J. Math. Soc. Japan
Vol. 37, No. 1, 1985

L^p-estimates of solutions of some nonlinear degenerate diffusion equations

Dedicated to Professor Sigeru Mizohata on his sixtieth birthday

By Mitsuhiro NAKAO

(Received July 29, 1982)
(Revised Dec. 2, 1983)

0. Introduction.

The object of this paper is to show the existence, uniqueness and L^p estimates (including $p=\infty$) of global solutions of some nonlinear degenerate diffusion equations.

The first problem we are concerned with is the following initial-boundary value problem of the perturbed porous medium equation;

$$(\mathrm{P}_1)\qquad \begin{aligned} &\frac{\partial}{\partial t}u-\Delta u^{m+1}+f(x, t, u)=0 \quad \text{in} \quad \Omega\times(0, \infty) \\ &u(x, 0)=u_0(\geqq 0), \quad u|_{\partial\Omega}=0 \quad \text{and} \quad u\geqq 0 \end{aligned}$$

where Ω is a bounded domain in R^N with smooth boundary $\partial\Omega$ (C^3 class is sufficient), m is a positive constant and $f(x, t, u)$ is a function satisfying;

ASSUMPTION 1. (i) $f(x, t, u)$ is locally Hölder continuous in $\bar{\Omega}\times R^+\times R^+$ ($R^+=[0, \infty)$) and locally Lipschitz continuous with respect to u uniformly in (x, t).

(ii) $f(x, t, u)\geqq -C_0u^{1+\alpha}$ on $\bar{\Omega}\times R^+\times R^+$ for some $C_0>0$ and $\alpha\geqq 0$.

It should be noted that the theory of nonlinear semi-groups does not apply to (P_1) for the existence of global solution since we do not assume that $f(x, t, u)$ is monotone with respect to u.

To treat the problem (P_1) it is convenient to compare it with the problem;

$$(\mathrm{P}_2)\qquad \begin{aligned} &\frac{\partial}{\partial t}u-\Delta u^{m+1}-C_0u^{1+\alpha}=0 \quad \text{in} \quad \Omega\times(0, \infty) \\ &u(x, 0)=u_0(\geqq 0), \quad u|_{\partial\Omega}=0 \quad \text{and} \quad u\geqq 0. \end{aligned}$$

Recently in [13] we have discussed the existence, nonexistence and some

---

This paper was written during the author's visit to Heriot-Watt University, Edinburgh, U.K. (1981/1982) under the support of the Japan Society for the Promotion of Science.

asymptotic behaviour of global solutions to the problem (P_2).

THEOREM 0. (i) *If* $m>\alpha$, (P_2) *admits a solution* $u(t)$ *for each* u_0 *with* $u_0^{m+1}\in H_0^1$, *and we have*

$$\left(\int_t^{t+1}\left\|\frac{\partial}{\partial t}(u^{m/2+1}(s))\right\|_2^2 ds\right)^{1/2}+\|\nabla u^{m+1}(t)\|_2\leqq C(\|\nabla u_0^{m+1}\|_2) \tag{0.1}$$

for $t\geqq 0$.

(ii) *If* $m<\alpha<m^*\equiv\{m(N+2)+4\}/(N-2)$ $(N>2)$ *or* ∞ $(N=1, 2)$ *the same assertion as above holds for each* u_0 *with* $u_0^{m+1}\in\mathfrak{W}$ (for the definition of the potential well $\mathfrak{W}$ see the section 1). *Moreover, the solution satisfies the decay property;*

$$\left(\int_t^{t+1}\left\|\frac{\partial}{\partial t}(u^{m/2+1}(s))\right\|_2^2 ds\right)^{1/2}+\|\nabla u^{m+1}(t)\|_2\leqq C(d-J(u_0^{m+1}))(1+t)^{-(m+1)/m} \tag{0.2}$$

for $t\geqq 0$. (For the definitions of J and d see also the section 1.)

In the above and hereafter $\|\cdot\|_p$ denotes L^p norm on Ω and $C(a)$ denotes various constants depending on a and other known constants. We mean by a solution of (P_2) a function u such that $u\geqq 0$, $(\partial/\partial t)(u^{m/2+1})\in L^2_{\text{loc}}(R^+;\ L^2(\Omega))$, $u^{m+1}\in L^\infty_{\text{loc}}(R^+;\ H_0^1(\Omega))$ and the following equation holds for any $\phi\in C_0^1([0, \infty);\ H_0^1)$;

$$\int_0^\infty\int_\Omega\{-u(x, t)\phi_t(x, t)+\nabla u^{m+1}(x, t)\cdot\nabla\phi(x, t)-C_0u^{1+\alpha}(x, t)\phi(x, t)\}\,dx\,dt$$

$$=\int_\Omega u_0(x)\phi(x, 0)dx\,. \tag{0.3}$$

The solutions of the problem (P_1) are defined similarly;

DEFINITION 1. A measurable function u on $\Omega\times R^+$ is called a solution of (P_1) if $u\geqq 0$, $(\partial/\partial t)(u^{m/2+1})\in L^2_{\text{loc}}(R^+;\ L^2)$, $u^{m+1}\in L^\infty_{\text{loc}}(R^+;\ H_0^1)$, $f(x, t, u)\in L^1_{\text{loc}}(R^+;\ L^1)$ and the equation (0.3) holds with $-C_0u^{1+\alpha}$ replaced by $f(x, t, u)$ for all ϕ as above.

We want to estimate the solutions of (P_1) by combining the result for (P_2) with the comparison principle (Aronson, Crandall and Peletier [2]). For this it is needed moreover to establish L^p estimates $(2\leqq p\leqq\infty)$ for the solutions of the problem (P_2). In particular, the L^∞ estimate will yield the uniqueness theorem in a certain class of functions where the existence is also assured. Such estimation is done in the sections 2 and 3 on the basis of Theorem 0 and a result of Alikakos [1]. Furthermore, the same estimates as in Theorem 0 will be derived for the solutions of (P_1).

Secondly, we apply the result for (P_1) to the following system of reaction diffusion equations;

$$
(\mathrm{P}_3)\qquad \begin{cases} \dfrac{\partial}{\partial t}u-\Delta u^{m+1}+f_1(x, t, u, v)=0\,, & u\geqq 0, \\ \dfrac{\partial}{\partial t}v-\Delta v^{n+1}+f_2(x, t, u, v)=0\,, & v\geqq 0, \end{cases}
$$

in $\Omega\times(0, \infty)$ with the initial-boundary conditions

$$u(x, 0)=u_0\ (\geqq 0)\,,\quad v(x, 0)=v_0\ (\geqq 0)\quad \text{and}\quad u|_{\partial\Omega}=v|_{\partial\Omega}=0\,.$$

We make the following assumptions on f_i $(i=1, 2)$.

ASSUMPTION 2. (i) f_i is locally Hölder continuous in $\bar{\Omega}\times R^+\times R^+\times R^+$ and locally Lipschitz continuous with respect to (u, v) uniformly in (x, t).

(ii) $f_1(x, t, u, v)\geqq -C_0(r)u^{1+\alpha}$ if $0\leqq v\leqq r$, $r>0$.

(iii) $f_2(x, t, u, v)\geqq 0$ on $\Omega\times R^+\times R^+\times R^+$.

The system (P_3) is a generalization of the so-called Martin's problem ($m=n=0$ and $f_1=-f_2=-u^{1+\alpha}v$) and also related to Rosenweig-MacArther equation which arises in the theory of ecology (Alikakos [**1**], Conway and Smoller [**6**], Masuda [**11**]). The definition of solutions of (P_3) is given quite similarly to that of (P_1) and may be omitted.

Finally we briefly discuss another typical problem;

$$
(\mathrm{P}_4)\qquad \begin{aligned} &\frac{\partial}{\partial t}u-\sum_{i=1}^{N}\frac{\partial}{\partial x_i}\left(\left|\frac{\partial}{\partial x_i}u\right|^m\frac{\partial}{\partial x_i}u\right)-|u|^{\alpha}u=0 \\ &u(x, 0)=u_0 \quad\text{and}\quad u|_{\partial\Omega}=0. \end{aligned}
$$

The existence and nonexistence problem for (P_4) has been investigated by many authors. Among others we refer to Fujita [**8**] (the case $m=0$) and Tsutsumi [**18**]. The decay or boundedness of $u(t)$ in $W_0^{1,m+2}$ norm was obtained in [**14**]. Using the same method as in sections 2 and 3 we show L^p estimates (including $p=\infty$) for the solutions of (P_4). As a consequence of L^∞ estimate we can easily derive a uniqueness theorem in a certain function space where the existence holds, which is given in [**18**] only for the case $1\leqq N<m+2$.

1. Approximate solutions for the problems (P_1) and (P_2).

To show the existence and L^p estimates of global solutions of (P_1) and (P_2) it is convenient to construct smooth approximate solutions which exist globally. In [**13**] we employed the Galerkin method together with the regularization of singularity. As will be seen below this method is not suitable to get L^p estimates for large p. Here we adopt a slightly different method.

Setting $u^{m+1}=U$ the problem (P_1) is equivalent to

$$
(\mathrm{P}_1')\qquad \begin{aligned} &\frac{1}{m+1}U^{-m/(m+1)}\frac{\partial}{\partial t}U-\Delta U+f(x, t, U^{1/(m+1)})=0 \\ &U(0)=U_0=u_0^{m+1},\quad U|_{\partial\Omega}=0 \quad\text{and}\quad U\geqq 0. \end{aligned}
$$

Then we consider the following approximate problem;

$$(\mathrm{P}_{1,\varepsilon})\quad \begin{aligned}&\frac{1}{m+1}((U+\varepsilon)^{-m/(m+1)}+\varepsilon)\frac{\partial}{\partial t}U-\Delta U+f(x,\,t,\,U^{1/(m+1)})+\varepsilon U^{(1+\alpha')/(m+1)}=0\\ &U(0)=U_0,\quad U|_{\partial\Omega}=0\quad\text{and}\quad U\geqq 0,\end{aligned}$$

with $\alpha'>\alpha$.

Since the problem $(\mathrm{P}_{1,\varepsilon})$ has no singularity in the coefficients and, by Assumption 1, (ii),

$$f(x,\,t,\,U^{1/(m+1)})+\varepsilon U^{(1+\alpha')/(m+1)}\geqq -C_\varepsilon \qquad \text{for}\quad U\geqq 0$$

with some $C_\varepsilon>0$, it has a classical (smooth) solution $U_\varepsilon(x,\,t)$ for each $U_0\in C_0^1(\Omega)$ and for $\varepsilon>0$ (Ladyzhenskaya, Solonnikov and Ural'tseva [**9**]). (The solution is in fact unique.) We shall call $(\mathrm{P}_{1,\varepsilon})$ as $(\mathrm{P}_{2,\varepsilon})$ when $f(x,\,t,\,u)=-C_0u^{1+\alpha}$.

Here we state the definition of the potential well $\mathfrak{W}$;

$$\mathfrak{W}=\left\{U\in H_0^1 \mid J(U)<d\ \text{and}\ \|\nabla U\|_2^2-C_0\int_\Omega U^{(m+\alpha+2)/(m+1)}dx>0\right\}\cup\{0\}$$

where

$$J(U)=\frac{1}{2}\|\nabla U\|_2^2-\frac{C_0(m+1)}{m+\alpha+2}\int_\Omega U^{(m+\alpha+2)/(m+1)}dx$$

and

$$d=\inf_{\substack{U\in H_0^1\\ U\neq 0}}\ \sup_{\lambda>0} J(\lambda U)$$

(see Sattinger [**16**] and Tsutsumi [**18**]).

PROPOSITION 1.1. *Let $U_0\in C_0^1(\Omega)$ (if $m>\alpha$) or $U_0\in\mathfrak{W}\cap C_0^1(\Omega)$ (if $0<m<\alpha<m^*$). Then the result of Theorem 0 is valid for $u_\varepsilon\equiv U_\varepsilon^{1/(m+1)}$ for sufficiently small $\varepsilon>0$, U_ε being the solutions of $(\mathrm{P}_{2,\varepsilon})$, with the estimates replaced by*

$$\left(\int_t^{t+1}\left\|\frac{\partial}{\partial t}(u_\varepsilon^{m/2+1}(s))\right\|_2^2ds\right)^{1/2}+\|\nabla u_\varepsilon^{m+1}(t)\|_2+\varepsilon\|u_\varepsilon(t)\|_{m+\alpha'+2}^{m+\alpha'+2}$$
$$\leqq\begin{cases} C(\|\nabla u_0^{m+1}\|_2,\ \varepsilon\|u_0\|_{m+\alpha'+2},\ \varepsilon) & \text{if}\quad \alpha<m\\ C(\|\nabla u_0^{m+1}\|_2,\ \varepsilon\|u_0\|_{m+\alpha'+2})(1+t)^{-(m+1)/m}+C\varepsilon & \text{if}\quad m<\alpha<m^*\end{cases}\tag{1.1}$$

where $\lim_{\varepsilon\to 0}C(a,\,\varepsilon b)=C(a)$.

PROOF. The proof is essentially the same as that of Theorem 0. For completeness, however, we reproduce it briefly in a slightly simpler way for the case $N\geqq 3$ and $0<m<\alpha<m^*$. We write $U_\varepsilon=U$ for simplicity.

Multiplying the equation in $(\mathrm{P}_{2,\varepsilon})$ by $(\partial/\partial t)U$ and integrating, we have

$$\frac{1}{m+1}\int_0^t\int_\Omega\{(U+\varepsilon)^{-m/(m+1)}+\varepsilon\}\,U_t^2\,dx\,ds+J_\varepsilon(U(t))=J_\varepsilon(U_0)\tag{1.2}$$

where

$$J_\varepsilon(U)=\frac{1}{2}\|\nabla U\|_2^2-\frac{C_0(m+1)}{m+\alpha+2}\int_\Omega U^{(m+\alpha+2)/(m+1)}dx$$
$$+\frac{\varepsilon(m+1)}{m+\alpha+2}\int_\Omega U^{(m+\alpha'+2)/(m+1)}dx.$$

Since $U_0\in\mathfrak{W}$, $J_\varepsilon(U_0)<d$ for sufficiently small $\varepsilon>0$. We assert that $U(t)\in\mathfrak{W}$ for any $t>0$. Indeed, if not, there exists $t^*>0$ such that $U(t^*)\in\partial\mathfrak{W}$ and $U(t)\in\mathfrak{W}$ for $0\leqq t<t^*$. Then, by the definition of $\mathfrak{W}$ we have either

$$J(U(t^*))=d \quad \text{or} \quad \|\nabla U(t^*)\|_2^2=C_0\int_\Omega U^{(m+\alpha+2)/(m+1)}(t^*)dx.$$

Both cases easily yield contradictions. Thus, by Lemma 3.1 in [**13**] we have

$$\|\nabla U\|_2^2-C_0\int_\Omega U^{(m+\alpha+2)/(m+1)}dx\geqq C(d-J(U_0))\|\nabla U\|_2^2 \tag{1.3}$$

for a certain $C(a)>0$ such that $\lim_{a\to 0}C(a)=0$, and

$$J_\varepsilon(U(t))\geqq C(d-J(U_0))\|\nabla U(t)\|_2^2. \tag{1.4}$$

Now, (1.2) is equivalent to

$$\frac{1}{m+1}\int_\Omega((U+\varepsilon)^{-m/(m+1)}+\varepsilon)U_t^2dx+\frac{d}{dt}J_\varepsilon(U(t))=0. \tag{1.2$'$}$$

Next, multiplying the equation of $(\mathrm{P}_{2,\varepsilon})$ by U we have

$$\|\nabla U\|_2^2-C_0\int U^{(m+\alpha+2)/(m+1)}dx+\varepsilon\int U^{(m+\alpha'+2)/(m+1)}dx$$
$$=\frac{1}{m+1}\int((U+\varepsilon)^{-m/(m+1)}+\varepsilon)UU_tdx$$
$$\leqq\frac{1}{m+1}\left\{\int((U+\varepsilon)^{-m/(m+1)}+\varepsilon)U_t^2dx\right\}^{1/2}\left\{\int((U+\varepsilon)^{-m/(m+1)}+\varepsilon)U^2dx\right\}^{1/2}$$

and, using (1.2)′,

$$J_\varepsilon(U(t))\leqq C\{J_\varepsilon(U(t))^{(m+2)/(m+1)}+\varepsilon J_\varepsilon(U(t))\}^{1/2}\left\{-\frac{d}{dt}J_\varepsilon(U(t))\right\}^{1/2}. \tag{1.5}$$

From (1.5) we can derive easily

$$\left\{-\frac{d}{dt}J_\varepsilon(U(t))\right\}^{2(m+1)/(3m+2)}\geqq C\{J_\varepsilon(U(t))-\varepsilon\},\qquad 0<\varepsilon\ll 1.$$

We may assume $J_\varepsilon(U_0)\geqq\varepsilon$ $(U_0\neq 0)$ and, as long as $J_\varepsilon(U(t))>\varepsilon$,

$$\frac{d}{dt}(J_\varepsilon(U(t))-\varepsilon)\leqq -C(J_\varepsilon(U(t))-\varepsilon)^{(3m+2)/2(m+1)}$$

which implies

$$J_\varepsilon(U(t))\leqq C\{t+(\|\nabla U_0\|_2^2+\varepsilon\|U_0\|_{\beta'}^{\beta'})^{-m/2(m+1)}\}^{-2(m+1)/m}+\varepsilon$$

where we set $\beta'=(m+\alpha'+2)/(m+1)$. This inequality is valid for all $t\geqq 0$, because $J_\varepsilon(U(t))$ is a nonincreasing function of t.

From the above we can conclude, for $t\geqq 0$,

$$\|\nabla U(t)\|_2\leqq C(\|\nabla U_0\|_2,\ \varepsilon\|U_0\|_{\beta'})(1+t)^{-(m+1)/m}+\sqrt{\varepsilon}\,.$$

Integrating (1.2)' from t to $t+1$ we get the same estimates for

$$\left\{\int_t^{t+1}\int_\Omega((U+\varepsilon)^{-m/(m+1)}+\varepsilon)U_t^2\,dx\,ds\right\}^{1/2}.$$

PROPOSITION 1.2. *Let $\tilde{U}_\varepsilon(t)$ and $U_\varepsilon(t)$ be the solutions with the same initial value $U_0\in C_0^1(\Omega)$ of* $(\mathrm{P}_{1,\varepsilon})$ *and* $(\mathrm{P}_{2,\varepsilon})$, *respectively. Then*

$$0\leqq\tilde{U}_\varepsilon(t)\leqq U_\varepsilon(t)\,. \tag{1.6}$$

PROOF. The inequality (1.6) follows from the comparison principle in nonlinear parabolic equations (Aronson, Peletier and Crandall [**2**]).

From the estimate in Proposition 1.1 it easily follows that $u_\varepsilon(t)=U_\varepsilon^{1/(m+1)}(t)$ converges as $\varepsilon\to 0$ (along a subsequence) to a solution of (P_2). The proof is given in [**13**]. The resulting estimate contains no longer the norm $\|u_0\|_{m+\alpha'+2}$, and hence the assumption $U_0\in C_0^1(\Omega)$ is easily removed by approximation. Indeed, when $U_0\in\mathfrak{W}$ (consider the case $0<m<\alpha<m^*$) we can take a sequence $\{U_{0,\nu}\}\subset C_0^1(\Omega)$ such that $U_{0,\nu}\to U_0$ in H_0^1 as $\nu\to 0$. Since $\mathfrak{W}$ is open in H_0^1 we may assume $U_{0,\nu}\in\mathfrak{W}\cap C_0^1(\Omega)$. Corresponding solutions $u_\nu(t)$ with $u_\nu(0)=U_{0,\nu}^{1/(m+1)}$ satisfy essentially the same estimates as in Theorem 0, and we can apply again the compactness argument to $\{u_\nu\}$ to obtain the desired solution with the initial value $u_0=U_0^{1/(m+1)}$. Thus we arrive at Theorem 0 through a slightly different way from [**13**]. In what follows we regard the solution u in Theorem 0 as the one just obtained. The nonnegativity of solutions is inessential in sections 1-3.

2. L^p estimates ($p<\infty$) for the solutions of the problem (P_2).

When $N=1$, 2 the estimate of $\|\nabla u^{m+1}(t)\|_2$ in Theorem 0 implies L^p $(2\leqq p<\infty)$ estimates of $u(t)$. But, this is not true for the case $N\geqq 3$. Here we shall derive L^p estimates for such case. First we assume $0<m<\alpha<m^*$ (this case is more complicated than the other case).

For our purpose it suffices to treat the approximate solutions $U_\varepsilon(t)$ or $u_\varepsilon(t)$ defined in the previous section. We write again $U_\varepsilon=U$.

Now, multiplying the equation of $(\mathrm{P}_{2,\varepsilon})$ by $U^{(p+1)/(m+1)}$ $(p\geqq m)$ we have

$$\frac{d}{dt}\int\beta_{p,\varepsilon}(U(t))dx+\frac{4(m+1)(p+1)}{(p+m+2)^2}\|\nabla U^{(p+m+2)/2(m+1)}(t)\|_2^2$$
$$-C_0\int U^{(p+\alpha+2)/(m+1)}dx+\varepsilon\int U^{(p+\alpha'+2)/(m+1)}dx=0\,, \tag{2.1}$$

where we set

$$\beta_{p,\varepsilon}(U)\equiv\int_0^U\{(\eta+\varepsilon)^{-m/(m+1)}+\varepsilon\}\,\eta^{(p+1)/(m+1)}d\eta\,. \tag{2.2}$$

We shall denote by $\bar{C}_\varepsilon$ various constants depending on $\|\nabla U_0\|_2^2+\varepsilon\|U_0\|_{\beta'}^{\beta'}$ and ε, and by $k(\varepsilon)$ constants depending on ε in such a way that $\lim_{\varepsilon\to 0}k(\varepsilon)=0$.

By Proposition 1.1 we know

$$\|\nabla U(t)\|_2\leqq C(d)<\infty \tag{2.3}$$

and applying Hölder's and Sobolev's inequalities we have

$$\begin{aligned}\int U^{(p+\alpha+2)/(m+1)}dx&\leqq\|U\|_{(m+\alpha+2)/(m+1)}^{(p+\alpha+2)/(m+1)-\theta_p}\|U\|_q^{\theta_p}\\&\leqq C\|\nabla U\|_2^{(p+\alpha+2)/(m+1)-\theta_p}\|\nabla U^{(p+m+2)/2(m+1)}\|_2^{2\theta'_p}\\&\leqq C\|\nabla U^{(p+m+2)/2(m+1)}\|_2^{2\theta'_p}\end{aligned} \tag{2.4}$$

where

$q=N(p+m+2)/(N-2)(m+1)$,

$$\theta_p=\frac{N(p-m)(p+m+2)}{(m+1)\{N(p+m+2)-(N-2)(m+\alpha+2)\}}\quad\text{and}\quad\theta'_p=\frac{\theta_p(m+1)}{p+m+2}.$$

By our assumption on α we see $\theta'_p<1$ for any $p\geqq m$, and applying Young's inequality we have from (2.1) and (2.4) that

$$\begin{aligned}\frac{d}{dt}\int\beta_{p,\varepsilon}(U(t))dx&+\frac{2(m+1)(p+1)}{(p+m+2)^2}\|\nabla U^{(p+m+2)/2(m+1)}(t)\|_2^2\\&+\varepsilon\int U^{(p+\alpha'+2)/(m+1)}dx\leqq C(p)\,.\end{aligned} \tag{2.5}$$

But, from the definition of $\beta_{p,\varepsilon}(U)$ (see (2.2)),

$$\begin{aligned}\int\beta_{p,\varepsilon}(U)dx&\leqq\frac{1}{p+2}\int U^{(p+2)/(m+1)}dx+\frac{\varepsilon}{p+m+2}\int U^{(p+m+2)/(m+1)}dx\\&\leqq\frac{1}{p+2}\Big\{\|\nabla U^{(p+m+2)/2(m+1)}\|_2^{2(p+2)/(p+m+2)}\\&\qquad+\varepsilon\Big(\int U^{(p+\alpha'+2)/(m+1)}dx\Big)^{(p+m+2)/(p+\alpha'+2)}\Big\}\,.\end{aligned}$$

Here we choose α' as $\alpha'>\max(\alpha,(m^2+4m)/2)$. Then $(p+m+2)^2/(p+\alpha'+2)(p+2)<1$, and

$$\begin{aligned}&\Big(\int\beta_{p,\varepsilon}(u)dx\Big)^{(p+m+2)/(p+2)}\\&\leqq\frac{C}{p}\Big\{\|\nabla U^{(p+m+2)/2(m+1)}\|_2^2+\varepsilon^{(p+m+2)/(p+2)}\Big(\int U^{(p+\alpha'+2)/(m+1)}dx\Big)^{(p+m+2)^2/(p+\alpha'+2)(p+2)}\Big\}\\&\leqq C\Big\{\frac{1}{p}\|\nabla U^{(p+m+2)/2(m+1)}\|_2^2+\varepsilon\int U^{(p+\alpha'+2)/(m+1)}dx+k_p(\varepsilon)\Big\}\,.\end{aligned} \tag{2.6}$$

It follows from (2.5) and (2.6) that

$$\frac{d}{dt}\int\beta_{p,\varepsilon}(U(t))dx+C\Big(\int\beta_{p,\varepsilon}(U(t))dx\Big)^{(p+m+2)/(p+2)}\leqq C(p)+k_p(\varepsilon) \tag{2.7}$$

which implies (see Simon [**17**])

$$\int\beta_{p,\varepsilon}(U(t))dx\leqq\Big\{\Big(\Big(\int\beta_{p,\varepsilon}(U_0)dx\Big)^{-m/(p+2)}+\frac{mC}{p+2}t\Big\}^{-(p+2)/m}+\{C(p)+k_p(\varepsilon)\}t. \tag{2.8}$$

Moreover, we see

$$\begin{aligned}\beta_{p,\varepsilon}(U)&\geqq\frac{1}{m+1}\int_0^U(\eta+\varepsilon)^{-m/(m+1)}\eta^{(p+1)/(m+1)}d\eta\\ &=(U+\varepsilon)^{1/(m+1)}U^{(p+2)/(m+1)}-\frac{p+1}{p+2}(U+\varepsilon)^{(p+2)/(m+1)}+\frac{p+1}{p+2}\varepsilon^{(p+2)/(m+1)}\\ &\geqq\frac{1}{2(p+2)}U^{(p+2)/(m+1)}-k_p(\varepsilon)\end{aligned}$$

and hence, by (2.8),

$$\|u_\varepsilon(t)\|_{p+2}\leqq C(p)\Big\{\Big(\Big(\int\beta_{p,\varepsilon}(U_0)dx\Big)^{-m/(p+2)}+t\Big\}^{-1/m}+(C(p)+k_p(\varepsilon))t^{1/(p+2)}+k_p(\varepsilon) \tag{2.9}$$

for $0<t<\infty$. We obtain also from (2.9) that

$$\|u_\varepsilon(t)\|_{p+2}\leqq C(p)t^{-1/m}+C(p)t^{1/(p+2)}+k_p(\varepsilon) \tag{2.10}$$

for $0<t<\infty$.

The inequalities (2.9) and (2.10) are useful to know the behaviour of $u(t)$ near $t=0$. In order to see the decay property of $u(t)$ in L^p norm as $t\to\infty$ we must make a further device. We use another inequality instead of (2.4). First, by Hölder's inequality

$$\|U\|_{(p+\alpha+2)/(m+1)}\leqq\|U\|_r^{1-\theta}\|U\|_q^{\theta} \tag{2.11}$$

where we set $q=N(p+m+2)/(N-2)(m+1)$, $r=N(\alpha-m)/2(m+1)$ and $\theta=(p+m+2)/(p+\alpha+2)$. Since $r<2N/(N-2)$ by our assumption on α, it follows from (2.11) that

$$\int U^{(p+\alpha+2)/(m+1)}dx\leqq C\|\nabla U\|_2^{(p+\alpha+2)(1-\theta)/(m+1)}\|\nabla U^{(p+m+2)/2(m+1)}\|_2^2 \tag{2.12}$$

which we shall employ instead of (2.4).

By Proposition 1.1 we already know

$$\|\nabla U(t)\|_2\leqq\bar{C}_\varepsilon(1+t)^{-(m+1)/m}+k(\varepsilon)$$

and (2.12) implies

$$\int U^{(p+\alpha+2)/(m+1)}dx\leqq\bar{C}_\varepsilon\{(1+t)^{-(\alpha-m)/m}+k(\varepsilon)\}\|\nabla U^{(p+m+2)/2(m+1)}\|_2^2. \tag{2.13}$$

From (2.1) and (2.13) we see that there exists $T_p>0$ such that, if $t\geqq T_p$,

$$\int\beta_{p,\varepsilon}(U(t))dx\leqq\left\{\left(\int\beta_{p,\varepsilon}(U(T_p))dx\right)^{-m/(p+2)}+\frac{C}{p}(t-T_p)\right\}^{-(p+2)/m}+k_p(\varepsilon)(t-T_p)$$

or

$$\|u_\varepsilon(t)\|_{p+2}\leqq\left\{\left(\int\beta_{p,\varepsilon}(U(T_p))dx\right)^{-m/(p+2)}+\frac{C}{p}(t-T_p)\right\}^{-1/m}$$
$$+k_p(\varepsilon)(t-T_p)^{1/(p+2)}+k_p(\varepsilon)\,. \tag{2.14}$$

Combining the estimate in Proposition 1.1 with (2.9), (2.10) and (2.14) we can prove the following;

THEOREM 2.1. *Let* $0<m<\alpha<m^*$ $(N\geqq3)$ *and let* $u_0^{m+1}\in\mathfrak{W}$. *Then the problem* (P_2) *admits a solution in the sense of Theorem* 0 *which satisfies, in addition to* (0.3),

$$\|u(t)\|_{p+2}\leqq C(p)t^{-1/m}+C(p,\ \|\nabla u_0^{m+1}\|_2)(1+t)^{-1/m} \tag{2.15}$$

on $(0,\infty)$ *for any* $p\geqq m$. *Moreover, if we assume* $u_0^{m+1}\in\mathfrak{W}\cap L^{(p+2)/(m+1)}$ *the solution satisfies*

$$\|u(t)\|_{p+2}\leqq C(p)(\|u_0\|_{p+2}^{-m}+t)^{-1/m}+C(p,\ \|\nabla u_0^{m+1}\|_2)(1+t)^{-1/m} \tag{2.16}$$

on $[0,\infty)$ *for any* $p\geqq m$.

At this stage the proof of Theorem 2.1 is routine (see the section 1) and omitted.

Next, we consider the case $0\leqq\alpha<m$. In this case the situation concerning the global existence and L^p boundedness of solutions is much simpler. Indeed, we can estimate the third term in (2.1) as follows;

$$\int U^{(p+\alpha+2)/(m+1)}dx\leqq\frac{2(m+1)(p+1)}{(p+m+2)}\|\nabla U^{(p+m+2)/2(m+1)}\|_2^2+C(p)$$

and we obtain again (2.9) and (2.10) through (2.5)-(2.8). Furthermore, we have from (2.7)

$$\int\beta_{p,\varepsilon}(U(t))dx\leqq\max\left\{\int\beta_{p,\varepsilon}(U(s))dx,\ (C(p)+k_p(\varepsilon))^{(p+2)/(p+m+2)}\right\} \tag{2.17}$$

for $t>s$, and also

$$\overline{\lim_{t\to\infty}}\int\beta'_{p,\varepsilon}(U(t))dx\leqq(C(p)+k_p(\varepsilon))^{(p+2)/(p+m+2)}.$$

Thus we can assert;

THEOREM 2.2. *Let* $0\leqq\alpha<m$ *and* $u_0^{m+1}\in H_0^1$. *Then the solution of* (P_2) *in Theorem* 0 *satisfies, in addition to* (0.1),

$$\|u(t)\|_{p+2}\leqq C(p)t^{-1/m}+C(p,\ \|\nabla u_0^{m+1}\|_2)\qquad on\quad(0,\infty)$$

and $\overline{\lim}_{t\to\infty}\|u(t)\|_{p+2}\leqq C(p)$ *for any* $p\geqq m$. *If we make additional assumption* $u_0\in L^{p+2}$, *then*

$$\|u(t)\|_{p+2}\leqq\max(\|u_0\|_{p+2},\ \|\nabla u_0^{m+1}\|_2,\ C(p)) \qquad on \quad [0,\infty)$$

for any $p\geqq m$.

We close this section by noting that the estimates (2.9), (2.10), (2.14) and (2.17) remain valid for the approximate solutions $u_\varepsilon(t)$ (or $U_\varepsilon(t)$) of the problem $(P_{1,\varepsilon})$ because of the relation (1.6).

3. L^∞ estimate of the solutions of the problem (P_2).

Taking the limits as $p\to\infty$ of the estimates obtained in the section 2 all of the right hand sides are divergent and we can not get any result on L^∞ norm of the solutions. Here we shall show that adopting the Moser's technique as in Alikakos [**1**] this difficulty is overcomed to yield the boundedness or decay property of the solutions of $(P_{2,\varepsilon})$ (and consequently of (P_2)) in L^∞ norm under the condition

$$u_0^{m+1}\in H_0^1\cap L^\infty\ (0\leqq\alpha<m) \quad \text{or} \quad u_0^{m+1}\in\mathfrak{W}\cap L^\infty\ (0<m<\alpha<m^*).$$

LEMMA 3.1. *Let* $w(t)$ *be a function defined on* $\Omega\times[0,\infty]$ *(appropriately smooth) satisfying*

$$\frac{d}{dt}\|w(t)\|_{\lambda+1}^{\lambda+1}+C_0(1+\lambda)^{-\theta_0}\|\nabla w(t)^{(\lambda+m+1)/r}\|_r^r\leqq C_1(1+\lambda)^{\theta_1}\|w(t)\|_{\lambda+1}^{\lambda+1}$$

for any $\lambda\geqq\lambda_0>\max(0,\ r-m-1,\ (m-r+1)/(r-1))$ *with some constants* C_0 (>0), C_1 (>0), θ_0 $(\geqq0)$, θ_1 $(\geqq0)$ *and* $r>1$. *Suppose that* $w_0\equiv w(0)\in L^\infty(\Omega)$ *and* $\sup_{t\geq0}\|w(t)\|_{\lambda_0+1}<\infty$. *Then, there exist constants a, b, c and d such that*

$$\sup_{t\geq0}\|w(t)\|_\infty\leqq a^\eta r^{[\theta_1+(\theta_0+\theta_1)b]r\eta}\max\{1,\ \sup_{t\geq0}\|w(t)\|_{\lambda_0+1}^d,\ c\|w_0\|_\infty\}$$

where $\eta=[\lambda_0-(m-r+1)/(r-1)]^{-1}$.

The special case; $r=2$, $\theta_0=1$ and $\theta_1=0$ is proved in Alikakos [**1**] and the general case is also proved quite similarly. For completeness, however, we shall give the outline of the proof later in the Appendix.

Now, our result is;

THEOREM 3.1. *Let* $u(t)$ *be the solution of the problem* (P_2) *obtained in the section* 1.

(i) *If* $0\leqq\alpha<m$ *and* $u_0^{m+1}\in H_0^1\cap L^\infty$, *then we have* $u\in L^\infty(R^+\times\Omega)$ *and*

$$\|u(t)\|_\infty\leqq C(\|\nabla u_0^{m+1}\|_2,\ \|u_0\|_\infty) \qquad on \quad [0,\infty).$$

(ii) *If* $0<m<\alpha<m^*$ *and* $u_0^{m+1}\in\mathfrak{W}\cap L^\infty$, *then we have also* $u\in L^\infty(\Omega\times R^+)$ *and*

$$\|u(t)\|_\infty\leqq C(\|\nabla u_0^{m+1}\|_2,\ \|u_0\|_\infty)(1+t)^{-1/m}.$$

PROOF. We shall give a formal proof, which may be made rigorous by the use of approximate solutions u_ε or the theory of nonlinear semi-group (Evans [7], Benilan and Crandall [3] etc.). We consider the case $N\geqq 3$. The proof is valid for $N=2$ with slight modification.

By (2.1) (with $\varepsilon=0$)

$$\frac{1}{p+2}\frac{d}{dt}\|u(t)\|_{p+2}^{p+2}+\frac{4(m+1)(p+1)}{(p+m+2)^2}\|\nabla u^{(p+m+2)/2}\|_2^2=C_0\int u^{p+\alpha+2}dx. \tag{3.1}$$

The right hand side is estimated as follows.

$$C_0\int u^{p+\alpha+2}dx\leqq C_0\|u\|_{p+2}^{\theta_1}\|u\|_{m+\alpha+2}^{\theta_2}\|u\|_{N(p+m+2)/(N-2)}^{\theta_3} \tag{3.2}$$

where we set

$$\theta_1=(p+2)\left\{1-\frac{N\alpha}{2(m+\alpha+2)+mN}\right\},\quad \theta_2=\alpha\left\{1-\frac{mN}{2(m+\alpha+2)+mN}\right\}$$

$$\text{and}\quad \theta_3=\frac{N\alpha(p+m+2)}{2(m+\alpha+2)+mN}.$$

Noting that $\|u(t)\|_{m+\alpha+2}\leqq C(\|\nabla u_0^{m+1}\|_2)$, and applying Sobolev's Lemma and Young's inequality we have

$$\begin{aligned}C_0\int u^{p+\alpha+2}dx&\leqq\frac{\theta_1}{p+2}\{C_0\bar{C}C^{2\theta_3/(p+m+2)}\}^{(p+2)/\theta_1}\\&\quad\times\left\{\frac{2(m+1)(p+1)}{(p+m+2)^2}\cdot\frac{2(m+\alpha+2)+mN}{N\alpha}\right\}^{-N\alpha(p+2)/\theta_1(2(m+\alpha+2)+mN)}\\&\quad\times\|u(t)\|_{p+2}^{p+2}+\frac{2(m+1)(p+1)}{(p+m+2)^2}\|\nabla u^{(p+m+2)/2}\|_2^2\\&\leqq\bar{C}p^{N\alpha/(2(m+\alpha+2)+mN-N\alpha)}\|u(t)\|_{p+2}^{p+2}+\frac{2(m+1)(p+1)}{(p+m+2)^2}\|\nabla u^{(p+m+2)/2}\|_2^2\end{aligned}$$

where $\bar{C}$ denotes constants depending on $\|\nabla u_0^{m+1}\|_2$.

It follows from (3.1) and the above that

$$\frac{d}{dt}\|u(t)\|_{p+2}^{p+2}+\frac{2(m+1)(p+1)(p+2)}{(p+m+2)^2}\|\nabla u^{(p+m+2)/2}(t)\|_2^2\leqq\bar{C}p^{\nu}\|u(t)\|_{p+2}^{p+2} \tag{3.3}$$

with $\nu=N\alpha/[2(m+\alpha+2)+mN-N\alpha]$.

Applying Lemma 3.1 with $r=2$, $\theta_0=0$, $\theta_1=\nu$, $\lambda_0=m+1$ and $\lambda=p+1$ to (3.3) we obtain

$$\|u(t)\|_\infty\leqq C(\|\nabla u_0^{m+1}\|_2,\ \|u_0\|_\infty)<\infty.$$

To show the further estimate for the case $0<m<\alpha<m^*$ we set $(1+t)^{1/m}u(t)=w(t)$. Then, $w(t)$ satisfies

$$\frac{\partial}{\partial t}w(t)=\frac{1}{(1+t)}\left\{\Delta w^{m+1}(t)+C_0(1+t)^{1-\alpha/m}w^{1+\alpha}+\frac{1}{m}w(t)\right\}$$

and, changing the scale as $\tau=\log(1+t)^{(m+1)/m}$, we have

$$\frac{\partial}{\partial t}w(\tau)=\frac{m}{m+1}\Delta w^{m+1}+\frac{1}{m+1}w+\frac{mC_0}{m+1}(1+t)^{-\alpha/m}w^{1+\alpha}$$

$$\leqq\frac{m}{m+1}\Delta w^{m+1}+\frac{1}{m+1}w+Cw^{1+\alpha}. \tag{3.4}$$

From (3.4) we can get an inequality similar to (3.3) for $w(t)$. Since we know already

$$\|w(\tau)\|_{m+2}\leqq C(1+t)^{1/m}\|\nabla u^{m+1}(t)\|_2^{1/(m+1)}\leqq C(\|\nabla u_0^{m+1}\|_2)<\infty\,,$$

Lemma 3.1 yields

$$\|w(t)\|_\infty\leqq C(\|\nabla u_0^{m+1}\|_2,\ \|u_0\|_\infty)$$

which proves the assertion of (ii).

4. Existence, uniqueness and behaviour of solutions of the problem (P_1).

On the basis of the results in the sections 1-3 we shall treat here the problem (P_1). We make the same assumptions on u_0 as in the section 1. The smoothness assumption can be removed easily at the last stage as is usual (see the section 1). Let u_ε and $\tilde{u}_\varepsilon$ be the approximate solutions of $(P_{2,\varepsilon})$ and $(P_{1,\varepsilon})$, respectively.

By (1.6) and (2.9) we know

$$\|\tilde{u}_\varepsilon(t)\|_{p+2}\leqq\|u_\varepsilon(t)\|_{p+2}\leqq C(p,\ \|u_0\|_{p+2},\ \|\nabla u_0^{m+1}\|_2,\ k(\varepsilon)\|u_0\|_{m+\alpha'+2},\ \varepsilon) \tag{4.1}$$

where the right hand side tends to a constant $C(p,\ \|u_0\|_{p+2},\ \|\nabla u_0^{m+1}\|_2)$ as $\varepsilon\to 0$.

In order to see the convergence of $\tilde{u}_\varepsilon(t)$, let us consider the problem;

$$\begin{gathered}\frac{1}{m+1}((U+\varepsilon)^{-m/(m+1)}+\varepsilon)U_t-\Delta U+\varepsilon U^{(1+\alpha')/(m+1)}=g(x,\ t)\\ U(x,\ t)=U_0\,,\quad U|_{\partial\Omega}=0\quad\text{and}\quad U\geqq 0\end{gathered} \tag{4.2}$$

where $g(x,\ t)$ is an appropriately smooth function. We may assume $U_0\in C_0^1(\Omega)$. By (4.2) we have

$$\frac{1}{m+1}\int\{(U+\varepsilon)^{-m/(m+1)}+\varepsilon\}U_t^2dx+\frac{d}{dt}F_\varepsilon(U(t))=\int gU_tdx$$

where we set

$$F_\varepsilon(U)=\frac{1}{2}\|\nabla U\|_2^2+\varepsilon\frac{(m+1)}{m+\alpha+2}\int U^{(m+\alpha'+2)/(m+1)}dx\,.$$

Now,

$$\int gU_tdx\leqq\frac{1}{2(m+1)}((U+\varepsilon)^{-m/(m+1)}+\varepsilon)U_t^2dx+C\int g^2((U+\varepsilon)^{-m/(m+1)}+\varepsilon)^{-1}dx$$

$$\leqq \frac{1}{2(m+1)}\int((U+\varepsilon)^{-m/(m+1)}+\varepsilon)U_t^2dx$$
$$+C\{\|g\|_{q_0}^2\|\nabla U\|_2^{m/(m+1)}+\varepsilon^{m/(m+1)}\|g\|_2^2\}$$

where we set

$$q_0=\begin{cases} 4N(m+1)/(mN+2N+2m) & \text{if } N>2 \\ \text{arbitrary } (>2) & \text{if } N=2 \\ 2 & \text{if } N=1. \end{cases} \tag{4.3}$$

Thus we have

$$\frac{1}{2(m+1)}\int((U+\varepsilon)^{-m/(m+1)}+\varepsilon)U_t^2dx+\frac{d}{dt}F_\varepsilon(U(t))$$
$$\leqq C(\|g(t)\|_{q_0}^{4(m+1)/(m+2)}+\varepsilon^{m/(m+1)}\|g(t)\|_2^2+F_\varepsilon(U(t)))$$

and hence

$$\|\nabla U(t)\|_2^2+\varepsilon\int U^{(m+\alpha'+2)/(m+1)}(t)dx$$
$$+\int_0^T\int\left|\frac{\partial}{\partial t}\int_0^{U(s)}(\eta+\varepsilon)^{-m/2(m+1)}d\eta\right|^2dx\,ds+\varepsilon\int_0^T\int U_t^2\,dx\,ds$$
$$\leqq C(T,\ U_0,\ g,\ \varepsilon)<\infty \tag{4.4}$$

for any $t\in[0,\ T]$, $T>0$, where the above constant depends on

$$\int_0^T(\|g(s)\|_{q_0}^{4(m+1)/(m+2)}+\varepsilon^{m/(m+1)}\|g(s)\|_2^2)ds,\quad T\quad \text{and}\quad \|\nabla U_0\|_2^2+\varepsilon\|U_0\|_{\beta'}^{\beta'}$$

$(\beta'=(m+\alpha'+2)/(m+1))$.

Here, we make an additional assumption;

$$|f(x,\ t,\ u)|\leqq C_1(1+u^{1+\bar{\alpha}})\qquad \text{for}\quad u\geqq 0 \tag{4.5}$$

with some $\bar{\alpha}$ $(\geqq\alpha)$.

Then, we see by (4.1) that

$$\|f(\cdot,\ t,\ \tilde{u}_\varepsilon)\|_{q_0}\leqq C(1+\|\tilde{u}_\varepsilon\|_{q_0(1+\bar{\alpha})}^{1+\bar{\alpha}})\leqq\tilde{C} \tag{4.6}$$

where $\tilde{C}$ denotes constants depending on $\|\nabla u_0^{m+1}\|_2$ and $\|u_0\|_{q_0(1+\bar{\alpha})}$. Thus, setting $g(x,\ t)=f(x,\ t,\ \tilde{u}_\varepsilon(x,\ t))$, we obtain from (4.4) and (4.6)

$$\|\nabla u^{m+1}(t)\|_2^2+\varepsilon\int u^{m+\alpha'+2}(t)dx+\int_0^T\left\|\frac{\partial}{\partial t}u^{m/2+1}(s)\right\|_2^2ds\leqq C(T,\ \varepsilon) \tag{4.7}$$

where $C(T,\ \varepsilon)$ is a constant depending on u_0, T and ε. Note that $\lim_{\varepsilon\to 0}C(T,\ \varepsilon)$ depends on only $\|\nabla u_0^{m+1}\|_2$ and $\|u_0\|_{q_0(1+\bar{\alpha})}$. The estimates (4.1) and (4.7) are sufficient for the convergence of $\tilde{u}_\varepsilon(t)$ to the required solution of (P_1).

THEOREM 4.1. *Suppose that the Assumption* 1 *and* (4.5) *are valid, and let* $u_0^{m+1}\in H_0^1\cap L^{q_0(1+\bar{\alpha})/(m+1)}$ (*if* $0\leqq\alpha<m$) *or* $u_0^{m+1}\in\mathfrak{W}\cap L^{q_0(1+\bar{\alpha})/(m+1)}$ (*if* $0<m<\alpha<m^*$) *with* q_0 *defined by* (4.3). *Then the problem* (P_1) *has a solution* $u(t)$ *such that*

$$\|u(t)\|_{(1+\bar{\alpha})q_0} \leqq \tilde{C} \qquad for \quad t \in R^+$$

and

$$\int_t^{t+1} \left\| \frac{\partial}{\partial t}(u^{m/2+1}(s)) \right\|_2^2 ds + \|\nabla u^{m+1}(t)\|_2^2 \leqq \tilde{C} \qquad for \quad t \in R^+. \tag{4.8}$$

Moreover, if $0<m<\alpha<m^*$ *and* f *satisfies the stronger condition*

$$|f(x, t, u)| \leqq C_0 u^{1+\alpha} \qquad for \quad u \geqq 0, \tag{4.9}$$

the right hand side of (4.8) *can be replaced by* $\tilde{C}(1+t)^{-2(m+1)/m}$.

PROOF. It remains to prove (4.8) and the final assertion. By (4.7) we know already

$$\int_0^T \left\| \frac{\partial}{\partial t}(u^{m/2+1}(s)) \right\|_2^2 ds + \|\nabla u^{m+1}(t)\|_2^2 \leqq C(T) \tag{4.10}$$

for each $T>0$. Now, let us consider the problem

$$u_t - \Delta u^{m+1} = g(x, t) \tag{4.11}$$

with the conditions $u(0)=u_0$, $u|_{\partial\Omega}=0$. It is known that for $g \in L^1_{\mathrm{loc}}(R^+; L^1(\Omega))$ and $u_0 \in L^1(\Omega)$ the problem (4.11) has a unique solution in the sense of semi-group theory (Evans [7], Benilan and Crandall [3]). By the general uniqueness theorem (Brézis [4], Brézis and Crandall [5]) the solution of (4.11) must coincide with our solution $u(t)$ when $g(x, t)=-f(x, t, u(x, t))$. We shall give somewhat formal proof of our estimates, which can be made rigorous by use of semi-group theory ([3]) or appropriate approximate solution ([12]). (In fact our argument is a simplified version of the proof of the theorem in [12].)

Multiplying (4.11) (with $g(x, t)=-f(x, t, u)$) by $(u^{m+1})_t$ we have

$$\frac{(m+1)}{2}\int u^m u_t^2 dx + \frac{1}{2}\frac{d}{dt}\|\nabla u^{m+1}(t)\|^2 \leqq C\|g(t)\|_{q_0}^2 \|\nabla u^{m+1}(t)\|_2^{m/(m+1)} \tag{4.12}$$

with q_0 defined in (4.3) (note that (4.11) is equivalent to (4.2) with $\varepsilon=0$). Next, multiplying (4.11) by u^{m+1} and using (4.12) we have (cf. (1.5))

$$\begin{aligned}\|\nabla u^{m+1}(t)\|_2^2 &\leqq \left(\int u^m u_t^2 dx\right)^{1/2} \|u^{m+1}(t)\|_{(m+2)/(m+1)}^{(m+2)/(m+1)} + \|g(t)\|_2 \|u^{m+1}(t)\|_2 \\ &\leqq C\|\nabla u^{m+1}(t)\|_2^{(m+2)/2(m+1)} \Big\{ C\|g(t)\|_{q_0}^2 \|\nabla u^{m+1}(t)\|_2^{m/(m+1)} \\ &\qquad - \frac{1}{2}\frac{d}{dt}\|\nabla u^{m+1}(t)\|^2 \Big\}^{1/2} + C\|g(t)\|_{q_0}\|\nabla u^{m+1}(t)\|_2\end{aligned}$$

and hence

$$\begin{aligned}&\|\nabla u^{m+1}(t)\|_2^{(m+2)/(m+1)} \Big\{ C\|\nabla u^{m+1}(t)\|_2^{(3m+2)/(m+1)} \\ &\qquad + \frac{d}{dt}\|\nabla u^{m+1}(t)\|^2 - C\|g(t)\|_{q_0}^2 \|\nabla u^{m+1}(t)\|_2^{m/(m+1)} \Big\} \leqq 0 .\end{aligned}$$

Thus we obtain

$$\frac{d}{dt}\|\nabla u^{m+1}(t)\|_2^2+C\|\nabla u^{m+1}(t)\|_2^{(3m+2)/(m+1)}\leqq C\|g(t)\|_{q_0}^{(3m+2)/(m+1)} \tag{4.13}$$

as long as $\|\nabla u^{m+1}(t)\|_2\neq 0$. Comparing $\|\nabla u^{m+1}(t)\|_2^2$ with a solution of ordinary differential inequality

$$y'+Cy^{(3m+2)/2(m+1)}\geqq h(t)$$

with $h(t)=\|g(t)\|_{q_0}^{(3m+2)/(m+1)}$ we conclude from (4.13) that

$$\|\nabla u^{m+1}(t)\|_2^2\leqq\{\|\nabla u^{m+1}(s)\|_2^{-m/(m+1)}+C(t-s)\}^{-2(m+1)/m}$$
$$+C\int_s^t\|g(\eta)\|_{q_0}^{(3m+2)/(m+1)}d\eta \tag{4.14}$$

for $t\geqq s\geqq 0$.

Setting $g(x, t)=-f(x, t, u)$ and taking $s=t-1$, $t\geqq 1$, it follows from (4.14) that

$$\|\nabla u^{m+1}(t)\|_2^2\leqq C^{-2(m+1)/m}+C\int_{t-1}^t\|g(\eta)\|_{q_0}^{(3m+2)/(m+1)}d\eta\leqq\tilde{C}$$

under the assumption (4.5) (see (4.6)). We have also by (4.12)

$$\int_t^{t+1}\|u^{m/2}u_t\|_2^2ds\leqq\tilde{C}\,.$$

If $m<\alpha<m^*$ and (4.9) holds, we see from Theorem 2.1 and (1.6) that

$$\|f(x, t, u)\|_{q_0}\leqq C_0\|u(t)\|_{q_0(1+\alpha)}^{1+\alpha}\leqq\tilde{C}(1+t)^{-(1+\alpha)/m}.$$

Thus, taking $s=t/2$ in (4.14) we have

$$\|\nabla u^{m+1}(t)\|_2^2\leqq\tilde{C}t^{-2(m+1)/m}. \tag{4.15}$$

Combining (4.15) with (4.10) we obtain the desired estimate for $\|\nabla u^{m+1}(t)\|_2$. Finally, integrating (4.12) we see easily

$$\int_t^\infty\|u^{m/2}u_t\|_2^2ds\leqq\tilde{C}(1+t)^{-2(m+1)/m}$$

which is a slightly stronger version than required. q. e. d.

We do not know whether the uniqueness is valid or not under the assumptions of Theorem 4.1. However, if we can derive L^∞ estimate of solutions the uniqueness follows rather easily. We establish the following;

THEOREM 4.2. *Let* $u_0^{m+1}\in H_0^1\cap L^\infty$ *if* $0\leqq\alpha<m$ *and let* $u_0^{m+1}\in\mathfrak{W}\cap L^\infty$ *if* $0<m<\alpha<m^*$. *Then, under the Assumption* 1, *the problem* (P_1) *has a unique solution* $u(t)$ *such that*

$$\|u(t)\|_\infty\leqq\begin{cases}\bar{C}<\infty & \textit{if}\ \ 0<\alpha<m\\ \bar{C}(1+t)^{-1/m} & \textit{if}\ \ 0<m<\alpha<m^*\end{cases}$$

and (4.8) *is satisfied with* C *replaced by* $\bar{C}$, *where* $\bar{C}$ *denotes constants depending on* $\|\nabla u_0^{m+1}\|_2$ *and* $\|u_0\|_\infty$. *Moreover, if* $0<m<\alpha<m^*$ *and* f *satisfies*

$$|f(x, t, u)| \leqq C(r)u^{1+\alpha} \qquad for \quad 0 \leqq u \leqq r, \quad r>0,$$

the right hand side of (4.8) *can be replaced by* $\bar{C}(1+t)^{-(m+1)/m}$.

PROOF. The existence and the estimates except for $\|u(t)\|_\infty$ are clear from the proof of Theorem 4.1. The L^∞ estimate follows from Theorem 3.1 and Proposition 1.2. To show the uniqueness we let u_1 and u_2 be two solutions. Then, by the theory of nonlinear semi-groups we have

$$\begin{aligned}\|u_1(t)-u_2(t)\|_1 &\leqq \|u_1(0)-u_2(0)\|_1+\int_0^t \|f(\cdot, s, u_1)-f(\cdot, s, u_2)\|_1 ds \\ &\leqq C\int_0^t \|u_1(s)-u_2(s)\|_1 ds \end{aligned} \tag{4.16}$$

where we have used the L^∞ boundedness of u_i $(i=1, 2)$ and the Lipschitz continuity of f with respect to u. The inequality implies $u_1=u_2$. q.e.d.

REMARK. Quite recently Levine and Sacks [**10**] have proved essentially the same result as Theorem 4.2 for the case $0 \leqq \alpha < m$. Their proof is different from ours. The existence and decay for the case $0<m<\alpha$, which is more difficult, is not investigated there. Subsequently, Sacks [**15**] has established closely related result to our Theorem 4.2. His method is also quite different from ours and does not seem to be applicable to the problem (P_4).

5. Solutions to the system (P_3).

In this section we discuss the problem (P_3). Since the case $m=0$ or $n=0$ is simpler we assume m and n are positive.

Let U_ε and V_ε be a pair of solutions of the approximate equations

$$(P_{3,\varepsilon}) \quad \begin{cases} \dfrac{1}{m+1}((U+\varepsilon)^{-m/(m+1)}+\varepsilon)U_t-\Delta U+f_{1,\varepsilon}(x, t, U^{1/(m+1)}, V^{1/(n+1)})=0 \\ \dfrac{1}{n+1}((V+\varepsilon)^{-n/(n+1)}+\varepsilon)V_t-\Delta V+f_2(x, t, U^{1/(m+1)}, V^{1/(n+1)})=0 \end{cases}$$

with $U(x, 0)=U_0$, $V(x, 0)=V_0$, $U|_{\partial\Omega}=V|_{\partial\Omega}=0$, and $U, V \geqq 0$, where we set

$$f_{1,\varepsilon}=f_1+\varepsilon u^{1+\alpha'} \qquad (\alpha'>\alpha).$$

First, we assume $U_0 \in C_0^1(\Omega)$ and $V_0 \in C_0^1(\Omega)$ which can be weakened at the last step by approximation. Then it is easy to see that $(P_{3,\varepsilon})$ has a local classical solution $(U_\varepsilon(t), V_\varepsilon(t))$ and for the continuation of it to $[0, \infty)$ it suffices to derive a priori L^∞ bounds for $U_\varepsilon(t)$ and $V_\varepsilon(t)$ on $[0, \infty)$. To prove the convergence of them as $\varepsilon \to 0$ we have only to show the estimate like (4.4), which follows immediately from L^∞ boundedness of them. Denoting the limits as $\varepsilon \to 0$ of U_ε

and V_ε by U and V, respectively, $u=U^{1/(m+1)}$ and $v=V^{1/(n+1)}$ should be a pair of desired solutions of the problem (P_3) with $u_0=U_0^{1/(m+1)}$ and $v_0=V_0^{1/(n+1)}$. Thus our task is to derive L^∞ estimates for U_ε and V_ε. To make the essential feature clear we consider the original problem (P_3) (or $(P_{3,\varepsilon})$ with $\varepsilon=0$) and treat the equations formally. The result can be made rigorous through U_ε and V_ε as is done in previous sections.

Multiplying the second equation of (P_3) by v^{p+1} and integrating we have, by Assumption 2, (ii),

$$\frac{1}{p+2}\frac{d}{dt}\|v(t)\|_{p+2}^{p+2}+\frac{4(n+1)(p+1)}{(p+n+2)^2}\|\nabla v^{(p+n+2)/2}(t)\|_2^2\leqq 0$$

which yields as in the section 3

$$\|v(t)\|_\infty\leqq\|v_0\|_\infty\equiv K \tag{5.1}$$

and

$$\|v(t)\|_{p+2}\leqq C(\|v_0\|_{p+2})(1+t)^{-1/n} \tag{5.2}$$

for $0\leqq p\leqq\infty$. By (5.1) and the Assumption 2, (i), we have

$$f_1(x,t,u(t),v(t))\geqq -C_0(K)u^{1+\alpha} \tag{5.3}$$

with the above K. We denote by $\mathfrak{W}_K$ the potential well $\mathfrak{W}$ defined by (1.1) with C_0 replaced by $C_0(K)$. Then, the estimates in Theorem 4.3 are valid for $u(t)$ under the assumption $u_0^{m+1}\in\mathfrak{W}_K\cap L^\infty$ $(0<m<\alpha<m^*)$ or $u_0^{m+1}\in H_0^1\cap L^\infty$ $(0\leqq\alpha<m)$. From the L^∞ boundedness of $u(t)$ and $v(t)$, $f_i(x,t,u,v)$, $i=1,2$, are uniformly bounded and we obtain the estimates like (4.8) for u and v. We state our conclusion.

THEOREM 5.1. *Let $v_0^{n+1}\in H_0^1\cap L^\infty$ and let $u_0^{m+1}\in\mathfrak{W}_K\cap L^\infty$ $(K=\|v_0\|_\infty)$ (if $0<m<\alpha<m^*$) or $u_0^{m+1}\in H_0^1\cap L^\infty$ (if $0\leqq\alpha<m$). Then, under Assumption 2, the system (P_3) admits a unique pair of solutions $(u(t),v(t))$ such that*

$$\|u(t)\|_\infty\leqq\begin{cases} C_1(1+t)^{-1/m} & \text{if } 0<m<\alpha<m^* \\ C_1<\infty & \text{if } 0\leqq\alpha<m, \end{cases}$$

$$\|v(t)\|_\infty\leqq C_2(1+t)^{-1/n}$$

and

$$\int_t^{t+1}\left(\left\|\frac{\partial}{\partial t}u^{m/2+1}(s)\right\|_2^2+\left\|\frac{\partial}{\partial t}v^{n/2+1}(s)\right\|_2^2\right)ds + \|\nabla u^{m+1}(t)\|_2^2+\|\nabla v^{n+1}(t)\|_2^2\leqq C_3<\infty \tag{5.4}$$

for any $t\in[0,\infty)$, where the definitions of solutions are similar to that of Theorem 0. Moreover, if we assume

$$|f_i(x,t,u,v)|\leqq C(r)(u^{\beta_i}+v^{\bar\beta_i}),\qquad i=1,2,$$

for $0 \leqq u,\ v \leqq r$ *with* $\beta_1 \geqq n(m+1)/m$, $\bar{\beta}_1 \geqq m+1$, $\beta_2 > m(n+1)/n$ *and* $\bar{\beta}_2 > n+1$, *we have the following decay properties instead of* (5.4);

$$\left(\int_t^{t+1} \left\| \frac{\partial}{\partial t} u^{m/2+1}(s) \right\|_2^2 ds\right)^{1/2} + \|\nabla u^{m+1}(t)\|_2 \leqq C_4(1+t)^{-(m+1)/m} \tag{5.5}$$

and

$$\left(\int_t^{t+1} \left\| \frac{\partial}{\partial t} v^{n/2+1}(s) \right\|_2^2 ds\right)^{1/2} + \|\nabla v^{n+1}(t)\|_2 \leqq C_5(1+t)^{-(n+1)/n} \tag{5.5$'$}$$

where (5.5) *is valid only for the case* $0 < m < \alpha < m^*$. *The dependence of the constants* C_i *are as follows;*

$$C_1 = C_1(\|\nabla u_0^{m+1}\|_2,\ \|u_0\|_\infty,\ \|v_0\|_\infty), \qquad C_2 = C_2(\|v_0\|_\infty) \quad \text{and}$$

$$C_i = C_i(\|\nabla u_0^{m+1}\|_2,\ \|\nabla v_0^{n+1}\|_2,\ \|u_0\|_\infty,\ \|v_0\|_\infty), \qquad i=3,\ 4,\ 5.$$

PROOF. It remains to prove the uniqueness and the estimates (5.5), (5.5)′.

Applying (4.14) to $v(t)$ with $g(x, t) = f_2(x, t, u(x, t), v(x, t))$ we have easily

$$\|\nabla v^{n+1}(t)\|_2 \leqq C_3(1+t)^{-\eta} \tag{5.6}$$

with

$$\eta = \min\left(\frac{n+1}{n},\ \frac{\beta_2(3n+2)}{2m(n+1)} - 1,\ \frac{\bar{\beta}_2(3n+2)}{2n(n+1)} - 1\right)$$

where we have used the decay properties of $\|u(t)\|_\infty$ and $\|v(t)\|_\infty$. The inequality (5.6) implies (5.5)′ immediately by our assumptions on β_2 and $\bar{\beta}_2$. (5.5) is also proved quite similarly.

Next we shall prove the uniqueness. Let (u_i, v_i), $i=1, 2$, be two pairs of solutions of the problem. Then by the theory of nonlinear semi-groups we have

$$\|u_1(t) - u_2(t)\|_1 \leqq \int_0^t \|f_1(x, s, u_1, v_1) - f_1(x, s, u_2, v_2)\|_1 ds$$

$$\leqq \text{Const.} \int_0^t \|u_1 - u_2\|_1 + \|v_1 - v_2\|_1 ds$$

where we have used the Lipschitz continuity of f_1 and the uniform boundedness of u_i and v_i, $i=1, 2$. Similar inequality holds for $\|v_1 - v_2\|_1$ and we have, for $w(t) = \|u_1(t) - u_2(t)\|_1 + \|v_1(t) - v_2(t)\|_1$,

$$w(t) \leqq \text{Const.} \int_0^t w(s) ds$$

which implies $u_1 = u_2$ and $v_1 = v_2$. q. e. d.

REMARK. If we consider the case $m=0$ or (and) $n=0$ the all assertions are valid with the right hand sides of (5.5) or (and) (5.5)′ replaced by $\exp(-\lambda t)$ for some $\lambda > 0$. The proof is clear from the derivation of (5.5) and (5.5)′.

6. L^∞ estimate and uniqueness of solution to the problem (P_4).

In this section we shall discuss briefly on solutions to the problem;

$$(P_4)\qquad \begin{aligned} & u_t-\sum_{i=1}^{N}(|u_{x_i}|^m u_{x_i})_{x_i}-u^{1+\alpha}=0 \quad \text{in} \quad \Omega\times(0, \infty) \\ & u(x, 0)=u_0, \quad u|_{\partial\Omega}=0 \quad \text{and} \quad u\geqq 0, \end{aligned}$$

where m is a positive constant.

The existence and nonexistence of global solutions to (P_4) were investigated by Tsutsumi [**18**] and the results are generalized by several authors. Decay property of global solutions for the case $m<\alpha<m^*$ (see below) is included in [**14**]. The condition imposed to the initial data in [**14**] is somewhat stronger than that in [**18**]. But, this gap is easily buried by a method similar to the proof of Proposition 1.1 (see [**13**]). For comparison with Theorem 0 and for later use we state the following;

THEOREM 0′. (i) *If* $0\leqq\alpha<m$ *the problem* (P_4) *admits a solution* $u(t)$ (*in a standard generalized sense*) *for each* $u_0(\geqq 0)\in W_0^{1,m+2}$ *such that*

$$u(t)\in L^\infty(R^+; W_0^{1,m+2}), \qquad u_t\in L^2_{\text{loc}}(R^+; L^2(\Omega))$$

and

$$\int_t^{t+1}\|u_t(s)\|_2^2 ds+\|u(t)\|_{1,m+2}^{m+2}\leqq C(\|u_0\|_{1,m+2}) \tag{6.1}$$

where $\|\cdot\|_{1,m+2}$ *denotes* $W_0^{1,m+2}$ *norm.*

(ii) *If* $0<m<\alpha<m^*$, *the same assertion holds for* $u_0(\geqq 0)\in\mathfrak{W}$, *and moreover* $u(t)\in\mathfrak{W}$ *for any* $t\geqq 0$ *and we have*

$$\int_t^{t+1}\|u_t(s)\|_2^2 ds+\|u(t)\|_{1,m+2}^{m+2}\leqq C(d-J(u_0))(1+t)^{-(m+2)/m}. \tag{6.2}$$

In the above theorem we define m^*, d, J and $\mathfrak{W}$ as follows;

$$m^*=\begin{cases} (m(N+2)+4)/(N-(m+2)) & \text{if} \quad N>m+2 \\ \infty & \text{if} \quad 1\leqq N\leqq m+2, \end{cases}$$

$$J(u)=\frac{1}{m+2}\sum_{i=1}^{N}\int_\Omega |u_{x_i}|^{m+2}dx-\frac{1}{\alpha+2}\int_\Omega |u|^{\alpha+2}dx,$$

$$d=\inf_{\substack{u\in W_0^{1,m+2}\\ u\neq 0}}\ \sup_{\lambda>0} J(\lambda u)$$

and

$$\mathfrak{W}=\left\{u\in W_0^{1,m+2}\mid J(u)<d \text{ and } \sum_{i=1}^{N}\int_\Omega |u_{x_i}|^{m+2}dx-\int_\Omega |u|^{\alpha+2}dx>0\right\}\cup\{0\}.$$

Our result in this section reads as follows.

THEOREM 6.1. (i) *If* $0\leqq\alpha<m$ *and* $u_0\in W_0^{1,m+2}\cap L^{p+2}$ $(m\leqq p\leqq\infty)$, *the solution of Theorem* 0′ *may be assumed to satisfy the additional estimate*

$$\|u(t)\|_{p+2}\leqq C(p,\ \|u_0\|_{1,m+2},\ \|u_0\|_{p+2})<\infty\,. \tag{6.3}$$

(ii) *If* $0<m<\alpha<m^*$ *and* $u_0\in\mathfrak{W}\cap L^{p+2}$ $(m\leqq p\leqq\infty)$, *then the solution may be assumed to satisfy the additional estimate*

$$\|u(t)\|_{p+2}\leqq C(p,\ \|u_0\|_{1,m+2},\ \|u_0\|_{p+2})(1+t)^{-1/m}. \tag{6.4}$$

(iii) *If* $p=\infty$ *in the above* (i) *and* (ii) *such solution is unique.*

PROOF. It suffices to derive the estimates formally. The detailed proof, which we omit, can be given by virtue of standard compactness and monotonicity arguments through approximate solutions $u_\varepsilon(t)$, say, of the problem;

$$(\mathrm{P}_{4,\varepsilon})\qquad \begin{aligned} &u_t-\sum_{i=1}^{N}((|u_{x_i}|^2+\varepsilon)^{m/2}u_{x_i})_{x_i}+\varepsilon u^{1+\alpha'}-u^{1+\alpha}=0\\ &u(0)=u_0\,,\quad u|_{\partial\Omega}=0\quad\text{and}\quad u\geqq 0\,.\end{aligned}$$

Since the case (i) is treated in a similar and simpler manner we restrict ourselves to the case (ii). We have only to show the estimate for $\|u(t)\|_{p+2}$. As is seen from the treatment of the problem (P_2) the case $p<\infty$ is easier and we consider the case $p=\infty$ only.

Now, multiplying the equation by u^{p+1} $(p\geqq 0)$ and integrating we have

$$\frac{1}{p+2}\frac{d}{dt}\|u(t)\|_{p+2}^{p+2}+\frac{(p+1)(m+2)^2}{(p+m+2)^{m+2}}\sum_{i=1}^{N}\int_\Omega\left|\frac{\partial}{\partial x_i}(u^{(p+m+2)/(m+2)})\right|^{m+2}dx$$
$$=\int_\Omega u^{p+\alpha+2}dx\,. \tag{6.5}$$

By Hölder's inequality and the boundedness of $\|u(t)\|_{1,m+2}$,

$$\int u^{p+\alpha+2}dx\leqq\|u(t)\|_{p+2}^{\theta_1}\|u(t)\|_{\alpha+2}^{\theta_2}\|u(t)\|_{\gamma}^{\theta_3}\leqq C\|u(t)\|_{p+2}^{\theta_1}\|\nabla u^{(p+m+2)/(m+2)}\|_{m+2}^{(m+2)\theta_3/(p+m+2)}$$

with $\gamma=(p+m+2)N/(N-2)$ (we consider the case $N\geqq 3$). Applying Young's inequality we have

$$\int u^{p+\alpha+2}dx\leqq\frac{(p+1)(m+2)^2}{2(p+m+2)^{m+2}}\sum_{i=1}^{N}\int\left|\frac{\partial}{\partial x_i}(u^{(p+m+2)/(m+2)})\right|^{m+2}dx+C_p\|u(t)\|_{p+2}^{p+2} \tag{6.6}$$

where we set in the above

$$\theta_1=(p+2)\left\{1-\frac{N\alpha}{mN+(m+2)(\alpha+2)}\right\},\qquad \theta_2=\alpha\left\{1-\frac{mN}{mN+(m+2)(\alpha+2)}\right\}$$
$$\text{and}\quad \theta_3=\frac{N\alpha(p+m+2)}{mN+(m+2)(\alpha+2)},$$

and C_p is a constant such that

$$C_p \leqq C(\|u_0\|_{1,m+2})(p+1)^{(m+1)\{mN+(m+2)(\alpha+2)-N\alpha\}/N\alpha} \equiv C(p+1)^{\tilde{\theta}}.$$

Thus, from (6.5) and (6.6) we obtain

$$\frac{d}{dt}\|u(t)\|_{p+2}^{p+2}+C(p+1)^{-m}\|\nabla u^{(p+m+2)/(m+2)}(t)\|_{m+2}^{m+2} \leqq C(p+1)^{\tilde{\theta}+1}\|u(t)\|_{p+2}^{p+2}. \tag{6.7}$$

This together with (6.1) implies, by Lemma 3.1, the boundedness of $\|u(t)\|_\infty$. The decay property for the case $0<m<\alpha<m^*$ is also proved quite similarly to the proof of Theorem 3.1, (ii), and the details are omitted.

After the uniform boundedness is known the uniqueness of such solution easily follows from the monotonicity of the principal elliptic term and the Lipschitz continuity of $u^{1+\alpha}$. Indeed, letting u_i, $i=1, 2$, be two solutions we have

$$\frac{1}{2}\frac{d}{dt}\|u_1(t)-u_2(t)\|_2^2 \leqq \int (u_1^{1+\alpha}-u_2^{1+\alpha})(u_1-u_2)dx \leqq C(M)\|u_1(t)-u_2(t)\|_2^2$$

with $M=\sup_{t,i}(\|u_i(t)\|_\infty)$, which implies $u_1=u_2$. q. e. d.

REMARK. If we assume only $u_0 \in W_0^{1,m+2}$ $(0\leqq\alpha<m)$ or $u_0 \in \mathfrak{W}$ $(0<m<\alpha<m^*)$ we can obtain (see Theorems 2.1 and 2.2)

$$\|u(t)\|_{p+2} \leqq \begin{cases} C_p t^{-1/m}+C_p(\|u_0\|_{1,m+2}) & \text{if } 0\leqq\alpha<m \\ C_p t^{-1/m}+C_p(\|u_0\|_{1,m+2})(1+t)^{-1/m} & \text{if } 0<m<\alpha<m^* \end{cases}$$

for any $0\leqq p<\infty$, which is a kind of regularizing effect. We can conjecture that these estimates should be valid for the case $p=\infty$. The nonnegativity of solutions is inessential in this section.

Appendix. Outline of the proof of Lemma 3.1.

We follow Alikakos [1]. Setting

$$\lambda_k=\left[\lambda_0-\frac{m-r+1}{r-1}\right]r^k+\frac{m-r+1}{r-1}, \quad \alpha_k=C_1\lambda_k^{\theta_1}, \quad \nu_k=C_0\lambda_k^{-\theta_0},$$

$$\beta_k=\frac{\lambda_k+1}{\lambda_{k-1}+1} \quad \text{and} \quad v=w^{\lambda_{k-1}+1} \qquad (k=1, 2, 3, \cdots),$$

the inequality takes the form

$$\frac{d}{dt}\int v^{\beta_k}dx \leqq -\nu_k\int|\nabla v|^r dx+\alpha_k\int v^{\beta_k}dx. \tag{A.1}$$

Noting $1<\beta_k<r$ and applying the Gabliardo-Nirenberg inequality, we know

$$\|v\|_{\beta_k}^{\beta_k} \leqq \varepsilon_k\|\nabla v\|_r^r+C_{\varepsilon_k}\|v\|_1^{\gamma_k} \tag{A.2}$$

where ε_k is a positive constant to be chosen later, $C_\varepsilon=\varepsilon^{-\tau}$ for a certain $\tau>0$, and

$$r_k=\frac{r(1-\gamma_k)\beta_k}{r-\gamma_k\beta_k}, \qquad \gamma_k=\frac{rN(\beta_k-1)}{\beta_k(r-N+rN)}.$$

Now, choosing ε_k so small that we may have $\alpha_k\varepsilon_k+\varepsilon_k^2\leqq\nu_k$, it follows from (A.1) and (A.2) that

$$\frac{d}{dt}\int w^{\lambda_k+1}dx\leqq-\varepsilon_k\int w^{\lambda_k+1}dx+(\alpha_k+\varepsilon_k)C_{\varepsilon_k}\left[\sup_{t\geqq0}\int w^{\lambda_{k-1}+1}dx\right]^{r_k}$$

and hence

$$\int w^{\lambda_k+1}dx\leqq\max\left\{\delta_k\left[\sup_{t\geqq0}\int w^{\lambda_{k-1}+1}dx\right]^{r_k}, \int w^{\lambda_k+1}(x, 0)dx\right\} \tag{A.3}$$

where $\delta_k\equiv(\alpha_k+\varepsilon_k)C_{\varepsilon_k}/\varepsilon_k$ (>1).

Thus, inductively from (A.3), we can obtain

$$\int w^{\lambda_k+1}dx\leqq\delta_k\delta_{k-1}^{r_k}\cdots\delta_1^{r_2\cdots r_k}K^{\lambda_k+1} \tag{A.4}$$

with

$$K=\max\{1, c\|w_0\|_\infty, \sup_{t\geqq0}\|w(t)\|_{\lambda_0+1}^d\}, \qquad c, d>0.$$

Since $r_k<r$ and $\delta_k\leqq r^{k\theta_1+(\theta_0+\theta_1)(1+\mu)}a$ for certain a and $\mu>0$, we have from (A.4) that

$$\|w(t)\|_{\lambda_k+1}\leqq a^{p_k}r^{q_k}K \tag{A.5}$$

where we set

$$p_k=\frac{r^k-1}{\lambda_k+1} \quad\text{and}\quad q_k=\frac{(r^{k+1}-k-1)\theta_1+(\theta_0+\theta_1)(1+\mu)(r^k-1)}{\lambda_k+1}.$$

Taking the limit in (A.5) as $k\to\infty$, we obtain

$$\sup_{t\geqq0}\|w(t)\|_\infty\leqq a^\eta Kr^{[\theta_1+(\theta_0+\theta_1)(1+\mu)]r\eta}$$

with $\eta=[\lambda_0-(m-r+1)(r-1)^{-1}]^{-1}$, which completes the proof.

ACKNOWLEDGEMENT. The author would like to thank Professor N.D. Alikakos for many interesting discussions. He would like to thank also Professors R.J. Knops, J.M. Ball and other staff of the Department of Mathematics of Heriot-Watt University for their generous hospitality.

References

[1] N. D. Alikakos, L^p-bounds of solutions of reaction diffusion equations, Comm. Partial Differential Equations, **4** (1979), 827-868.
[2] D. Aronson, M. G. Crandall and L. A. Peletier, Stabilization of solutions of a degenerate nonlinear diffusion problem, MRC Technical Summary Report, **2230** (1981).
[3] Ph. Benilan and M. G. Crandall, The continuous dependence on ϕ of solutions of $u_t-\Delta\phi(u)=0$, Indiana Univ. Math. J., **30** (1978), 161-177.
[4] H. Brézis, Monotonicity method in Hilbert spaces and some applications to nonlinear partial differential equations, Contributions to Nonlinear Funct. Anal., Academic

Press, New York-London, 1971, p. 101-156.

[5] H. Brézis and M. G. Crandall, Uniqueness of solutions of the initial-value problem for $u_t - \Delta\phi(u) = 0$, J. Math. Pures Appl., **58** (1979), 153-163.

[6] E. D. Conway and J. A. Smoller, Diffusion and the predictor-prey interaction, SIAM J. Appl. Math., **33** (1979), 673-686.

[7] L. C. Evans, Application of nonlinear semi-group theory to certain partial differential equations, Nonlinear Evolution Equation, M. G. Crandall ed., Academic Press, New York, 1978.

[8] H. Fujita, On some nonexistence and nonuniqueness theorem for nonlinear parabolic equations, Proc. Symp. Pure Math., **18**, Nonlinear Funct. Anal., Ann. Amer. Math. Soc., 1968.

[9] O. A. Ladyzhenskaya, V. A. Solonnikov and N. N. Ural'tseva, Linear and Quasilinear Equations of Parabolic Type, Amer. Math. Soc., Providence, R. I., 1968.

[10] H. A. Levine and P. E. Sacks, Some existence and nonexistence theorems for solutions of degenerate parabolic equations, to appear.

[11] K. Masuda, On the global existence and asymptotic behavior of solutions of reaction-diffusion equations, Hokkaido Math. J., **12** (1983), 360-370.

[12] M. Nakao, On solutions to the initial-boundary value problem for $(\partial/\partial t)u - \Delta\beta(u) = f$, J. Math. Soc. Japan, **35** (1983), 71-83.

[13] M. Nakao, Existence, nonexistence and some asymptotic behaviour of global solutions of a nonlinear degenerate parabolic equation, Math. Rep., College Gen. Ed. Kyushu Univ., 1983, 1-21.

[14] M. Nakao and T. Narazaki, Existence and decay of solutions of some nonlinear parabolic variational inequalities, Internat. J. Math. and Math. Sci., **2** (1980), 79-102.

[15] P. E. Sacks, Global behavior for a class of nonlinear evolution equations, to appear.

[16] D. H. Sattinger, On global solution of nonlinear hyperbolic equations, Arch. Rational Mech. Anal., **30** (1968), 148-172.

[17] J. Simon, Quelques propriétés de solutions d'équations et d'inéquations d'évolution paraboliques non linéaires, Annali Scuola Norm. Sup. Pisa Ser. IV, **11** (1975), 585-609.

[18] M. Tsutsumi, Existence and nonexistence of global solutions for nonlinear parabolic equations, Publ. RIMS Kyoto Univ., **8** (1972-73), 211-229.

Mitsuhiro NAKAO
Department of Mathematics
College of General Education
Kyushu University
Fukuoka 810
Japan