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LQG Control under Limited Communication

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Abstract— **We discuss Kalman filtering and LQ optimal control of a networked control system (NCS) whose sensors and actuators exchange information with a remote controller over a shared communication medium. Access to that medium is governed by a pair of periodic communication sequences. Under the proposed model, the controller and plant handle communication disruptions by "ignoring" sensors and actuators that are not actively communicating. We show that Kalman filtering and LQ optimal control for NCSs can be formulated as a standard LQG problem for an equivalent periodic system. Moreover, under mild conditions, there always exist periodic communication sequences that preserve the detectability and observability of the NCS and thus make it possible to guarantee the existence of a stabilizing LQG controller.**

I. Introduction

This paper presents an LQG (Linear Quadratic Gaussian) design approach for Networked Control Systems (NCSs) in which sensors and actuators of an MIMO plant exchange information with a remote controller via a shared communication medium. In contrast to traditional control systems, NCSs are subject to medium access constraints because of the limited capacity provided by the communication medium. Consequently, any controller must be accompanied by a Medium Access Control (MAC) policy that determines which sensors and actuators gain access to the shared medium at any given time.

A well known MAC policy for NCS ([1], [2]) is to schedule access for the different sensors and actuators off-line, according to a periodic *communication sequence*. The task of designing effective communication sequences is quite challenging, despite their intuitive appeal. Previously proposed methods [3], [4], [5] only handle simple NCSs that consist of a number of "uncoupled" plants, or assume that medium access constraints exist for input or output signals, but not for both. With regard to feedback stabilization in particular, the question of whether a stabilizing constant-gain feedback controller exists, is NP-hard (see [2] and references therein), even if the communication sequence is fixed in advance. Likewise, the complexity of LQ optimal control problems involving NCS is significant; the cases studied [5], [6] involve simple NCS configurations and rely on dynamic programming or exhaustive search in order to identify optimal sequences.

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It will become apparent in the sequel that the difficulties encountered in these works are partly due to the use of a zero order hold (ZOH) at the "receiving end" of a communication medium (i.e. at the plant's and controller's input stages). Doing so has the effect of greatly increasing the system's complexity because it introduces time-varying delays and leads to closed-loop dynamics in which communication and control are tightly coupled (see, for example, the "extensive form" in [2]). Moreover, holding input (output) signals whose actuators (sensors) are not actively communicating may not necessarily improve performance.

The contribution of this paper is to propose an alternative NCS architecture, whereby the plant and controller forgo the use of a ZOH and instead choose to "ignore" (in a manner to be made precise) the actuators and sensors that are not actively communicating. We are motivated by recent work on NCS stabilization, that has shown [7] that a similar choice leads to a decoupling of the design of communication sequences from that of a feedback controller, and reduces the complexity of the problem to a level that is amenable to analysis. Here, we investigate state estimation and LQ optimal control of NCS under periodic communication. We show that by avoiding the use of ZOH elements: i) the communication sequences that govern controller-plant interaction can be designed easily and separately from the controller, and ii) by proper choice of communication, the problem of state estimation and LQ optimal control for a NCS is no more complicated than a standard LQG problem, which can be addressed by composing existing results. To the authors' knowledge, this work is the first to provide a solution to the LQG problem under medium access constraints. For related work on state estimation for simple NCS configurations see [8], [9], and [10].

The proposed approach avoids the complexity associated with previously proposed models and addresses MIMO NCS whose dynamics are "fully coupled". Furthermore, the estimation error covariance and optimal feedback gain associated with the Kalman filter and LQ controller used to steer the NCS, converge to periodic solutions. Thus, the LQG controller can be easily implemented in practice.

The remainder of this paper is structured as follows: In Section II we show that under periodic communication, a MIMO NCS can be modeled as a periodic time varying system with a reduced number of inputs and outputs. The stabilizability and detectability of a NCS can be preserved by properly choosing communication sequences. Section III, discusses an LQG problem for NCSs, including a discussion of sufficient conditions that ensure the convergence of the Kalman filter and the optimal LQ gains associated with the

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LQG controller. A numerical example is given in Section IV.

II. MODELING NCSS

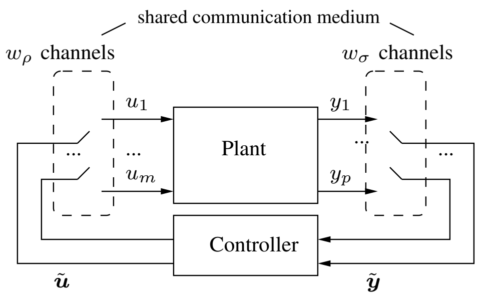

Fig. 1. A Networked Control System with medium access constraints

We begin with a deterministic model for NCSs with medium access constraints. This model, inspired by [7], will be generalized to a stochastic setting in Section III.

Consider a NCS (Fig. 1) in which the dynamics of the plant are given by the discrete-time linear time-invariant (LTI) system

$$\begin{aligned} \boldsymbol{x}(k+1) &= A\boldsymbol{x}(k) + B\boldsymbol{u}(k) \\ \boldsymbol{y}(k) &= C\boldsymbol{x}(k) \end{aligned} \tag{1}$$

where $\boldsymbol{x} = [x_1, \cdots, x_n]^T \in \mathbb{R}^n$, $\boldsymbol{u} = [u_1, \cdots, u_m]^T \in \mathbb{R}^m$, and $\boldsymbol{y} = [y_1, \cdots, y_p]^T \in \mathbb{R}^p$ are the plant's states, inputs, and outputs, respectively. Suppose that outputs are transmitted to the remote controller via a communication medium which can only carry w_σ signals, with $1 \le w_\sigma < p$. That is, only w_σ of the p outputs can be sent to the controller at any one time, while others must wait. Similarly, at the plant's input stage, the communication medium can only accommodate w_ρ ($1 \le w_\rho < m$) signals, with only w_ρ of the m inputs being updated by the controller at any time k.

A. Communication sequences and the Extended Plant

For $i = 1, \cdots, p$, let the binary-valued function $\sigma_i(k)$ denote the medium access status of the i-th output y_i at time k, i.e., $\sigma_i(k) : \mathbb{Z} \mapsto \{0, 1\}$, where 1 means "accessing" and 0 means "not accessing". The medium access status of all p outputs will be represented by a "*p-to-w_σ communication sequence*" [2], [7],

$$\boldsymbol{\sigma}(k) = [\sigma_1(k), \cdots, \sigma_p(k)]^T.$$

Definition 1: Let $M, N \in \mathbb{N}$ with $N \le M$. An M-to-N *communication sequence* is a map, $\boldsymbol{\sigma}(k) : \mathbb{Z} \mapsto \{0, 1\}^M$, satisfying $\|\boldsymbol{\sigma}(k)\|^2 = N$, $\forall k$.

As we have previously indicated, the controller is to choose plant inputs based only on the w_σ output elements which have been granted medium access at any time k. All other outputs will be effectively ignored. Let the output information received by the controller at time k be denoted by $\tilde{\boldsymbol{y}}(k) = [\tilde{y}_1(k), \cdots \tilde{y}_{w_\sigma}(k)]^T$. For all k, $\tilde{\boldsymbol{y}}(k)$ contains those elements from $\boldsymbol{y}(k)$ for which $\sigma_i(k) = 1$. To establish the relationship between $\boldsymbol{y}(k)$ and $\tilde{\boldsymbol{y}}(k)$, we will make use of the following definition:

Definition 2: Let $\eta(k)$ be an M-to-N communication sequence. Then, for all $k \in \mathbb{N}$, the $N \times M$ matrix $\mu_\eta(k)$ is obtained by removing the $M - N$ all-zero rows from the $M \times M$ matrix $diag(\eta(k))$.

Example 1: Let $\eta(1) = [1, 1, 0, 1]^T$, then

$$\mu_\eta(1) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Using the last definition, we can express $\tilde{\boldsymbol{y}}(k)$ as

$$\tilde{\boldsymbol{y}}(k) = \mu_\sigma(k)\boldsymbol{y}(k), \tag{2}$$

where $\boldsymbol{\sigma}(k)$ is the output communication sequence.

Similarly, the medium access status of the plant's m inputs is represented by an m-to-w_ρ communication sequence $\boldsymbol{\rho}(k)$. When an input u_j loses its access to the communication medium, the plant ignores that input until the corresponding actuator regains medium access. This is equivalent to setting $u_j = 0$ while $\rho_j = 0$. Let $\tilde{\boldsymbol{u}}(k) = [\tilde{u}_1(k), \cdots \tilde{u}_{w_\rho}(k)]^T$ denote the elements of $\boldsymbol{u}(k)$ whose actuators were granted medium access and received updated values from the controller at time k. Under the protocol outlined above,

$$\boldsymbol{u}(k) = \mu_{\boldsymbol{\rho}}(k)^T \tilde{\boldsymbol{u}}(k). \tag{3}$$

Combining (1)-(3), we obtain a linear time-varying system with w_ρ inputs and w_σ outputs:

$$\begin{aligned} \boldsymbol{x}(k+1) &= A\boldsymbol{x}(k) + B\mu_\rho(k)^T \tilde{\boldsymbol{u}}(k) \\ \tilde{\boldsymbol{y}}(k) &= \mu_\sigma(k) C\boldsymbol{x}(k). \end{aligned} \tag{4}$$

These equations describe the NCS "from the controller's point of view". We call (4) the *extended plant*; it incorporates the dynamics of the plant together with the access status of the communication medium. In the sequel, $\boldsymbol{\rho}$ and $\boldsymbol{\sigma}$ will be referred to as the "input" and "output" communication sequences, respectively.

We remark that the choice of "removing" the ZOH elements which were included in previous NCS models has the effect of avoiding any "enlargement" of the state vector and leaves us with a LTV system whose parameters are a function of the input and output communication sequences. This situation is to be compared with those in [1], [2], among others. Also, the choice of representation for the signals $\tilde{\boldsymbol{u}}(k)$ and $\tilde{\boldsymbol{y}}(k)$ as well as the sequences $\mu_\rho(k)$, $\mu_\sigma(k)$ differs from that in [7]. It will become clear in Sections III-A, III-B, that the new model is necessary in order to avoid singularities that would arise when solving the matrix Riccati equations associated with the LQG problem.

Designing an effective NCS controller requires the selection not only of an output-to-input map, but also of the input and output communication sequences, as they are the ones that determine the kinds of time-variation that will be present in the dynamics of the extended plant. In previous works, this coupling has proved to be a significant problem, especially if one requires optimality with respect to the control and communication policies jointly. Here, we will not attempt to solve the joint problem, but instead will identify classes of communication sequences which make the extended plant suitable for the application of existing design tools from LTV systems theory. This task is made straightforward by our choice of protocol for controller-plant interaction and will be discussed next.

B. Choosing effective communication sequences

Clearly, the controllability and observability of the extended plant (4) are both crucial when it comes to estimation and control of the underlying NCS. For a related NCS model [7], it has been shown that the controllability and observability of the extended plant can be preserved by properly choosing communication sequences $\boldsymbol{\rho}(\cdot)$ and $\boldsymbol{\sigma}(\cdot)$. These results can be easily applied to the NCS model in (4).

To simplify the analysis, we assume that the matrix A in (4) is invertible. This can be guaranteed if A is obtained by sampling a continuous-time plant, and implies that the controllability and reachability of (4) are equivalent, as are observability and reconstructability. The case where A is singular will not be included here because of space constraints.

Definition 3: The system (4) is *controllable on* $[k_0, k_f]$ if, given any $\boldsymbol{x}_0$, there exists a control signal $\tilde{\boldsymbol{u}}(\cdot)$ that steers (4) from $\boldsymbol{x}(k_0) = \boldsymbol{x}_0$ to the origin at time k_f. We say that (4) is *controllable* if, for any k, there exists a positive integer l such that (4) is controllable on $[k, k+l]$.

Definition 4: The system (4) is *observable on* $[k_0, k_f]$ if any initial condition at k_0 can be uniquely determined by the corresponding response $\tilde{\boldsymbol{y}}(k)$ for $k \in [k_0, k_f]$. We say that (4) is *observable* if, for any k, there exists a positive integer l such that (4) is observable on $[k, k+l]$.

Theorem 1: Let A be invertible and the pair (A, B) be controllable. For any integer $1 \le w_\rho < m$, there exist integers $l, N > 0$ and an N-periodic[1] m-to-w_ρ communication sequence $\boldsymbol{\rho}(\cdot)$ such that the extended plant (4) is controllable on $[k, k+l]$ for all k, and thus controllable.

Proof: Let

$$R = [A^{N-1}B\mu_\rho^T(0), A^{N-2}B\mu_\rho^T(1), \cdots, B\mu_\rho^T(N-1)]. \quad (5)$$

The system (4) is controllable on $[0, N]$ iff $\mathbf{rank}(R) = n$. Notice that, at each step k, $\mu_\rho^T(k)$ has the effect of "selecting" w_ρ columns from the m columns of the term $A^{N-k-1}B$ on the RHS of (5). Also notice that the matrices $\Gamma_i = [A^{ni+n-1}B, A^{ni+n-2}B, \cdots, A^{ni}B]$, contain n linearly independent columns, for all $i = 0, 1, \cdots$, because A is invertible and (A, B) is controllable. Let $\gamma_i^0, \cdots, \gamma_i^{n-1}$ be any n linearly independent columns from Γ_i and let $L_i \triangleq \{\gamma_i^0, \cdots, \gamma_i^{n-1}\}$. Then, $\boldsymbol{\rho}(\cdot)$ can be designed using the following algorithm:

1) Let $L = L_0$.
2) Replace γ_0^1 in L by a column from L_1 while maintaining $\mathbf{rank}(L) = n$. Such a replacement can always be found because $\mathbf{rank}(L_1) = n$.
3) For $i = 2, \cdots, n-1$, replace γ_0^i in L by a column from L_i while keeping the rank of L fixed.

[1]A discrete-time communication sequence $\boldsymbol{\sigma}(\cdot)$ is called N-periodic if $\boldsymbol{\sigma}(k) = \boldsymbol{\sigma}(k+N)$ for all k.

The resulting L has one column from each Γ_i ($i = 0, \cdots, n-1$) and has rank n. The algorithm ensures that it is possible to select n linearly independent columns as long as one can select one column from each Γ_i. Notice that, for a m-to-w_ρ communication sequence $\boldsymbol{\rho}(\cdot)$, on the RHS of (5), $\mu_\rho^T(\cdot)$ selects a total of $n \cdot w_\rho$ columns from each Γ_i. Therefore, there always exists a $\mu_\rho^T(\cdot)$ (equivalently, a sequence $\boldsymbol{\rho}(\cdot)$) that selects n independent columns from the RHS of (5) for some $N \le n^2$.

The algorithm described above yields a communication sequence $\boldsymbol{\rho}(k)$, for $k = 0, \cdots, N-1$, such that (4) is controllable on $[0, N]$. Now, extend $\boldsymbol{\rho}(k)$ for $k \ge N$ by setting $\boldsymbol{\rho}(k+N) = \boldsymbol{\rho}(k)$, $\forall k$. It can be shown (Th. 1 and Cor. 1 in [7]) that this choice of $\boldsymbol{\rho}$ makes (4) controllable on $[k, k+l]$ for all k, provided that $l \ge 2N-1$. ■

The bound for k_f in Th. 1 is conservative and, as one can observe in practice, a set of linearly independent columns can be found in far fewer steps. If optimality is required, it may be possible to find the minimum k_f by searching (off-line) over all possible communication sequences $\boldsymbol{\rho}(\cdot)$ in the interval $k = \left[\left\lceil \frac{n}{w_\rho} \right\rceil, \left\lceil \frac{n}{w_\rho} \right\rceil \cdot n\right]$, when computationally feasible. The search can be expedited through the use of a number of heuristics that will not be detailed here because of space constraints.

By switching from column manipulations to row manipulations, the duality of controllability and observability gives the following result whose proof is similar to that of Th. 1.

Theorem 2: Let A be invertible and the pair (A, C) be observable. For any integer $1 \le w_\sigma < p$, there exist integers $l, N > 0$ and an N-periodic p-to-w_σ communication sequence $\boldsymbol{\sigma}(\cdot)$ such that the system (4) is observable on $[k, k+l]$ for all k, and thus observable.

The above results can be generalized to stabilizable and detectable systems. The construction of the extended plant (4) ensures that its stabilizability and detectability do not change under change of coordinates is stabilizable. Suppose that its Kalman canonical decomposition has the form: $\begin{bmatrix} A_c & \bar{A}'_c \\ 0 & \bar{A}_c \end{bmatrix}, \begin{bmatrix} B_c \\ 0 \end{bmatrix}$, where (A_c, B_c) is the controllable part of (A, B). Then, any m-to-w_ρ communication sequence $\boldsymbol{\rho}(\cdot)$ that preserves the controllability of the subsystem (A_c, B_c) will also guarantee the stabilizability of the extended plant (4). The same holds for detectability.

III. Problem formulation and solution

Consider a NCS whose plant is described by the discrete-time stochastic LTI system:

$$\begin{aligned} \boldsymbol{x}(k+1) &= A\boldsymbol{x}(k) + Bu(k) + v(k) \quad (6) \\ \boldsymbol{y}(k) &= C\boldsymbol{x}(k) + w(k), \qquad k = 0, 1, \cdots N-1 \end{aligned}$$

where $\boldsymbol{x} \in \mathbb{R}^n$, $\boldsymbol{u} \in \mathbb{R}^m$, $\boldsymbol{y} \in \mathbb{R}^p$. The disturbances, $v(\cdot)$, $w(\cdot)$, are both taken to be Gaussian, iid, with $v(\cdot) \sim N(0, G)$ and $w(\cdot) \sim N(0, I_{p\times p})$, where $I_{p\times p}$ is the $p \times p$ identity matrix and G is a positive definite $n \times n$ matrix. Let the initial condition $\boldsymbol{x}(0)$ be Gaussian as well, with $\boldsymbol{x}(0) \sim N(x_0, \Sigma_0)$.

As before, suppose that the communication medium connecting the plant and the controller provides w_ρ ($1 \le w_\rho < m$) input channels, and w_σ ($1 \le w_\sigma < p$) output channels. Using the model presented in the previous Section,

$$\boldsymbol{x}(k+1) = A\boldsymbol{x}(k) + \tilde{B}(k)\tilde{\boldsymbol{u}}(k) + v(k) \quad (7)$$
$$\tilde{\boldsymbol{y}}(k) = \tilde{C}\boldsymbol{x}(k) + \tilde{w}(k),$$

where $\tilde{B}(k) = B\mu_\rho(k)^T$, $\tilde{C} = \mu_\sigma(k)C$, and $\tilde{w}(k) = \mu_\sigma(k)w(k)$. Suppose that the m-to-w_ρ communication sequence $\boldsymbol{\rho}(\cdot)$, and the p-to-w_σ communication sequence $\boldsymbol{\sigma}(\cdot)$ have been selected off-line, so that the stabilizability and detectability of (6) are preserved in (7). This can be accomplished using the algorithm given in the proof of Th. 1.

Problem 1: Design an optimal controller for (7), such that the quadratic cost function

$$J = \mathcal{E}\left\{\boldsymbol{x}^T(N)Q\boldsymbol{x}(N) + \sum_{k=0}^{N-1} \boldsymbol{x}^T(k)Q\boldsymbol{x}(k) + \tilde{\boldsymbol{u}}^T(k)\tilde{\boldsymbol{u}}(k)\right\} \quad (8)$$

is minimized.

Notice that if $\boldsymbol{\sigma}(\cdot)$ is determined off-line, $\tilde{w}(0), \cdots \tilde{w}(N-1), \boldsymbol{x}(0), v(0), \cdots v(N-1)$ are independent random variables because $\tilde{w}(k)$ is a sub-vector of $w(k)$ for all k. Moreover, because of the linearity of $\mu_\sigma(k)$, $\tilde{w}(k)$ is Gaussian, with $\tilde{w}(k) \sim N(0, I_{w_\sigma \times w_\sigma})$ for all k. Thus, the optimal control of the NCS described in Section II becomes a standard LQG problem for the stochastic extended plant (7). It is well known (e.g., [11]) that the solution to this LQG problem has two components:

1) A Kalman filter that gives the optimal state estimate $\hat{\boldsymbol{x}}(k)$ based on the outputs $\tilde{\boldsymbol{y}}(\cdot)$.
2) An LQ optimal feedback gain $L(k)$, obtained by solving a deterministic LQ problem with perfect state information and without the presence of noise $v(\cdot)$ and $w(\cdot)$.

The separation principle ensures that the two subproblems can be solved independently. The resulting optimal controller that minimizes J is given by the feedback law

$$\tilde{\boldsymbol{u}}^*(k) = -L(k)\hat{\boldsymbol{x}}(k)$$

In the following, we give analytical expressions for the Kalman filter and optimal control law; the proofs of these results can be found in most stochastic control textbooks (e.g., [11]).

A. Kalman filtering

The optimal estimator for the extended plant (7) is the discrete time Kalman filter, described in the following recursive steps, with $\hat{\boldsymbol{x}}(0) = x_0$, $\Sigma(0) = \Sigma_0$:

1) Time update

$$\hat{\boldsymbol{x}}(k^-) = A\hat{\boldsymbol{x}}(k-1) + \tilde{B}(k-1)\tilde{\boldsymbol{u}}(k-1) \quad (9)$$
$$P(k) = A\Sigma(k-1)A^T + G, \quad (10)$$

where $\hat{\boldsymbol{x}}(k^-)$ is the conditional mean of state variable $\boldsymbol{x}(k)$ prior to the measurement of $\tilde{\boldsymbol{y}}(k)$, and $P(k)$ is the variance of the prediction error.

$$\hat{\boldsymbol{x}}(k^-) \triangleq \mathcal{E}\{\boldsymbol{x}(k)|\tilde{\boldsymbol{y}}(0)\cdots\tilde{\boldsymbol{y}}(k-1)\},$$
$$P(k) \triangleq \mathcal{E}\{(\boldsymbol{x}(k) - \hat{\boldsymbol{x}}(k^-))(\boldsymbol{x}(k) - \hat{\boldsymbol{x}}(k^-))^T\}.$$

2) Measurement update

$$H(k) = P(k)\tilde{C}^T(k)(\tilde{C}(k)P(k)\tilde{C}^T(k) + I)^{-1} \quad (11)$$
$$\hat{\boldsymbol{x}}(k) = \hat{\boldsymbol{x}}(k^-) + H(k)(\tilde{\boldsymbol{y}}(k) - \tilde{C}(k)\hat{\boldsymbol{x}}(k^-)) \quad (12)$$
$$\Sigma(k) = (I - H(k)\tilde{C}(k))P(k). \quad (13)$$

where $\hat{\boldsymbol{x}}(k) \triangleq \mathcal{E}\{\boldsymbol{x}(k)|\tilde{\boldsymbol{y}}(0)\cdots\tilde{\boldsymbol{y}}(k)\}$ is the state estimate, and $\Sigma(k)$ is the covariance of the estimation error.

From (10), (11), and (13), it is easy to verify that the sequence $P(k+1)$ satisfies the time-varying discrete time Riccati equation

$$P(k+1) = AP(k)A^T + G-$$
$$AP(k)\tilde{C}^T(k)[I + \tilde{C}(k)P(k)\tilde{C}^T(k)]^{-1}\tilde{C}(k)P(k)A^T \quad (14)$$

The "one-step prediction" error, $e(k) = \boldsymbol{x}(k) - \hat{\boldsymbol{x}}(k^-)$, satisfies

$$e(k+1) = (A - \Gamma(k)\tilde{C}(k))e(k) + v(k) - \Gamma(k)w(k),$$

and

$$\mathcal{E}\{e(k+1)\} = (A - \Gamma(k)\tilde{C}(k))\mathcal{E}\{e(k)\}, \quad (15)$$

where $\Gamma(k) = AH(k)$ is the Kalman gain.

B. LQ Optimal Control

The optimal control law for the LQG problem can be obtained by solving a standard LQ problem for the deterministic linear system

$$\boldsymbol{x}(k+1) = A\boldsymbol{x}(k) + \tilde{B}(k)\tilde{u}(k) \quad (16)$$

while assuming precise state feedback of $\boldsymbol{x}(k)$ at each step k. The optimal controller is given by the feedback law

$$\tilde{\boldsymbol{u}}^*(k) = -L(k)\boldsymbol{x}(k). \quad (17)$$

The gain matrix $L(k)$ can be obtained from

$$L(k) = (\tilde{B}^T K(k+1)\tilde{B}(k) + I)^{-1}\tilde{B}(k)^T K(k+1)A, \quad (18)$$

where the symmetric positive semidefinite matrices $K(k)$ satisfy the backwards Riccati equation

$$K(N) = Q, \quad (19)$$
$$K(k) = A^T K(k+1)A + Q - A^T K(k+1)\tilde{B}(k)\cdot$$
$$\cdot(\tilde{B}^T(k)K(k+1)\tilde{B}(k) + I)^{-1}\tilde{B}^T(k)K(k+1)A. \quad (20)$$

The closed loop dynamics of the system (16) under the optimal control law $\tilde{\boldsymbol{u}}^*$ are

$$\boldsymbol{x}(k+1) = (A - \tilde{B}(k)L(k))\boldsymbol{x}(k). \quad (21)$$

C. Periodic Riccati Equations

From the discussion in Section II-B, it follows that if the plant (6) is invertible, it is always possible to design periodic communication sequences $\boldsymbol{\sigma}(\cdot)$ and $\boldsymbol{\rho}(\cdot)$ such that the stabilizability and detectability of (6) are preserved in the extended plant (7). Notice that, under periodic communication, $\tilde{B}(k)$, $\tilde{C}(k)$ are both periodic. Therefore, the Riccati equations (14), (20) associated with the LQG problem will both become Discrete-time Periodic Riccati Equations (DPREs). DPREs have been studied extensively (e.g., [12] and references therein). We go on to review some basic facts.

Definition 5: [12] A Discrete-time Periodic Riccati Equation (DPRE) is a difference equation of the form

$$\begin{aligned}\mathcal{P}(k+1) = \mathcal{A}(k)\mathcal{P}(k)\mathcal{A}^T(k) + \mathcal{B}(k)\mathcal{B}(k)^T - \\ \mathcal{A}(k)\mathcal{P}(k)\mathcal{C}^T(k)[I+\mathcal{C}(k)\mathcal{P}(k)\mathcal{C}^T(k)]^{-1}\mathcal{C}(k)\mathcal{P}(k)\mathcal{A}^T(k),\end{aligned} \quad (22)$$

where $\mathcal{A}(k) : \mathbb{Z} \mapsto \mathbb{R}^{n\times n}$, $\mathcal{B}(k) : \mathbb{Z} \mapsto \mathbb{R}^{n\times m}$, $\mathcal{C}(k) : \mathbb{Z} \mapsto \mathbb{R}^{p\times n}$ and $\mathcal{A}(\cdot)$, $\mathcal{B}(\cdot)$, and $\mathcal{C}(\cdot)$ are T-periodic.

Theorem 3: ([12] Th. 5) Consider the Kalman gain $\mathcal{K}(k) = \mathcal{A}(k)\mathcal{P}(k)\mathcal{C}^T(k)(\mathcal{C}(k)\mathcal{P}(k)\mathcal{C}^T(k)+I)^{-1}$ associated with *any* symmetric positive semidefinite solution $\mathcal{P}(\cdot)$ of (22). If $(\mathcal{A}(\cdot), \mathcal{B}(\cdot))$ is stabilizable and $(\mathcal{A}(\cdot), \mathcal{C}(\cdot))$ detectable, then the corresponding closed-loop matrix $\hat{\mathcal{A}}(\cdot) = \mathcal{A}(\cdot) - \mathcal{K}(\cdot)\mathcal{C}(\cdot)$ is exponentially stable.

Theorem 4: ([12] Th. 6) There exists a unique Symmetric Periodic Positive Semidefinite (SPPS) solution $\bar{\mathcal{P}}(\cdot)$ of the DPRE (14) and $\hat{\bar{\mathcal{A}}}(\cdot) = \mathcal{A}(\cdot) - \bar{\mathcal{K}}(\cdot)\mathcal{C}(\cdot)$ is asymptotically stable iff $(\mathcal{A}(\cdot), \mathcal{B}(\cdot))$ is stabilizable and $(\mathcal{A}(\cdot), \mathcal{C}(\cdot))$ is detectable, where $\bar{\mathcal{K}}(\cdot)$ is the Kalman gain associated with $\bar{\mathcal{P}}(\cdot)$.

Theorem 5: ([12] Th. 7) Suppose that $(\mathcal{A}(\cdot), \mathcal{B}(\cdot))$ is stabilizable and $(\mathcal{A}(\cdot), \mathcal{C}(\cdot))$ detectable. Then, every symmetric and positive semidefinite solution of the DPRE converges to the unique SPPS solution.

The above results give a necessary and sufficient condition (Th. 4) for the existence and uniqueness of an SPPS solution as well a stability condition (Th. 3) for the closed-loop system. Theorem 5 guarantees the asymptotic convergence of the DPRE to the unique SPPS solution.

D. Convergence of the LQG optimal controller

Theorem 1, combined with the results of Sec. III-C and applied to the Riccati equations (14) and (20), implies the following:

Theorem 6: Suppose that:

1) The communication sequence $\boldsymbol{\sigma}(\cdot)$ is chosen to be T-periodic and such that the detectability of the the plant (6) is preserved in the extended plant (7).
2) The pair (A, g) is stabilizable, where $G = gg^T$.

Then, starting from any positive definite initial conditions, the Riccati equation associated with the Kalman filter (14) converges to a unique T-periodic solution $\bar{\Sigma}(k)$ as $k \to \infty$. Moreover, the error dynamics (15) are exponentially stable.

Theorem 7: Suppose that:

1) The communication sequence $\boldsymbol{\rho}(\cdot)$ is T-periodic and such that the stabilizability of the the plant (6) is preserved in the extended plant (7).
2) The pair (A, q^T) is detectable, where $Q = qq^T$.

Then, starting from any positive definite initial conditions, the Riccati equation associated with the LQ problem (20) converges to a unique T-periodic solution $\bar{K}(k)$ as $k \to \infty$. Moreover, the closed loop dynamics (21) are exponentially stable.

IV. A NUMERICAL EXAMPLE

Consider the 2-input, 2-output, 4th order unstable LTI plant with parameters

$$A = \begin{bmatrix} 1 & 1/5 & 0 & 0 \\ 0 & 11/4 & 0 & 1/5 \\ 1 & 1/5 & 1/3 & 3/4 \\ 0 & -1 & 0 & 1/4 \end{bmatrix}, \; B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix},$$
$$C = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

The disturbances were chosen to be $v(\cdot) \sim N(0, 4I_{4\times4})$ and $w(\cdot) \sim N(0, I_{2\times2})$. The plant was controlled via a shared communication medium which had only one input and one output channels (i.e., $w_\rho = w_\sigma = 1$). Using the algorithm provided in the proof of Th. 1, we found that under the period-2 communication sequences

$$\begin{aligned}\{\boldsymbol{\sigma}(0), \boldsymbol{\sigma}(1), \cdots\} = \{[0,\ 1]^T, [1,\ 0]^T, \cdots\}, \\ \{\boldsymbol{\rho}(0), \boldsymbol{\rho}(1), \cdots\} = \{[0,\ 1]^T, [1,\ 0]^T, \cdots\},\end{aligned}$$

the extended plant was controllable and observable.

We formulated the LQG problem described in Sec. III, with $Q = 25I_{4\times4}$ and initial conditions $\boldsymbol{x}(0) = [100, 50, 7, 6]^T$, $\hat{\boldsymbol{x}}(0) = [1, 1, 3, 4]^T$, and $\Sigma(0) = 4I_{4\times4}$. The solution of the periodic Riccati equation (14), associated with the Kalman filter error covariance, converged to a 2-periodic SPPS solution $\bar{P}(\cdot)$ in 5 steps. The solution of the periodic backwards Riccati equation (20), associated with the LQ optimal gain, converged to a 2-periodic SPPS solution $\bar{K}(\cdot)$ in 15 steps. The evolutions of $tr(P(k))$ and $tr(K(k))$ are shown in Fig. 2. The SPPS solutions of (14) and (20) are, for $i \in \mathbb{Z}^+$,

$$\bar{P}(2i) = \begin{bmatrix} 8.92 & 1.15 & 5.00 & -0.40 \\ 1.15 & 267.59 & -21.96 & -105.34 \\ 5.00 & -21.96 & 14.12 & 10.06 \\ -0.40 & -105.34 & 10.06 & 46.61 \end{bmatrix},$$
$$\bar{P}(2i+1) = \begin{bmatrix} 9.53 & -17.70 & 9.01 & 7.29 \\ -17.70 & 71.87 & -29.46 & -27.60 \\ 9.01 & -29.46 & 23.52 & 13.57 \\ 7.29 & -27.60 & 13.57 & 15.79 \end{bmatrix},$$
$$\bar{K}(2i) = \begin{bmatrix} 456.09 & 62.55 & 12.61 & 34.65 \\ 62.55 & 79.70 & 0.33 & -7.34 \\ 12.61 & 0.33 & 28.10 & 7.54 \\ 34.65 & -7.34 & 7.54 & 45.73 \end{bmatrix},$$
$$\bar{K}(2i+1) = \begin{bmatrix} 496.33 & 284.55 & 11.30 & 39.56 \\ 284.55 & 715.87 & 3.60 & 53.81 \\ 11.30 & 3.60 & 27.98 & 6.83 \\ 39.56 & 53.81 & 6.83 & 43.83 \end{bmatrix}.$$

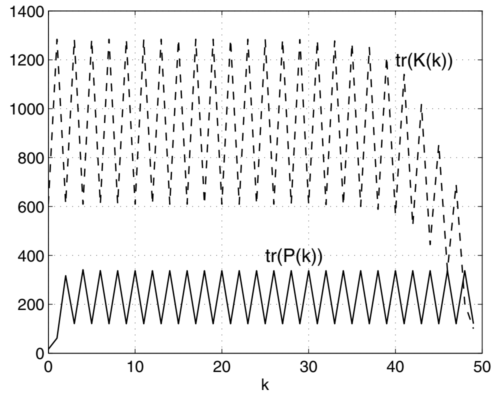

Fig. 2. Evolution of $tr(P(k))$ and $tr(K(k))$. The $P(k)$ satisfy the Riccati equation (14), while $K(k)$ satisfy the backwards Riccati equation (20).

Using the solutions for $P(\cdot)$ and $K(\cdot)$, the Kalman filter and the LQ optimal feedback gain were constructed from the formulae in Sec.s III-A, III-B. The state evolution of the closed-loop system is shown in Fig. 3. The evolution of the Kalman filter's one-step prediction error, $e(k)$, is shown in Fig. 4.

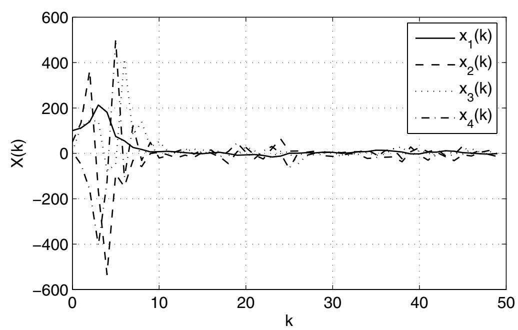

Fig. 3. State evolution of the closed-loop NCS under the LQG controller.

V. Conclusions

We presented an LQG design method for NCSs which are subject to medium access constraints. Our approach forgoes the use of ZOH elements in the loop; instead, the controller and plant "ignore" sensors and actuators which are not granted medium access. The benefits of doing so are twofold. First, the complexity of the closed-loop dynamics is lower than that of previously-proposed architectures. Second, the selection of communication sequences is decoupled from the choice of controller, thereby simplifying the identification of useful communication patterns and allowing us to bring existing tools to bear. Specifically, for a reversible plant, it is always possible to design periodic communication sequences that preserve detectability and stabilizability. Having doing so, Kalman filtering and LQ optimal control of a NCS can be formulated as a standard LQG problem for a periodic time-varying system. Our choice of communication sequences

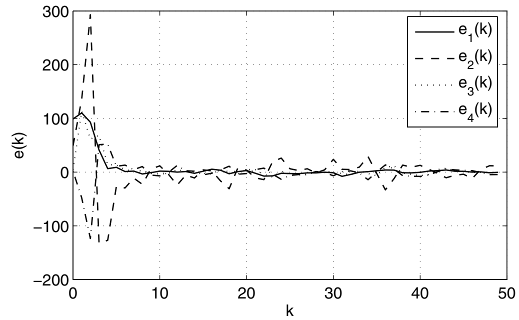

Fig. 4. Evolution of Kalman filter's one-step prediction error $e(k) = \boldsymbol{x}(k) - \hat{\boldsymbol{x}}(k^-)$.

ensures that the Riccati equations associated with the Kalman filter and the LQ optimal gain both converge to periodic solutions regardless of initial conditions. One can thus construct a (sub-optimal) LQG controller by implementing the two periodic solutions.

The proposed approach to LQG control of NCS is comprised of two components:

- A pair of periodic communication sequences which are designed off-line, independently of the controller. Periodic communication sequences can be easily implemented via MAC level network protocols such as polling, token passing, or Time Division Multiple Access (TDMA).
- A periodic time-varying linear controller whose parameters can be computed off-line.

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