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LQR-Based Least-Squares Output Feedback Control of Rotor Vibrations Using the Complex Mode and Balanced Realization Methods

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ABSTRACT

The complex mode and balanced realization methods are used separately to obtain reduced-order models for general linear asymmetric rotor systems. The methods are outlined and then applied to a typical rotor system which is represented by a 52 degree-of-freedom finite element model. The accuracy of the two methods is compared for this model and the complex mode method is found to be more accurate than the balanced realization method for the desired frequency bandwidth and for models of the same reduced order. However, with some limitations, it is also shown that the balanced realization method can be applied to the reduced-order complex mode model to obtain further order reduction without loss of model accuracy. An "Linear-Quadratic-Regulator-based least-squares output feedback control" procedure is developed for the vibration control of rotor systems. This output feedback procedure eliminates the requirement of an observer for the use of an LQ regulator, and provides the advantage that the rotor vibration can be effectively controlled by monitoring only one single location along the rotor shaft while maintaining an acceptable performance. The procedures presented are quite general and may be applied to a large class of vibration problems including rotor-dynamics.

NOMENCLATURE

A system matrix (2nx2n)
B control distribution matrix (2nxr)
c bearing damping coefficient
C output matrix (mx2n)
D dissipation/gyro matrix (nxn)
E elastic modulus
f unbalance force vector (nx1)
G optimal gain matrices (rx2n), system transfer function
I identity matrices (nxn)
J performance index
k bearing stiffness
K stiffness/circulation matrix (nxn)
m number of outputs
M mass matrix (nxn)
n system order
P control distribution matrix (nxr)
q physical coordinate vector (nx1)
r measurement vector (mx1), number of control forces
R solution to Ricatti equations (2nx2n)
S state weighting matrix (2nx2n)
T time, transformation matrix (4nx2n)
U control weighting matrix (rxr)
u control force vector (rx1), weighting constant
V,W Y,Z-translations
W controllability and observability grammiam
x state vector (2nx1)
X,Y,Z fixed reference
y right eigenvector
Y system right eigenvectors (modal matrix)
z left eigenvector
Z system left eigenvectors (modal matrix)

Greek
Ξ controllability and observability grammiam
B,Γ Y,Z-rotations
λ eigenvalue
σ singular value
ζ complex mode coordinate vector
α state vector (subset of ζ)
ϕ eigenvector
Ψ complex mode transformation matrix

Subscript
1,2,3 bearing indices
b balanced system
c controllability
j station indices
o observability
r balanced reduced-order system

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L left inverse
ζ corresponding terms associated with complex mode coordinate
α corresponding terms associated with α

Superscript
-1 inverse
i imaginary part
r real part
t transpose
· d/dt
^ number of coordinates retained in complex mode method
~ number of coordinates retained in balanced truncation method

INTRODUCTION

The control system design for an asymmetric rotordynamic system initially requires the development of a discrete mathematical model. For most realistic rotor systems these models are typically tens of degrees of freedom. Active control of lateral vibrations in rotordynamic systems has been studied by many researchers within the last several years. Several researchers have successfully used the normal mode approach to obtain a reduced-order system from an original high-order model. These normal modes do not include gyroscopic and/or circulation effects and as a result they do not decouple the asymmetric system equations. This work introduces a consecutive application approach of the complex mode procedure (Fan and Nelson, 1991) and the balanced truncation method (Moore, 1981) to deal with such systems in the process of LQR design.

A full-state feedback control system does not require an observer and this is a primary motivation for using an optimal output feedback technique. In addition, the use of an observer has the following disadvantages: a) Very high speed computation is needed for on-line state estimation, and this is especially difficult to implement for high speed rotor systems. b) The cost is much higher for fabrication of an observer than for implementation of an optimal output feedback procedure. c) Many more components are required for an observer than for an output feedback controller, and this results in less accuracy and less reliability. This work introduces an "LQR-based least-squares output feedback control" procedure for the vibration control of rotor systems to eliminate the requirement of an observer while using the LQR control.

Moore (1981) has shown that every system can be brought to a balanced form via a balancing transformation, and then a reduced-order model can be obtained by truncating the states associated with the smaller controllability (or observability) grammiams. This balancing procedure was tested on two example systems which possess real eigenvalues, and was proved to produce accurate reduced-order models. Lauh (1987) presented an algorithm for computing the balancing transformation matrix directly from any given state space realization. Safonov (1988) introduced another algorithm for implementing the balancing and truncation model reduction procedure presented by Moore (1981), but unlike Moore's original algorithm, these algorithms bypass the numerically delicate preliminary step of computing a balanced realization of the system. In other words, a not-necessarily- balanced state-space realization of the Moore's reduced model can be computed directly without balancing the system first.

Palazzolo (1989) experimentally demonstrated the success of an output feedback procedure with the limitation that the number of measured states has to be equal to the number of eigenvalues retained in the reduced-order model. He studied the active control of transient lateral rotor vibrations due to a sudden change in imbalance, such as caused by loss of a blade, by utilizing optimal control methods. A seven-mass rotor system was considered and a normal mode approach was used to obtain a reduced-order model. Comparisons of responses were made for various configurations of sensors and actuators. The spillover effect was examined by comparing results from collocated and noncollocated sensor configurations. Results were verified by experimental work. A filter was used in the experiments to prevent spillover effects for the noncollocated case. In the particular example used in his work, the number of measured states is 12 which is equal to the number of eigenvalues in the reduced-order rotor model. The least-squares output feedback procedure presented in this work requires only 4 measured ststes (instead of 12) from one single location while providing an acceptable performance.

Salm and Schweitzer (1984) discussed the modeling and control of a flexible rotor using magnetic bearings. They used a normal mode approach to obtain a reduced-order system, and, based on this system, designed an output feedback controller which guarantees the closed-loop stability for the original high-order system despite some spillover effects. Ulsoy (1984) investigated an optimal active control scheme by pole assignment and the effects of observation and control spill-over in a translating elastic system. His method was verified by using simulation results of two examples based on both physical coordinates and normal mode coordinates, respectively.

Cannon and Schmitz (1984) studied the precise control of a one-arm flexible manipulator without gyroscopic effects. They used a normal mode transformation to obtain a reduced-order model, and then designed an optimal controller based on this model. The system modal parameters in their work were measured and they were successful in experimentally verifying the validity of the proposed approach.

In this present work, the LQR-based least-squares output feedback controllers are designed based on the reduced-order model by using the complex mode and balanced realization methods. Controlled responses of the original high-order system with the reduced-order controllers are shown for comparison purposes. A fifty-two degree-of-freedom finite element based model for a rotordynamic system is used as an example to illustrate the proposed procedure.

SYSTEM EQUATIONS

The general linear matrix equation of motion of an nth order rotordynamic system, in terms of physical coordinates, is of the form

$$M_q \ddot{q} + D_q \dot{q} + K_q q = f + Pu \tag{1}$$

The M_q, D_q, and K_q arrays represent the mass, dissipation/gyroscopic, and stiffness/circulation properties of the rotor system. The M_q array is usually symmetric and positive definite while the D_q and K_q arrays are generally asymmetric. These asymmetric arrays can be written as the sum of a symmetric and skew-symmetric component. The symmetric part of the D_q array is referred to as the dissipation matrix while the skew-symmetric part is the gyroscopic matrix. The symmetric part of the K_q array is the system stiffness matrix while the skew-symmetric part is referred to as the circulation matrix. In addition, D_q and K_q are generally a function of the rotor spin speeds. The excitation force f is usually dominated by harmonic terms associated with the synchronous rotating unbalance of the rotating assembly. The P array identifies the physical location of the external forces included in the control vector u. The speed dependency of the unbalance forces, gyroscopic moments, and the bearing forces also requires spin speed dependency for the control action. In first order state form, eq. (1) is written as

$$\dot{x} = A_x x + B_x u + W_x f \tag{2}$$

where $x = [\, \dot{q}^t \;\; q^t \,]^t$ is the system state and $B_x = W_x P$ is the control matrix with

$$A_x = \begin{bmatrix} -M_q^{-1}D_q & -M_q^{-1}K_q \\ I & 0 \end{bmatrix}, \text{ and } W_x = \begin{bmatrix} M_q^{-1} \\ 0 \end{bmatrix} \tag{3}$$

The partial state measurements represented by the vector r are expressed by an output equation of the form

$$r = C_x x \tag{4}$$

MODAL REDUCTION

Complex Mode Approach

A mode transformation, based on the physical coordinate state form of the system equations,

$$x = Y\zeta = \sum_{j=1}^{2\hat{n} < 2n} y_j \zeta_j \tag{5}$$

is defined where the columns of Y contain the complex right eigenvectors y_j ($j=1,2,...,2\hat{n}$) of the system. These vectors are obtained from the eigenvalue problem associated with the homogeneous form of eq. (2), i.e.,

$$(A_x - \lambda_j I)\; y_j = 0 \tag{6}$$

where λ_j is the jth eigenvalue of the system associated with the right eigenvector y_j. Let Z represent a matrix of complex left eigenvectors z_j ($j=1,2,...,2\hat{n}$) which are obtained from the adjoint eigenvalue problem

$$(A_x^t - \lambda_j^* I)\, z_j = 0 \tag{7}$$

where λ_j^* is the conjugate of λ_j. It is convenient to binormalize the dual set of right and left eigenvectors so that $Z^t Y = I$, then this set also satisfies another biorthogonality condition

$$Z^t A_x Y = A_\zeta \tag{8}$$

where A_ζ is a diagonal array of the system eigenvalues, λ_j.

The eigenvalues from eq. (6) appear as complex conjugate pairs for underdamped modes and are purely real for overdamped modes. For stable linear systems, the real parts of all eigenvalues are non-positive. The selection of the modes to be retained in the transformation of eq. (5) requires careful consideration. The underdamped right eigenvectors appear in complex conjugate pairs and the superposition of the corresponding eigensolutions provides a real precessional mode solution. Thus, the transformation of eq. (5) would generally contain the complex conjugate pair right eigenvectors associated with the lower frequency precession modes. In addition, any purely real modes with relatively large time constants would also be retained. A verification of the accuracy of a particular choice, however, can only be obtained by comparison with other choices and experience with particular types of applications.

The substitution of the transformation of eq. (5) into the state equations, eq. (2), and premultiplication by Z^t, and use of the biorthogonality conditions gives the reduced-order ($2\hat{n} < 2n$) state model

$$\begin{aligned} \dot{\zeta} &= A_\zeta \zeta + B_\zeta u + W_\zeta f \\ r &= C\, Y\, \zeta \end{aligned} \tag{9}$$

where $A_\zeta = Z^t A_x Y$, $B_\zeta = Z^t B_x$, $W_\zeta = Z^t W_x$. The control u and excitation f are real while the other elements of this state equation are in terms of complex quantities. Since most readily available computational procedures are written in real form, it is convenient to separate eq. (9) into real and imaginary parts. Equation (9) may then be written as

$$\begin{bmatrix} \dot{\zeta}^r \\ \dot{\zeta}^i \end{bmatrix} = \begin{bmatrix} A_\zeta^r & -A_\zeta^i \\ A_\zeta^i & A_\zeta^r \end{bmatrix} \begin{bmatrix} \zeta^r \\ \zeta^i \end{bmatrix} + \begin{bmatrix} B_\zeta^r \\ B_\zeta^i \end{bmatrix} u + \begin{bmatrix} W_\zeta^r \\ W_\zeta^i \end{bmatrix} f$$

$$r = C\,[\,Y^r \; -Y^i\,] \begin{bmatrix} \zeta^r \\ \zeta^i \end{bmatrix}$$

or

$$\begin{aligned} \dot{\zeta}' &= A_{\zeta'} \zeta' + B_{\zeta'} u + W_{\zeta'} f \\ r &= C\, Y_{\zeta'}\, \zeta' \end{aligned} \tag{10}$$

where the order has been increased to $4\hat{n}$.

At this point, the analysis is restricted to only those systems which possess solely underdamped modes. In this case the modes appear only in complex conjugate pairs and it is possible to reduce the $4\hat{n}$ order of eq. (10) back to $2\hat{n}$. This is accomplished by noting the following relation

$$\begin{bmatrix} \zeta^r \\ \zeta^i \end{bmatrix} = \begin{bmatrix} T^r & 0 \\ 0 & T^i \end{bmatrix} \begin{bmatrix} \alpha^r \\ \alpha^i \end{bmatrix}$$

or

$$\zeta' = T\alpha \tag{11}$$

Both T^r and T^i arrays consist of $2\hat{n}$ rows and $\hat{n}$ columns of zero elements except for

$$T^r_{(2j-1,j)} = 1 = T^r_{(2j,j)} \quad \text{for } j = 1, 2, 3, ..., \hat{n},$$

and

$$T^i_{(2j-1,j)} = 1 = -\,T^i_{(2j,j)} \quad \text{for } j = 1, 2, 3, ..., \hat{n}.$$

The elements of ζ^r and ζ^i consist of the real and imaginary parts of the elements ζ for the odd j integers (i.e., $j=1,3,...,(2\hat{n}-1)$). Thus, the relation of eq. (11) is simply a method of retaining only one of the modal coordinates associated with each retained complex conjugate pair. In this application, the modal coordinates with a positive imaginary part are retained. The introduction of relation (11) into eq. (10) and premultiplication by the left inverse of T (see Nohle 1977, pp 15-17) gives

$$\begin{aligned} \dot{\alpha} &= A_\alpha \alpha + B_\alpha u + W_\alpha f \\ r &= C_\alpha \alpha \end{aligned} \tag{12}$$

where $A_\alpha = T_L^{-1} A_\zeta T$, $B_\alpha = T_L^{-1} B_\zeta$, $W_\alpha = T_L^{-1} W_\zeta$, and $C_\alpha = C\, Y_\zeta T$.

It is useful here to also note the identity $2T_L^{-1} = T^t$.

Balanced Realization Method

The model reduction procedure outlined below is summarized from the papers by Moore (1981), Laub (1987), and Safonov(1988). Consider the physical state form of the equations of motion, eq. (2), without including the unbalance forces,

$$\dot{x} = A_x x + B_x u \tag{13}$$

$$r = C_x x \tag{14}$$

Assume that the system is asymptotically stable and define the controllability grammiam as

$$W_c = \lim_{T \to \infty} \int_0^T (e^{A\tau} BB^t e^{A^t\tau}) d\tau \tag{15}$$

and the observability grammiam as

$$W_0 = \lim_{T \to \infty} \int_0^T (e^{A^t\tau} C^t C e^{A\tau}) d\tau \tag{16}$$

Also, W_c and W_0 satisfy, respectively, the following algebraic Lyapunov equations:

$$A_x W_c + W_c A_x^t + B_x B_x^t = 0 \tag{17}$$

$$A_x^t W_0 + W_0 A_x + C_x^t C_x = 0 \tag{18}$$

The state space representation is said to be balanced over the interval $[0,\infty]$ if $W_c = W_0 = \Xi$ for some diagonal matrix Ξ. Moore (1981) has shown that every system can be brought to a balanced form via a matrix transformation. In other words, eqs. (13) and (14) can be rewritten, by introducing $x = T_b x_b$, as

$$\dot{x}_b = A_b x_b + B_b u \tag{19}$$

$$r = C_b x_b \tag{20}$$

where x_b is the balanced state, $A_b = T_b^{-1} A_x T_b$, $B_b = T_b^{-1} B_x$, $C_b = C_x T_b$, and T_b is the balanced transformation matrix from the balanced states x_b to the physical states x, and is obtained by using the procedure proposed by Laub (1987).Furthermore, the following model reduction scheme is proposed. Assume that the diagonal entries of Ξ are ordered as $\sigma_1 \geq \sigma_2 \geq \sigma_3 \geq \dots \geq \sigma_{2n}$ and that it is possible to partition Ξ as $\Xi = \mathrm{diag}(\Xi_1, \Xi_2)$, where the dimension of Ξ_1 is $2(n - \tilde{n})$, Ξ_2 is $2\tilde{n}$, and $\sigma_{2(n-\tilde{n})}$ is much larger than $\sigma_{2(n-\tilde{n})+1}$. Let the balanced system matrices in eqs. (19) and (20) be partitioned as

$$A_b = \begin{bmatrix} A_{b11} & A_{b12} \\ A_{b21} & A_{b22} \end{bmatrix}, B_b = \begin{bmatrix} B_{b1} \\ B_{b2} \end{bmatrix}, \text{and } C_b = | \; C_{b1} \;\; C_{b2} \; | \tag{21}$$

The balanced reduced-order model, by neglecting all the states associated with $\sigma_{2(n-\tilde{n})+1} \dots \sigma_{2n}$, is defined as below

$$\dot{x}_{br} = A_{br} x_{br} + B_{br} u \tag{22}$$

$$r = C_{br} x_{br} \tag{23}$$

where $A_{br} = A_{b11}$, $B_{br} = B_{b1}$, and $C_{br} = C_{b1}$. Physically, this procedure neglects the $2\tilde{n}$ weakly controllable/observable states in the system. Moore (1981) proved that the reduced-order model is minimal and balanced with both controllability and observability Grammians equal to Ξ_1. Glover (1984) proved that the reduced-order model $G_{br}(s)$ satisfies the frequency domain infinity-norm (Noble 1977) error bound

$$\| G_x(s) - G_{br}(s) \|_\infty \leq 2 \sum_{j=2\tilde{n}+1}^{2n} \sigma_j \quad \text{for all } s\ (= j\omega) \tag{24}$$

where $G_x(s) = C_x(sI - A_x)^{-1} B_x$, $G_{br}(s) = C_{br}(sI - A_{br})^{-1} B_{br}$, σ_j are the diagonal elements of Ξ and are equal to

$$\sigma_j = [\lambda_j \{\Xi^t \Xi\}]^{1/2}$$

where $\lambda_j\{\Xi^t \Xi\}$ denotes the j-th eigenvalues of $(\Xi^t \Xi)$. Note that eq. (24) implies that the infinity norm of the error of the impulse responses between the reduced-order and original-order models is limited to the trace of Ξ, in other words, the reduced-order model has an impulse response which approximates the impulse response of the original-order model under the condition defined in eq. (24).

Unfortunately, the balanced model reduction procedure is, in general, not a good choice for high-order lightly-damped mechanical systems. The simulation result of this method on a 52 degree-of-freedom rotor system shows that the resulting reduced-order model is not accurate within the operating frequency range at all. However, the balancing method can be applied to a complex mode (or normal mode) reduced-order model to obtain a further model reduction without loss of model accuracy if the complex mode reduced-order model possesses weakly controllable/observable states. The following section outlines the formulation for this consecutive application of the complex mode and balanced realization methods. The "combined method" will be used to represent this type of procedure in the rest of this work. The consecutive application of normal mode and balanced realization methods can be done in a similar process, and will not be presented here.

Combined Method

Starting with complex mode reduced-order model as defined in eq. (12), without the unbalance force term included, and introducing

$$\alpha = T_{b\alpha} x_{b\alpha} \tag{25}$$

where $T_{b\alpha}$ is the balanced transformation from the balanced states to the complex modal states. Then, the balanced system based on the complex mode reduced-order model can be written as

$$\dot{x}_{b\alpha} = A_{b\alpha} x_{b\alpha} + B_{b\alpha} u \tag{26}$$

$$r = C_{b\alpha} x_{b\alpha} \tag{27}$$

where $A_{b\alpha} = T_{b\alpha}^{-1} A_\alpha T_{b\alpha}$, $B_{b\alpha} = T_{b\alpha}^{-1} B_\alpha$, and $C_{b\alpha} = C_\alpha T_{b\alpha}$. By neglecting the $2\tilde{n}$ weakly controllable/observable states, the further reduced model of order $(2\hat{n} - 2\tilde{n})$ can be obtained in a similar process as eqs. (22) and (23), and can be written as

$$\dot{x}_r = A_r x_r + B_r u \tag{28}$$

$$r = C_r x_r \tag{29}$$

where A_r, B_r, and C_r are the retained parts in $A_{b\alpha}$, $B_{b\alpha}$, and $C_{b\alpha}$, respectively, and are associated with the retained states in $x_{b\alpha}$.

The physical state, x, is related to the complex modal state through the transformation of eqs. (5) and (11). Equation (5) may be separated into the real part and imaginary components and written equivalently, using eq. (11), as

$$x \doteq \Psi_\alpha \alpha \tag{30}$$

where

$$\Psi_\alpha = | \; Y^r \;\; -Y^i \; | \, T. \tag{31}$$

The transformation between x_r defined in eq. (28) and the original physical state x can be obtained from eq. (25) and (30), and can be written as

$$x \doteq T_{xr}\, x_{b\alpha} \doteq T_{xr}\begin{bmatrix} x_r \\ 0 \end{bmatrix}_{2\hat{n}\times 1} = T_r\, x_r \tag{32}$$

where $T_{xr} = \Psi_\alpha T_{b\alpha}^{-1}$, and T_r consists of first $(2\hat{n} - 2\bar{n})$ columns of T_{xr}.

OPTIMAL OUTPUT FEEDBACK CONTROLLER DESIGN

Basic Concept

The following explanation of the procedure used in designing the control system is presented in terms of physical coordinates. The same procedure, however, can also be developed in terms of the complex mode coordinates and balanced coordinates. Equation (2), without the unbalance forcing term W_x f, is

$$\dot{x} = A_x x + B_x u \tag{33}$$

For a full state linear feedback control, the control input, associated with the physical state x, can be written as

$$u_x = -K_x x \tag{34}$$

where K_x is a state feedback gain matrix which is obtained based on minimization of the selected performance index to provide a desired performance. Practically, we can either measure all required states or use an observer to reconstruct the required states, however, the former involves a large number of sensors for large order systems and the latter requires high speed computation for on-line state estimation. The following provides a procedure to avoid the above practical problems. The objective here is to obtain a control input, associated with the system output r,

$$u_r = -K_r r \tag{35}$$

such that $u_r = u_x$. An error vector $e = u_x - u_r$ is defined and the introduction of relations from eqs. (4), (34) and (35) with the assumption that the C_x is properly selected such that the output of the least-squares controller approaches that of the full state feedback controller, then provides

$$e \doteq (K_x - K_r C_x)\, x \tag{36}$$

It is seen here that for the case when

$$K_r C_x = K_x \tag{37}$$

the error vector e is null. Unfortunately eq. (37) does not hold unless C_x is an identity matrix (or equivalently r = x), and K_r and K_x are of the same size. Practically, the size of K_r is much smaller than that of K_x, because the number of sensor locations is usually severely limited. A least-squares solution based on minimizing the quadratic error $(K_r C_x - K_x)(K_r C_x - K_x)^t$ can be obtained (Noble, 1988, p.69), however, as shown below

$$K_r = K_x C_x^t (C_x C_x^t)^{-1}. \tag{38}$$

It is possible to arrange the order of the elements of the state vector x such that the observation matrix C_x has the following form

$$C_x = [\, I_m \;\; 0 \,] \tag{39}$$

where m is the number of independent measured states. Substituting eqs. (38)-(39) into eq. (36), and utilizing the fact that $C_x C_x^t = m$, we have

$$e^t e \doteq \frac{1}{m^2} x^t \begin{bmatrix} 0_m & 0 \\ 0 & -I_{(2n-m)} \end{bmatrix} K_x^t K_x \begin{bmatrix} 0_m & 0 \\ 0 & -I_{(2n-m)} \end{bmatrix} x \tag{40}$$

Equation (40) provides two ways to reduce the error e. The first is to increase the number of measurements m. As m increases toward 2n the error e decreases, and the case of m=2n corresponds to all states being measured with e=0. The second method is to design K_x such that the elements associated with the measured states are much larger in magnitude than those associated with the unmeasured states while the desired performance is maintained. The latter can be achieved by minimizing the following performance index which includes the system measurements, r, instead of system states, x, for certain areas such as rotor systems

$$J = \lim_{T\to\infty} \frac{1}{2T}\int_0^T (r^t S_r r + u_w u^t U u)\, dt \tag{41}$$

where S_r is a positive semidefinite state weighting matrix, U is a positive definite control weighting matrix, u_w is the weighting constant and system output r is defined in eq. (4).

Least-Squares Output Feedback Design

1. Complex Mode Method: The physical state is related to the modal state through the transformation of eq. (5). Equation (5) may be separated into the real part and imaginary components and written equivalently, using eq. (11), as

$$x \doteq \Psi_\alpha\, \alpha \tag{42}$$

where

$$\Psi_\alpha = [\, Y^r \;\; -Y^i \,]\, T \tag{43}$$

with this relation, the performance index from eq. (41) may be rewritten as

$$J = \lim_{T\to\infty} \frac{1}{2T}\int_0^T (\alpha^t S_{\alpha r}\, \alpha + u_w u^t U u)\, dt \tag{44}$$

where $S_{\alpha r} = \Psi_\alpha^t C^t S_r C \Psi_\alpha$ is a weighting matrix associated with the retained complex coordinates. The steady state Ricatti equation associated with eq. (44) is

$$R_\alpha A_\alpha + A_\alpha^t R_\alpha - \frac{1}{u_w} R_\alpha B_\alpha U^{-1} B_\alpha^t R_\alpha + S_{\alpha r} = 0 \tag{45}$$

and the optimal control law can be written as

$$u = -G_\alpha\, \alpha \tag{46}$$

where

$$G_\alpha = U^{-1} B_\alpha^t R_\alpha$$

Introducing eq. (42) into eq. (46), we have

$$u = -G_\alpha\, \alpha \tag{47}$$

Using eq. (4) and (42), we can rewrite eq. (35) as

$$u_r = -K_r C_x \Psi_\alpha\, \alpha \tag{48}$$

or

$$u_r = -K_r C_\alpha\, \alpha \tag{49}$$

where $C_\alpha = C_x \Psi_\alpha$. Similar to the procedure used to obtain eq. (38), the objective here is to determine K_r such that u_r in eq. (49) is a good approximation of u in eq. (47) for any selected observation matrix C. The least square solution associated with eq. (47) and (49) is

$$K_r = G_\alpha C_\alpha^t (C_\alpha C_\alpha^t)^{-1} \tag{50}$$

Note that the number of measured states associated with matrix C_x in eq. (32) should generally be equal to that of matrix C in eq. (44) for best performance, however, one can use less measured states than the output r to be minimized in the performance index, eq. (41), but at a price of degrading the controller performance.

2. Combined Method: The physical state x is related to the balanced reduced model state x_r through the transformation of eq. (32). With this relation, the performance index from eq. (41) may be rewritten as

$$J = \lim_{T\to\infty} \frac{1}{2T}\int_0^T (x_r^t S_{xr} x_r + \frac{1}{u_w} u^t U u)dt \tag{51}$$

where $S_{xr} = T_r^t C^t S_r C T_r$ is a weighting matrix associated with the retained complex coordinates. The steady state Ricatti equation associated with eq. (51) is

$$R_r A_r + A_r^t R_r - \frac{1}{u_w} R_r B_r U^{-1} B_r^t R_r + S_{xr} = 0 \tag{52}$$

and the optimal control law can be written as

$$u = -G_{xr} x_r \tag{53}$$

where $G_{xr} = U^{-1} B_r^t R_r$. Using eq. (4) and (32), we can rewrite eq. (35) as

$$u_r = -K_{xr} r \doteq -K_{xr} C_x T_r x_r \tag{54}$$

or

$$u_r \doteq -K_{xr} C_{xr} x_r \tag{55}$$

where $C_{xr} = C_x T_r$. Similar to the procedure used to obtain eq. (38), the objective here is to determine K_{xr} such that u_r in eq. (55) is a good appruximation of u in eq. (53) for any selected observation matrix C_x. The least square solution associated with eq. (53) and (55) is

$$K_{xr} = G_{xr} C_{xr}^t (C_{xr} C_{xr}^t)^{-1} \tag{56}$$

Similar to the complex mode procedure, the number of measured states of matrix C_x in eq. (54) should generally be equal to that of matrix C in eq. (51) for best performance, however, one can use less measured states than the output r to be minimized in the performance index, eq. (44) or (51), but at a price of degrading the controller performance. In the next section, an example is given for demonstration of the proposed procedure.

EXAMPLE

Simulations are performed on a large order rotordynamic system so as to test the optimal active control procedures presented above. The test rotor configuration is shown in Fig. 1 and it consists of 4 rigid disks, 3 isotropic bearings, and 12 rotating shafts elements. The discrete model of this system is a finite element based model (Nelson and McVaugh, 1976), and it utilizes 13 stations with 2 translation (V_j,W_j) and 2 rotation (B_j,Γ_j) coordinates at station j (j=1,2,...,13) for a total of 52 degree-of freedom. The geometric and material properties are listed in Tables I and II. The mass unbalance for this system is assumed to be localized at each disk and are chosen in phase and of equal cg eccentricity. A plot of the natural frequencies of whirl versus the rotor spin speed is presented in Fig. 2. The first three forward critical speeds are 649, 872, 1320 rad/sec and the first three backward critical speeds are 479, 718, and 949 rad/sec.

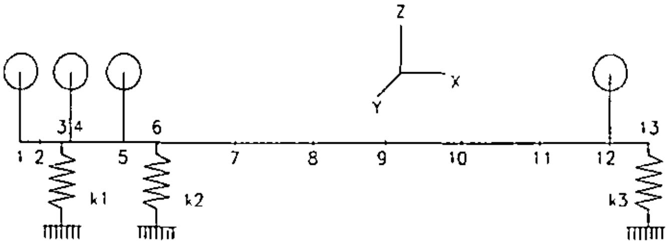

Fig. 1 Rotor Schematic of Three Bearing System

The control inputs are taken as translational forces at station 7 in the perpendicular Y,Z-directions. The modal displacement of station 7 is relatively large for each of the first several modes. In the simulations presented here, a spin speed of 649 rad/sec is chosen such that it coincides with the first forward critical speed. The precessional modes are circular here since the bearings are isotropic and the first three backward and forward modes are illustrated in Fig. 3 corresponding to the spin speed of 649 rad/sec. The right eigenvectors associated with these precessional modes are used in designing the controller for the complex mode approach. The normal modes are independent of the rotor spin speed and are obtained from the undamped symmetric eigenvalue problem related to eq. (1). The first three planar modes are illustrated in Fig. 3 and provide a graphical comparison with the gyroscopic influenced precessional modes. It should be noted that the superposition of two planar normal modes of the same frequency and amplitude can form a forward or backward whirling mode of the same frequency depending on the phasing. The three planar normal modes in Fig. 3 are retained for both the horizontal and vertical planes for the controller design using normal modes. This is equivalent to retaining three forward and three backward precessional modes where the forward-backward pair are of the same frequency.

The frequency response functions of the Y-direction displacement at station 12 due to the Y-direction force input at station 7, associated with the reduced-order models, are plotted in Fig. 4 for the complex mode (12 states retained), the balanced realization (12 and 16 states retained), and the combined approaches (10 states retained). The corresponding frequency response function for the original 52 degree-of-freedom system is also included in the graph. Figure 4a shows that the mode associated with natural frequency at 852 rad/s has a zero very close to the pole, i.e., the zero tends to cancel the pole. Under this condition, the balanced truncation method seems to be a reasonable choice to eliminate that pairs of pole and zero to obtain a further order reduction from a complex mode reduced-order model. Figure 4d shows the combined reduced-order frequency response function versus the complex mode reduced-order transfer function. The reduced-order frequency response functions are both quite accurate at the low frequency range for the complex mode and the combined methods. As the frequency increases above the highest retained frequency mode, both reduced-order functions deteriorate as expected. The reduced-order transfer functions associated with the balanced method along are not good, in general, even at the low frequency range for both cases of 12 and 16 retained-states. This is because the balanced truncation method neglects the weaker controllable/observable states first, and tends to give a reduced-order model with an approximating impulse response to the impulse response of the original high-order model over entire frequency.

The controlled and uncontrolled time responses of the original-order system from stations 1, 7 and 12, and the control input forces (Y-direction) from station 7, for measurements from different number of stations are shown in Figures 5 and 6 for the complex mode and the combined methods, respectively. In this work, four measurements are made per selected station, and they are two translational displacements and two translational velocities in Y-Z directions. The weighting matrices S_X and U are chosen to be an identity matrix and u_w is 10^{-6} in this example. Figures 5 and 6 show that the controller performance associated with 8 measured states (stations 7 and 12) are almost the same as those from 12 measured states (stations 1, 7 and 12), while the controller performance from 4 measurements (station 7) shows only slight degradation for both the complex mode and the combined methods. Also, the performance of the combined reduced-order controller is almost the same as that of the complex mode reduced-order controller. Figure 7 shows the eigenvalues of the closed-loop systems for different numbers of measurements. Apparently, the "LQR-based least-squares output feedback control" can be considered as an alternative way of implementing a full state feedback control for a general high-order system without using an observer, in other words, figures 7a and 7b demonstrate that a desirable closed-loop response may be obtained by using this least-squares output feedback procedure without having to use a full state feedback, which is important for high-order systems.

Table I Mass Properties of Rigid Disks

Station No.	Mass (Kg)	Polar Inertia ($Kg\text{-}m^2$)	Transverse Inertia ($Kg\text{-}m^2$)	cg Eccentricity (μm)
1	11.38	19.53	9.82	25.4
4	7.88	16.70	8.35	25.4
5	7.70	17.61	8.80	25.4
12	21.70	44.48	22.24	25.4

Table II Shaft and Bearing Information

Station No.	Axial Distance to Station (cm)	Other Properties
1	0.0	shaft : Inner Radius = 1.68 cm
2	4.29	Outer Radius = 2.95 cm
3	8.89	Elastic modulus = 2.069×10^{11} N/m^2
4	10.49	Density = 8193.0 Kg/m^3
5	20.17	bearing :
6	27.69	$k_1 = k_2 = k_3 = 175 \times 10^5$ N/m
7	44.20	$c_1 = c_2 = c_3 = 900$ N-Sec/m
8	59.44	
9	74.68	
10	89.92	
11	105.16	
12	120.14	
13	127.94	

CONCLUSIONS

This work introduces coordinate reduction procedure by consecutively applying the complex mode method and the balanced truncation method for high-order asymmetric dynamical systems, and for their optimal control system design. Whether the balancing method should be applied to a complex mode reduced-order model will depend on the number of weak controllable/observable states in the reduced model and on the desired reduced-model accuracy. The coordinate transformation matrices are derived so that an LQ regulator can be designed.

The balanced truncation method alone is not a good choice for a reduced-order model in this particular application of a high-order under damped mechanical systems due to the fact that it produces an approximating model over the entire frequency range. However, it can be applied to a complex mode reduced-order model to obtain a further order reduction.

An LQR-based least-squares output feedback control procedure is presented and is shown to be effective based on the simulation results, however, one of the disadvantage is that the required accuracy cannot be taken into consideration before performing the design procedure, and improper selection of the measurements may destabilize the closed-loop system.

Full-state measurement for high-order systems is usually not possible. In order to realistically utilize full state feedback control, either an observer or an "optimal output feedback" procedure can be used. The advantage of using an observer is that the control system only requires the measurement of a small number of system responses so long as the open-loop system is observable. This is desirable for physical systems with limited sensor accessibility. A disadvantage, however, is that the closed-loop system might be quite sensitive to certain system parameter variations (Ridgy and Banda, 1986), and the process of obtaining the estimation of all the states requires particularly fast computational equipment. This is especially difficult for high-speed rotor systems. An LQR-based least-squares output feedback procedure presented here makes a full state feedback of a high-order system possible without using an observer, and eliminates the limitation of sensor's number (Palazzolo, 1989). Either the use of an observer or an "optimal output feedback" method may introduce additional stability problems such as the stability robustness for the closed-loop control system which need to be considered in the optimal control design of a particular problem.

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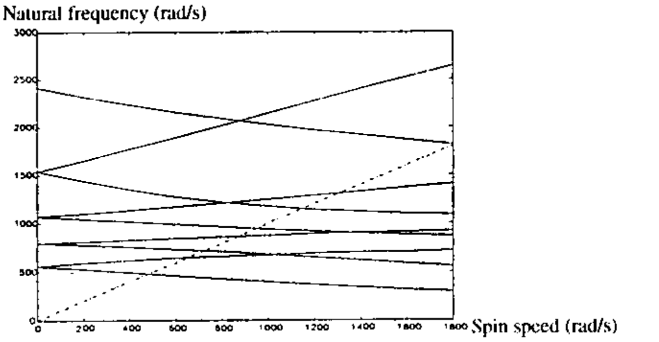
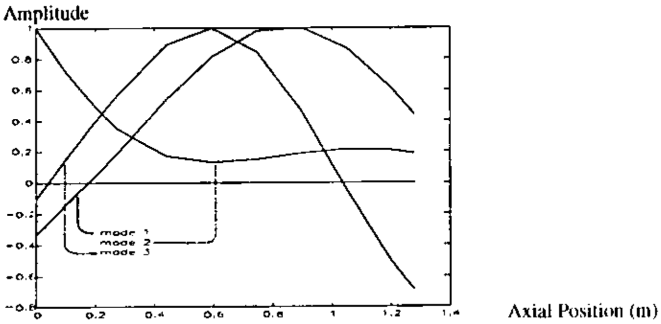
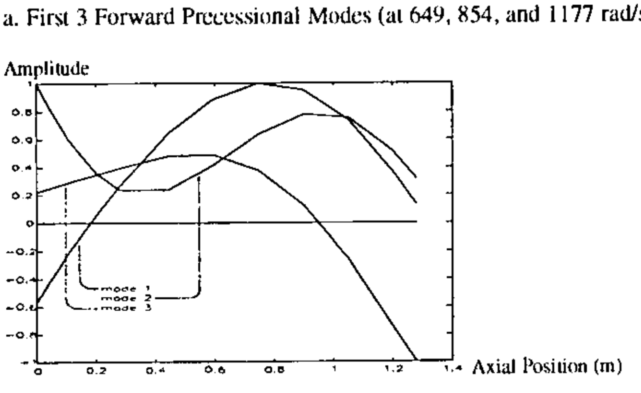


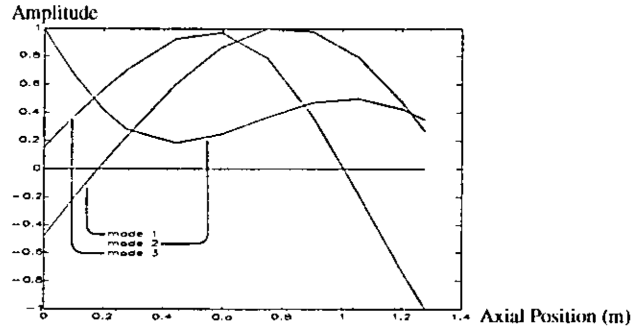
Fig. 2 Whirl Speed Map



a. First 3 Forward Precessional Modes (at 649, 854, and 1177 rad/s)

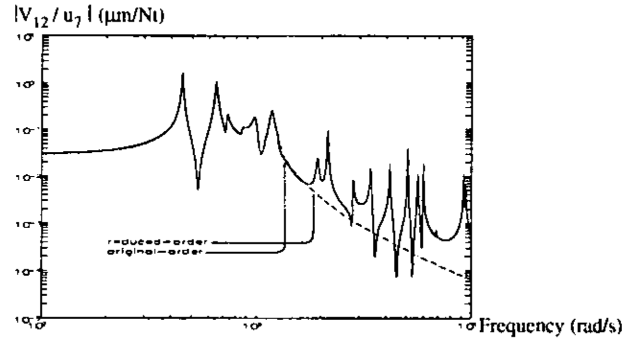


b. First 3 Backward Precessional Modes (at 450, 728, and 985 rad/s)

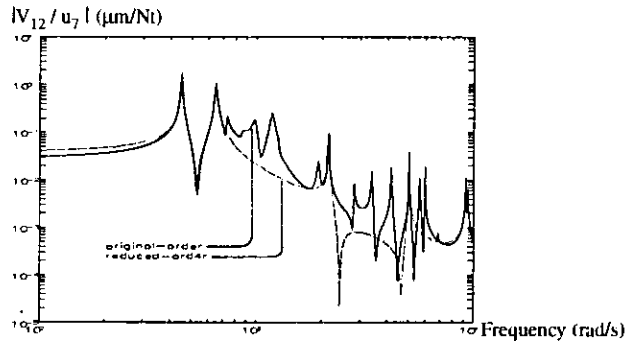


c. First 3 Normal Modes (at 561, 796, and 1070 rad/s)

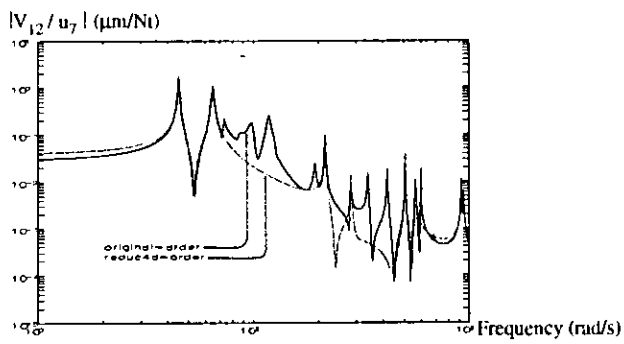
Fig. 3 System Mode Shapes



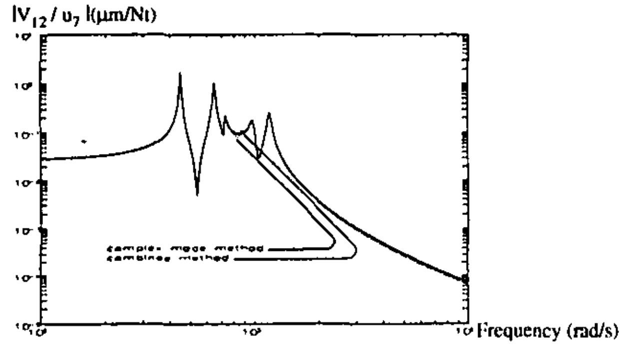
a. Complex Mode Reduced Model (with 12 eigenvalues retained)



b. Balanced Reduced Model (with 12 eigenvalues retained)

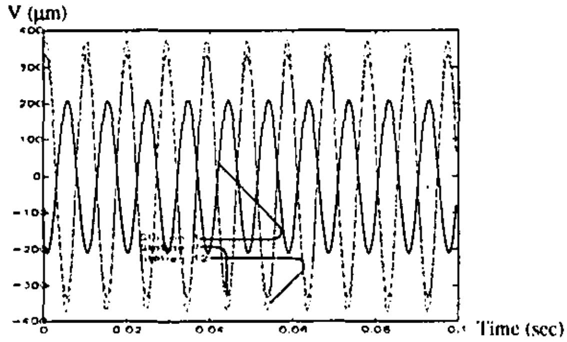


c. Balanced Reduced Model (with 16 eigenvalues retained)

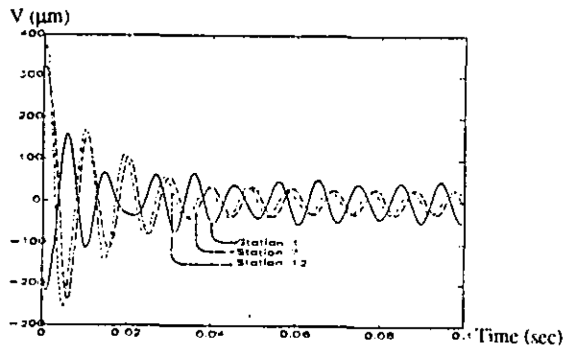


d. Combined Reduced Model (with 10 eigenvalues retained)

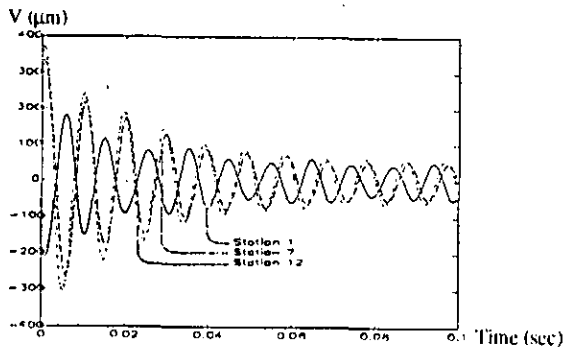
Fig. 4 Frequency Response Functions of Reduced Models



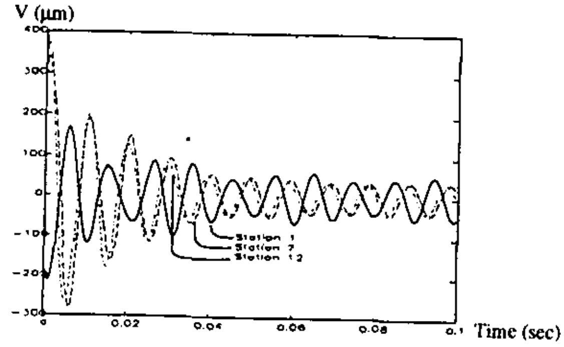
a. Uncontrolled Responses at Stations 1, 7, and 12



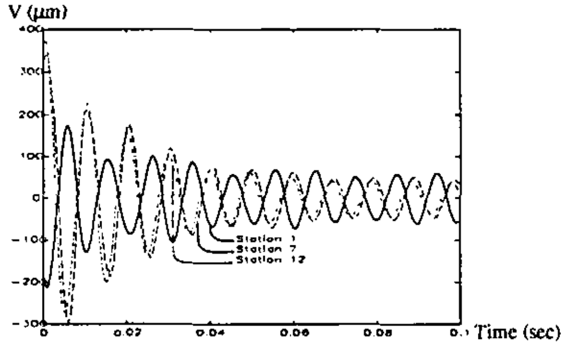
b. Controlled Responses of Full Measurements



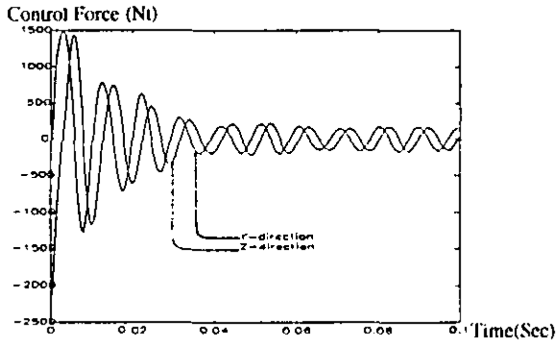
c. Controlled Responses of 4 Measurements (Station 7)



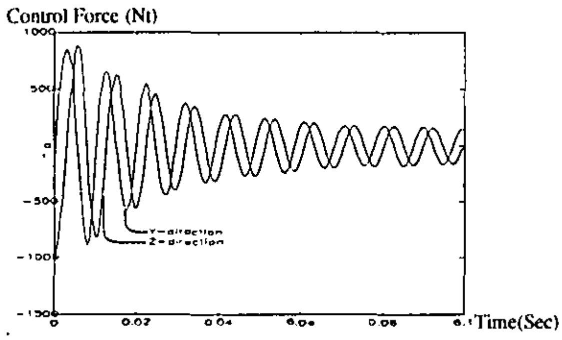
d. Controlled Responses of 8 Measurements (Stations 7, and 12)



e. Controlled Responses of 12 Measurements (Stations 1, 7, and 12)

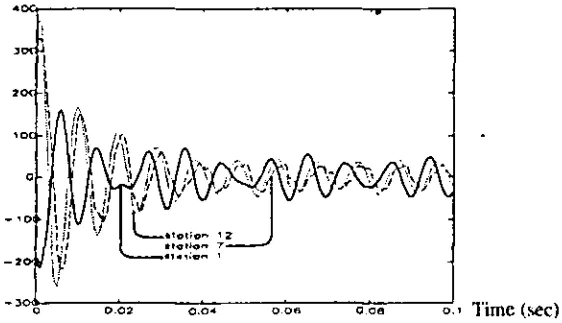


f. Control Forces associated with Full Measurements

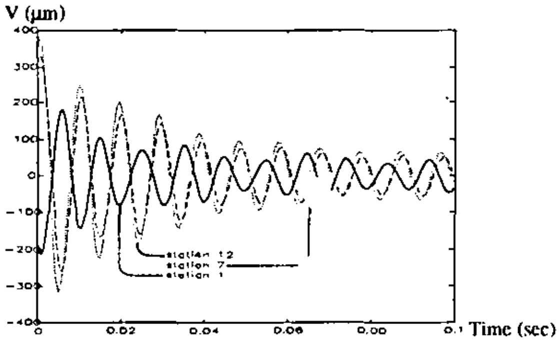


g. Control Forces associated with 4 Measurements (Station 7)

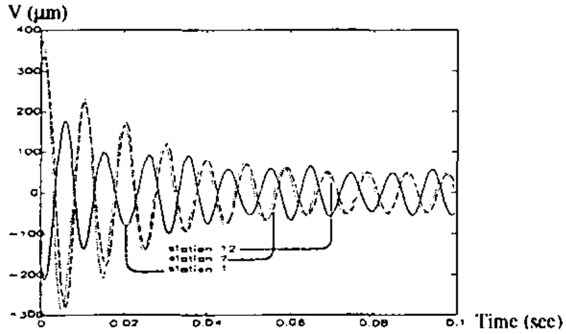
Fig. 5 Responses of the Original System
with Complex Mode Reduced-Order Controller



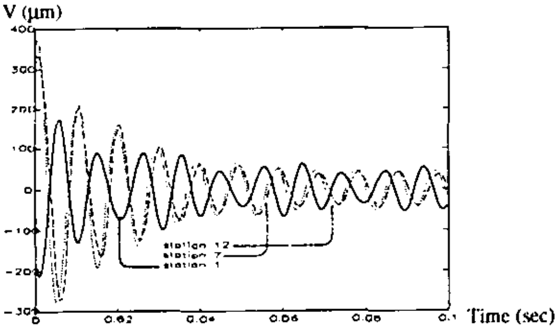
a. Controlled Responses of Full Measurements



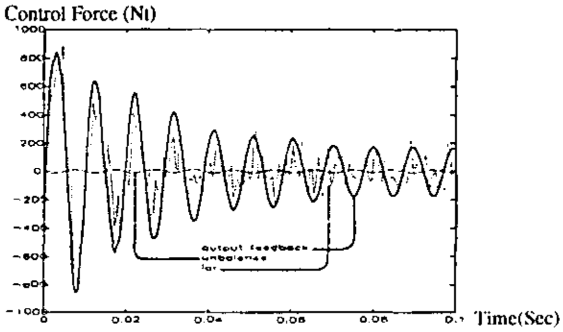
b. Controlled Responses of 4 Measurements (Station 7)



c. Controlled Responses of 8 Measurements (Stations 7 and 12)

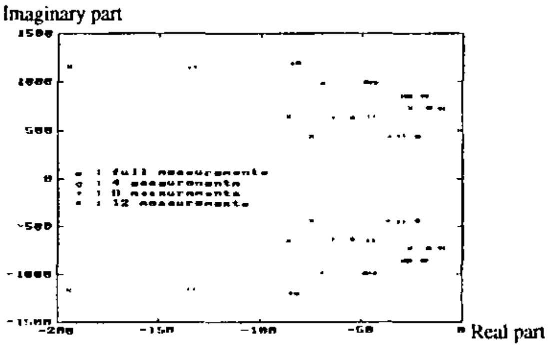


d. Controlled Responses of 12 Measurements (Stations 1, 7, and 12)

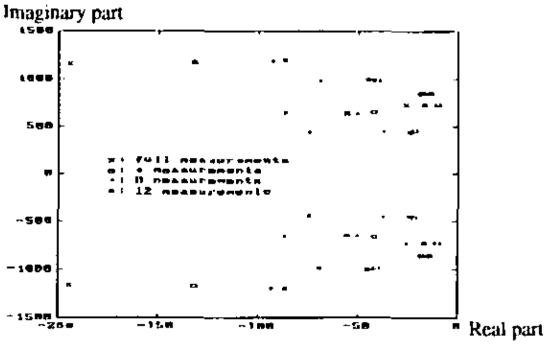


c. Y-Control Forces at Station 7
(LQR: Full Measurements, Output Feedback: 4 Measurements)

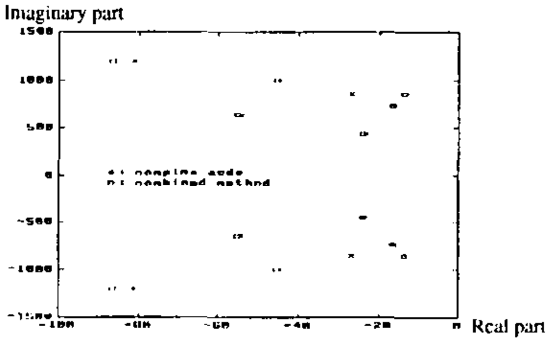
Fig. 6 Responses of the Original System with Combined Reduced-Order Controller



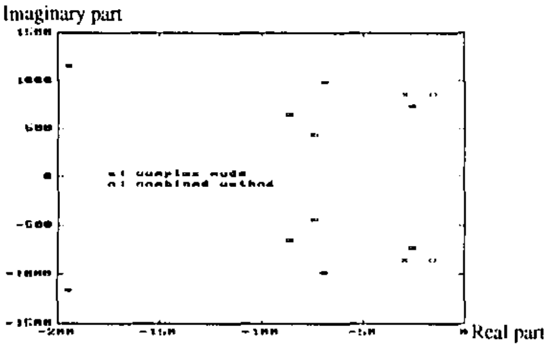
a. Complex Mode Method



b. Combined Method



c. Complex Mode Versus Combined Method: 12 Measurements



d. Complex Mode Versus Combined Method: Full Measurements

Fig. 7 Eigenvalues of Closed-loop System associated with Various Number of Measurements