\frac{a_{-jj^0}^{(2k+1)}(a_{jj^0}^{(2k+1)}+2)a_j^{(2k+1)}}{a_{jk}^{(2k+1)}(a_{j-k}^{(2k+1)}+2)}\right|^{\frac12} \left|\begin{matrix} m_{2k+1,j} \\ m_{2k,j}\end{matrix}\right\rangle$$
$$\times \left\langle\begin{matrix} m_{2k+1,j} \\ m_{2k,j}\end{matrix}\right| \prod_{\alpha=1}^{j-1}\left|\frac{a_{-j\alpha^0}^{(2k+1)}(a_{j\alpha^0}^{(2k+1)}+2)}{a_{-j\alpha}^{(2k+1)}(a_{j\alpha}^{(2k+1)}+2)}\right|^{\frac12} \prod_{\beta=j+1}^{k}\left|\frac{a_{j-\beta^0}^{(2k+1)}(a_{j\beta^0}^{(2k+1)}+2)}{a_{j-\beta}^{(2k+1)}(a_{j\beta}^{(2k+1)}+2)}\right|^{\frac12} \left|\begin{matrix} m_{2k+1,j} \\ m_{2k,j}\end{matrix}\right\rangle, \quad j = 1, 2, \cdots, k. \quad (5.14)$$

B. Evaluation of $\left\langle\begin{matrix} m'_{n-1,i} \\ m_{n-2,i}\end{matrix}\right| V^{(n-1)} \left|\begin{matrix} m_{n-1,i} \\ m_{n-2,i}\end{matrix}\right\rangle$

$V^{(n-1)}$ has the transformation properties of

$$\left|\begin{matrix} 1 & 0 & \cdots \\ 0 & 0 & \cdots\end{matrix}\right\rangle$$

and is to be normalized such that

$$\left\langle\begin{matrix} m_{n-1,1} & m_{n-1,2} & m_{n-1,3} & \cdots & m'_{n-1,i} & m_{n-1,i+1} & \cdots \\ m_{n-1,1} & m_{n-1,2} & m_{n-1,3} & \cdots & m_{n-1,i} & m_{n-1,i+1} & \cdots\end{matrix}\right| V^{(n-1)} \left|\begin{matrix} m_{n-1,1} & m_{n-1,2} & \cdots & m_{n-1,i} & \cdots \\ m_{n-1,1} & m_{n-1,2} & \cdots & m_{n-1,i} & \cdots\end{matrix}\right\rangle = 1. \quad (5.15)$$

It is convenient to introduce the following shorthand notation. Change $m_{ni} \to \alpha_i$ $m_{n-1,i} \to \beta_i$ $m_{n-2,i} \to \gamma_i \le \beta_i$ and define

$$\left|\overline{\begin{matrix}\beta_i \\ \gamma_i\end{matrix}}\right\rangle = \left|\begin{matrix} \beta_1 & \beta_2 & \cdots & \beta_{i-1} & \beta_i & \beta_{i+1} & \beta_{i+2} & \cdots \\ \beta_1 & \beta_2 & \cdots & \beta_{i-1} & \gamma_i & \gamma_{i+1} & \gamma_{i+2} & \cdots\end{matrix}\right\rangle, \quad (5.16)$$

$$\left|\overline{\begin{matrix}\beta_j+1 & \beta_i \\ \beta_j & \gamma_i\end{matrix}}\right\rangle = \left|\begin{matrix} \beta_1 & \beta_2 & \cdots & \beta_j+1 & \beta_{j+1} & \beta_{j+2} & \cdots & \beta_{i-1} & \beta_i & \beta_{i+1} & \beta_{i+2} & \cdots \\ \beta_1 & \beta_2 & \cdots & \beta_j & \beta_{j+1} & \beta_{j+2} & \cdots & \beta_{i-1} & \gamma_i & \gamma_{i+1} & \gamma_{i+2} & \cdots\end{matrix}\right\rangle, \quad (5.17)$$

$$\left|\overline{\begin{matrix}\beta_i & \beta_j+1 \\ \gamma_i & \gamma_j\end{matrix}}\right\rangle = \left|\begin{matrix} \beta_1 & \beta_2 & \cdots & \beta_{i-1} & \beta_i & \beta_{i+1} & \cdots & \beta_{j-1} & \beta_j+1 & \beta_{j+1} & \cdots \\ \beta_1 & \beta_2 & \cdots & \beta_{i-1} & \gamma_i & \gamma_{i+1} & \cdots & \gamma_{j-1} & \gamma_j & \gamma_{j+1} & \cdots\end{matrix}\right\rangle, \quad (5.18)$$

$$\left|\begin{matrix}1 \\ 0\end{matrix}\right\rangle = \left|\begin{matrix} 1 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots\end{matrix}\right\rangle. \quad (5.19)$$

(1) *The n = 2k Case*

Define coefficients Γ_{ij} by the relation

$$\left|\begin{matrix}1 \\ 0\end{matrix}\right\rangle \left|\overline{\begin{matrix}\beta_i \\ \gamma_i\end{matrix}}\right\rangle = \sum_{j=1}^{i-1}\Gamma_{ij}\begin{pmatrix}\beta_j & \beta_i \\ \beta_j & \gamma_i\end{pmatrix}\left|\overline{\begin{matrix}\beta_j+1 & \beta_i \\ \beta_j & \gamma_i\end{matrix}}\right\rangle + \sum_{j=i}^{k-1}\Gamma_{ij}\begin{pmatrix}\beta_i & \beta_j \\ \gamma_i & \gamma_j\end{pmatrix}\left|\overline{\begin{matrix}\beta_i & \beta_j+1 \\ \gamma_i & \gamma_j\end{matrix}}\right\rangle + \sum_{j=i}^{k-1}\Gamma_{i-j}\begin{pmatrix}\beta_i & \beta_j-1 \\ \gamma_i & \gamma_j\end{pmatrix}\left|\overline{\begin{matrix}\beta_i & \beta_j-1 \\ \gamma_i & \gamma_j\end{matrix}}\right\rangle + \Gamma_{i0}\begin{pmatrix}\beta_i \\ \gamma_i\end{pmatrix}\left|\overline{\begin{matrix}\beta_i \\ \gamma_i\end{matrix}}\right\rangle, \quad (5.20)$$

where the Γ_{ij} are generalized Wigner coefficients for the Kronecker product $[100\cdots] \times [\beta_1\beta_2\beta_3\cdots]$ of $O(2k-1)$. Note that Γ_{1j}, the coefficient with *all* $\gamma_i < \beta_i$ starting with γ_1, is equal to the matrix element of $V^{(2k-1)}$ provided Γ_{k-1j}, the coefficient with *all* $\gamma_i = \beta_i$, satisfies the normalization condition $\Gamma_{k-1j} = 1$ required by Eq. (5.15). The coefficients Γ_{ij} can be related to the coefficients $\Gamma_{i+1,j}$ by recursion techniques, leading after repeated recursion to a relation between Γ_{1j} and $\Gamma_{k-1,j}$. Since the recursion is to be established through the raising generator Q_i, it is necessary to define further coefficients, Ω_{ij}, by the relation

$$\left\{Q_{2k-1,i}\left|\begin{matrix}1\\0\end{matrix}\right\rangle\right\}\left|\begin{matrix}\bar{\beta}_i\\\gamma_i\end{matrix}\right\rangle=\sum_{j=1}^{i-1}\Omega_{ij}\begin{pmatrix}\beta_j&\beta_i\\\beta_j&\gamma_i\end{pmatrix}\left|\overline{\begin{matrix}\beta_j+1&\beta_i\\\beta_j&\gamma_i+1\end{matrix}}\right\rangle+\sum_{j=i}^{k-1}\Omega_{ij}\begin{pmatrix}\beta_i&\beta_j\\\gamma_i&\gamma_j\end{pmatrix}\left|\overline{\begin{matrix}\beta_i&\beta_j+1\\\gamma_i+1&\gamma_j\end{matrix}}\right\rangle+\sum_{j=i}^{k-1}\Omega_{i,-j}\begin{pmatrix}\beta_i&\beta_j\\\gamma_i&\gamma_j\end{pmatrix}\left|\overline{\begin{matrix}\beta_i&\beta_j-1\\\gamma_i+1&\gamma_j\end{matrix}}\right\rangle+\Omega_{i0}\begin{pmatrix}\beta_i\\\gamma_i\end{pmatrix}\left|\overline{\begin{matrix}\beta_i\\\gamma_i+1\end{matrix}}\right\rangle+\sum_{j=1}^{i-1}\Lambda(\rho)\left|\overline{\begin{matrix}\beta_j+1&\beta_i\\\beta_j+1&\gamma_i\end{matrix}}\right\rangle. \tag{5.21}$$

The operators $\Lambda(\rho)$ of the last term of the equation, when acting on states (5.16), create states outside the basis $[\mathcal{M}_{2k-1,\mu}^{(2k-2)}]$. These are orthogonal to the states of present interest so that the last term of Eq. (5.21) plays no further role in the discussion.

Applying $Q_{2k-1,i}$ again to Eq. (5.21), a set of recursion relations is established for the coefficients Ω_{ij}

$$\frac{\Omega_{ij}\begin{pmatrix}\beta_i&\beta_j\\\gamma_i&\gamma_j\end{pmatrix}}{\Omega_{ij}\begin{pmatrix}\beta_i&\beta_j\\\gamma_i+1&\gamma_j\end{pmatrix}}=\frac{q_i\begin{pmatrix}\beta_i\\\gamma_i\end{pmatrix}}{q_i\begin{pmatrix}\beta_i&\beta_j+1\\\gamma_i+1&\gamma_j\end{pmatrix}}, \tag{5.22}$$

$$\frac{\Omega_{i0}\begin{pmatrix}\beta_i\\\gamma_i\end{pmatrix}}{\Omega_{i0}\begin{pmatrix}\beta_i\\\gamma_i+1\end{pmatrix}}=\frac{q_i\begin{pmatrix}\beta_i\\\gamma_i\end{pmatrix}}{q_i\begin{pmatrix}\beta_i\\\gamma_i+1\end{pmatrix}}, \tag{5.23}$$

where

$$\left\langle\overline{\begin{matrix}\beta_i\\\gamma_i+1\end{matrix}}\right|Q_{2k-1,i}\left|\begin{matrix}\bar{\beta}_i\\\gamma_i\end{matrix}\right\rangle=q_i\begin{pmatrix}\beta_i\\\gamma_i\end{pmatrix}. \tag{5.24}$$

Applying $Q_{2k-1,i}$ also to Eq. (5.20), another set of recursion relations is established,

$$\Gamma_{ij}\begin{pmatrix}\beta_i&\beta_j\\\gamma_i&\gamma_j\end{pmatrix}q_i\begin{pmatrix}\beta_i&\beta_j+1\\\gamma_i&\gamma_j\end{pmatrix}=q_i\begin{pmatrix}\beta_i\\\gamma_i\end{pmatrix}\Gamma_{ij}\begin{pmatrix}\beta_i&\beta_j\\\gamma_i+1&\gamma_j\end{pmatrix}+\Omega_{ij}\begin{pmatrix}\beta_i&\beta_j\\\gamma_i&\gamma_j\end{pmatrix}, \tag{5.25}$$

$$\Gamma_{i0}\begin{pmatrix}\beta_i\\\gamma_i\end{pmatrix}q_i\begin{pmatrix}\beta_i\\\gamma_i\end{pmatrix}=q_i\begin{pmatrix}\beta_i\\\gamma_i\end{pmatrix}\Gamma_{i0}\begin{pmatrix}\beta_i\\\gamma_i+1\end{pmatrix}+\Omega_{i0}\begin{pmatrix}\beta_i\\\gamma_j\end{pmatrix}. \tag{5.26}$$

The recursion process for Γ_{ij} can be started if the coefficients Γ_{ij}, Ω_{ij} can be related for a particular value of γ_i. The cases Γ_{ij} and Γ_{i0} are somewhat different.

a. *The Case* Γ_{ij}. From Eqs. (4.1)

$$q_i\begin{pmatrix}\beta_i&\beta_j\\\gamma_i&\gamma_j\end{pmatrix}=\left|(\beta_i-\gamma_i)(\beta_i+\gamma_i+2k-2i-1)\times\prod_{\lambda=i+1}^{k-1}\frac{a_{i,-\lambda}{}^0(a_{i,\lambda}{}^0+2)}{(a_{i,\lambda}+2)(a_{i,-\lambda}+2)}\right|^{\frac{1}{2}}. \tag{5.27}$$

For $m_i=\beta_i-j+i+1$

$$0=q_i\begin{pmatrix}\beta_i&\beta_j+1\\m_i&\gamma_j\end{pmatrix}\neq q_i\begin{pmatrix}\beta_i\\m_i\end{pmatrix} \tag{5.28}$$

and Eq. (5.25) reduces to

$$q_i\begin{pmatrix}\beta_i\\m_i\end{pmatrix}\Gamma_{ij}\begin{pmatrix}\beta_i&\beta_j\\m_i+1&\gamma_j\end{pmatrix}+\Omega_{ij}\begin{pmatrix}\beta_i&\beta_j\\m_i&\gamma_j\end{pmatrix}=0. \tag{5.29}$$

With this starting relation and the recursion relations (5.22) and (5.25),

$$\Gamma_{ij}\begin{pmatrix}\beta_i&\beta_j\\\gamma_i&\gamma_j\end{pmatrix}=\frac{\gamma_i-\beta_j+j-i-1}{\beta_i-\beta_j+j-i-1}\times\prod_{\lambda_i=0}^{\beta_i-\gamma_i-1}\frac{q_i\begin{pmatrix}\beta_i\\\gamma_i+\lambda_i\end{pmatrix}}{q_i\begin{pmatrix}\beta_i&\beta_j+1\\\gamma_i+\lambda_i&\gamma_j\end{pmatrix}}\Gamma_{ij}\begin{pmatrix}\beta_i&\beta_j\\\beta_i&\gamma_j\end{pmatrix}, \tag{5.30}$$

where Γ_{ij} has been related to

$$\Gamma_{i+1,j}\left[\equiv\Gamma_{ij}\begin{pmatrix}\beta_i&\beta_j\\\beta_i&\gamma_j\end{pmatrix}\right].$$

In the same way the relationship $\Gamma_{1j}\to\Gamma_{2j}\to\Gamma_{3j}\to\cdots\to\Gamma_{jj}$ can be established, leading to

$$\Gamma_{1j}\begin{pmatrix}\beta_1&\beta_2&\cdots&\beta_j&\cdots&\beta_{k-1}\\\gamma_1&\gamma_2&\cdots&\gamma_j&\cdots&\gamma_{k-1}\end{pmatrix}=\Gamma_{jj}\begin{pmatrix}\beta_1&\beta_2&\cdots&\beta_j&\beta_{j+1}&\beta_{j+2}&\cdots\\\beta_1&\beta_2&\cdots&\beta_j&\gamma_{j+1}&\gamma_{j+2}&\cdots\end{pmatrix}\times\left|\prod_{i=1}^{j}\frac{(\gamma_i-\beta_j+j-i-1)(\gamma_i+\beta_j+2k-i-j-1)}{(\beta_i-\beta_j+j-i-1)(\beta_i+\beta_j+2k-i-j-1)}\right|^{\frac{1}{2}}. \tag{5.31}$$

So far the recursive chain stops at $i=j$ since Eqs. (5.22) and (5.25) are valid only if $i\leq j$. To complete the recursive chain, the relationship $\Gamma_{jj}\to\Gamma_{j+1,j}\to\cdots\to\Gamma_{k-1,j}$ must be established. For this purpose consider

$$\text{(a)}\quad \left|\begin{matrix}1\\0\end{matrix}\right\rangle \left|\overline{\begin{matrix}\beta_j & \beta_{j+1}\\ \beta_j & \gamma_{j+1}\end{matrix}}\right\rangle = \gamma_{jj}\begin{pmatrix}\beta_j & \beta_{j+1}\\ \beta_j & \gamma_{j+1}\end{pmatrix}\left|\overline{\begin{matrix}\beta_j+1 & \beta_{j+1}\\ \beta_j & \gamma_{j+1}\end{matrix}}\right\rangle + \cdots, \tag{5.32}$$

$$\text{(b)}\quad \left|\begin{matrix}1\\0\end{matrix}\right\rangle \left|\overline{\begin{matrix}\beta_j & \beta_{j+1}\\ \beta_j & \gamma_{j+1}+1\end{matrix}}\right\rangle = \gamma_{jj}\begin{pmatrix}\beta_j & \beta_{j+1}\\ \beta_j & \gamma_{j+1}+1\end{pmatrix}\left|\overline{\begin{matrix}\beta_j+1 & \beta_{j+1}\\ \beta_j & \gamma_{j+1}+1\end{matrix}}\right\rangle + \cdots. \tag{5.33}$$

[The omitted states are similar to those of Eq. (5.21). They are orthogonal to the states of present interest.] Operating with $Q_{2k-1,i}$ on (b), and with $Q_{2k-1,j+1}Q_{2k-1,j}$ on (a) and comparing the two equations

$$\frac{\Gamma_{jj}\begin{pmatrix}\beta_j & \beta_{j+1}\\ \beta_j & \gamma_{j+1}\end{pmatrix}}{\Gamma_{jj}\begin{pmatrix}\beta_j & \beta_{j+1}\\ \beta_j & \gamma_{j+1}+1\end{pmatrix}} = \frac{q_{j+1}\begin{pmatrix}\beta_j & \beta_{j+1}\\ \beta_j & \gamma_{j+1}\end{pmatrix} q_j\begin{pmatrix}\beta_j+1 & \beta_{j+1}\\ \beta_j & \gamma_{j+1}+1\end{pmatrix}}{q_j\begin{pmatrix}\beta_j+1 & \beta_{j+1}\\ \beta_j & \gamma_{j+1}\end{pmatrix} q_{j+1}\begin{pmatrix}\beta_j+1 & \beta_{j+1}\\ \beta_j+1 & \gamma_{j+1}\end{pmatrix}}. \tag{5.34}$$

Then

$$\Gamma_{j+1,j}\begin{pmatrix}\beta_j & \beta_{j+1}\\ \beta_j & \gamma_{j+1}\end{pmatrix} = \Gamma_{j+2,j}\begin{pmatrix}\beta_{j+1} & \beta_{j+2}\\ \beta_{j+1} & \gamma_{j+2}\end{pmatrix}\frac{q_j\begin{pmatrix}\beta_j+1 & \beta_{j+1}\\ \beta_j & \beta_{j+1}\end{pmatrix}}{q_j\begin{pmatrix}\beta_j+1 & \beta_{j+1}\\ \beta_j & \gamma_{j+1}\end{pmatrix}} \times \prod_{\lambda_{j+1}=0}^{\beta_{j+1}-\gamma_{j+1}-1}\frac{q_{j+1}\begin{pmatrix}\beta_j & \beta_{j+1}\\ \beta_j & \gamma_{j+1}+\lambda_{j+1}\end{pmatrix}}{q_{j+1}\begin{pmatrix}\beta_j+1 & \beta_{j+1}\\ \beta_j+1 & \gamma_{j+1}+\lambda_{j+1}\end{pmatrix}}. \tag{5.35}$$

By repeated application of Eq. (5.35), finally

$$\Gamma_{jj}\begin{pmatrix}\beta_j & \beta_{j+1} & \cdots\\ \beta_j & \gamma_{j+1} & \cdots\end{pmatrix} = \Gamma_{k-1,j}\begin{pmatrix}\beta_{k-1}\\ \beta_{k-1}\end{pmatrix} \times \prod_{i=j+1}^{k-1}\frac{q_j\begin{pmatrix}\beta_j+1 & \beta_i\\ \beta_j & \beta_i\end{pmatrix}}{q_j\begin{pmatrix}\beta_j+1 & \beta_i\\ \beta_j & \gamma_i\end{pmatrix}}\prod_{\lambda_i=0}^{\beta_i-\gamma_i-1}\frac{q_i\begin{pmatrix}\beta_j & \beta_i\\ \beta_j & \gamma_i+\lambda_i\end{pmatrix}}{q_i\begin{pmatrix}\beta_j+1 & \beta_i\\ \beta_j+1 & \gamma_i+\lambda_i\end{pmatrix}}. \tag{5.36}$$

Combining Eqs. (5.31) and (5.36) with the restriction $\Gamma_{k-1,j} = 1$,

$$\left\langle\begin{matrix}\beta_1 & \beta_2 & \cdots & \beta_j+1 & \beta_{j+1} & \beta_{j+2} & \cdots & \beta_{k-1}\\ \gamma_1 & \gamma_2 & \cdots & \gamma_j & \gamma_{j+1} & \gamma_{j+2} & \cdots & \gamma_{k-1}\end{matrix}\right| V^{(2k-1)} \left|\begin{matrix}\beta_1 & \beta_2 & \cdots & \beta_j & \beta_{j+1} & \cdots & \beta_{k-1}\\ \gamma_1 & \gamma_2 & & \gamma_j & \gamma_{j+1} & \cdots & \gamma_{k-1}\end{matrix}\right\rangle = \left|\prod_{i=1}^{k-1}\frac{(\beta_j-\gamma_i+i-j+1)(\beta_j+\gamma_i+2k-i-j-1)}{(\beta_j-\beta_i+i-j+1)(\beta_j+\beta_i+2k-i-j-1)}\right|^{\frac{1}{2}}. \tag{5.37}$$

b. *The Case* Γ_{i0}. From (5.23) and (5.26)

$$\Gamma_{i0}\begin{pmatrix}\beta_i\\ \gamma_i\end{pmatrix} = \Gamma_{i0}\begin{pmatrix}\beta_i\\ \beta_i\end{pmatrix} + (\beta_i-\gamma_i)\frac{\Omega_{i0}\begin{pmatrix}\beta_i\\ \beta_i-1\end{pmatrix}}{q_i\begin{pmatrix}\beta_i\\ \beta_i-1\end{pmatrix}}. \tag{5.38}$$

In order to start the recursion process, the relation between $\Gamma_{i0}\begin{pmatrix}\beta_i\\ \beta_i\end{pmatrix}$ and $\Omega_{i0}\begin{pmatrix}\beta_i\\ \beta_i-1\end{pmatrix}\Big/q_i\begin{pmatrix}\beta_i\\ \beta_i-1\end{pmatrix}$ must be known. The technique used for the case Γ_{ij} cannot be applied here. However, by applying the quadratic invariant to both sides of

$$\left\{Q_{2k-1,i}\left|\begin{matrix}1\\0\end{matrix}\right\rangle\right\}\left|\overline{\begin{matrix}\beta_i\\ \beta_i-1\end{matrix}}\right\rangle = \Omega_{i0}\begin{pmatrix}\beta_i\\ \beta_i-1\end{pmatrix}\left|\overline{\begin{matrix}\beta_i\\ \beta_i\end{matrix}}\right\rangle + \cdots, \tag{5.39}$$

the desired relation is obtained (details are given in Appendix B) as

$$\Gamma_{i0}\begin{pmatrix}\beta_i\\ \beta_i\end{pmatrix} = -(\beta_i+k-i-1) \times \Omega_{i0}\begin{pmatrix}\beta_i\\ \beta_i-1\end{pmatrix}\Big/q_i\begin{pmatrix}\beta_i\\ \beta_i-1\end{pmatrix}, \tag{5.40}$$

$$\Gamma_{10}\begin{pmatrix}\beta_1\\ \gamma_1\end{pmatrix} = \prod_{i=1}^{k-1}\frac{\gamma_i+k-i-1}{\beta_i+k-i-1}\,\Gamma_{k-1,0}\begin{pmatrix}\beta_{k-1}\\ \beta_{k-1}\end{pmatrix}, \tag{5.41}$$

so that

$$\left\langle\begin{matrix}\beta_1 & \beta_2 & \cdots & \beta_i & \cdots & \beta_{k-1}\\ \gamma_1 & \gamma_2 & & \gamma_i & & \gamma_{k-1}\end{matrix}\right| \times V^{(2k-1)}\left|\begin{matrix}\beta_1 & \beta_2 & \cdots & \beta_i & \cdots & \beta_{k-1}\\ \gamma_1 & \gamma_2 & \cdots & \gamma_i & \cdots & \gamma_{k-1}\end{matrix}\right\rangle = \prod_{i=1}^{k-1}\frac{\gamma_i+k-i-1}{\beta+k-i-1}. \tag{5.42}$$

(2) *The Case $n = 2k + 1$*

The procedure is exactly the same, except that the term Γ_{i0} does not exist ($V^{(2k)}$ has no diagonal matrix element). The result is

$$\left\langle \begin{matrix}\beta_1 & \beta_2 & \cdots & \beta_{j-1} & \beta_j+1 & \beta_{j+1} & \cdots & \beta_{k-1} & \beta_k \\ \gamma_1 & \gamma_2 & \cdots & \gamma_{j-1} & \gamma_j & \gamma_{j+1} & & \gamma_{k-1} & \end{matrix}\right| V^{(2k)} \left| \begin{matrix}\beta_1 & \beta_2 & \cdots & \beta_{j-1} & \beta_j+1 & \beta_{j+1} & \cdots & \beta_{k-1} & \beta_k \\ \gamma_1 & \gamma_2 & & \gamma_{j-1} & \gamma_j & \gamma_{j+1} & \cdots & \gamma_{k-1} & \end{matrix}\right\rangle$$

$$= \left| \prod_{i=1}^{k-1} \frac{(\beta_j-\gamma_i+i-j+1)(\beta_j+\gamma_i+2k-i-j)}{(\beta_j-\beta_i+i-j+1)(\beta_j+\beta_i+2k-i-j)}\right|^{\frac{1}{2}}. \quad (5.43)$$

C. Evaluation of $\left\langle \begin{matrix} m_{nj} \\ m'_{n-1,n} \\ m_{n-2,j}\end{matrix} \right| J_{n-1,n} \left| \begin{matrix} m_{nj} \\ m_{n-1,j} \\ m_{n-2,j}\end{matrix}\right\rangle$

Combining the results of subsections A and B above, Eqs. (5.2), (5.9), (5.10), (5.14), (5.37), (5.42), and (5.43), the Gel'fand–Zetlin matrix elements are obtained. With

$$l_{2k,\alpha} = m_{2k,\alpha} + k - \alpha,$$
$$l_{2k-1,\alpha} = m_{2k-1,\alpha} + k - \alpha,$$

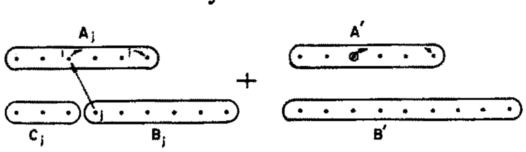
$$\left\langle \begin{matrix} m_{2k,j} \\ m_{2k-1,j}+1 \\ m_{2k-2,j}\end{matrix} \right| J_{2k-1,2k} \left| \begin{matrix} m_{2k,j} \\ m_{2k-1,j} \\ m_{2k-2,j}\end{matrix}\right\rangle = -i \left| \frac{\prod_{\alpha=1}^{k-1} (l^2_{2k-2,\alpha} - l^2_{2k-1,j}) \prod_{\beta=1}^{k} (l^2_{2k,\beta} - l^2_{2k-1,j})}{l^2_{2k-1,j}(4l^2_{2k-1,j}-1) \prod_{\alpha\neq j}^{k-1} (l^2_{2k-1,\alpha} - l^2_{2k-1,j})[(l_{2k-1,\alpha}-1)^2 - l^2_{2k-1,j}]} \right|^{\frac{1}{2}}, \quad (5.44)$$

$$\left\langle \begin{matrix} m_{2k,j} \\ m_{2k-1,j} \\ m_{2k-2,j}\end{matrix} \right| J_{2k-1,2k} \left| \begin{matrix} m_{2k,j} \\ m_{2k-1,j} \\ m_{2k-2,j}\end{matrix}\right\rangle = \frac{\prod_{\alpha=1}^{k-1} l_{2k-2,\alpha} \prod_{\beta=1}^{k} l_{2k,\beta}}{\prod_{\alpha=1}^{k-1} l_{2k-1,\alpha}(l_{2k-1,\alpha}-1)}, \quad (5.45)$$

$$\left\langle \begin{matrix} m_{2k+1,j} \\ m_{2k,j}+1 \\ m_{2k-1,j}\end{matrix} \right| J_{2k,2k+1} \left| \begin{matrix} m_{2k+1,j} \\ m_{2k,j} \\ m_{2k-1,j}\end{matrix}\right\rangle$$
$$= \frac{-i}{2} \left| \frac{\prod_{\alpha=1}^{k-1} (l_{2k-1,\alpha} - l_{2k,j} - 1)(l_{2k-1,\alpha} + l_{2k,j}) \prod_{\beta=1}^{k} (l_{2k+1,\beta} - l_{2k,j} - 1)(l_{2k+1,\beta} + l_{2k,j})}{\prod_{\alpha\neq j}^{k} (l^2_{2k\alpha} - l^2_{2kj})[l^2_{2k\alpha} - (l_{2kj}+1)^2]} \right|^{\frac{1}{2}}. \quad (5.46)$$

APPENDIX A. EVALUATION OF $\langle i \mid Q_{2k+1,i} O_{2k+1,-i} \mid i\rangle$

There are many graphs in $O_{2k+1,-i}$. For some types of calculation certain ways of grouping them are more convenient than others. The following example demonstrates one way.



$$O_{2k+1,-i} = \sum_{j=1}^{k} \sum_{p=1}^{j} \sum_{l=i+1}^{k} \{-i, l\}(-\rho_{-l,-j})\{j, -p\} Q_{2k+1,p} \times \prod_{\gamma=j+1}^{k} a_{i,\gamma} \prod_{\gamma=l+1}^{k} a_{i,-\gamma} \prod_{\gamma=1}^{p-1} a_{i,\gamma} + \sum_{j=i}^{k} \{-i, j\} Q_{2k+1,-j} \prod_{\gamma=1}^{k} a_{i\gamma} \prod_{\gamma=j+1}^{k} a_{i,-\gamma}, \quad (A1)$$

$$\left.\begin{aligned} A_j &= \sum_{l=i+1}^{k} \{-i, l\}(-\rho_{-l,-j}) \prod_{\gamma=l+1}^{k} a_{i,-\gamma}, \\ &\qquad \{-i, i\} = \{j, -j\} = 1 \\ B_j &= \sum_{p=1}^{j} \{j, -p\} Q_{2k+1,p} \prod_{\gamma=1}^{p-1} a_{i\gamma}, \\ C_j &= \prod_{\gamma=j+1}^{k} a_{i\gamma}, \\ A' &= \sum_{j=i}^{k} \{-i, j\} Q_{2k+1,-j} \prod_{\gamma=j+1}^{k} a_{i,-\gamma}, \\ B' &= \prod_{\gamma=1}^{k} a_{i\gamma}, \end{aligned}\right\} \quad (A2)$$

$$O_{2k+1,-i} = \sum_{j=1}^{k} A_j B_j C_j + A'B'; \quad (A3)$$

where $\{-\alpha, \beta\}$ means summation of all possible graphs, which have a chain away from the $|\beta|$th point on the top when β is positive and on the bottom when β is negative; and a chain ending at the $|\alpha|$th point on the top when α is positive and on the bottom when α is negative.

Example:

$\{-1, 5\}$ includes the following graphs in the top row:

(With one arrow link of the chain and three points.)

\+ 

(All the possible distinct graphs generated by removing any one free point from the first graph, with the appropriate chains.)

\+ 

(All the possible distinct graphs generated by removing any two free points from the first graph, with the appropriate chains.)

\+

(All the possible distinct graphs generated by removing three free points from the first graph.)

Since the distribution of free points uniquely determines the graph, it is sometimes more convenient to define the graph by its free points.

With

$$\langle i|\, Q_{2k+1,\alpha}(-\rho_{-\alpha\beta}) = \langle i|\, (-2)Q_{2k+1,\beta} \quad 0 < \alpha < |\beta| \tag{A4}$$

each chain contraction gives a factor (-2), and

$$\begin{aligned}\langle i|\, Q_{2k+1,i}\{-i, l\} &= -2\langle i|\, Q_{2k+1,l} \prod_{\alpha=i+1}^{l-1} a_{i,-\alpha}\left[1 + (-2)\sum_{\beta=i+1}^{l-1} \frac{1}{a_{i,-\beta}}\right.\\ &\left.+ (-2)^2 \sum_{\substack{\beta,\gamma=i+1\\ \beta\neq\gamma}}^{l-1} \frac{1}{a_{i,-\beta}a_{i,-\gamma}} + \cdots (-2)^{l-i-1}\frac{1}{\prod_{\alpha=i+1}^{l-1} a_{i,-\alpha}}\right].\end{aligned} \tag{A5}$$

The second term in the parenthesis comes from the removal of one free point from the first graph which is

$$(-\rho_{-il}) \prod_{\alpha=i+1}^{l-1} a_{i,-\alpha},$$

and the last term in the parenthesis comes from the removal of all free points from the first graph. A similar removal of free points gives the intermediate terms. By summing up all the terms,

$$\langle i|\, Q_{2k+1,i}\{-i, l\} = -2\langle i|\, Q_{2k+1,l} \prod_{\alpha=i+1}^{l-1} (a_{i,-\alpha} - 2). \tag{A6}$$

Similarly

$$\begin{aligned}\langle i|\, Q_{2k+1,i}A_j = \langle i|\, (-2)Q_{2k+1,-j}&\left[\prod_{\gamma=i+1}^{k} (a_{i,-\gamma} - 2)\right.\\ &\left.+ 2\prod_{\gamma=i+1}^{j-1} (a_{i,-\gamma} - 2) \prod_{\gamma=j+1}^{k} a_{i,-\gamma}\right].\end{aligned} \tag{A7}$$

The first term is the summation of the contribution of all possible graphs with any number of free points from $i+1$ to k in the A_j; but it has included the graphs

$$\{-i, j\}(-\rho_{-j,-j}) \prod_{\alpha=j+1}^{k} a_{i,-\alpha},$$

which should be zero, since $\rho_{-j-j} = 0$. The second term is therefore needed to take away the improper contribution of

$$\{-i, j\}(-\rho_{-j,-j}) \prod_{\alpha=j+1}^{k} a_{i,-\alpha}.$$

Similarly

$$\begin{aligned}\langle i|\, Q_{2k+1,-j}B_j &= \langle i|\, \sum_{p=1}^{j-1}(-2)Q_{2k+1,-p}Q_{2k+1,p}\\ &\times \prod_{\alpha=p+1}^{j-1} (a_{i\alpha} - 2) \prod_{\alpha=1}^{p-1} a_{i\alpha} + Q_{2k+1,-j}Q_{2k+1,j}\prod_{\alpha=1}^{j-1} a_{i\alpha}.\end{aligned} \tag{A8}$$

With Eq. (4.12) and Eq. (A2)

$$\langle i|\, Q_{2k+1,-j}B_j\,|i\rangle = \langle i|\, Q_{2k+1,-i}Q_{2k+1,i}\frac{2\prod_{\alpha=1}^{i-1} a_{i\alpha} \prod_{\alpha=i+1}^{j} a_{i\alpha} \prod_{\alpha=i+1}^{j-1} (a_{i,-\alpha} + 2)}{\prod_{\alpha=i+1}^{j} a_{i,-\alpha}}\,|i\rangle. \tag{A9}$$

By combining with (A7), (A2), and (A9)

$$\langle i|\, Q_{2k+1,i}\sum_{j=i}^{k} A_jB_jC_j\,|i\rangle = \sum_{j=i}^{k} -4\langle i|\,\frac{Q_{2k+1,-i}Q_{2k+1,i}\prod_{\alpha=i+1}^{j-1}(a_{i-\alpha}^2 - 4)\prod_{\alpha=1}^{k} a_{i\alpha}\left[\prod_{\alpha=j}^{k}(a_{i-\alpha} - 2) + 2\prod_{\alpha=j+1}^{k} a_{i,-\alpha}\right]}{a_{ii}\prod_{\alpha=i+1}^{j} a_{i,-\alpha}}\,|i\rangle. \tag{A10}$$

Similarly, with Eq. (4.12), Eq. (A2), and Eq. (A6)

$$\langle i|\, Q_{2k+1,i}A'B'\,|i\rangle = \sum_{j=i}^{k} -4\langle i|\, J_{2j-1,2j}\prod_{\alpha=i+1}^{j-1}(a_{i,-\alpha}-2)\prod_{\alpha=j+1}^{k}a_{i,-\alpha}\prod_{\alpha=1}^{k}a_{i\alpha}\,|i\rangle + \sum_{j=i}^{k} -4\langle i|\, Q_{2k+1,-i}Q_{2k+1,i}\frac{\prod_{\alpha=i+1}^{j-1}(a_{i,-\alpha}^2-4)\prod_{\alpha=j+1}^{k}a_{i,-\alpha}\prod_{\alpha=1}^{k}a_{i\alpha}}{\prod_{\alpha=i+1}^{j}a_{i,-\alpha}}\,|i\rangle. \tag{A11}$$

By summing up Eq. (A10) and (A11), finally

$$\langle i|\, Q_{2k+1,i}O_{2k+1,-i}\,|i\rangle = \langle i|\left[\prod_{\alpha=1}^{k}a_{i\alpha}\prod_{\alpha=i+1}^{k}(a_{i,-\alpha}-2)\right](m_{2k+1,i}-J_{2i-1,2i}+1)(m_{2k+1,i}+J_{2i-1,2i}+2k-2i)\,|i\rangle. \tag{A12}$$

A similar process works for $O(2k)$, and gives

$$\langle i|\, Q_{2k,i}O_{2k,-i}\,|i\rangle = \langle i|\,(m_{2k,i}+J_{2i-1,2i}+2k-2i-1)(m_{2k,i}-J_{2i-1,2i}+1)\,|i\rangle \times \langle i|\prod_{\alpha=i+1}^{k}(a_{i,-\alpha}-2)\prod_{\alpha=1}^{k}b_{i\alpha}\,|i\rangle\langle i|\frac{1}{b_{ii}}\,|i\rangle. \tag{A13}$$

APPENDIX B. DERIVATION OF EQ. (5.40)

To derive Eq. (5.40),

$$\Gamma_{i0}\begin{pmatrix}\beta_i\\ \beta_i\end{pmatrix} = -(\beta_i+k-i-1) \times \Omega_{i0}\begin{pmatrix}\beta_i\\ \beta_i-1\end{pmatrix}\Big/ q_i\begin{pmatrix}\beta_i\\ \beta_i-1\end{pmatrix}, \tag{5.40}$$

the quadratic Casimir invariant C_{2k-1} is applied to Eq. (5.39). In order to simplify the evaluation of these terms, the following points are useful:

(1) $$Q_{\alpha i}\left|\begin{matrix}\beta_i\\ \gamma_i\end{matrix}\right\rangle = 0 \quad i>0,\quad \alpha<2k-1 \tag{B1}$$

since $\left|\begin{matrix}\beta_i\\ \gamma_i\end{matrix}\right\rangle$ belongs to $[\mathcal{M}^{(2k-2)}_{2k-1\mu}]$

(2)a. $$Q_{2k-1,j}Q_{2k-1,i}\left|\begin{matrix}1\\ 0\end{matrix}\right\rangle = 0,\quad i\neq -j. \tag{B2}$$

The net result of the two Q operations in succession would either have to change one of the $m_{2k-2,\alpha}$ by two or two of the $m_{2k-2,\alpha}$ by one each. Both cases are impossible since $m_{2k-1,\alpha}$ is $[1000\cdots]$.

b. $$Q_{2k-1,-i}Q_{2k-1,i}\left|\begin{matrix}1\\ 0\end{matrix}\right\rangle = 2\left|\begin{matrix}1\\ 0\end{matrix}\right\rangle, \tag{B3}$$

a direct consequence of Eq. (4.14) and Eq. (4.15).

(3) $$C_{2k-1} = \sum_{i<j}^{2k-1}J_{ij}^2 = \sum_{j=1}^{k-1}Q_{2k-1,-j}Q_{2k-1,j} + \sum_{j<\alpha\le k-1}Q_{2\alpha,-j}Q_{2\alpha,j} + \sum_{j<\alpha\le k-1}Q_{2\alpha-1,-j}Q_{2\alpha-1,j} + \sum_{\alpha=1}^{k-1}J_{2\alpha-1,2\alpha}^2 + \sum_{\alpha=1}^{k-1}(2k-2\alpha-1)J_{2\alpha-1,2\alpha}. \tag{B4}$$

Proof: Operating on Eq. (5.39):

$$\left\{Q_{2k-1,i}\left|\begin{matrix}1\\ 0\end{matrix}\right\rangle\right\}\left|\overline{\begin{matrix}\beta_i\\ \beta_i-1\end{matrix}}\right\rangle = \Omega_{i0}\begin{pmatrix}\beta_i\\ \beta_i-1\end{pmatrix}\left|\begin{matrix}\beta_i\\ \beta_i\end{matrix}\right\rangle + \cdots \tag{5.39}$$

with C_{2k-1} in the form of Eq. (B4), bearing in mind (B1), (B2),

$$\left\{Q_{2k-1,-i}Q_{2k-1,i}\left|\begin{matrix}1\\ 0\end{matrix}\right\rangle\right\}\left\{Q_{2k-1,i}\left|\overline{\begin{matrix}\beta_i\\ \beta_i-1\end{matrix}}\right\rangle\right. + \left\{Q_{2k-1,i}\left|\begin{matrix}1\\ 0\end{matrix}\right\rangle\right\}\left[\sum_{j=1}^{k-1}Q_{2k-1,-j}Q_{2k-1,j} + \sum_{\alpha=1}^{k-1}J_{2\alpha-1,2\alpha}^2 + \sum_{\alpha=1}^{k-1}(2k-2\alpha-1)J_{2\alpha-1,2\alpha}\right]\left|\overline{\begin{matrix}\beta_i\\ \beta_i-1\end{matrix}}\right\rangle + (2k-2i+2\beta_i-2)\left\{Q_{2k-1,i}\left|\begin{matrix}1\\ 0\end{matrix}\right\rangle\right\}\left|\begin{matrix}\beta_i\\ \beta_i-1\end{matrix}\right\rangle = \Omega_{i0}\begin{pmatrix}\beta_i\\ \beta_i-1\end{pmatrix}C_{2k-1}\left|\begin{matrix}\beta_i\\ \beta_i\end{matrix}\right\rangle. \tag{B5}$$

The second term on the left-hand side cancels the term on the right-hand side, and with Eqs. (B3), (5.20), and (5.24) the derivation of Eq. (5.40) is attained.