

Lukasiewicz's 4-valued Logic and Normal Modal Logics

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Abstract. In this paper, we investigate the Lukasiewicz's 4-valued modal logic based on the Aristotele's modal syllogistic. We present a new interpretation of the set of algebraic truth values by introducing the truth and knowledge orderings similar to those in Belnap's 4-valued bilattice but by replacing the original Belnap's negation with the lattice pseudo-complement instead. Based on it, we develop a formal modal Boolean algebra for Lukasiewicz's system. We show that this modal algebra corresponds to the standard normal modal logic and develop an autoreferential Kripke-style semantics for it, where Lukasiewicz/Aristotele's "necessity" operator is an existential additive instead of an universal (standard) multiplicative modal operator. Moreover, we show that the standard (modern) necessity modal operator based on S4 accessibility relation (the truth partial order in Lukasiewicz/Belnap's bilattice) is an identity and consequently is not useful in Lukasiewicz's system.

1 Introduction

Since from the beginning, the modal and many-valued logic notions were part of Jan Lukasiewicz's (1878-1956) research motivations. He came across these notions in 1910 through his study of the principle of contradiction in Aristotele's work. Lukasiewicz demonstrated that this principle is not so self-evident as it was believed to be and proved that in addition to the true and false propositions, there are *possible* propositions to which objective possibility corresponds as a third logic value. In order to clarify the origin and significance of this three-valued logical system, he published his paper [1] that was devoted to the study of modal propositions. He examined Aristotele's ideas of necessity and possibility from the standpoint of the known systems of modal logic and then developed his new system where he was able to explain the difficulties and correct the errors of the Aristotelian modal syllogistic.

In Lukasiewicz's conception, the real definition of a logic must be semantic and truth-functional (the logic connectives are to be truth functions operating on these logical values): "logic is the science of objects of a specific kind, namely a science of *logical values*". Thus, in order to support the necessity and possibility, he introduced the modal unary logic connectives L and M respectively.

After a long time from the first considerations mentioned above, Lukasiewicz completed his approach by introducing the 4-valued logic which satisfies all his intuitions concerning the modalities and then presented his L -system [2] with four values $\mathcal{B}_4 =$

$\{0, 1, 2, 3\}$, where the *truth* is represented by 1 and *falsehood* by 0, with the usual non-modal connectives $\Rightarrow, \neg, \wedge$ and \vee for logic implication, negation, conjunction and disjunction respectively and with specific unary modal connectives L and M , as shown in the following truth-table:

\Rightarrow	1	2	3	0		\neg	M	L
1	1	2	3	0		0	1	2
2	1	1	3	3		3	1	2
3	1	2	1	2		2	3	0
0	1	1	1	1		1	3	0

and with conjunction and disjunction as those used in the classical logic (holds the De Morgan laws) defined by, $\phi \wedge \psi = \neg(\phi \Rightarrow \neg\psi)$ and $\phi \vee \psi = \neg\phi \Rightarrow \psi$.

It is easy to verify that the negation $\neg : \mathcal{B}_4 \rightarrow \mathcal{B}_4$ is an involution (that is, $\neg\neg = id$ is an identity mapping) such that two modal operators $M, L : \mathcal{B}_4 \rightarrow \mathcal{B}_4$ satisfy the classic interdependency $M = \neg L \neg$ or $L = \neg M \neg$ holds as well.

In the literature also, the denotation 11, 10, 01, 00 (pairs of zeros and ones) is used for four values, instead of using the original Lukasiewicz's denotation 1, 2, 3, 0 respectively. However, both the semantics and many theorems of this modal logic have a difficult interpretation in terms of any intuitive notions of possibility and necessity where, actually, the modal operator M is interpreted, in a Kripke style modal semantics (which was established one decade after this Lukasiewicz's work), as an *existential* modal operator, and L as an *universal* modal operator. In fact, for this interpretation of modal operators, we obtain a strange modal logic where the $\frac{\phi}{L\phi}$ does not hold *Rule of Necessitation*, and this fact explains why such a logic cannot be characterized by Kripke frames.

Because of such considerations, in view of the enormous development of relational semantics (Kripke style) for modal logics since Lukasiewicz's book appeared, the current opinion is that Lukasiewicz's system is rather a *dead end* [3] whose value is just its historical rarity. For instance, this system is nowhere mentioned in the historical introduction of Lemmon [4] that was originally written in 1966 nor in the historical part of Bull and Segerberg [5] that was published in 1984 nor in the historical notes of Chagrov and Zacharyashev's [6] in 1997. It didn't appear in the book [7] by Hughes and Cresswell in 1996 while it was mentioned, as an extremely non-standard system. in their first book [8] in 1968. This opinion (i.e., that it is a strange, non-standard, yet legitimate system) was also expressed in the last study dedicated to this system [3], where the reader can find an exhaustive overview and discussion about these topics.

Based on the previous considerations, the L -system erroneously did not influence the mainstream development of modern logic. But the authors of this paper are not convinced by this long time opinion and consider that it is strange if Lukasiewicz, after a lot of years of his work, could produce these highly irrelevant results.

Consequently, our contribution to the Lukasiewicz's L -system is to demonstrate that not only from the point of view of the abstract algebraic logic this modal 4-valued logic enjoys the best possible behavior but *also from the relational* Kripke style point of view there exists a coherent interpretation for his system which shows that it is a perfect *normal* modal logic (where the universal modal algebraic operator is multiplicative, while the existential modal algebraic operator is additive, as specified in Proposition 3). In this

coherent interpretation, given a new meaning and truth ordering to the four logic values in \mathcal{B}_4 , the interpretation of modal operators are opposite w.r.t. the current relational interpretation, that is $M = \Box$ is *universal* and $L = \Diamond$ is *existential* modal operator. Thus, the Lukasiewicz's "L-necessity" corresponds to the existential modal operator based on Kripke style semantics and his "L-possibility" corresponds to the universal modal operator.

This result confirms the following remark by Hughes and Cresswell ([8], p. 310): "So M and L in the L -modal system do seem to express senses of 'possibility' and 'necessity' considerably different from the usual ones; and perhaps, if by a 'modal logic' we mean a logic of possibility and necessity, this system takes us to the limit of what we should regard as a modal logic at all". But also Lukasiewicz closed his 1953 paper [2] by saying: "I am fully aware that other systems of modal logic are possible based on different concepts of necessity and possibility.... I hope that the L -modal system expounded above will be a useful instrument, and deserves a further investigation and development" (Quoted from the reprinting of [2], in [9], pp.378-379).

Consequently, by changing the old Kripke style interpretation for L -system, we reinterpret the Lukasiewicz's "L-necessity" by modern relational (Kripke style) existential modal operator and "L-possibility" by universal modal operator that is not Lewis-Kripke necessity operator, and demonstrate that this coherent (mathematically founded) interpretation leads to the relational semantics of *normal modal* logic for original algebraic L -system. Other important point is that the negation Lukasiewicz's operator can be obtained from the consistent extension of a set of Lukasiewicz's algebraic truth values into negation-less fragment of Belnap's *bilattice*, as a pseudo-complement, i.e., for any $x \in \mathcal{B}_4$, $\neg x = \vee\{y \in \mathcal{B}_4 \mid x \wedge y = 0\}$. This implication can be defined in a standard classical way by $x \Rightarrow y = \neg x \vee y$.

The main results of our research are presented in Section 4: we provided an *autoreferential* relational Kripke semantics based on the semantics for many-valued modal logics presented in [10] (with the set of possible worlds $\widehat{\mathcal{B}}_4 = \{0, 2, 3\}$) for the obtained Galois algebra of Lukasiewicz's 4-valued logic. We demonstrated that, even if non-modal fragment of Lukasiewicz's logic is a Boolean algebra (i.e., complemented distributive lattice where $\neg x \vee x = 1$ for each $x \in \mathcal{B}_4$), for a Kripke model \mathcal{M}_K and a world $w \in \widehat{\mathcal{B}}_4$ it does not hold that $\mathcal{M}_K \models_w \neg\phi$ iff $\mathcal{M}_K \not\models_w \phi$. That is, the standard Kripke semantics for negation *does not hold* for the set $\widehat{\mathcal{B}}_4$ of possible worlds. Such a standard result can be obtained instead if we reduce the set of possible worlds to be exactly the set of join-irreducible elements $\{2, 3\}$ of Lukasiewicz's algebraic values.

The plan of this paper is the following:

In Section 2, we present a short introduction to complete lattices and modal truth-functional algebraic logics based on Galois connections for modal operators. The original contribution is presented in the next two sections: in Section 3, we present a new interpretation of the truth values of this Lukasiewicz's 4-valued modal system and the direct relationships with the distributive Belnap's bilattice and Boolean algebras. Finally, in Section 4, we develop the autoreferential Kripke style semantics proposed by Majkić in [10] for this Lukasiewicz's system and show that this system corresponds to the standard normal modal logic, where Lukasiewicz/Aristotele's "necessity" operator corresponds to the existential additive modal operator instead to the standard

universal multiplicative modal operator. After that, we present an autoreferential representation for the Lukasiewicz's algebra of this modal system and show that the standard necessity modal operator, based on the S4 accessibility relation \preceq (the truth ordering in Lukasiewicz/Belnap's bilattice), is an identity and as a result this standard modal operator is not useful for Lukasiewicz's system. Even if Lukasiewicz's negation is a Boolean negation, it is shown that its relational semantics is different from the standard relational Kripke semantics for classical 2-valued negation.

2 Preliminaries

Posets and lattices (note that a poset means for all its elements x and y , the set $\{x, y\}$ has both a join (lub - least upper bound) and a meet (glb - greatest lower bound)) with a partial order \leq play an important role in what follows. A *bounded* lattice has a greatest (top) and least (bottom) element, denoted by convention by 1 and 0 , respectively. Finite meets in a poset will be written as $1, \wedge$, and finite joins as $0, \vee$. A lattice (poset) X is *complete* if each (also infinite) subset $S \subseteq X$ (or, $S \in \mathcal{P}(X)$ where \mathcal{P} is the symbol for powerset and $\emptyset \in \mathcal{P}(X)$ denotes the empty set) has the least upper bound (supremum) denoted by $\bigvee S \in X$ (when S has only two elements and the supremum corresponds to the join operator \vee). Each finite bounded lattice is a complete lattice. Each subset S has the greatest lower bound (infimum) denoted by $\bigwedge S \in X$, given as $\bigvee\{x \in X \mid \forall y \in S. x \leq y\}$. The complete lattice is bounded and has the bottom element $0 = \bigwedge X \in X$ and the top element $1 = \bigvee X \in X$.

A function $l : X \rightarrow Y$ between posets X and Y is *monotone* if $x \leq x'$ implies $l(x) \leq l(x')$ for all $x, x' \in X$.

Such a function $l : X \rightarrow Y$ is said to have a right (or upper) adjoint if there is a function $r : Y \rightarrow X$ in the reverse direction such that $l(x) \leq y$ iff $x \leq r(y)$ for all $x \in X, y \in Y$. Such a situation forms a Galois connection and will often be denoted by $l \dashv r$. In that case, l is called a left (or lower) adjoint of r . If X and Y are complete lattices (posets) then $l : X \rightarrow Y$ has a right adjoint iff l preserves all the joins (it is *additive*, i.e., $l(x \vee y) = l(x) \vee l(y)$ and $l(0_X) = 0_Y$ where $0_X, 0_Y$ are bottom elements in the complete lattices X and Y respectively). The right adjoint is then $r(y) = \bigvee\{z \in X \mid l(z) \leq y\}$. Similarly, a monotone function $r : Y \rightarrow X$ is a right adjoint (it is *multiplicative*, i.e., has a left adjoint) iff r preserves all meets; the left adjoint is then $l(x) = \bigwedge\{z \in Y \mid x \leq r(z)\}$.

Each monotone function $l : X \rightarrow Y$ on a complete lattice (poset) X has both a *least* fixed point $\mu l \in X$ and a *greatest* fixed point $\nu l \in X$. These can be described explicitly as: $\mu l = \bigwedge\{x \in X \mid l(x) \leq x\}$ and $\nu l = \bigvee\{x \in X \mid x \leq l(x)\}$.

We denote as $y < x$ iff $y \leq x$ and not $x \leq y$. We denote as $x \bowtie y$ if x and y are two unrelated elements in X (i.e., neither $x \leq y$ nor $y \leq x$). An element $x \neq 0$ in a lattice is a *join-irreducible* element iff $x = a \vee b$ implies $x = a$ or $x = b$ for any $a, b \in X$. An element $x \in X$ in a lattice is an *atom* iff $x > 0$ and $\nexists y. (x > y > 0)$.

A lower set (down closed) is any subset Y of a given poset (X, \leq) such that for all elements x and y , if $x \leq y$ and $y \in Y$ then $x \in Y$.

A *Heyting algebra* is a bounded lattice X with finite meets and joins such that for each element $x \in X$, the function $(-) \wedge x : X \rightarrow X$ has a right adjoint $x \rightarrow (-)$, also

called an algebraic implication. An equivalent definition can be given by considering a bounded lattice such that for all x and y in X there is a greatest element z in X , denoted by $x \multimap y$, such that $z \wedge x \leq y$, i.e., $x \multimap y = \bigvee \{z \in X \mid z \wedge x \leq y\}$ (relative pseudo-complement). In Heyting algebra, we can define a negation $\neg x$ as a pseudo-complement $x \multimap 0$. Then $x \leq \neg \neg x$. A complete Heyting algebra is a Heyting algebra which is complete as a poset. A complete lattice is thus a complete Heyting algebra iff the *distributivity* $x \wedge (\bigvee S) = \bigvee_{y \in S} (x \wedge y)$ holds.

The negation and implication operators can be represented as the following monotone functions: $\neg : X \rightarrow X^{OP}$ and $\Rightarrow : X \times X^{OP} \rightarrow X^{OP}$, where X^{OP} is the lattice with inverse partial ordering and $\wedge^{OP} = \vee$, $\vee^{OP} = \wedge$.

The smallest complete distributive lattice is denoted by $\mathbf{2} = \{0, 1\}$ with classical two values, false and true respectively. It is also a complemented Heyting algebra and as a result, it is Boolean. A *Galois algebra* is a complete Heyting algebra enriched with a "nexttime" monotone function From X to X that preserves all meets (i.e., right adjoint). Such Galois algebras are often called as Heyting algebras with (unary) modal operators.

3 Interpretation of Lukasiewicz's 4-valued logic: Belnap's bilattice and Boolean algebra

A modality is any word or phrase that can be applied to a given statement ϕ to create a new statement that makes *assertion* about the mode of truth of ϕ : about when, where or how ϕ is true, or about the *circumstances* under which ϕ may be true.

Here, the modalities are *doxastic*. The mode of truth for a proposition is made up of the circumstances, how much is believed, or what is believed, that can make the proposition true.

The criteria for a modal logic from Aristotele and Leibniz, which Lukasiewicz considered to be self-evident, is: "if it is not possible that ϕ , than not ϕ ".

Let us consider now the case when we are dealing with incomplete knowledge, where different evidences may contradict each other, so that it is convenient to explicitly represent both the belief and doubt, as it was argued in [11,12].

The main concern in this investigation is to find the coherent interpretation for Lukasiewicz's logic values and to define the correct truth ordering between them. We consider that the values 0, 1, 2, 3 (that is, (0, 0), (1, 1), (1, 0) and (0, 1) respectively) do not represent the degree of truth (with the truth ordering corresponding to the order of natural numbers) but plausibility or *degree of confidence* (or, confidence level) in their truth, which ranges from complete lack of the confidence 0 = (0, 0) (entirely doubted) to the certainty 1 = (1, 1) (the complete confidence), so that $(0, 0) \preceq (1, 1)$.

Thus, a proposition is plausible when it is doubted but not entirely doubted (for example, when it has a degree of confidence 2 or 3, i.e., (1, 0) or (0, 1) respectively) and a proposition is impossible (it has a degree of confidence (0, 0)) when it is entirely doubted, that is, when something is believed that makes it logically unobtainable. This interpretation, based on the confidence level (a, b) , corresponds to the doxastic possibility based on the degree $a \in \mathbf{2}$ of belief in the truth and on the degree $b \in \mathbf{2}$ of belief in the falsity, where $(\mathbf{2}, \preceq) = \{0, 1\}$ is a complete lattice of truth values for standard 2-valued propositional logic. It is easy to see that the 4-valued algebra \mathcal{B}_4 is the Cartesian

product $\mathbf{2} \times \mathbf{2}$.

This approach is analog to the more sophisticated case of the fuzzy-based confidence level propositional logic [13], where instead of using the finite complete lattice $\mathbf{2}$, the infinite complete lattice of reals $[0, 1]$ is used. It can be explained if we assume that the belief of the subject in the truth of a proposition ϕ is *independent* from the belief of the same subject in the falsity of this proposition and that for each of these two beliefs, the subject uses the consistent 2-valued propositional logic.

Clearly, we obtain a paraconsistent logic, where it is possible that the subject believes both in the truth and in the falsity. From the fact that Lukasiewicz gives the value $(0, 0)$ to the bottom confidence level (degree of confidence) and the value $(1, 1)$ to the certainty (top confidence level), the unique coherent interpretation for a confidence level (x, y) is that x represents the logic value for subject assertion "believe in the truth of ϕ ", while y represents the logic value for the independent subject assertion "not believe in the falsity of ϕ ". Such confidence level (x, y) is *consistent* [14] if $x + (1 - y) \leq 1$. In fact, with this interpretation $(0, 0)$ means that the subject does not believe in truth and believes in the falsity of ϕ , while $(1, 1)$ means that the subject believes in the truth and does not believe in the falsity of ϕ .

Based on the above discussion, we can conclude that the value $2 = (1, 0)$ means that the subject believes in the truth and (contemporarily) believes in the falsity of ϕ : thus, this confidence level corresponds to the "inconsistent" value \top of Belnap's 4-valued logic [15]. Consequently, the value $3 = (0, 1)$ means that the subject does not believe in the truth and does not believe in the falsity, i.e., the subject does not know in what to believe, so that this confidence level corresponds to the "unknown" value \perp of Belnap's 4-valued logic.

Remark: The obtained result is that Belnap's ideas apply to non-modal and negation-less fragment of Lukasiewicz's algebra in a consistent way. Differently from the epistemic Belnap's negation \sim (defined by $\sim 0 = 3$, $\sim 3 = 0$, but $\sim 1 = 1$ and $\sim 2 = 2$) which is antitonic w.r.t. the truth ordering \preceq but *monotonic* modal operator w.r.t. the knowledge ordering \preceq_K , the Lukasiewicz's negation \neg is a pseudo-complement and antitonic for both orderings in Belnap's bilattice. The monotonicity of Belnap's bilattice negation w.r.t. the knowledge ordering is used for fixpoint semantics of logic programs based on Belnap's bilattice. This is because the fixpoint operator for such a logic can be defined for this knowledge ordering. Obviously, such a fixpoint semantics for logic programs, based on Lukasiewicz's 4-valued logic, is impossible.

□

In this Lukasiewicz-Belnap's lattice, we are interested in the ordering \preceq , which increases the belief and decreases the doubt of facts (as in the case of the *truth* ordering in Ginsberg's bilattices [16]), because it is used to define the monotonic fixpoint operator for model theoretic characterization of confidence level logic programs (see [16,14,17] for more details). That is, $(x, y) \preceq (x_1, y_1)$ iff $x \leq x_1$ and $y \leq y_1$. The join operation (conjunction) and the meet operation (disjunction) for this ordering \preceq correspond to the disjunction and conjunction operator in Lukasiewicz's 4-valued logic, that is,

$$(x, y) \wedge (x_1, y_1) = (x \wedge x_1, y \wedge y_1), \quad (x, y) \vee (x_1, y_1) = (x \vee x_1, y \vee y_1).$$

The bottom and the top members of this lattice \mathcal{B}_4 are $0 = (0, 0)$ and $1 = (1, 1)$, exactly

as in the truth ordering of Lukasiewicz's 4-valued logic.

Notice that the knowledge ordering \preceq_K of Belnap's bilattice

$(x, y) \preceq_K (x_1, y_1)$ iff $x \leq x_1$ and $y_1 \leq y$ is not used in the Lukasiewicz's 4-valued logic.

The Lukasiewicz's definition for the implication in this lattice \mathcal{B}_4 satisfies the following important property.

Proposition 1 *The Lukasiewicz's 4-valued implication is the relative pseudo-complement w.r.t. the complete distributive lattice (\mathcal{B}_4, \preceq) , that is,*

$$(x, y) \Rightarrow (x_1, y_1) = \bigvee \{(z, v) \mid (z, v) \wedge (x, y) \preceq (x_1, y_1)\},$$

such that $\neg(x, y) = (x, y) \Rightarrow (0, 0)$.

Proof: From the thesis we obtain $(x, y) \Rightarrow (x_1, y_1) = \bigvee \{(z, v) \mid (z, v) \wedge (x, y) \preceq (x_1, y_1)\} = \bigvee \{(z, v) \mid z \wedge x \leq x_1 \text{ and } v \wedge y \leq y_1\} = (\bigvee \{z \mid z \wedge x \leq x_1\}, \bigvee \{v \mid v \wedge y \leq y_1\}) = (x \rightarrow x_1, y \rightarrow y_1)$, where $x \rightarrow x_1 = \neg x \vee x_1$ corresponds to classical material 2-valued implication.

Similarly, $\neg(x, y) = (x, y) \Rightarrow (0, 0) = (x \rightarrow 0, y \rightarrow 0) = (\neg x \vee 0, \neg y \vee 0) = (\neg x, \neg y)$ corresponds to Lukasiewicz's definition of logic negation.

It is easy to verify that $(x \rightarrow x_1, y \rightarrow y_1)$ and $(\neg x, \neg y)$ satisfy the truth-tables for Lukasiewicz's implication and negation in \mathcal{B}_4 .

□

Remark: It was previously mentioned that the negation in this Lukasiewicz's 4-valued logic is different from the epistemic negation used in the Belnap's 4-valued logic. It is easy to verify that the strong negation used in Proposition 1 both with conjunction, disjunction and implication makes this 4-valued algebra a Boolean algebra (or more generally a Heyting algebra: in what follows we will consider this Lukasiewicz's logic in this more general framework also, in order to use the results for Kripke frames developed in [18,19]).

Alternatively to the definition of the implication given by Proposition 1, we can first define the negation as a pseudo-complement, $\neg(x, y) = \bigvee \{(z, v) \mid (z, v) \wedge (x, y) = (0, 0)\}$, and then the implication in standard way by $(x, y) \Rightarrow (x_1, y_1) = \neg(x, y) \vee (x_1, y_1)$.

□

We say that ϕ is *stronger* than ψ , denoted by $\phi \Vdash \psi$, if for all valuations (homomorphisms) $v : \mathcal{L} \rightarrow \mathcal{B}_4$, $v(\phi) \preceq v(\psi)$, where \mathcal{L} is a set of all formulae obtained as free grammar algebra from a set of propositional variables P and logic connectives. That is, if $v(\phi \Rightarrow \psi) = (1, 1)$ for each valuation v , so that $\Vdash \phi \Rightarrow \psi$, i.e., $\phi \Rightarrow \psi$ is a theorem w.r.t. the matrix (\mathcal{B}_4, D) where $D = \{(1, 1)\}$ is a subset of designated elements.

Clearly, \Vdash is an entailment relation and $\phi \Vdash \psi$ iff $\Vdash \phi \Rightarrow \psi$ is the deduction theorem.

It is easy to verify that, as in Lukasiewicz's 4-valued logic, we can define the conjunction and disjunction based on the implication, as in standard logic. For conjunction, it is:

$$(x, y) \wedge (z, v) = (x \wedge z, y \wedge v) = (\neg(x \rightarrow \neg z), \neg(y \rightarrow \neg v)) = \neg(x \rightarrow \neg z, y \rightarrow \neg v) = \neg((x, y) \Rightarrow (\neg z, \neg v)) = \neg((x, y) \Rightarrow \neg(z, v)).$$

Similarly for disjunction. Thus, the Lukasiewicz's 4-valued logic without modal operators $(\mathcal{B}_4, \preceq, \wedge, \vee, \Rightarrow, \neg)$ is

a Boolean algebra. More about the relationship between bilattices with truth-knowledge duality and Boolean algebras with even number of algebraic truth value can be found in [17].

So, from Proposition 1, we deduce that for this interpretation of Lukasiewicz's 4-valued logic, all logic operators are derived from the meet and join operators of complete *distributive* lattice (\mathcal{B}_4, \preceq) .

Proposition 2 *The 2-valued reduction $(\{(0, 0), (1, 1)\}, \preceq, \wedge, \vee, \Rightarrow, \neg)$ is a subalgebra of the Lukasiewicz's 4-valued Boolean algebra $(\mathcal{B}_4, \preceq, \wedge, \vee, \Rightarrow, \neg)$.*

Proof: It is easy to verify that all algebraic operators are closed with respect to the subset $\{(0, 0), (1, 1)\} \subseteq \mathcal{B}_4$.

□

Thus, the Lukasiewicz's 4-valued modal system contains the classical bivalent logic. Consequently, a formula which is valued true or false in classical logic, for a given specific assignment to variables, should have the same value in this 4-valued modal logic.

In the rest of the paper, we will consider the Lukasiewicz's modal operators M and L w.r.t. this lattice and will explain why they correspond to the normal modal operators, and we will also modify their actual attribution to the universal and existential Kripke style modal operators.

4 Autoreferential Kripke style semantics for Lukasiewicz's 4-valued modal logic

In Lukasiewicz's notes, the following requirements have to be satisfied for any modal logic:

1. The system must be built from or contain classical bivalent logic. In fact, Proposition 2 demonstrates this property.
2. The "L-necessity" $L\phi$ should be stronger than ϕ , and ϕ must be stronger than the "L-possibility" $M\phi$. That is, the following should be theorems $L\phi \Rightarrow \phi$ and $\phi \Rightarrow M\phi$.
3. The "L-necessity" L and "L-possibility" M should be inter-definable as follows: $L = \neg M \neg$, $M = \neg L \neg$.
4. The following should *not* be theorems in the system: $M\phi \Rightarrow \phi$, $\phi \Rightarrow L\phi$, $M\phi$, $\neg L\phi$.
5. The system must contain unary operators for the necessity and possibility which are not definable in terms of non-modal operators. This requirement presupposes that a modal logic has to be intensional. That is, a formula with certain structures should (or should not) be valid by virtue of their structure -necessarily so- regardless of the logic values of their components.

In fact, it is easy to verify that the above properties are satisfied by Lukasiewicz's 4-valued modal logic. But from the Galois connection for monotone operators L and M , $L\phi \Rightarrow \phi$ iff $\phi \Rightarrow M\phi$, we deduce that the "L-necessity" L corresponds to the *existential* modal operator while the "L-possibility" M corresponds to the dual universal modal operator.

Proposition 3 *The algebraic "L-necessity" L operator is an additive normal modal operator, while the algebraic "L-possibility" M operator is a multiplicative normal modal operator. Thus they correspond to the existential \diamond and universal \square logical modal Kripke style operators respectively.*

Proof: It is easy to verify that they are monotone operators, and that $L(\alpha \vee \beta) = L\alpha \vee L\beta$ and $L(0, 0) = (0, 0)$, that is, the "L-necessity" L is an additive algebraic modal operator, and its logical correspondent connective is an existential modal operator \diamond . Dual multiplicative property holds for M .

□

With this correct Kripke style interpretation of Lukasiewicz's modal operators, we have that the theorems of the point 2 above correspond to the following Galois connection, $\diamond\phi \Rightarrow \phi$ iff $\phi \Rightarrow \square\phi$ (this is a trivial result because both sides of this Galois connection are tautologies of Lukasiewicz's system), and the second theorem is in fact the Necessity role of normal Kripke modal logic, $\frac{\phi}{\square\phi}$.

Based on this correct Kripke style interpretation of Lukasiewicz's 4-valued modal operators, the following facts can be obtained:

1. The "L-possibility" of ϕ , i.e., $\diamond\phi$, is *true* in \mathcal{B}_4 when it is believed (in the standard 2-valued logic) in the truth of ϕ .
2. The "L-possibility" of ϕ is *unknown* in \mathcal{B}_4 when it is not believed (in the standard 2-valued logic) in the truth of ϕ .
3. The "L-necessity" of ϕ , i.e., $\square\phi$, is *false* in \mathcal{B}_4 when is not believed (in standard 2-valued logic) in truth of ϕ .
4. The "L-necessity" of ϕ is *inconsistent* (both true and false) in \mathcal{B}_4 when it is believed (in the standard 2-valued logic) in the truth of ϕ .

In the rest of this paper, we consider the Lukasiewicz-Belnap's complete distributive lattice of truth values $(\mathcal{B}_4, \preceq, \wedge, \vee)$ and its *isomorphic* representation which has to be a complete sublattice of the powerset lattice $(\mathcal{P}(\mathcal{B}_4), \subseteq, \cap, \cup)$, closed under intersection \cap and union \cup . That is, we will find the isomorphism $is : (\mathcal{B}_4, \preceq, \wedge, \vee) \simeq (C_L(\mathcal{P}(\mathcal{B}_4)), \subseteq, \cap, \cup)$.

For this Lukasiewicz-Belnap's complete distributive lattice, we will obtain that the carrier set \mathcal{B}_4 , of this many-valued logic, is the set of possible worlds of the Kripke frame (which is defined for the dual relational representation of the algebraic semantics): this is an *autoreferential assumption* [10] where the possible world 0 is so called inconsistent world (the world with explosive inconsistency), such that each formula in this world is true. The relational semantic of Lukasiewicz's modal operators (different from the meet and join operators) of the algebra $\mathbf{A} = ((\mathcal{B}_4, \preceq, \wedge, \vee), \Rightarrow, \neg, L)$ will be obtained successively by correct definition of the accessibility relations of the Kripke frame.

The closure property for the intersection and union holds in the more general case of *hereditary subsets*. A set $B \in \mathcal{P}(\mathcal{B}_4)$ is hereditary if it is closed downwards under \preceq , i.e., if we have that whenever $x \in B$ and $y \preceq x$ then $y \in B$.

From the Birkhoff's representation theorem [20] for distributive lattices, we obtain that every finite (thus complete) distributive lattice A is isomorphic to the lattice of lower sets of the poset of join-irreducible elements. Lower set (down closed) is any subset Y

of a given poset (A, \preceq) such that, for all elements x and y , if $x \preceq y$ and $y \in Y$ then $x \in Y$.

Proposition 4 [20] 0-LIFTED BIRKHOFF ISOMORPHISM: *Let A be a complete distributive lattice. Then we define the following mapping $\downarrow^+ : A \rightarrow \mathcal{P}(A)$:*

for any $x \in A$,

$$\downarrow^+ x = \downarrow x \cap \widehat{A},$$

where $\widehat{A} = \{y \mid y \in A \text{ and } y \text{ is join-irreducible}\} \cup \{0\}$.

We define the set $A^+ = \{\downarrow^+ a \mid a \in A\} \subseteq \mathcal{P}(A)$, so that $\downarrow^+ \bigvee = id_{A^+} : A^+ \rightarrow A^+$ and $\bigvee \downarrow^+ = id_A : A \rightarrow A$. Thus, the operator \downarrow^+ is inverse of the supremum operation $\bigvee : A^+ \rightarrow A$. The set (A^+, \subseteq) is a complete lattice, such that there is the following 0-lifted Birkhoff isomorphism $\downarrow^+ : (A, \preceq, \wedge, \vee) \simeq (A^+, \subseteq, \cap, \cup)$.

Proof: Let us show the homomorphic property of \downarrow^+ :

$$\begin{aligned} \downarrow^+(x \wedge y) &= \downarrow(x \wedge y) \cap \widehat{A} = (\downarrow x \cap \downarrow y) \cap \widehat{A} = (\downarrow x \cap \widehat{A}) \cap (\downarrow y) \cap \widehat{A} = \\ &= \downarrow^+ x \cap \downarrow^+ y, \quad \text{and} \end{aligned}$$

$$\begin{aligned} \downarrow^+(x \vee y) &= \downarrow(x \vee y) \cap \widehat{A} = (\downarrow x \cup \downarrow y) \cap \widehat{A} = (\downarrow x \cap \widehat{A}) \cup (\downarrow y) \cap \widehat{A} = \\ &= \downarrow^+ x \cup \downarrow^+ y. \end{aligned}$$

The isomorphic property holds from Birkhoff's representation theorem.

□

The name lifted is used to denote the difference from the original Birkhoff's isomorphism: that is, for any $x \in A$, $0 \in \downarrow^+ x$, so that $\downarrow^+ x$ is never be an empty set (it is lifted by bottom element 0).

Notice that $(A^+, \subseteq, \cap, \cup)$ is a *subalgebra* of the powerset algebra $(\mathcal{P}(A), \subseteq, \cap, \cup)$.

In this case, for the Lukasiewicz-Belnap's lattice, when $A = \mathcal{B}_4$, which is a distributive lattice w.r.t. the \preceq ordering with two join-irreducible elements $2 = (1, 0)$ and $3 = (0, 1)$, so that $C_L(\mathcal{P}(\mathcal{B}_4)) = \widehat{\mathcal{B}_4} = \{0, 2, 3\}$.

In that case, we have that $\downarrow^+ 1 = \downarrow^+(2 \vee 3) = \downarrow^+ 2 \cup \downarrow^+ 3 = \{0, 2\} \cup \{0, 3\} = \{0, 2, 3\} = \widehat{\mathcal{B}_4} \neq \mathcal{B}_4$. It is easy to verify that $\downarrow^+ 0 = \{0\}$ is the bottom element in $\mathcal{B}_4^+ = \{\{0\}, \{0, 2\}, \{0, 3\}, \{0, 2, 3\}\}$.

Based on this results, we are able to extend the complete distributive lattice \mathcal{B}_4 with other unary algebraic operators $\{o_i\}_{i \in N} : \mathcal{B}_4 \rightarrow \mathcal{B}_4$ and binary operators $\{\otimes_i\}_{i \in N} : \mathcal{B}_4 \times \mathcal{B}_4 \rightarrow \mathcal{B}_4$ in order to obtain the following set-based canonical representation:

Proposition 5 CANONICAL REPRESENTATION: *Let $\mathbf{A} = ((\mathcal{B}_4, \preceq, \wedge, \vee), \Rightarrow, \neg, L)$ be a complete distributive lattice based Lukasiewicz's algebra. It is a Galois algebra.*

Its set based canonical representation is the algebra $\mathbf{A}^+ = ((\mathcal{B}_4^+, \subseteq, \cap, \cup), \Rightarrow^+, \neg^+, L^+)$, such that $o_i^+ = \downarrow^+ o_i \bigvee : \mathcal{B}_4^+ \rightarrow \mathcal{B}_4^+$, for $o_i \in \{\neg, L\}$, and $\Rightarrow^+ = \downarrow^+ \Rightarrow \bigvee : \mathcal{B}_4^+ \times \mathcal{B}_4^+ \rightarrow \mathcal{B}_4^+$ are the unary and binary set-based operators.

Proof: \mathbf{A} is a Boolean algebra (thus also an Heyting algebra) with unary modal operator L which satisfies the Galois connection, thus from Section 2 it is a Galois algebra.

We have that for any $x, y \in A$, $\downarrow^+ o_i(x) = \downarrow^+ o_i(\bigvee \downarrow^+)(x) = (\downarrow^+ o_i \bigvee) \downarrow^+ x = o_i^+(\downarrow^+ x)$ and $\downarrow^+ \otimes_i(x, y) = \otimes_i^+(\downarrow^+ x, \downarrow^+ y)$. Thus \downarrow^+ is the isomorphism $\downarrow^+ : \mathbf{A} \simeq \mathbf{A}^+$.

□

Let us consider the binary implication operator \Rightarrow (equal to the relative pseudo-complement over the complete distributive lattice \mathcal{B}_4). Then we have that $(\downarrow^+ x) \Rightarrow^+ (\downarrow^+ y) = \Rightarrow^+ (\downarrow^+ x, \downarrow^+ y) = \downarrow^+ \Rightarrow (x, y) = \downarrow^+ (x \Rightarrow y) = \downarrow^+ (\bigvee \{z \mid z \wedge x \preceq y\}) = \bigcup \{\downarrow^+ z \mid z \wedge x \preceq y\}$ (from the homomorphism \downarrow^+ w.r.t. the join operator of this lattice) $= \bigcup \{\downarrow^+ z \mid \downarrow^+ (z \wedge x) \subseteq \downarrow^+ y\}$ (from $\downarrow^+ v \subseteq \downarrow^+ w$ iff $v \preceq w$) $= \bigcup \{\downarrow^+ z \mid \downarrow^+ z \cap \downarrow^+ x \subseteq \downarrow^+ y\}$ (from the homomorphism \downarrow^+ w.r.t. the meet operator of this lattice) $= \bigcup \{S \in A^+ \mid S \cap \downarrow^+ x \subseteq \downarrow^+ y\}$.

That is, we obtain that the corresponding operator $\Rightarrow^+ : \mathcal{B}_4^+ \times \mathcal{B}_4^+ \rightarrow \mathcal{B}_4^+$ is the relative pseudo-complement for the lattice \mathcal{B}_4^+ , and, consequently, $\neg^+(\downarrow^+ x) = (\downarrow^+ x) \Rightarrow^+ \{0\}$ is the pseudo-complement in \mathcal{B}_4^+ .

Consequently, the canonical set-based Lukasiewicz's algebra \mathbf{A}^+ is also a Boolean algebra. Thus, the relational Kripke style semantics for Lukasiewicz's 4-valued logic based on the modal Boolean algebras in Proposition 5 is given by:

Definition 1. Let $(A, I : Var \rightarrow \mathcal{B}_4)$ be an algebraic model of the Lukasiewicz's 4-valued modal logic, then its Kripke style model is $\mathcal{M}_K = (K, I_K)$, such that $K = \langle (\widehat{\mathcal{B}}_4, \preceq), \mathcal{R}_L \rangle$ is a frame, where $\mathcal{R}_L = \{(x, y) \mid y \in \widehat{\mathcal{B}}_4, \text{ and } x \in \downarrow^+ L(y)\}$ is the accessibility relation for a modal operator L , and $I_K : Var \rightarrow \mathcal{B}_4^+$ is a canonical valuation, such that for any atomic formula (propositional variable) $p \in Var$, $I_K(p) = \downarrow^+ (I(p))$. Then, for any world $x \in \widehat{\mathcal{B}}_4$, and formulae ψ, ϕ ,

$\mathcal{M}_K \models_x p$ iff $x \in I_K(p)$,

$\mathcal{M}_K \models_x \phi \wedge \psi$ iff $\mathcal{M}_K \models_x \phi$ and $\mathcal{M}_K \models_x \psi$,

$\mathcal{M}_K \models_x \phi \vee \psi$ iff $\mathcal{M}_K \models_x \phi$ or $\mathcal{M}_K \models_x \psi$,

$\mathcal{M}_K \models_x \phi \Rightarrow \psi$ iff $\forall y \in \widehat{A}(y \preceq x \text{ implies } (\mathcal{M}_K \models_y \phi \text{ implies } \mathcal{M}_K \models_y \psi))$,

$\mathcal{M}_K \models_x \neg \phi$ iff $\mathcal{M}_K \models_x \phi \Rightarrow 0$,

$\mathcal{M}_K \models_x \diamond \phi$ iff $\exists y \in \widehat{A}((x, y) \in \mathcal{R}_L \text{ and } \mathcal{M}_K \models_y \phi)$, for modal operator L denoted by standard existential modal symbol \diamond as well.

Notice that in the world $x = 0$ (bottom element in \mathcal{B}_4) each formula ϕ of Lukasiewicz's 4-valued modal logic is satisfied: because of that, we will denominate this world by inconsistent or *trivial* world [10]. The semantics for the implication operator is analogous to the Kripke modal semantics for the implication of intuitionistic logic (but with inverted ordering for the accessibility relation \sqsubseteq) because, in both cases, the negation is expressed by pseudo-complement and the implication by relative pseudo-complement.

Theorem 1 [18] Let $K = \langle (\widehat{\mathcal{B}}_4, \preceq), \mathcal{R}_L \rangle$ be a frame, and $I_K : Var \rightarrow \mathcal{B}_4^+$ be a canonical valuation on it, given by Definition 1. Then, for any propositional formula ϕ , the set of worlds where ϕ holds is $\|\phi\| = \bar{I}_K(\phi) = \downarrow^+ (\bar{I}(\phi)) \in \mathcal{B}_4^+$, where \bar{I}_K and \bar{I} are unique homomorphic extensions of I_K and I respectively, over all the formulae.

Proof: the full proof for any complete distributive lattice of algebraic truth values can be find in [18].

□

As we can see, the accessibility relation for Lukasiewicz's modal operator "L-necessity" L (which is an existential logic modal operator \diamond) is given by $R_L = \{(0, 0), (0, 2), (2, 2)\}$,

$(0, 3)$ which is transitive but *not reflexive* (the formula $M\phi \Rightarrow \phi$ is not an axiom), so that this modal logic is not S4 Kripke normal modal logic, as is intended for Kripke style modal logic with necessity modal operator. But it is explicitly required by Lukasiewicz (see the point 4 at the beginning of this Section), so that his meaning of "L-necessity" (but also of "L-possibility") is different from the standard assumption for the necessity modal operator of Lewis used to define the intensional implication [21] (that is, intuitionistic implication in S4 modal logic [22] used to translate intuitionistic logic into modal Kripke style logic).

The Lukasiewicz's "L-necessity" is an existential modal operator \diamond , and in his 4-valued modal logic we have both "necessity" operators: His modal operator L , and the standard universal necessity operator, here denoted by \Box , with the accessibility relation corresponding to the inverted poset \preceq^{-1} of algebraic truth values in \mathcal{B}_4 (so that $(x, y) \in \preceq^{-1}$ corresponds to $y \preceq x$). This second operator is implicitly used to define the intuitionistic implication of his 4-valued logic, that is, by $\phi \Rightarrow \psi = \Box(\phi \Rightarrow_C \psi)$, where \Rightarrow_C is the standard material implication, with

$$\mathcal{M}_K \models_x \Box\phi \text{ iff } \forall y \in \hat{A}((x, y) \in \preceq^{-1} \text{ implies } \mathcal{M}_K \models_y \phi)$$

while for the Lukasiewicz's universal modal operator \Box (corresponding to his algebraic modal operator M) we have that

$$\mathcal{M}_K \models_x \Box\phi \text{ iff } \mathcal{M}_K \models_x \neg\diamond\neg\phi \text{ iff } \mathcal{M}_K \models_x \diamond\neg\phi \Rightarrow 0.$$

But for the Lukasiewicz-Belnap's lattice (\mathcal{B}_4, \preceq) , we have the following property:

Proposition 6 *The standard necessity modal operator \Box based on the S4 accessibility relation \preceq^{-1} is an identity modal operator for the Lukasiewicz-Belnap's lattice (\mathcal{B}_4, \preceq) .*

Proof: It is easy to verify that for any $\alpha \in \mathcal{B}_4$ we have that

$$\begin{aligned} \|\Box\alpha\| &= \{x \mid \mathcal{M}_K \models_x \Box\alpha\} = \{x \mid \forall y \in \hat{A}((x, y) \in \preceq^{-1} \text{ implies } \mathcal{M}_K \models_y \alpha)\} = \\ &= \{x \mid \forall y \in \hat{A}(y \preceq x \text{ implies } y \in \downarrow^+ \alpha)\} = \downarrow^+ \alpha. \end{aligned}$$

Or, algebraically, $\Box = id : \mathcal{B}_4 \rightarrow \mathcal{B}_4$.

□

Thus, the Lukasiewicz's implication \Rightarrow corresponds to material implication (in fact we have that $\phi \Rightarrow \psi = \neg\phi \vee \psi$ as in the standard logic, so that both implication and negation in Lukasiewicz's 4-valued logic are standard (the relative pseudo-complement corresponds to standard material implication in the Lukasiewicz-Belnap's lattice (\mathcal{B}_4, \preceq)), and the satisfaction relation \models in Kripke style semantics, presented in Definition 1, for the implication can also be given in the standard form:

$$(a) \mathcal{M}_K \models_x \phi \Rightarrow \psi \text{ iff } \mathcal{M}_K \models_x \phi \text{ implies } \mathcal{M}_K \models_x \psi,$$

and for negation, we obtain:

$$(b) \mathcal{M}_K \models_x \neg\phi \text{ iff } \mathcal{M}_K \models_x \phi \Rightarrow 0 \text{ iff } \mathcal{M}_K \models_x \phi \text{ implies } \mathcal{M}_K \models_x 0) \\ \text{iff } \mathcal{M}_K \not\models_x \phi \text{ or } \mathcal{M}_K \models_x 0.$$

Remark: Notice that $\mathcal{M}_K \models_x \neg\phi$ is different from $\mathcal{M}_K \not\models_x \phi$ (as in standard relational semantics for Boolean negation where $\mathcal{M}_K \models_x 0$ is false for any x and can be omitted). Here, the second component $\mathcal{M}_K \models_x 0$, which is true only for $x = 0$, is *necessary* in order to guarantee that $0 \in \|\neg\phi\|$ and $0 \in \|\phi\|$, that is, the fact that in the autoreferential semantics, any formula is satisfied in the world 0 (called inconsistent world, or world with explosive inconsistency as explained in [10]).

It is a direct consequence of the fact that for pseudo-complements in this autoreferential

semantics holds that $\|\neg\phi\| = \bigcup\{X \in \mathcal{B}_4^+ \mid X \cap \|\phi\| = \{0\}\} = \{0\} \cup \widehat{\mathcal{B}}_4 \setminus \|\phi\|$, where \setminus is a set difference.

But if we reduce the set of possible worlds into the set of join-irreducible elements, $\mathcal{W} = \{2, 3\} \subset \widehat{\mathcal{B}}_4$, with $\mathcal{W}^+ = \{\emptyset, \{2\}, \{3\}, \{2, 3\}\}$, then we have the following bijective mapping of algebraic truth values,

Values in \mathcal{B}_4	Autoreferential semantics in \mathcal{B}_4^+	Reduced semantics in \mathcal{W}^+
0	$\{0\}$	\emptyset
2	$\{0, 2\}$	$\{2\}$
3	$\{0, 3\}$	$\{3\}$
1	$\{0, 2, 3\}$	$\{2, 3\}$

It is easy to verify that the Definition 1 for a Kripke semantics of Lukasiewicz's logic is valid also for this reduced semantics with a set of worlds \mathcal{W} , and that in this semantics we obtain the standard result, $\|\neg\phi\| = \bigcup\{X \in \mathcal{W}^+ \mid X \cap \|\phi\| = \emptyset\} = \mathcal{W} \setminus \|\phi\|$, i.e., $(b_1) \mathcal{M}_K \models_x \neg\phi$ iff $\mathcal{M}_K \not\models_x \phi$.

□

The consequence of the Proposition 6 is that in the Lukasiewicz-Belnap's 4-valued lattice (\mathcal{B}_4, \preceq) , the standard necessity modal operator is banal and not useful, because it corresponds to the identity.

Consequently, the introduction of "L-necessity" operator L is the way to overcome this singularity in this Lukasiewicz's 4-valued logic, and explains our consistent opinion why Lukasiewicz introduced this non-standard "necessity" operator in his modal logic. Here we demonstrated these hypothesis by introducing our interpretation for modal operators so that Lukasiewicz's 4-valued logic still remains to be a *normal* Kripke style modal logic.

5 Conclusion

We believe that this non-standard "necessity" operator in Lukasiewicz's modal logic, which is an existential additive modal operator, should be attributed directly to Aristotele and also to Lukasiewicz's efforts to obtain a system that could faithfully represent Aristotele's ideas about necessity and possibility (obviously reinforced by his commitment to the truth-functionality).

In Lukasiewicz's own words (p.133 in [23]): "There are two reasons why Aristotele's modal logic is so little known. The first is due to the author himself: in contrast to the assertoric syllogistic which is perfectly clear and nearly free of errors, Aristotele's modal syllogistic is almost incomprehensible because of its many faults and inconsistencies..... The second reason is that modern logicians have not as yet been able to construct a universally acceptable system of modal logic which would yield a solid basis for the interpretation and appreciation of Aristotele's work. I have tried to construct such a system, different from those hitherto known, and built upon Aristotele's ideas."

We understand and accept only that the Aristotele's ideas about "necessity" are *different* than the Kripke style ideas based on the universal multiplicative modal operator.

But this Lukasiewicz's algebraic translation of the Aristotele's work is coherent and self-consistent and results in a *normal modal* logic. Consequently, this Lukasiewicz's

modal system is an algebraic precursor for the modern development of (normal) modal logics and to the introduction of the possible world semantics, based on the relational Kripke-style representations. Definitely, it cannot be considered as a kind of "historical rarity and a dead end" as erroneously considered in [3].

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