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L^p-nuclearity, traces, and Grothendieck–Lidskii formula on compact Lie groups

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Abstract

Given a compact Lie group G, in this paper we give symbolic criteria for operators to be nuclear and r-nuclear on $L^p(G)$-spaces, with applications to distribution of eigenvalues and trace formulae. Since criteria in terms of kernels are often not effective in view of Carleman's example, in this paper we adopt the symbolic point of view. The criteria here are given in terms of the concept of matrix symbols defined on the noncommutative analogue of the phase space $G \times \hat{G}$, where $\hat{G}$ is the unitary dual of G. No regularity of the kernel (or of the symbol) is assumed so that several of the obtained criteria extend to the more general setting of compact topological groups.

Résumé

Soit G un groupe de Lie compact, dans cet article on établit des critères sur les symboles pour assurer qu'un opérateur est nucléaire et r-nucléaire sur les espaces $L^p(G)$, avec des applications à la distribution des valeurs propres et la formule pour la trace. Depuis des critères en termes de noyaux ne sont souvent pas efficase en vue de l'exemple de Carleman, dans cet article, on adopte le point de vue symbolique. Les critères ici sont donnés en termes du concept de symboles matriciels définis sur l'analogue non commutatif de l'espace des phases $G \times \hat{G}$, où $\hat{G}$ est le dual unitaire de G. Aucune régularité sur le noyau (ou du symbole) n'est supposée de sorte que plusieurs des critères obtenus s'étendent au cas plus général des groupes topologiques compacts.

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1. Introduction

Let G be a compact Lie group. In this paper we address the following problems:

- to find criteria for operators to be nuclear on $L^p(G)$, for $1 \leqslant p < \infty$;

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- since in the Banach spaces, due to Grothendieck's work [9], we know that in order to have the operator trace to agree with the spectral trace, the notion of nuclearity is not sufficient, to find criteria for the r-nuclearity ($0 < r \leqslant 1$) and to apply this to derive information on the spectral behaviour and on the traces of operators on $L^p(G)$.

Our analysis will be based on the global quantisation recently developed in [19] and [21] as a noncommutative analogue of the Kohn–Nirenberg quantisation of operators on $\mathbb{R}^n$.

In general, for trace class operators in Hilbert spaces, the trace of an operator given by integration of its integral kernel over the diagonal is equal to the sum of its eigenvalues. However, this property fails in Banach spaces. The notion of r-nuclear operators becomes useful, and Grothendieck [9] proved that $\frac{2}{3}$-nuclear operators in this scale satisfy the Lidskii trace formula on L^p-spaces. The question of finding good criteria for ensuring the r-nuclearity of operators arises but this has to be formulated in terms different from those on Hilbert spaces and has to take into account the impossibility of certain kernel formulations in view of Carleman's example [2] recalled below.

The main results of the paper (in the setting of operators on a compact Lie group G) do not impose any conditions on the regularity of the kernel (or of the symbol), and include:

- sufficient conditions for operators from $L^{p_1}(G)$ to $L^{p_2}(G)$ to be nuclear, for $1 \leqslant p_1, p_2 < \infty$;
- sufficient conditions for operators from $L^{p_1}(G)$ to $L^{p_2}(G)$ to be r-nuclear, for $0 < r \leqslant 1$, $1 \leqslant p_1, p_2 < \infty$;
- a new trace formula relating the operator trace to an expression involving the matrix-symbol of an r-nuclear operator (for $0 < r \leqslant 1$), which in turn is equal to the sum of eigenvalues by the Lidskii formula (for $0 < r \leqslant \frac{2}{3}$);
- an application to the trace formula for the heat kernel.

In order to get an efficient criterion for the r-nuclearity, the application of the notion of a matrix symbol of an operator on a compact Lie group will be instrumental. We also give several further applications. A special feature of our criteria is that we do not assume any regularity condition on the symbols, which shows a certain advantage in comparison with the traditional Kohn–Nirenberg quantisation in the manifold setting. Here we will completely drop regularity assumption on the symbol as a consequence of the technique of noncommutative quantisation that we are using.

As a result, several of our criteria are valid on compact topological groups, without assuming the differential structure of a Lie group.

While Grothendieck's result yields the same index $2/3$ for all L^p-spaces, we relate the index r of the r-nuclear operators with the index p of L^p-spaces in which the trace formula holds. Nuclearity criteria for operators on L^2 with smooth symbols in Hörmander classes have been analysed, see e.g. Shubin [24, Section 27]. The problem of finding criteria for Schatten classes in terms of symbols with lower regularity has been of interest in the last years, see e.g. [26,27,1].

Symbolic criteria for the L^p-boundedness of operators on compact Lie groups, the Mikhlin–Hörmander multiplier theorem and its extension to non-invariant operators for $1 < p < \infty$, are presented in [23].

To formulate the notions more precisely, let E and F be two Banach spaces and let $0 < r \leqslant 1$. A linear operator T from E to F is called *r-nuclear* if there exist sequences (x_n') in E' and (y_n) in F so that

$$Tx = \sum_{n=1}^{\infty} x_n'(x) y_n \tag{1.1}$$

and

$$\sum_{n=1}^{\infty} \|x_n'\|_{E'}^r \|y_n\|_F^r < \infty. \tag{1.2}$$

The class of r-nuclear operators is usually endowed with the quasi-norm

$$n_r(T) = \inf\left\{\sum_{n=1}^{\infty} \|x_n'\|_{E'}^r \|y_n\|_F^r\right\}^{\frac{1}{r}}, \tag{1.3}$$

where the infimum is taken over the representations (1.1) of T such that (1.2) holds. When $r = 1$ we obtain the ideal of *nuclear operators* and $n_1(\cdot)$ is a norm. In this case the definition above agrees with the concept of trace class operators in the setting of Hilbert spaces ($E = F = H$).

Since we are also interested in the distribution of eigenvalues we shall consider the case $E = F$ and the notion of the trace. In order to ensure the existence of a good definition of the trace on the ideal of nuclear operators $\mathfrak{N}(E)$ one is led to consider the Banach spaces E enjoying the so-called *approximation property* (cf. [16,4]). It is well known that the spaces $L^p(\Omega, \mathcal{M}, \mu)$ satisfy the approximation property for any measure μ and $1 \leqslant p \leqslant \infty$ (cf. [14, Lemma 19.3.5]). Thus, a Banach space E is said to have the *approximation property* if for every compact subset K of E and every $\epsilon > 0$ there exists a finite rank bounded operator B on E such that

$$\|x - Bx\|_E < \epsilon \quad \text{for all } x \in K.$$

On such spaces, if $T : E \to E$ is nuclear, the trace is well-defined by

$$\mathrm{Tr}(T) = \sum_{n=1}^{\infty} x_n'(y_n),$$

where $T = \sum_{n=1}^{\infty} x_n' \otimes y_n$ is a representation of T as in (1.1). It can be shown that this definition is independent of the choice of the representation.

In the setting of Hilbert spaces the class of r-nuclear operators agrees with the Schatten–von Neumann ideal of order r, a result due to R. Oloff (cf. [13]). When $r = \frac{2}{3}$, Grothendieck proved (cf. [9]) that the trace in Banach spaces agrees with the sum of all the eigenvalues with multiplicities counted. In Hilbert spaces this holds for nuclear (i.e. trace class) operators, the result which is known as the Lidskii formula (cf. [12]). It has been proven by A. Pietsch [15] that if $r > 1$ the class of operators having decomposition (1.1) and satisfying (1.2) is essentially reduced to the null operator. The question about the sharpness of the index $r = \frac{2}{3}$ for trace formulae in the case of L^p-spaces has been recently considered by Reinov and Laif [17]. Being in the class of r-nuclear operators can be used to deduce properties concerning the asymptotic behaviour of the corresponding operators. The statement relating Grothendieck's r-nuclearity result to the Lidskii formula in L^p-spaces is known as a Grothendieck–Lidskii formula (see e.g. [17]) and we give its variant on compact Lie groups in Theorem 4.2 for $0 < r \leqslant 2/3$ and in Corollary 4.4 for $0 < r \leqslant 1$ with $\frac{1}{r} = 1 + |\frac{1}{2} - \frac{1}{p}|$. The r-nuclear operators are sometimes known as p-nuclear operators, but here we will reserve the index p to indicate the L^p-spaces. A description of the current state of the art of the general theory of p-nuclear operators has recently appeared in Hinrichs and Pietsch [10].

Among other things, in this paper we establish sufficient conditions on the matrix-valued symbol of an operator in order to ensure the r-nuclearity in L^p-spaces. The nuclearity of pseudo-differential operators on the circle $\mathbb{T}^1$ has been recently analysed in [7] but the situation in the present paper is much more subtle because of the necessarily appearing multiplicities of the eigenvalues of the Laplacian on the noncommutative compact Lie groups; moreover, due to the commutativity of the torus, the symbol there is scalar and hence all of its "matrix"-norms are uniformly equivalent which is not the case if the group G is noncommutative.

We shall now briefly recall a classical result of Carleman [2] which will be helpful to clarify the significance of our symbolic criteria. In 1916 Torsten Carleman constructed a periodic continuous function $\varkappa(x) = \sum_{n=-\infty}^{\infty} c_n e^{2\pi i n x}$, i.e. a continuous function on the commutative Lie group $\mathbb{T}^1$, for which the Fourier coefficients c_n satisfy

$$\sum_{n=-\infty}^{\infty} |c_n|^r = \infty \quad \text{for any } r < 2.$$

Now, considering the normal operator

$$Tf = f * \varkappa \tag{1.4}$$

acting on $L^2(\mathbb{T}^1)$ one obtains that the sequence $(c_n)_n$ forms a complete system of eigenvalues of this operator corresponding to the complete orthonormal system $\phi_n(x) = e^{2\pi i n x}$, $T\phi_n = c_n \phi_n$. The system ϕ_n is also complete for T^*, $T^*\phi_n = \overline{c_n}\phi_n$, the singular values of T are given by $s_n(T) = |c_n|$ and hence

$$\sum_{n=-\infty}^{\infty} s_n(T)^r = \infty,$$

for $r < 2$. Hence, the operator T is not nuclear. Moreover, due to the aforementioned Oloff's result the operator T is not r-nuclear for $0 < r \leqslant 1$. However, the continuous integral kernel $k(x, y) = \varkappa(x - y)$ satisfies any kind of integral condition of the form $\iint |k(x, y)|^s \, dx \, dy < \infty$ due to the boundedness of k. This shows that it is impossible to formulate a sufficient condition of this type for the kernel ensuring nuclearity on the torus $\mathbb{T}^1$.

In this work we will establish conditions imposed on symbols instead of kernels ensuring the r-nuclearity of the corresponding operators. The criteria that we will obtain in the general case for the nuclearity from $L^{p_1}(G)$ to $L^{p_2}(G)$ will depend on whether $p_1 \leqslant 2$ or $p_1 \geqslant 2$. The formulation for the left-invariant operators is simpler than that in the general case, and in this respect, the result on the left-invariant operators with symmetric symbols is that in Theorem 3.4, saying that if $1 \leqslant p_1, p_2 < \infty$ and $0 < r \leqslant 1$, and if $A : L^{p_1}(G) \to L^{p_2}(G)$ is a left-invariant linear continuous operator, formally self-adjoint, with matrix-valued symbol $\sigma_A(\xi)$ satisfying

$$\sum_{[\xi] \in \hat{G}} d_\xi^{1+(\frac{1}{\tilde{p}_1} - \frac{1}{\tilde{p}_2})r} \left\| \sigma_A(\xi) \right\|_{S_r}^r < \infty, \tag{1.5}$$

where $\tilde{p}_1 = \min\{2, p_1\}$, $\tilde{p}_2 = \max\{2, p_2\}$, then the operator $A : L^{p_1}(G) \to L^{p_2}(G)$ is r-nuclear. We also analyse the general case of non-self-adjoint non-invariant operators but in this case conditions analogous to (1.5) become more complicated. This is in contrast to criteria for Schatten classes in the case of $p_1 = p_2 = 2$ (see [6]) when the conditions do not depend on whether the operator is self-adjoint or not.

For $p_1 = p_2$ we will apply this to the question of the convergence of the series of eigenvalues of operators and to the validity of the Lidskii formula. Thus, for $1 \leqslant p < \infty$ and $0 < r \leqslant 1$, if $A : L^p(G) \to L^p(G)$ is a linear continuous r-nuclear operator with the matrix-valued symbol $\sigma_A(x, \xi)$, then (under certain conditions) in Theorem 4.2 we prove the trace formula

$$\operatorname{Tr} A = \int_G \sum_{[\xi] \in \hat{G}} d_\xi \operatorname{Tr}\big(\sigma_A(x, \xi)\big) \, dx \tag{1.6}$$

relating the operator trace to the matrix symbol of the operator. If in addition $0 < r \leqslant \frac{2}{3}$, we have by the Lidskii formula also the equality

$$\operatorname{Tr} A = \sum_{n=1}^{\infty} \lambda_n(A), \tag{1.7}$$

where $\lambda_n(A)$ denote the eigenvalues of A counted with multiplicities.

We give examples applying our results to the heat kernel on general compact Lie groups (Section 4.1) as well as to the Laplacian and the sub-Laplacian on $\mathrm{SU}(2) \simeq \mathbb{S}^3$ and on $\mathrm{SO}(3)$ (Section 3.2).

In Section 2 we discuss and formulate the known criteria for nuclearity as well as make a short introduction to the noncommutative matrix quantisation on compact Lie groups. In Section 3 we move to the setting of L^p-spaces and formulate our criteria for the r-nuclearity. There are different possibilities of how to impose conditions on the symbol. We will discuss both the cases of invariant and non-invariant operators, and give examples of our results on the tori, on the group SU(2) and on SO(3). In Section 4 we give applications to summability of eigenvalues, trace formulae and the Lidskii theorem. In particular, in Section 4.1 we give the example of the heat kernel and its trace.

2. Preliminaries

In this section we recall some basic facts about the concepts of nuclear and r-nuclear operators, and the notion of the trace on Banach spaces. In particular, we consider the trace of nuclear operators on $L^p(\mu)$. The fact that these spaces satisfy the approximation property is a classical result (cf. [9,14]). We refer the reader to [14] and to [16, Chapter 4.2] for the general theory of traces on operator ideals and the notation used in this section, see also [8] for an exposition on the distribution of the eigenvalues. For the theory of pseudo-differential operators on compact Lie groups the we refer to [19] and [21].

In the case of L^p-spaces we first record the following characterisation of nuclear operators (cf. [3]). In the statement below we shall consider $(\Omega_1, \mathcal{M}_1, \mu_1)$ and $(\Omega_2, \mathcal{M}_2, \mu_2)$ to be two σ-finite measure spaces.

Theorem 2.1. *Let $1 \leqslant p_1, p_2 < \infty$ and let q_1 be such that $\frac{1}{p_1} + \frac{1}{q_1} = 1$. An operator $T : L^{p_1}(\mu_1) \to L^{p_2}(\mu_2)$ is nuclear if and only if there exist sequences $(g_n)_n$ in $L^{p_2}(\mu_2)$, and $(h_n)_n$ in $L^{q_1}(\mu_1)$ such that $\sum_{n=1}^{\infty} \|g_n\|_{L^{p_2}} \|h_n\|_{L^{q_1}} < \infty$, and such that for all $f \in L^{p_1}(\mu_1)$ we have*

$$Tf(x) = \int \left(\sum_{n=1}^{\infty} g_n(x) h_n(y) \right) f(y)\, d\mu_1(y), \quad \textit{for a.e. } x.$$

Remark 2.2. An analogue of the characterisation above holds for r-nuclear operators, $0 < r \leqslant 1$, replacing the terms $\|g_n\|_{L^{p_2}} \|h_n\|_{L^{q_1}}$ by $\|g_n\|_{L^{p_2}}^r \|h_n\|_{L^{q_1}}^r$ in the sum, i.e. under the condition that

$$\sum_{n=1}^{\infty} \|g_n\|_{L^{p_2}}^r \|h_n\|_{L^{q_1}}^r < \infty.$$

A distribution of the eigenvalues for r-nuclear operators can be obtained from the next theorem relating the eigenvalues and the class of r-nuclear operators (cf. [9, Chap. II, p. 16], [8, Chap. 5, Theorem 4.2]):

Theorem 2.3. *Let E be a Banach space which has the approximation property. Let T be an r-nuclear operator from E into E for some $0 < r \leqslant 1$. Then*

$$\sum_{n=1}^{\infty} \left|\lambda_n(T)\right|^s \leqslant n_r^s(T), \qquad \frac{1}{s} = \frac{1}{r} - \frac{1}{2},$$

where $\lambda_n(T)$ denote the eigenvalues of T with multiplicities counted, and where $n_r(T)$ is defined in (1.3).

Remark 2.4. (i) Note that from $\frac{1}{s} = \frac{1}{r} - \frac{1}{2}$ we obtain that $s = \frac{2r}{2-r}$ for $0 < r \leqslant 1$. In particular, the function $s(r) = \frac{2r}{2-r}$ has the range $(0, 2]$. It is clear that if $s > 2$ the series on the left in Theorem 2.3 also converges but the interesting situation is to find smaller values of s ensuring such convergence.

(ii) Theorem 2.3 was established by Grothendieck [9], and later extended by e.g. König [11, p. 107] to the scale of Lorentz sequences spaces; see also [16, Theorem 3.8.6].

(iii) If $r = 1$ we get $s = 2$, a classical result by Grothendieck (cf. [9]) establishing the square summability of eigenvalues for nuclear operators. It is also known by Grothendieck that $s = 2$ is the best possible exponent in this case.

Theorem 2.3 will be applied jointly with our sufficient conditions for r-nuclearity, to obtain estimates on the asymptotic behaviour of the eigenvalues. From this point of view, the main goal of this paper becomes to find suitable criteria for ensuring the r-nuclearity of an operator.

Given a compact Lie group G, in this work we consider $\Omega_1 = \Omega_2 = G$ and $\mathcal{M} = \mathcal{M}_1 = \mathcal{M}_2$, the Borel σ-algebra associated to the topology of the smooth manifold G, with $\mu = \mu_1 = \mu_2$ the normalised Haar measure of G. The results of this paper concerning multiplier operators do not use the differential structure of G and hold on general compact groups. When talking about compact topological groups in this paper we will always assume that the set consisting of the unit element $\{e\}$ is closed, so that the group is Hausdorff.

Let $\hat{G}$ denote the set of equivalence classes of continuous irreducible unitary representations of G. Since G is compact, the set $\hat{G}$ is discrete. For $[\xi] \in \hat{G}$, by choosing a basis in the representation space of ξ, we can view ξ as a matrix-valued function $\xi : G \to \mathbb{C}^{d_\xi \times d_\xi}$, where d_ξ is the dimension of the representation space of ξ. By the Peter–Weyl theorem the collection

$$\left\{ \sqrt{d_\xi}\, \xi_{ij} \colon\ 1 \leqslant i, j \leqslant d_\xi,\ [\xi] \in \hat{G} \right\}$$

is an orthonormal basis of $L^2(G)$. If $f \in L^1(G)$ we define its global Fourier transform at ξ by

$$\mathcal{F}_G f(\xi) \equiv \hat{f}(\xi) := \int_G f(x) \xi(x)^*\, dx, \tag{2.1}$$

where dx is the normalised Haar measure on G. Thus, if ξ is a matrix representation, we have $\hat{f}(\xi) \in \mathbb{C}^{d_\xi \times d_\xi}$. The Fourier inversion formula is a consequence of the Peter–Weyl theorem, so that

$$f(x) = \sum_{[\xi]\in\hat{G}} d_\xi \operatorname{Tr}\big(\xi(x)\hat{f}(\xi)\big). \tag{2.2}$$

Given a sequence of matrices $a(\xi) \in \mathbb{C}^{d_\xi \times d_\xi}$, we can define

$$\big(\mathcal{F}_G^{-1}a\big)(x) := \sum_{[\xi]\in\hat{G}} d_\xi \operatorname{Tr}\big(\xi(x)a(\xi)\big), \tag{2.3}$$

where the series can be interpreted distributionally or absolutely depending on the growth of (the Hilbert–Schmidt norms of) $a(\xi)$. For a further discussion we refer the reader to [19].

For each $[\xi] \in \hat{G}$, the matrix elements of ξ are the eigenfunctions for the Laplacian $\mathcal{L}_G$ (or the Casimir element of the universal enveloping algebra), with the same eigenvalue which we denote by $-\lambda^2_{[\xi]}$, so that

$$-\mathcal{L}_G \xi_{ij}(x) = \lambda^2_{[\xi]}\xi_{ij}(x) \quad \text{for all } 1 \leqslant i, j \leqslant d_\xi. \tag{2.4}$$

The weight for measuring the decay or growth of Fourier coefficients in this setting is $\langle\xi\rangle := (1 + \lambda^2_{[\xi]})^{\frac{1}{2}}$, the eigenvalues of the elliptic first-order pseudo-differential operator $(I - \mathcal{L}_G)^{\frac{1}{2}}$. The Parseval identity takes the form

$$\|f\|_{L^2(G)} = \bigg(\sum_{[\xi]\in\hat{G}} d_\xi \big\|\hat{f}(\xi)\big\|^2_{\mathrm{HS}}\bigg)^{1/2}, \quad \text{where } \big\|\hat{f}(\xi)\big\|^2_{\mathrm{HS}} = \operatorname{Tr}\big(\hat{f}(\xi)\hat{f}(\xi)^*\big), \tag{2.5}$$

which gives the norm on $\ell^2(\hat{G})$.

For a linear continuous operator A from $C^\infty(G)$ to $\mathcal{D}'(G)$ we define its *matrix-valued symbol* $\sigma_A(x,\xi) \in \mathbb{C}^{d_\xi \times d_\xi}$ by

$$\sigma_A(x,\xi) := \xi(x)^*(A\xi)(x) \in \mathbb{C}^{d_\xi \times d_\xi}, \tag{2.6}$$

where $A\xi(x) \in \mathbb{C}^{d_\xi \times d_\xi}$ is understood as $(A\xi(x))_{ij} = (A\xi_{ij})(x)$, i.e. by applying A to each component of the matrix $\xi(x)$. Then one has [19,21] the global quantisation

$$Af(x) = \sum_{[\xi]\in\hat{G}} d_\xi \operatorname{Tr}\big(\xi(x)\sigma_A(x,\xi)\hat{f}(\xi)\big) \tag{2.7}$$

in the sense of distributions, and the sum is independent of the choice of a representation ξ from each equivalence class $[\xi] \in \hat{G}$. If A is a linear continuous operator from $C^\infty(G)$ to $C^\infty(G)$, the series (2.7) is absolutely convergent and can be interpreted in the pointwise sense. We will also write $A = \operatorname{Op}(\sigma_A)$ for the operator A given by the formula (2.7). The symbol σ_A can be interpreted as a matrix-valued function on $G \times \hat{G}$. We refer to [19,21] for the consistent development of this quantisation and the corresponding symbolic calculus. If the operator A is left-invariant then its symbol σ_A does not depend on x. We often call such operators simply invariant.

We now record simple inequalities on the norms of the representation coefficients which will be essential for the analysis of the r-nuclearity. The result holds in a more general setting of compact (Hausdorff) groups without assuming the differential structure:

Lemma 2.5. *Let G be a compact group and let $[\xi] \in \hat{G}$. Then for all $1 \leqslant i, j \leqslant d_\xi$ we have*

$$\|\xi_{ij}\|_{L^q(G)} \leqslant \begin{cases} d_\xi^{-\frac{1}{q}}, & 2 \leqslant q \leqslant \infty, \\ d_\xi^{-\frac{1}{2}}, & 1 \leqslant q \leqslant 2, \end{cases} \tag{2.8}$$

where for $q = \infty$ we adopt the usual convention $d_\xi^{-\frac{1}{q}} = 1$.

Proof. If $q=\infty$, for any $y\in G$ we have $|\xi(y)_{ij}|\leqslant\|\xi(y)\|_{op}=1$ by the unitarity of representations in $\hat{G}$. If $2\leqslant q<\infty$ we apply the inequality

$$\|f\|_{L^q}\leqslant\|f\|_{L^\infty}^{\frac{q-2}{q}}\|f\|_{L^2}^{\frac{2}{q}}.$$

Using that $\sqrt{d_\xi}\,\xi_{ij}$ is an orthonormal set in $L^2(G)$, i.e. that $\|\xi_{ij}\|_{L^2}=d_\xi^{-\frac{1}{2}}$, and that we have just showed that $\|\xi_{ij}\|_{L^\infty}\leqslant 1$, we obtain

$$\|\xi_{ij}\|_{L^q(G)}\leqslant\|\xi_{ij}\|_{L^\infty}^{\frac{q-2}{q}}\|\xi_{ij}\|_{L^2}^{\frac{2}{q}}\leqslant\|\xi_{ij}\|_{L^2}^{\frac{2}{q}}\leqslant d_\xi^{-\frac{1}{q}}.$$

Finally, for $1\leqslant q\leqslant 2$, using Hölder's inequality, we get

$$\|\xi_{ij}\|_{L^q(G)}^q=\int_G|\xi_{ij}(y)|^q\,dy\leqslant\left(\int_G 1\,dy\right)^{1-\frac{q}{2}}\left(\int_G|\xi_{ij}(y)|^{q\frac{2}{q}}\,dy\right)^{\frac{q}{2}}\leqslant\|\xi_{ij}\|_{L^2(G)}^q=d_\xi^{-\frac{q}{2}},$$

where we have used the fact that the Haar measure on G is normalised. □

Our criteria will be formulated in terms of norms of the matrix-valued symbols. In order to justify the appearance of them, we recall that if $A\in\Psi^m(G)$ on a compact Lie group G is a pseudo-differential operators in Hörmander's class $\Psi^m(G)$, i.e. if all of its localisations to $\mathbb{R}^n$ are pseudo-differential operators with symbols in the class $S^m_{1,0}(\mathbb{R}^n)$, then the matrix-symbol of A satisfies

$$\left\|\sigma_A(x,\xi)\right\|_{op}\leqslant C\langle\xi\rangle^m\quad\text{for all } x\in G,\ [\xi]\in\hat{G}.$$

Here $\|\cdot\|_{op}$ denotes the operator norm of the matrix multiplication by the matrix $\sigma_A(x,\xi)$. For this fact, see e.g. [19, Lemma 10.9.1] or [21], and for the complete characterisation of Hörmander classes $\Psi^m(G)$ in terms of matrix-valued symbols see also [22]. In particular, this motivates the usage of the operator norms of the matrix-valued symbols. However, since σ_A is in general a matrix, other matrix norms become useful as well.

3. r-Nuclearity on $L^p(G)$ and examples

In this and next sections we analyse the r-nuclearity and trace formulae in L^p-spaces. We recall that the case $r=1$ corresponds to the class of nuclear operators. One of the features of the obtained criteria is that they require the integrability (in some L^p-spaces) of symbols $\sigma_A(x,\xi)$ with respect to x but do not assume any regularity of the symbol.

We start by proving the following sufficient condition for the r-nuclearity of operators on $L^2(G)$ with symbols depending only on ξ. We note that the property that the symbol depends only on ξ means that the operator is left-invariant, that is, it commutes with the left translations on the group G.

Theorem 3.1. *Let G be a compact group and let $0<r\leqslant 1$. Let $A:L^2(G)\to L^2(G)$ be a linear continuous operator with matrix-valued symbol $\sigma_A(\xi)$ depending only on ξ. Then A is r-nuclear provided that its symbol σ_A satisfies*

$$\sum_{[\xi]\in\hat{G}}d_\xi\left\|\sigma_A(\xi)\right\|_{S_r}^r<\infty. \tag{3.1}$$

Here $\|\sigma_A(\xi)\|_{S_r}=(\mathrm{Tr}(|\sigma_A(\xi)|^r))^{1/r}$ is the Schatten-norm of order r of the matrix $\sigma_A(\xi)\in\mathbb{C}^{d_\xi\times d_\xi}$.

Although this result is known (even in the 'if and only if' form, see [6]), we give here a different proof from the one in that paper as it is more suitable for consequent extensions of this paper.

Proof of Theorem 3.1. Let us suppose that the symbol σ_A satisfies (3.1). We note that the kernel of the operator A is given by

$$k(x,y)=\sum_{[\xi]\in\hat{G}} d_\xi \operatorname{Tr}\bigl(\xi(x)\sigma_A(\xi)\xi(y)^*\bigr),$$

and we will show that it is well-defined and has the tensor product form of Theorem 2.1 that is required for the nuclearity. To abbreviate the notation, we will write $\sigma(\xi)$ for $\sigma_A(\xi)$. We begin by writing

$$\operatorname{Tr}\bigl(\xi(x)\sigma(\xi)\xi(y)^*\bigr)=\sum_{i,j=1}^{d_\xi}\bigl(\xi(x)\sigma(\xi)\bigr)_{ij}\overline{\xi(y)}_{ij},$$

and we set

$$g_{\xi,ij}(x):=d_\xi\bigl(\xi(x)\sigma(\xi)\bigr)_{ij},\qquad h_{\xi,ij}(y):=\bigl(\xi(y)^*\bigr)_{ji}=\overline{\xi(y)}_{ij}.$$

For $g_{\xi,ij}(x)$ we have

$$\begin{aligned}
\|g_{\xi,ij}\|_{L^2(G)}&=\sqrt{d_\xi}\,\bigl\|\sqrt{d_\xi}\bigl(\xi(x)\sigma(\xi)\bigr)_{ij}\bigr\|_{L^2(G)}\\
&=\sqrt{d_\xi}\left\|\sum_{k=1}^{d_\xi}\sqrt{d_\xi}\bigl(\xi(x)\bigr)_{ik}\sigma(\xi)_{kj}\right\|_{L^2(G)}\\
&=\sqrt{d_\xi}\left(\sum_{k=1}^{d_\xi}\sqrt{d_\xi}\bigl(\xi(x)\bigr)_{ik}\sigma(\xi)_{kj},\sum_{k'=1}^{d_\xi}\sqrt{d_\xi}\bigl(\xi(x)\bigr)_{ik'}\sigma(\xi)_{k'j}\right)_{L^2(G)}^{\frac12}\\
&=\sqrt{d_\xi}\left(\sum_{k=1}^{d_\xi}\sigma(\xi)_{kj}\overline{\sigma(\xi)_{kj}}\right)^{\frac12}\\
&=\sqrt{d_\xi}\left(\sum_{k=1}^{d_\xi}\bigl(\sigma(\xi)^*\bigr)_{jk}\sigma(\xi)_{kj}\right)^{\frac12}\\
&=\sqrt{d_\xi}\bigl(\sigma(\xi)^*\sigma(\xi)\bigr)_{jj}^{\frac12}\\
&=\sqrt{d_\xi}\bigl|\sigma(\xi)\bigr|_{jj}.
\end{aligned}\tag{3.2}$$

Hence $\|g_{\xi,ij}\|_{L^2(G)}^r=d_\xi^{\frac r2}|\sigma(\xi)|_{jj}^r$.

Now, since $\{d_\xi^{\frac12}\xi_{ij}\}$ is an orthonormal set in $L^2(G)$, we have

$$\|\bar{\xi}_{ij}\|_{L^2(G)}^r=d_\xi^{-\frac r2}.$$

Therefore,

$$\begin{aligned}
\sum_{\xi,ij}\bigl\|g_{\xi,ij}(\cdot)\bigr\|_{L^2(G)}^r\bigl\|h_{\xi,ij}(\cdot)\bigr\|_{L^2(G)}^r&\leqslant\sum_\xi d_\xi^{\frac r2}d_\xi^{-\frac r2}\sum_{i,j=1}^{d_\xi}\bigl|\sigma(\xi)\bigr|_{jj}^r=\sum_\xi\sum_{j=1}^{d_\xi}d_\xi\bigl|\sigma(\xi)\bigr|_{jj}^r\\
&=\sum_\xi d_\xi\operatorname{Tr}\bigl(\bigl|\sigma(\xi)\bigr|^r\bigr)=\sum_\xi d_\xi\bigl\|\sigma(\xi)\bigr\|_{S_r}^r<\infty,
\end{aligned}$$

completing the proof. □

Remark 3.2. We point out that one can prove that the condition (3.1) ensuring the r-nuclearity for left-invariant operators on $L^2(G)$ is also necessary. We recall that in the Hilbert space setting, the Schatten class of order r agrees with the class of r-nuclear operators whenever $0<r\leqslant 1$ by Oloff's result [13]. For the details of Schatten classes of invariant operators on compact Lie groups we refer the reader to the recent work [6].

We will now extend Theorem 3.1 to the setting of $L^p(G)$-spaces. We shall require the following notation: ℓ^∞ denotes the L^∞-norm on $\mathbb{C}^{d_\xi}$ and $\|\cdot\|_{op(\ell^\infty,\ell^\infty)}$ denotes the operator norm with respect to ℓ^∞ on $\mathbb{C}^{d_\xi}$. More precisely, for each $d\in\mathbb{N}$, let $B\in\mathbb{C}^{d\times d}$ and $u\in\mathbb{C}^d$. Denoting

$$\|B\|_{op(\ell^\infty,\ell^\infty)} := \max_{1\leqslant i\leqslant d}\sum_{j=1}^{d}|B_{ij}|,$$

we have $|(Bu)_i|\leqslant\sum_{j=1}^{d}|B_{ij}|\max_{1\leqslant j\leqslant d}|u_j|\leqslant\|B\|_{op(\ell^\infty,\ell^\infty)}\|u\|_{\ell^\infty}$, so that we get

$$\|Bu\|_{\ell^\infty}\leqslant\|B\|_{op(\ell^\infty,\ell^\infty)}\|u\|_{\ell^\infty},$$

justifying the notation $\|\cdot\|_{op(\ell^\infty,\ell^\infty)}$, and the appearance of this norm. The transpose of the matrix M will be denoted by M^t. We first deal with left-invariant operators.

Theorem 3.3. *Let G be a compact group and let $1\leqslant p_1,p_2<\infty$, $0<r\leqslant 1$. Let $A:L^{p_1}(G)\to L^{p_2}(G)$ be a linear continuous operator with matrix-valued symbol $\sigma_A(\xi)$ depending only on ξ. If $1\leqslant p_2\leqslant 2$ and*

$$\sum_{[\xi]\in\hat{G}} d_\xi^{1+(\frac{1}{\tilde{p}_1}-\frac{1}{2})r}\|\sigma_A(\xi)\|_{S_r}^r<\infty,$$

where $\tilde{p}_1=\min\{2,p_1\}$, then the operator $A:L^{p_1}(G)\to L^{p_2}(G)$ is r-nuclear.

If $p_2>2$, and

$$\sum_{[\xi]\in\hat{G}} d_\xi^{1+(\frac{1}{\tilde{p}_1}-\frac{1}{p_2})r}\|(\sigma_A(\xi))^t\|_{op(\ell^\infty,\ell^\infty)}^{\frac{p_2-2}{p_2}r}\|\sigma_A(\xi)\|_{S_{\frac{2r}{p_2}}}^{\frac{2r}{p_2}}<\infty,$$

then $A:L^{p_1}(G)\to L^{p_2}(G)$ is r-nuclear.

Proof. Let q_1 and $\tilde{q}_1$ be such that $\frac{1}{p_1}+\frac{1}{q_1}=\frac{1}{\tilde{q}_1}+\frac{1}{\tilde{p}_1}=1$. Then, in particular, $\tilde{q}_1=\max\{2,q_1\}$. If $1\leqslant p_2\leqslant 2$ we have, using (3.2), and denoting $\sigma=\sigma_A$,

$$\|(\xi(x)\sigma(\xi))_{ij}\|_{L^{p_2}(G)}^r\leqslant\|(\xi(x)\sigma(\xi))_{ij}\|_{L^2(G)}^r\leqslant d_\xi^{-\frac{r}{2}}|\sigma(\xi)|_{jj}^r.$$

On the other hand

$$\|\overline{\xi}_{ij}\|_{L^{q_1}(G)}^r\leqslant d_\xi^{-\frac{r}{\tilde{q}_1}}.$$

Therefore,

$$\begin{aligned}\sum_{\xi,ij}d_\xi^r\|(\xi(x)\sigma(\xi))_{ij}\|_{L^{p_2}(G)}^r\|\overline{\xi(y)}_{ij}\|_{L^{q_1}(G)}^r&\leqslant\sum_\xi d_\xi^r d_\xi^{-\frac{r}{2}}d_\xi^{-\frac{r}{\tilde{q}_1}}\sum_{ij}|\sigma(\xi)|_{jj}^r=\sum_\xi d_\xi^{1+(\frac{1}{2}-\frac{1}{\tilde{q}_1})r}\|\sigma_A(\xi)\|_{S_r}^r\\&=\sum_\xi d_\xi^{1+(\frac{1}{\tilde{p}_1}-\frac{1}{2})r}\|\sigma_A(\xi)\|_{S_r}^r<\infty.\end{aligned}$$

Now, if $p_2>2$ we first observe that

$$(\xi(x)\sigma(\xi))_{ij}=\sum_{k=1}^{d_\xi}\xi(x)_{ik}\sigma(\xi)_{kj}=\sum_{k=1}^{d_\xi}(\sigma(\xi))_{jk}^t\xi(x)_{ik}.$$

Hence and taking into account that $|\xi(x)_{ik}|\leqslant 1$, we get

$$\begin{aligned}|(\xi(x)\sigma(\xi))_{ij}|&=\left|\sum_{k=1}^{d_\xi}(\sigma(\xi))_{jk}^t\xi(x)_{ik}\right|\\&\leqslant\|(\sigma(\xi))^t\|_{op(\ell^\infty,\ell^\infty)}\|(\xi(x)_{i1},\ldots,\xi(x)_{id_\xi})\|_{\ell^\infty}\\&\leqslant\|(\sigma(\xi))^t\|_{op(\ell^\infty,\ell^\infty)}.\end{aligned}\tag{3.3}$$

Then using (3.2) and (3.3) we obtain

$$
\begin{aligned}
\big\|\big(\xi(x)\sigma(\xi)\big)_{ij}\big\|^r_{L^{p_2}(G)} &= \Big(\int\limits_G \big|\big(\xi(x)\sigma(\xi)\big)_{ij}\big|^{p_2}\,dx\Big)^{\frac{r}{p_2}} \\
&= \Big(\int\limits_G \big|\big(\xi(x)\sigma(\xi)\big)_{ij}\big|^{p_2-2}\big|\big(\xi(x)\sigma(\xi)\big)_{ij}\big|^2\,dx\Big)^{\frac{r}{p_2}} \\
&\leqslant \sup_x \big|\big(\xi(x)\sigma(\xi)\big)_{ij}\big|^{\frac{p_2-2}{p_2}r}\big\|\big(\xi(x)\sigma(\xi)\big)_{ij}\big\|^{\frac{2r}{p_2}}_{L^2(G)} \\
&\leqslant \big\|\big(\sigma(\xi)\big)^t\big\|^{\frac{p_2-2}{p_2}r}_{op(\ell^\infty,\ell^\infty)} d_\xi^{-\frac{r}{p_2}}\big|\sigma(\xi)\big|^{\frac{2r}{p_2}}_{jj}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\sum_{\xi,ij} d_\xi^r\big\|\big(\xi(x)\sigma(\xi)\big)_{ij}\big\|^r_{L^{p_2}(G)}\big\|\overline{\xi(y)}_{ij}\big\|^r_{L^{q_1}(G)} \\
&\quad\leqslant \sum_\xi d_\xi^r d_\xi^{-\frac{r}{p_2}}, d_\xi^{-\frac{r}{\tilde{q}_1}}\big\|\big(\sigma(\xi)\big)^t\big\|^{\frac{p_2-2}{p_2}r}_{op(\ell^\infty,\ell^\infty)}\sum_{ij}\big|\sigma(\xi)\big|^{\frac{2r}{p_2}}_{jj} \\
&\quad= \sum_\xi d_\xi^r d_\xi^{-\frac{r}{p_2}} d_\xi^{-\frac{r}{\tilde{q}_1}}\big\|\big(\sigma(\xi)\big)^t\big\|^{\frac{p_2-2}{p_2}r}_{op(\ell^\infty,\ell^\infty)} d_\xi\sum_j\big|\sigma(\xi)\big|^{\frac{2r}{p_2}}_{jj} \\
&\quad= \sum_\xi d_\xi\, d_\xi^{(1-\frac{1}{\tilde{q}_1}-\frac{1}{p_2})r}\big\|\big(\sigma(\xi)\big)^t\big\|^{\frac{p_2-2}{p_2}r}_{op(\ell^\infty,\ell^\infty)}\operatorname{Tr}\Big(\big|\sigma(\xi)\big|^{\frac{2r}{p_2}}\Big) \\
&\quad= \sum_\xi d_\xi\, d_\xi^{(\frac{1}{\tilde{p}_1}-\frac{1}{p_2})r}\big\|\big(\sigma(\xi)\big)^t\big\|^{\frac{p_2-2}{p_2}r}_{op(\ell^\infty,\ell^\infty)}\big\|\sigma(\xi)\big\|^{\frac{2r}{p_2}}_{S_{\frac{2r}{p_2}}} \\
&\quad<\infty,
\end{aligned}
$$

completing the proof. □

In the particular case of diagonal symbols only depending on ξ we can improve the sufficient condition in the above theorem. An example of such behaviour are the left-invariant vector fields on a compact Lie group G, the Laplacian and the sub-Laplacian on G, which always have diagonal symbols in an appropriately chosen basis in the representation spaces. Moreover, symbols of general left-invariant self-adjoint operators can be chosen to be diagonal by choosing a particular representative from each equivalence class $[\xi]\in\hat{G}$. We formulate a general result now, and will give its application to the sub-Laplacian in Section 3.2.

Theorem 3.4. *Let G be a compact group, $1\leqslant p_1, p_2<\infty$ and $0<r\leqslant 1$. Let $A: L^{p_1}(G)\to L^{p_2}(G)$ be a left-invariant linear continuous operator which is formally self-adjoint. Assume that its matrix-valued symbol $\sigma_A(\xi)$ satisfies*

$$
\sum_{[\xi]\in\hat{G}} d_\xi^{1+(\frac{1}{\tilde{p}_1}-\frac{1}{\tilde{p}_2})r}\big\|\sigma_A(\xi)\big\|^r_{S_r}<\infty,
$$

where $\tilde{p}_1=\min\{2,p_1\}$, $\tilde{p}_2=\max\{2,p_2\}$. Then the operator $A: L^{p_1}(G)\to L^{p_2}(G)$ is r-nuclear.

Proof. First we observe that since A is left-invariant, indeed its matrix-valued symbol $\sigma_A(\xi)$ is independent of x, and since A is formally self-adjoint, the matrices $\sigma_A(\xi)$ are also self-adjoint for all ξ. Therefore, we can choose the basis in the representation spaces so that the symbol $\sigma_A(\xi)$ becomes diagonal. Since the Schatten norms $\|\sigma_A(\xi)\|_{S_r}$ do not change under a change of basis in the representation spaces, we may assume further, without loss of generality, that the matrices $\sigma_A(\xi)$ are diagonal for all ξ.

We consider q_1 such that $\frac{1}{p_1}+\frac{1}{q_1}=1$ and $\tilde{q}_1=\max\{2,q_1\}$, and we observe that $\frac{1}{\tilde{q}_1}+\frac{1}{\tilde{p}_1}=1$. Since $\sigma(\xi)=\sigma_A(\xi)$ is diagonal, and using the equality (3.2), we have

$$\left\|\left(\xi(x)\sigma(\xi)\right)_{ij}\right\|^r_{L^{p_2}(G)}=\left\|\xi(x)_{ij}\sigma(\xi)_{jj}\right\|^r_{L^{p_2}(G)}\leqslant d_\xi^{-\frac{r}{\tilde{p}_2}}\left|\sigma(\xi)\right|^r_{jj}.$$

On the other hand, by (2.8) we have

$$\|\bar{\xi}_{ij}\|^r_{L^{q_1}(G)}\leqslant d_\xi^{-\frac{r}{\tilde{q}_1}}.$$

Therefore,

$$\begin{aligned}\sum_{\xi,ij}d_\xi^r\left\|\left(\xi(x)\sigma(\xi)\right)_{ij}\right\|^r_{L^{p_2}(G)}\left\|\overline{\xi(y)}_{ij}\right\|^r_{L^{q_1}(G)}&\leqslant\sum_\xi d_\xi^r d_\xi^{-\frac{r}{\tilde{p}_2}}d_\xi^{-\frac{r}{\tilde{q}_1}}\sum_{ij}\left|\sigma(\xi)\right|^r_{jj}=\sum_\xi d_\xi^{-\frac{r}{\tilde{p}_2}}d_\xi^{\frac{r}{\tilde{p}_1}}\sum_j d_\xi\left|\sigma(\xi)\right|^r_{jj}\\&=\sum_\xi d_\xi^{1+(\frac{1}{\tilde{p}_1}-\frac{1}{\tilde{p}_2})r}\left\|\sigma(\xi)\right\|^r_{S_r}<\infty,\end{aligned}$$

completing the proof. □

We will sometimes give examples of our results on the torus, so we summarise its notation:

Remark 3.5. If $G=\mathbb{T}^n=\mathbb{R}^n/\mathbb{Z}^n$, we have $\hat{\mathbb{T}}^n\simeq\mathbb{Z}^n$, and the collection $\{\xi_k(x)=e^{2\pi ix\cdot k}\}_{k\in\mathbb{Z}^n}$ is the orthonormal basis of $L^2(\mathbb{T}^n)$, and all $d_{\xi_k}=1$. If an operator A is invariant on $\mathbb{T}^n$, its symbol becomes $\sigma_A(\xi_k)=\xi_k(x)^*A\xi_k(x)=A\xi_k(0)$. In general, on the torus we will often simplify the notation by identifying $\hat{\mathbb{T}}^n$ with $\mathbb{Z}^n$, and thus writing $\xi\in\mathbb{Z}^n$ instead of $\xi_k\in\mathbb{Z}^n$. The toroidal quantisation

$$Af(x)=\sum_{\xi\in\mathbb{Z}^n}e^{2\pi ix\cdot\xi}\sigma_A(x,\xi)\hat{f}(\xi) \tag{3.4}$$

has been analysed extensively in [20] (see also [18,19]) and it is a special case of (2.7), where we have identified, as noted, $\hat{\mathbb{T}}^n$ with $\mathbb{Z}^n$.

As a consequence of Theorem 3.1 on the torus, we obtain:

Corollary 3.6. *Let $1\leqslant p_1,p_2<\infty$ and $0<r\leqslant 1$. Let $A:L^{p_1}(\mathbb{T}^n)\to L^{p_2}(\mathbb{T}^n)$ be a linear continuous operator with symbol $\sigma_A(\xi)$ depending only on ξ. Then A is r-nuclear provided that its symbol σ_A satisfies*

$$\sum_{\xi\in\mathbb{Z}^n}\left|\sigma_A(\xi)\right|^r<\infty. \tag{3.5}$$

We shall now consider more general non-invariant operators so that the symbols may depend also on x.

Theorem 3.7. *Let G be compact Lie group and let $0<r\leqslant 1$. Let operator A have the matrix symbol $\sigma_A(x,\xi)$. Let $1\leqslant p_1,p_2<\infty$ and let us denote $\tilde{p}_1=\min\{2,p_1\}$. Suppose that the symbol σ_A satisfies*

$$\sum_{[\xi]\in\hat{G}}d_\xi^{2+\frac{r}{\tilde{p}_1}}\left\|\left\|\left(\sigma_A(x,\xi)\right)^t\right\|_{op(\ell^\infty,\ell^\infty)}\right\|^r_{L^{p_2}(G)}<\infty.$$

Then the extension $A:L^{p_1}(G)\to L^{p_2}(G)$ is r-nuclear.

Here we have denoted

$$\left\|\left\|\sigma_A(x,\xi)\right\|_{op(\ell^\infty,\ell^\infty)}\right\|_{L^{p_2}(G)}=\left(\int_G\left\|\sigma_A(x,\xi)\right\|^{p_2}_{op(\ell^\infty,\ell^\infty)}dx\right)^{\frac{1}{p_2}}.$$

Proof. The kernel of the operator A is given by

$$k(x,y)=\sum_{[\xi]\in\hat{G}} d_\xi \operatorname{Tr}\bigl(\xi(x)\sigma_A(x,\xi)\xi(y)^*\bigr),$$

and we will show that it is well-defined and has the tensor product form of Theorem 2.1 that is required for the nuclearity. As before, to abbreviate the notation, we will write $\sigma(x,\xi)$ for $\sigma_A(x,\xi)$. We begin by writing

$$\operatorname{Tr}\bigl(\xi(x)\sigma(x,\xi)\xi(y)^*\bigr)=\sum_{i,j=1}^{d_\xi}\bigl(\xi(x)\sigma(x,\xi)\bigr)_{ij}\overline{\xi(y)}_{ij},$$

and we set

$$g_{\xi,ij}(x)=d_\xi\bigl(\xi(x)\sigma(x,\xi)\bigr)_{ij},\qquad h_{\xi,ij}(y)=\bigl(\xi(y)^*\bigr)_{ji}=\overline{\xi(y)}_{ij}.$$

A similar argument like in (3.3) shows that

$$\bigl|\bigl(\xi(x)\sigma(x,\xi)\bigr)_{ij}\bigr|\leqslant\bigl\|\bigl(\sigma(x,\xi)\bigr)^t\bigr\|_{op(\ell^\infty,\ell^\infty)}.$$

Hence

$$\|g_{\xi,ij}(x)\|^r_{L^{p_2}(G)}=\bigl\|d_\xi\bigl(\xi(x)\sigma(x,\xi)\bigr)_{ij}\bigr\|^r_{L^{p_2}(G)}\leqslant d_\xi^r\bigl\|\bigl\|\bigl(\sigma(x,\xi)\bigr)^t\bigr\|_{op(\ell^\infty,\ell^\infty)}\bigr\|^r_{L^{p_2}(G)}.$$

Let q_1 be such that $\frac{1}{p_1}+\frac{1}{q_1}=1$. Now, if we denote $\tilde{q}_1=\max\{2,q_1\}$, we have $\frac{1}{\tilde{p}_1}+\frac{1}{\tilde{q}_1}=1$. According to (2.8), we have

$$\|\bar{\xi}_{ij}\|^r_{L^{q_1}(G)}\leqslant d_\xi^{-\frac{r}{\tilde{q}_1}}.$$

Therefore,

$$\begin{aligned}\sum_{\xi,ij}\|g_{\xi,ij}(\cdot)\|^r_{L^{p_2}(G)}\|h_{\xi,ij}(\cdot)\|^r_{L^{q_1}(G)}&\leqslant\sum_\xi d_\xi^r d_\xi^{-\frac{r}{\tilde{q}_1}}d_\xi^2\bigl\|\bigl\|\bigl(\sigma(x,\xi)\bigr)^t\bigr\|_{op(\ell^\infty,\ell^\infty)}\bigr\|^r_{L^{p_2}(G)}\\&=\sum_\xi d_\xi^{2+\frac{r}{\tilde{p}_1}}\bigl\|\bigl\|\bigl(\sigma(x,\xi)\bigr)^t\bigr\|_{op(\ell^\infty,\ell^\infty)}\bigr\|^r_{L^{p_2}(G)}<\infty,\end{aligned}$$

completing the proof. □

Remark 3.8. (i) If $G=\mathbb{T}^n=\mathbb{R}^n/\mathbb{Z}^n$, an invariant operator A is a Fourier multiplier, $\widehat{Af}(k)=a(k)\hat{f}(k)$ with symbol $\sigma_A(\xi_k)=a(k)$, see Remark 3.5. Theorem 3.7 implies that if $0<r\leqslant 1$ and $\sum_{k\in\mathbb{Z}^n}|a(k)|^r<\infty$, then the operator T is r-nuclear from $L^{p_1}(\mathbb{T}^n)$ to $L^{p_2}(\mathbb{T}^n)$ for all $1\leqslant p_1,p_2<\infty$.

(ii) For the convolution operator on $\mathbb{T}^1$ as in (1.4), we have $\hat{\mathbb{T}}^1\simeq\mathbb{Z}^1$ and $\sigma(n)=\hat{\varkappa}(n)=c_n$, or $\varkappa(x)=\sum_{n=-\infty}^{\infty}c_ne^{2\pi inx}$. In this case Theorem 3.7 implies that if $\sum_{n=-\infty}^{\infty}|c_n|^r<\infty$, the operator $Tf=f*\varkappa$ is r-nuclear from $L^{p_1}(\mathbb{T}^1)$ to $L^{p_2}(\mathbb{T}^1)$ for all $1\leqslant p_1,p_2<\infty$.

(iii) If $p_1=p_2=2$ and A is a left-invariant operator on a compact Lie group G, it follows from Theorem 3.1 that if $\sum_{[\xi]\in\hat{G}}d_\xi\|\sigma_A(\xi)\|_{S_1}<\infty$, then A is a trace class operator on $L^2(G)$.

(iv) We note that the condition of Corollary 3.12 required the integrability of the symbol with respect to x and does not require any regularity.

In order to deduce some interesting consequences we will apply the following lemma proved in [5]:

Lemma 3.9. *Let G be a compact Lie group. Then we have*

$$\sum_{[\xi]\in\hat{G}}d_\xi^2\langle\xi\rangle^{-s}<\infty$$

if and only if $s>\dim G$.

This yields the following corollary, and in Remark 3.14 we note that the following orders are in general sharp.

Corollary 3.10. *Let G be a compact Lie group of dimension n and let $0 < r \leqslant 1$. Let $1 \leqslant p_1, p_2 < \infty$ and let us denote $\tilde{p}_1 = \min\{2, p_1\}$. Assume that*

$$\left\| \left(\sigma_A(x, \xi)\right)^t \right\|_{op(\ell^\infty, \ell^\infty)} \leqslant C d_\xi^{-\frac{1}{\tilde{p}_1}} \langle \xi \rangle^{-\frac{s}{r}}$$

with some $s > n$. Then $A : L^{p_1}(G) \to L^{p_2}(G)$ is r-nuclear.

Proof. We have

$$d_\xi^{2+\frac{r}{\tilde{p}_1}} \left\| \left(\sigma(x, \xi)\right)^t \right\|_{op(\ell^\infty, \ell^\infty)}^r \leqslant C d_\xi^{2+\frac{r}{\tilde{p}_1}} d_\xi^{-\frac{r}{\tilde{p}_1}} \langle \xi \rangle^{-s} = C^r d_\xi^2 \langle \xi \rangle^{-s}.$$

The result now follows from Lemma 3.9 and Theorem 3.7. □

As consequence of Theorem 3.4 and Lemma 3.9 we have:

Corollary 3.11. *Let G be a compact Lie group, $1 \leqslant p_1, p_2 < \infty$ and $0 < r \leqslant 1$. Let $A : L^{p_1}(G) \to L^{p_2}(G)$ be a linear continuous formally self-adjoint operator with matrix-valued symbol $\sigma_A(\xi)$ depending only on ξ. Assume that*

$$\left\| \sigma_A(\xi) \right\|_{S_r} \leqslant C d_\xi^{\frac{1}{r} - (\frac{1}{\tilde{p}_1} - \frac{1}{\tilde{p}_2})} \langle \xi \rangle^{-\frac{s}{r}},$$

where $\tilde{p}_1 = \min\{2, p_1\}$, $\tilde{p}_2 = \max\{2, p_2\}$. Then the operator $A : L^{p_1}(G) \to L^{p_2}(G)$ is r-nuclear.

Proof. We can estimate

$$d_\xi^{1+(\frac{1}{\tilde{p}_1} - \frac{1}{\tilde{p}_2})r} \left\| \sigma_A(\xi) \right\|_{S_r}^r \leqslant C^r d_\xi^{1+(\frac{1}{\tilde{p}_1} - \frac{1}{\tilde{p}_2})r} d_\xi^{1-(\frac{1}{\tilde{p}_1} - \frac{1}{\tilde{p}_2})r} \langle \xi \rangle^{-s} = C^r d_\xi^2 \langle \xi \rangle^{-s}.$$

The result now follows from Lemma 3.9 and Theorem 3.4. □

3.1. Example on the torus

We observe that on the torus $\mathbb{T}^n$ criteria obtained in the above statements are in general sharp. In general, we recall that the relation of our setting to the special case of the torus was outlined in Remark 3.5, with examples given already in Corollary 3.6 and in Remark 3.8.

Indeed, as a consequence of Theorem 3.7, recalling the notation on the torus in Remark 3.5, for the torus group $G = \mathbb{T}^n$, we have:

Corollary 3.12. *Let $1 \leqslant p_1, p_2 < \infty$, $0 < r \leqslant 1$, and let $A : L^{p_1}(\mathbb{T}^n) \to L^{p_2}(\mathbb{T}^n)$ be a linear continuous operator with symbol $\sigma_A(x, \xi)$ satisfying*

$$\sum_{\xi \in \mathbb{Z}^n} \left\| \sigma_A(\cdot, \xi) \right\|_{L^{p_2}(\mathbb{T}^n)}^r < \infty,$$

then the operator $A : L^{p_1}(\mathbb{T}^n) \to L^{p_2}(\mathbb{T}^n)$ is r-nuclear.

To see the sharpness, we establish the following simple characterisation of the nuclearity for Bessel potentials on $L^2(\mathbb{T}^n)$.

Proposition 3.13. *Let Δ be the Laplacian on the torus $\mathbb{T}^n$ and let $0 < r \leqslant 1$. Then $(I - \Delta)^{-\frac{\alpha}{2}}$ is r-nuclear on $L^2(\mathbb{T}^n)$ if and only if $\alpha r > n$.*

Proof. The symbol of the operator $T = (I - \Delta)^{-\frac{\alpha}{2}}$ is positive, hence T being a multiplier operator, it is positive definite and $|T| = \sqrt{T^*T} = T$. Thus, the singular values of T agree with the values of its symbol $\langle \xi \rangle^{-\alpha}$. Therefore, $T \in S_r(L^2(\mathbb{T}^n))$ if and only if $\alpha r > n$. The result now follows from the identification of the Schatten class of order r and the class of r-nuclear operators [13]. □

Remark 3.14. In the case of the torus $\mathbb{T}^n$ we have $d_\xi = 1$. From Proposition 3.13 it follows that the index n in the sufficient condition in Corollary 3.10 cannot be improved.

Corollary 3.15. *Let $1 \leqslant p_1, p_2 < \infty$, $0 < r \leqslant 1$ and let $A : L^{p_1}(\mathbb{T}^n) \to L^{p_2}(\mathbb{T}^n)$ be a linear continuous operator with symbol $\sigma_A(x,\xi)$ satisfying*

$$\left\|\sigma_A(x,\xi)\right\|_{L^{p_2}(\mathbb{T}^n)} \leqslant C\langle\xi\rangle^{-s/r},$$

for some $s > n$. Then the operator $A : L^{p_1}(\mathbb{T}^n) \to L^{p_2}(\mathbb{T}^n)$ is r-nuclear (for all p_1, p_2).

Using compactness of $\mathbb{T}^n$, the following criterion can be practical:

Corollary 3.16. *Let $1 \leqslant p_1, p_2 < \infty$, $0 < r \leqslant 1$ and let $A : L^{p_1}(\mathbb{T}^n) \to L^{p_2}(\mathbb{T}^n)$ be a linear continuous operator with symbol $\sigma_A(x,\xi)$ satisfying*

$$\left|\sigma_A(x,\xi)\right| \leqslant C\langle\xi\rangle^{-s/r} \quad \textit{for all } x \in \mathbb{T}^n,$$

for some $s > n$. Then the operator $A : L^{p_1}(\mathbb{T}^n) \to L^{p_2}(\mathbb{T}^n)$ is r-nuclear (for all p_1, p_2).

3.2. *Examples on* SU(2) $\simeq \mathbb{S}^3$ *and on* SO(3)

Let us now show other examples of the above statements for some particular compact groups. We first consider the case of $G = \mathrm{SU}(2)$, the group of the unitary 2×2 matrices of determinant one. The same results as given below can be stated for the 3-sphere $\mathbb{S}^3$ by using of the identification $\mathrm{SU}(2) \simeq \mathbb{S}^3$, with the matrix multiplication in SU(2) corresponding to the quaternionic product on $\mathbb{S}^3$, with the corresponding identification of the symbolic calculus, see [19, Section 12.5].

We refer the reader to [19, Chapter 12] for the details of the global quantisation (2.7) on SU(2) an the details on the representation theory of the group $G = \mathrm{SU}(2)$. In this case, we can enumerate the elements of its dual as $\hat{G} \simeq \frac{1}{2}\mathbb{N}_0$, with $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$, so that

$$\widehat{\mathrm{SU}(2)} = \left\{\left[t^\ell\right]: t^\ell \in \mathbb{C}^{(2\ell+1)\times(2\ell+1)},\ \ell \in \frac{1}{2}\mathbb{N}_0\right\}.$$

The dimension of each t^ℓ is $d_{t^\ell} = 2\ell + 1$, and there are explicit formulae for t^ℓ as functions of Euler angles in terms of the so-called Legendre–Jacobi polynomials, see [19, Chapter 11]. The Laplacian on SU(2) has eigenvalues $\lambda^2_{t^\ell} = \ell(\ell+1)$, so that we have $\langle t^\ell\rangle \approx \ell$. With this, Corollary 3.10 becomes:

Corollary 3.17. *Let $0 < r \leqslant 1$ and $1 \leqslant p_1, p_2 < \infty$. Let $A : L^{p_1}(\mathrm{SU}(2)) \to L^{p_2}(\mathrm{SU}(2))$ be an operator with matrix symbol*

$$\sigma_A(x,\ell) \equiv \sigma_A\left(x,t^\ell\right) := t^\ell(x)^* A t^\ell(x), \quad \ell \in \frac{1}{2}\mathbb{N}_0.$$

Let $s > 3$ and $\tilde{p}_1 = \min\{2, p_1\}$. If there is a constant $C > 0$ such that

$$\left\|\left\|\left(\sigma_A(x,\ell)\right)^t\right\|_{op(\ell^\infty,\ell^\infty)}\right\|_{L^{p_2}(G)} \leqslant C\ell^{-\frac{1}{\tilde{p}_1}-\frac{s}{r}}$$

for all $\ell \in \frac{1}{2}\mathbb{N}$, then $A : L^{p_1}(\mathrm{SU}(2)) \to L^{p_2}(\mathrm{SU}(2))$ is r-nuclear.

For left-invariant operators with diagonalisable symbols on SU(2), as a consequence of Corollary 3.11 we have:

Corollary 3.18. *Let $1 \leqslant p_1, p_2 < \infty$, $0 < r \leqslant 1$ and let $\tilde{p}_1 = \min\{2, p_1\}$ and $\tilde{p}_2 = \max\{2, p_2\}$. Let $A : L^{p_1}(\mathrm{SU}(2)) \to L^{p_2}(\mathrm{SU}(2))$ be a formally self-adjoint operator with symbol $\sigma_A(\ell)$ such that*

$$\left\|\sigma_A(\ell)\right\|_{S_r} \leqslant C\ell^{\frac{1-s}{r}-(\frac{1}{\tilde{p}_1}-\frac{1}{\tilde{p}_2})},$$

for some $s > 3$. Then $A : L^{p_1}(\mathrm{SU}(2)) \to L^{p_2}(\mathrm{SU}(2))$ is r-nuclear.

In particular we will apply the above corollary to the Laplacian and the sub-Laplacian.

If $\mathcal{L}_{\mathrm{SU}(2)}$ denotes the Laplacian on SU(2), we have $\mathcal{L}_{\mathrm{SU}(2)} t^{\ell}_{mn}(x) = -\ell(\ell+1)t^{\ell}_{mn}(x)$ for all ℓ, m, n and $x \in \mathrm{SU}(2)$, so that the symbol of $I - \mathcal{L}_{\mathrm{SU}(2)}$ is given by

$$\sigma_{I-\mathcal{L}_{\mathrm{SU}(2)}}(x,\ell) = \big(1+\ell(\ell+1)\big)I_{2\ell+1},$$

where $I_{2\ell+1} \in \mathbb{C}^{(2\ell+1)\times(2\ell+1)}$ is the identity matrix. Hence, $\sigma_{I-\mathcal{L}_{\mathrm{SU}(2)}}(x,\ell)$ is diagonal and independent of x. Consequently, Corollary 3.18 applied to $1 \leqslant p = p_1 = p_2 < \infty$ says that the operator $(I - \mathcal{L}_{\mathrm{SU}(2)})^{-\frac{\alpha}{2}}$ is r-nuclear on $L^p(\mathrm{SU}(2))$ provided that $\ell^{\frac{1}{r}-\alpha} \leqslant C\ell^{\frac{1-s}{r}-|\frac{1}{p}-\frac{1}{2}|}$ for $s > 3$. Summarising, we obtain

Corollary 3.19. *For $\alpha > \frac{3}{r} + |\frac{1}{p} - \frac{1}{2}|$, $0 < r \leqslant 1$ and $1 \leqslant p < \infty$, the operator $(I - \mathcal{L}_{\mathrm{SU}(2)})^{-\frac{\alpha}{2}}$ is r-nuclear on $L^p(\mathrm{SU}(2))$.*

If $p = 2$, the order $\alpha r > 3$ is sharp, see [6, Section 4].

We shall now consider the group SO(3) of the 3×3 real orthogonal matrices of determinant one. For the details of the representation theory and the global quantisation of SO(3) we refer the reader to [19, Chapter 12]. The dual in this case can be identified as $\hat{G} \simeq \mathbb{N}_0$, so that

$$\widehat{\mathrm{SO}(3)} = \big\{\big[t^{\ell}\big]\colon\ t^{\ell} \in \mathbb{C}^{(2\ell+1)\times(2\ell+1)},\ \ell \in \mathbb{N}_0\big\}.$$

The dimension of each t^{ℓ} is $d_{t^{\ell}} = 2\ell + 1$. The Laplacian on SO(3) has eigenvalues $\lambda^2_{t^{\ell}} = \ell(\ell+1)$, so that we have $\langle t^{\ell}\rangle \approx \ell$. By the same argument as above, Corollary 3.19 also holds for the Laplacian on SO(3).

Let us fix three invariant vector fields D_1, D_2, D_3 on SO(3) corresponding to the derivatives with respect to the Euler angles. We refer to [19, Chapter 11] for the explicit formulae for these. However, for our purposes here we note that the sub-Laplacian $\mathcal{L}_{sub} = D_1^2 + D_2^2$, with an appropriate choice of basis in the representation spaces, has the diagonal symbol given by

$$\sigma_{\mathcal{L}_{sub}}(\ell)_{mn} = \big(m^2 - \ell(\ell+1)\big)\delta_{mn}, \quad m,n \in \mathbb{Z},\ -\ell \leqslant m,n \leqslant \ell, \tag{3.6}$$

where δ_{mn} is the Kronecker delta. The operator $\mathcal{L}_{sub}$ is a second order hypoelliptic operator and we can define the powers $(I - \mathcal{L}_{sub})^{-\alpha/2}$. These are pseudo-differential operators with symbols

$$\sigma_{(I-\mathcal{L}_{sub})^{-\alpha/2}}(\ell)_{mn} = \big(1+\ell(\ell+1)-m^2\big)^{-\alpha/2}\delta_{mn}.$$

We now have

$$\big\|\sigma_{(I-\mathcal{L}_{sub})^{-\alpha/2}}(\ell)\big\|_{S_r} = \big(\mathrm{Tr}\big(\sigma_{(I-\mathcal{L}_{sub})^{-\alpha/2}}(\ell)\big)^r\big)^{\frac{1}{r}} = \left(\sum_{m=-\ell}^{\ell}\big(1+\ell(\ell+1)-m^2\big)^{-\frac{\alpha r}{2}}\right)^{\frac{1}{r}},$$

where $\ell \in \mathbb{N}_0$. Comparing with the integral

$$\int_{-R}^{R}\big(1+R^2-x^2\big)^{-\frac{\alpha r}{2}}\,dx \approx CR^{-\frac{\alpha r}{2}}\int_0^R (1+R-x)^{-\frac{\alpha r}{2}}\,dx \approx CR^{-\frac{\alpha r}{2}},$$

for $\alpha r > 2$ and large R, it follows that $\sum_{m=-\ell}^{\ell}(1+\ell(\ell+1)-m^2)^{-\frac{\alpha r}{2}}$ is of order $\ell^{-\frac{\alpha r}{2}}$. Now, the inequality

$$\ell^{-\frac{\alpha}{2}} \leqslant C\ell^{(1-s)/r-|\frac{1}{p}-\frac{1}{2}|},$$

with $s > 3$ holds if and only $\alpha > \frac{4}{r} + 2|\frac{1}{p} - \frac{1}{2}|$. If $1 \leqslant p = p_1 = p_2 < \infty$, as a consequence of Corollary 3.18 we obtain the condition for the nuclearity of the operator $(1 - \mathcal{L}_{sub})^{-\alpha/2}$:

Corollary 3.20. *For $\alpha > \frac{4}{r} + 2|\frac{1}{p} - \frac{1}{2}|$ with $0 < r \leqslant 1$ and $1 \leqslant p < \infty$, the operator $(I - \mathcal{L}_{sub})^{-\frac{\alpha}{2}}$ is r-nuclear on $L^p(\mathrm{SO}(3))$. The same conclusion holds for the same powers (of sub-Laplacians) on $L^p(\mathrm{SU}(2))$ or on $L^p(\mathbb{S}^3)$.*

Again, if $p = 2$, the order $\alpha r > 4$ in Corollary 3.20 is sharp, see [6, Section 4].

4. Trace formulae on $L^p(G)$ and distribution of eigenvalues

We now turn to some applications of the r-nuclearity on $L^p(G)$-spaces for the trace formulae, the Lidskii formula and the distribution of eigenvalues. In the special case $1 \leqslant p_1 = p_2 = p < \infty$, applying Theorem 2.3 and Theorem 3.7 we obtain:

Corollary 4.1. *Let G be compact Lie group and $0 < r \leqslant 1$. Let $1 \leqslant p < \infty$ and let us denote $\tilde{p} = \min\{2, p\}$. Let $\sigma_A(x, \xi)$ be the matrix symbol of a bounded operator $A : L^p(G) \to L^p(G)$ such that*

$$\sum_{[\xi]\in\hat{G}} d_\xi^{2+\frac{r}{\tilde{p}}} \big\| \big\| \big(\sigma_A(x,\xi)\big)^t \big\|_{op(\ell^\infty,\ell^\infty)} \big\|_{L^p(G)}^r < \infty.$$

Then $A : L^p(G) \to L^p(G)$ is r-nuclear and

$$\sum_{n=1}^{\infty} \big|\lambda_n(A)\big|^{\frac{2r}{2-r}} < \infty.$$

We now derive another consequence relating the trace formulae with matrix-valued symbols. As we have already explained in the introduction, every nuclear operator acting from a Banach space E into E admits a trace provided that E satisfies the approximation property, which is the case here dealing with L^p-spaces. In the next proposition we show that, when $p = p_1 = p_2$, the sufficient condition in Theorem 3.7 ensures the existence of a formula for the trace in terms of the matrix-valued symbol.

Theorem 4.2. *Let G be a compact Lie group and $0 < r \leqslant 1$. Let $1 \leqslant p < \infty$ and $\tilde{p} = \min\{2, p\}$. Let $A : L^p(G) \to L^p(G)$ be a linear continuous operator with matrix-valued symbol $\sigma_A(x, \xi)$ such that*

$$\sum_{[\xi]\in\hat{G}} d_\xi^{2+\frac{r}{\tilde{p}}} \big\| \big\| \big(\sigma_A(x,\xi)\big)^t \big\|_{op(\ell^\infty,\ell^\infty)} \big\|_{L^p(G)}^r < \infty.$$

Then the operator $A : L^p(G) \to L^p(G)$ is r-nuclear and its trace is given by

$$\operatorname{Tr} A = \int_G \sum_{[\xi]\in\hat{G}} d_\xi \operatorname{Tr}\big(\sigma_A(x,\xi)\big)\, dx. \tag{4.1}$$

Moreover, if in addition $0 < r \leqslant \frac{2}{3}$, then

$$\operatorname{Tr} A = \sum_{n=1}^{\infty} \lambda_n(A), \tag{4.2}$$

with multiplicities taken into account.

Proof. The r-nuclearity is a consequence of Theorem 3.7 and we adopt the notation of the proof of Theorem 3.7, and denote $\sigma = \sigma_A$. Concerning the trace formula, for the sake of simplicity we will just consider $r = 1$, the general case follows from inclusion. As we have seen in the proof of Theorem 3.7, the formula

$$k(x, y) = \sum_{[\xi]\in\hat{G}} d_\xi \operatorname{Tr}\big(\xi(x)\sigma(x,\xi)\xi(y)^*\big)$$

represents the kernel of A. Moreover, it is well-defined on the diagonal: in fact for the terms of the decomposition of the kernel

$$g_{\xi,ij}(x) = d_\xi \big(\xi(x)\sigma(x,\xi)\big)_{ij}, \qquad h_{\xi,ij}(y) = \big(\xi(y)^*\big)_{ji} = \overline{\xi(y)}_{ij},$$

by Hölder's inequality we have on the diagonal

$$\int_G |g_{\xi,ij}(x)||h_{\xi,ij}(x)|\,dx \leqslant \|g_{\xi,ij}(\cdot)\|_{L^p(G)}\|h_{\xi,ij}(\cdot)\|_{L^q(G)}.$$

Hence, since $p = p_1 = p_2$ we have

$$\begin{aligned}\int_G \sum_{[\xi]\in\hat{G}} d_\xi \left|\mathrm{Tr}\big(\sigma(x,\xi)\big)\right| dx &\leqslant \sum_{\xi,ij}\|g_{\xi,ij}(\cdot)\|_{L^p(G)}\|h_{\xi,ij}(\cdot)\|_{L^q(G)}\\ &\leqslant \sum_{\xi} d_\xi^{2+\frac{1}{p}}\big\|\big\|\big(\sigma(x,\xi)\big)^t\big\|_{op(\ell^\infty,\ell^\infty)}\big\|_{L^p(G)} < \infty.\end{aligned}$$

Therefore,

$$\begin{aligned}\mathrm{Tr}\,A &= \int_G k(x,x)\,dx\\ &= \int_G \sum_{[\xi]\in\hat{G}} d_\xi\,\mathrm{Tr}\big(\xi(x)\sigma(x,\xi)\xi(x)^*\big)\,dx\\ &= \int_G \sum_{[\xi]\in\hat{G}} d_\xi\,\mathrm{Tr}\big(\sigma(x,\xi)\xi(x)^*\xi(x)\big)\,dx\\ &= \int_G \sum_{[\xi]\in\hat{G}} d_\xi\,\mathrm{Tr}\big(\sigma(x,\xi)\big)\,dx.\end{aligned}$$

We have employed the tracial property $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$ and the fact that $\xi(x)$ is unitary for every x. Finally, (4.2) follows from Theorem 4.2 and Grothendieck's theorem. □

Remark 4.3. We note that not for every kernel it is convenient to calculate the trace integrating along the diagonal due its degeneracy. When the kernel is representable by an expansion of the kind appearing in Theorem 2.1 one is allowed to proceed in such a way. For a general kernel the integration along the diagonal should be calculated involving an averaging processes, see e.g. [3].

Very recently it has been proved (cf. [17]) that if $\frac{1}{r} = 1 + |\frac{1}{2} - \frac{1}{p}|$, the Lidskii formula holds for r-nuclear operators on $L^p(\nu)$-spaces. The importance of this result for us is that it allows to move r along the interval $[\frac{2}{3}, 1]$ keeping the validity of Lidskii's formula for suitable values of p. If $r \in (\frac{2}{3}, 1)$ there exist two corresponding values of p solving the equation $\frac{1}{r} = 1 + |\frac{1}{2} - \frac{1}{p}|$ the first one with $p < 2$ and the other one with $p > 2$. As a consequence of this result and Theorem 4.2 we obtain an extension of (4.2) allowing now a larger range of r:

Corollary 4.4. *Let G be compact Lie group. Let $1 \leqslant p < \infty$ and let us denote $\tilde{p} = \min\{2, p\}$. Let $0 < r \leqslant 1$ be such that $\frac{1}{r} = 1 + |\frac{1}{2} - \frac{1}{p}|$. If*

$$\sum_{[\xi]\in\hat{G}} d_\xi^{2+\frac{r}{\tilde{p}}}\big\|\big\|\big(\sigma_A(x,\xi)\big)^t\big\|_{op(\ell^\infty,\ell^\infty)}\big\|^r_{L^p(G)} < \infty,$$

then A is r-nuclear on $L^p(G)$ and we have

$$\mathrm{Tr}\,A = \int_G \sum_{[\xi]\in\hat{G}} d_\xi\,\mathrm{Tr}\big(\sigma_A(x,\xi)\big)\,dx = \sum_{n=1}^{\infty}\lambda_n(A),$$

with multiplicities taken into account.

4.1. Heat kernels

We shall now establish some applications, in particular to the heat kernels on compact Lie groups. The heat kernel constructions, and the subsequent Poisson kernel constructions, are instrumental in the advances in the Littlewood–Paley theory on compact Lie groups, see e.g. [25]. However, our approach is more straightforward, making use of the symbol of the heat kernel. Indeed, taking into account that $\sigma_{e^{-t\mathcal{L}_G}}(x,\xi)=e^{-t|\xi|^2}I_{d_\xi}$, where $|\xi|^2=\lambda_{[\xi]}^2$ with $\lambda_{[\xi]}$ as in (2.4), we have

$$e^{-t\mathcal{L}_G}f(x)=\sum_{[\xi]\in\hat{G}}d_\xi\operatorname{Tr}\bigl(\xi(x)\sigma_{e^{-t\mathcal{L}_G}}(x,\xi)\hat{f}(\xi)\bigr)=\sum_{[\xi]\in\hat{G}}d_\xi e^{-t\lambda_{[\xi]}^2}\operatorname{Tr}\bigl(\xi(x)\hat{f}(\xi)\bigr).$$

We can now derive the nuclearity of the heat kernel on L^p-spaces.

Theorem 4.5. *Let G be compact Lie group. Then the heat operator $e^{-t\mathcal{L}_G}:L^{p_1}(G)\to L^{p_2}(G)$ is nuclear for every $t>0$ and all $1\leqslant p_1,p_2<\infty$. Moreover, if $0<r\leqslant 1$, then $e^{-t\mathcal{L}_G}:L^p(G)\to L^p(G)$ is r-nuclear for every $t>0$ and $1\leqslant p<\infty$. In particular, on each $L^p(G)$, due to the 1-nuclearity we have the trace formula*

$$\operatorname{Tr}e^{-t\mathcal{L}_G}=\sum_{[\xi]\in\hat{G}}d_\xi^2e^{-t\lambda_{[\xi]}^2}.$$

Proof. The kernel of $e^{-t\mathcal{L}_G}$ is given by

$$k_t(x,y)=\sum_{[\xi]\in\hat{G}}d_\xi e^{-t\lambda_{[\xi]}^2}\operatorname{Tr}\bigl(\xi(x)\xi(y)^*\bigr),$$

with

$$\operatorname{Tr}\bigl(\xi(x)\xi(y)^*\bigr)=\sum_{i,j=1}^{d_\xi}\xi(x)_{ij}\overline{\xi(y)}_{ij}.$$

We set

$$g_{\xi,ij}(x)=d_\xi e^{-t\lambda_{[\xi]}^2}\xi(x)_{ij},\qquad h_{\xi,ij}(y)=\bigl(\xi(y)^*\bigr)_{ji}=\overline{\xi(y)}_{ij}.$$

As before we shall consider q_1 such that $\frac{1}{p_1}+\frac{1}{q_1}=1$ and we denote $\tilde{q}_1=\max\{2,q_1\}$. Then by Lemma 2.5 we have

$$\|\bar{\xi}_{ij}\|_{L^{q_1}(G)}=d_\xi^{-\frac{1}{\tilde{q}_1}}.$$

On the other hand

$$\|g_{\xi,ij}\|_{L^{p_2}(G)}=\bigl\|d_\xi e^{-t\lambda_{[\xi]}^2}\xi_{ij}\bigr\|_{L^{p_2}(G)}\leqslant\bigl\|d_\xi e^{-t\lambda_{[\xi]}^2}\|\xi\|_{op}\bigr\|_{L^{p_2}(G)}\leqslant d_\xi e^{-t\lambda_{[\xi]}^2}.$$

Therefore,

$$\sum_{\xi,ij}\bigl\|g_{\xi,ij}(\cdot)\bigr\|_{L^{p_2}(G)}\bigl\|h_{\xi,ij}(\cdot)\bigr\|_{L^{q_1}(G)}\leqslant\sum_{\xi}d_\xi^2d_\xi^{\frac{1}{p_1}}e^{-t\lambda_{[\xi]}^2}<\infty,$$

the last convergence following, for example, from any of the Weyl formulae, see, for example [5].

The r-nuclearity follows in a similar way. The trace formula follows immediately from Lemma 4.2 and fact that the Haar measure on G is normalised:

$$\operatorname{Tr}e^{-t\mathcal{L}_G}=\int_G\sum_{[\xi]\in\hat{G}}d_\xi\operatorname{Tr}\bigl(e^{-t\lambda_{[\xi]}^2}I_{d_\xi}\bigr)=\sum_{[\xi]\in\hat{G}}d_\xi^2e^{-t\lambda_{[\xi]}^2}.$$

The proof is complete. $\square$

Remark 4.6. The Lidskii formula can be used to deduce lower bounds on the number of eigenvalues: Let E be a Banach space enjoying the approximation property. If $T : E \to E$ is a $\frac{2}{3}$-nuclear operator which possesses at least one eigenvalue and if $|\lambda_k(T)| \leqslant M$ for all k, then

$$\frac{|\operatorname{Tr}(T)|}{M} \leqslant N,$$

where N is the number of eigenvalues of T. Indeed, applying the Lidskii formula

$$\operatorname{Tr}(T) = \sum_{k=1}^{N} \lambda_k(T),$$

we can estimate

$$\left|\operatorname{Tr}(T)\right| = \left|\sum_{k=1}^{N} \lambda_k(T)\right| \leqslant \sum_{k=1}^{N} \left|\lambda_k(T)\right| \leqslant MN.$$

As a consequence of this observation, taking into account the trace formula in [3] we obtain the following estimate for integral operators. The symbol $\tilde{}$ will denote the averaging process for kernels described in [3]. Let μ be a Borel measure on a second countable topological space and let $T : L^p(\mu) \to L^p(\mu)$ be a $\frac{2}{3}$-nuclear operator with kernel $K(x, y)$. If T possesses at least one eigenvalue and if $|\lambda_k(T)| \leqslant M$ for all k, then

$$\frac{|\int_{\Omega} \tilde{K}(x, x)\, d\mu(x)|}{M} \leqslant N,$$

where N is the number of eigenvalues of T. The last inequality means that the better one can estimate the size of the trace the better lower bound one gets for the number of the eigenvalues.

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