J. Physique 45 (1984) 1283-1295 AOÛT 1984, PAGE 1283

Classification
Physics Abstracts
05.60 — 71.55 J

Lyapounov exponent of the one dimensional Anderson model : weak disorder expansions

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(*Reçu le 2 février 1984, accepté le 10 avril 1984*)

Résumé. — Nous présentons une méthode qui donne le développement de faible désordre ($\lambda \to 0$) de l'exposant de Lyapounov $\gamma(E)$ d'une équation de Schrödinger à une dimension $\psi_{n+1} + \psi_{n-1} + \lambda V_n \psi_n = E\psi_n$ avec un potentiel aléatoire V_n. Près du bord de bande du système pur ($E \to 2$), le développement de $\gamma(E)$ est non analytique et nous montrons que $\gamma(E) \sim \lambda^{2/3}$ pour $\lambda \to 0$. Au centre de bande ($E \to 0$) nous retrouvons l'anomalie qui a déjà été expliquée par Kappus et Wegner. Nous trouvons une autre anomalie à l'énergie $E = 2\cos(\pi/3)$ et nous pensons que des anomalies du même type se produisent pour toutes les énergies $E = 2\cos(\pi\alpha)$ où α est rationnel.

Abstract. — We describe a method which gives the weak disorder expansion ($\lambda \to 0$) of the Lyapounov exponent $\gamma(E)$ of a discretized one-dimensional Schrödinger equation $\psi_{n+1} + \psi_{n-1} + \lambda V_n \psi_n = E\psi_n$ with a random potential V_n. Near the band edge of the pure system ($E \to 2$), the weak disorder expansion of $\gamma(E)$ is non analytic and we show that $\gamma(E) \sim \lambda^{2/3}$ when $\lambda \to 0$. At the band centre ($E \to 0$), we recover the anomaly which has already been explained by Kappus and Wegner. We find another anomaly at the energy $E = 2\cos(\pi/3)$ and we believe that similar anomalies should occur at all energies $E = 2\cos(\alpha\pi)$ with α rational.

1. Introduction.

Products of random matrices appear very often in the study of disordered systems, in particular in the one-dimensional situations [1-5]. Usually, the first quantity that one would like to calculate is the Lyapounov exponent associated with a given product of random matrices. Several physical quantities can be deduced from the knowledge of the Lyapounov exponent : in a localization problem [6-7], the Thouless formula [8] relates directly the Lyapounov exponent to the density of states ; for the Ising chain in a random field [9-10], the Lyapounov exponent is nothing but the free energy.

Unfortunately, there does not exist any general method of calculating analytically the Lyapounov exponent of a given product of random matrices. In general, one can only calculate this Lyapounov exponent numerically or one has to expand around a well understood situation (product of random commuting matrices [9], weak disorder expansions [11], large coupling expansions [12]). It is therefore interesting to have available expansion methods which are as simple as possible.

In the present paper, we shall give a way of deriving the weak disorder expansion ($\lambda \to 0$) of the Lyapounov exponent $\gamma(E)$ associated with the following product of random matrices

$$\prod_{n=1}^{N} \begin{pmatrix} E - \lambda V_n & -1 \\ 1 & 0 \end{pmatrix} \qquad (1)$$

where the V_n are randomly distributed according to a given probability distribution $\rho(V)$ and the energy E is a fixed parameter. We shall limit ourselves to the case where the average potential $\langle V_n \rangle = 0$ since one can always incorporate this average in the energy E.

The product of random matrices (1) appears in several situations : first, if one considers the discretized Schrödinger equation in one dimension with a random potential λV_n on the site n, the wave function ψ_n obeys the following equation

$$\psi_{n+1} + \psi_{n-1} + \lambda V_n \psi_n = E\psi_n . \qquad (2)$$

One can easily relate (2) to (1) by considering the two-component vectors U_n defined by

$$U_n = \begin{pmatrix} \psi_{n+1} \\ \psi_n \end{pmatrix} \qquad (3)$$

and by noticing that the product (1) relates U_N to U_0.

The product (1) appears also in the calculation of the Lyapounov exponent of some dynamical systems [13] like the stadion or the diamond which are integrable systems for $\varepsilon = 0$ and have mixing properties for $\varepsilon \neq 0$.

In section 2, we shall first recall briefly a weak disorder expansion which was already presented in a previous work done in collaboration with C. Itzykson [11]. We shall explain why this expansion holds for all complex values of E except the interval $[-2, 2]$ and show why it breaks down in the neighbourhood of the band edge $E \to 2$ of the pure system. To describe correctly the region near of $E = 2$, we shall develop in section 3 an appropriate method and find explicit formulae for the density of states and the localization length. We shall recover several singular behaviours which had already been found in the neighbourhood of the band edge for continuous Schrödinger equations in a random potential [14, 5].

In section 4, we describe a method of finding the weak disorder expansion of $\gamma(E)$ which should be in principle valid in the neighbourhood of any energy $E = 2\cos(\pi\alpha)$ with α rational. In section 5, we shall apply this method to the case of the band centre ($\alpha = \frac{1}{2}$) where we shall recover the anomaly explained by Kappus and Wegner [15]. The λ^2 term in the Lyapounov exponent is different from that determined from the naive weak disorder expansion. In section 6, we shall consider the case $E = 1$ (i.e. $\alpha = \pi/3$) where we shall find a very similar anomaly at order λ^4. This anomaly has also been discussed recently by Lambert [18, 19].

2. Weak disorder expansion.

Let us start from the Schrödinger equation (2). If we define R_n by

$$R_n = \frac{\psi_n}{\psi_{n-1}} \tag{4}$$

the Lyapounov exponent $\gamma(E)$ is given by

$$\gamma(E) = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \log R_n . \tag{5}$$

Clearly, from (2) and (4), one finds that the R_n obey the following recursion relation

$$R_{n+1} = E - \lambda V_n - \frac{1}{R_n} . \tag{6}$$

Since the vector U_n was a two component vector, R_n is a way of measuring the direction of the vector U_n.

If we fix any complex value of the energy E, the R_n will be complex numbers. In equation (5) there is no ambiguity in defining the real part of γ since all the definitions of the logarithm give the same answer. On the contrary, to define the imaginary part of γ, we have to choose a definition of the logarithm. This can be done very easily by noticing that if E and R_n have a positive imaginary part, then R_{n+1} obtained from (6) has also a positive imaginary part. Therefore, if E and R_0 have positive imaginary parts, we are sure that all the R_n have also a positive imaginary part. So we can choose the logarithm of R_n to have an imaginary part between 0 and π when the imaginary part of E is positive and between $-\pi$ and 0 if $\operatorname{Im}(E) < 0$

$$\begin{aligned} 0 < \operatorname{Im}(\log R_n) < \pi \quad &\text{if} \quad \operatorname{Im} E > 0 \\ -\pi < \operatorname{Im}(\log R_n) < 0 \quad &\text{if} \quad \operatorname{Im} E < 0 . \end{aligned}$$

As usual, for real values of the energy E, one can always add an infinitesimal imaginary part $i\varepsilon$ to E and the imaginary part of γ in the limit $\varepsilon \to 0$ depends on the sign of ε.

For real values of E, all the R_n are real. If we choose ε to be positive, this means that we decide that the imaginary part of $\log R_n$ is π for all the negative R_n. We see that for real values of $E (E + i\varepsilon$ in the limit $\varepsilon \to 0^+)$, the imaginary part of the γ is just π times the density of negative R_n, i.e. the density of nodes of the wave function (see Eq. (4)). So it is clear that this imaginary part is equal to π times the integrated density of states.

Let us now recall a simple method for deriving the weak disorder expansion of γ [11]. For convenience, let us take a value of the energy E which does not belong to the spectrum of the pure system

$$E \neq 2\cos q \qquad \text{with } q \text{ real} . \tag{7}$$

We can choose any complex value for E or any real E with $|E| > 2$. Let us write R_n in the following way :

$$R_n = A\, e^{\lambda B_n + \lambda^2 C_n + \lambda^3 D_n + \cdots} \tag{8}$$

where $A, B_n, C_n, \ldots$ do not depend on λ. If we substitute this expansion into equation (6) and if we equate the two sides of the equation order by order in λ, we find recursion relations for $A, B_n, C_n, D_n, \ldots$

$$A = E - A^{-1} \tag{9}$$

$$AB_{n+1} = -V_n + A^{-1} B_n \tag{10}$$

$$A(C_{n+1} + \tfrac{1}{2} B_{n+1}^2) = A^{-1}(C_n - \tfrac{1}{2} B_n^2) . \tag{11}$$

etc....

It is not necessary to consider the dependence of A on n because for $\lambda = 0$, all the R_n are equal to the root A of equation (9) which has the largest modulus. (The two roots have different modulus because of condition (7).) The expansion of the Lyapounov exponent is then given by :

$$\gamma = \log A + \lambda \langle B \rangle + \lambda^2 \langle C \rangle + \lambda^3 \langle D \rangle + \cdots . \tag{12}$$

As explained in reference [11], it is easy to calculate the averages $\langle B \rangle$, $\langle C \rangle$, $\langle D \rangle$,... To do so, we have to notice that B_n, C_n, ... are functions of all the V_i for $i < n$ but do not depend on the V_i for $i \geqslant n$. This means that averages like $\langle B_n^p V_n^q \rangle$ can be replaced by $\langle B^p \rangle \langle V^q \rangle$. Using the fact that the averages of $\langle B \rangle$, $\langle C \rangle$, $\langle D \rangle$, etc... do not depend on n, one gets the following result

$$\begin{aligned}\gamma = \log A &- \frac{\lambda^2}{2} \frac{A^2}{(A^2-1)^2} \langle V^2 \rangle \\ &- \frac{\lambda^3}{3} \frac{A^3}{(A^2-1)^3} \langle V^3 \rangle - \frac{\lambda^4}{4} \frac{A^4}{(A^2-1)^4} \langle V^4 \rangle \\ &- \frac{\lambda^4}{2} \frac{(3+2A^2) A^4}{A^4-1} \frac{\langle V^2 \rangle^2}{(A^2-1)^4} + 0(\lambda^5) .\end{aligned} \tag{13}$$

The term linear in λ is not present because we have assumed that $\langle V \rangle = 0$. The expression (13) was already presented in reference [11] with a slightly different notation (one has to replace $(A-1)^2 z_n$ by $-AV_n$ in Eqs. (17) and (20) of reference [11]).

As we mentioned above, the R_n measure the directions of the vectors U_n. For $\lambda = 0$ and when condition (7) is fulfilled, the matrices (1) have 2 eigenvalues with different modulus. In the limit $n \to \infty$, the vectors U_n become parallel to the eigenvector $\overline{U}$ which has the largest eigenvalue (in modulus). The meaning of the expansions (8) and (13) is that for small λ, the vectors U_n have only small fluctuations around the direction of $\overline{U}$.

It is clear that, if for $\lambda = 0$ the two eigenvalues have the same modulus or if they are equal, then the vectors U_n have no reason to become parallel to a well defined direction. Therefore, for small λ, we can no longer consider that the U_n have small fluctuations around a direction $\overline{U}$. In that case the expansion (13) will not be valid. This can be seen in the expression (13) where one sees that if $E \to 2$, i.e. $A \to 1$, then each term in the expansion diverges.

It is interesting to notice that by looking at the expansion (13) of γ, one can guess its range of validity. If we want to approach the point $E = 2$, one finds that as long as $E - 2$ is large compared with $\lambda^{4/3}$, the first term (the term $\log A$) in the expansion (13) is dominant. On the other hand for $(E-2)/\lambda^{4/3}$ finite, the first term ($\log A$), the second term (which contains $\langle V^2 \rangle$) and the fifth term (which contains $\langle V^2 \rangle^2$) of the expansion (13) become of the same order. If we define x by

$$E - 2 = \lambda^{4/3} x . \tag{14}$$

Then

$$A - 1 \simeq \lambda^{2/3} \sqrt{x} . \tag{15}$$

And one finds that for large x, the expression (13) gives us

$$\gamma \simeq \lambda^{2/3} \left[\sqrt{x} - \frac{\langle V^2 \rangle}{8\,x} - \frac{5}{128} \frac{\langle V^2 \rangle^2}{x^{5/2}} + \cdots \right] + 0(\lambda) . \tag{16}$$

So we see that for $A \to 1$ (i.e. $E \to 2$), the expansion (13) becomes singular and the $\lambda^{2/3}$ can already be found. We should notice that $A \to \pm i$, i.e. at the band centre $E = 0$, is also a point where the expansion (13) breaks down because the fifth term diverges. The correct study of this band centre was done by Kappus and Wegner [15] and will be discussed in section 5. One would expect that, if the expansion (13) was pushed further, denominators like $A^6 - 1$, $A^8 - 1$, $A^{10} - 1$, ... would appear at higher orders and therefore that the expansion (13) would break down in the neighbourhood of any energy $E = 2 \cos \pi\alpha$ with α rational.

3. The neighbourhood of the band edge.

One can always formulate the problem of calculating the Lyapounov exponent γ as finding a stationary probability distribution for the R_n. This distribution, that we shall denote $P(R, E, \lambda)$ depends in principle on R, on the energy E, on the parameter λ and of course on the whole distribution $\rho(V)$ of the random potential V_n. $P(R, E, \lambda)$ obeys the following integral equation :

$$P(R, E, \lambda) = \int \rho(V)\, \mathrm{d}V \int P(R', E, \lambda) \times \delta\left(R - E + \lambda V + \frac{1}{R'} \right) \mathrm{d}R' \tag{17}$$

which can be rewritten as

$$P(R, E, \lambda) = \int \rho(V)\, \mathrm{d}V \frac{1}{(E - R - \lambda V)^2} \times P\left(\frac{1}{E - R - \lambda V}, E, \lambda \right) . \tag{18}$$

Of course, if we were able to find the complete solution $P(R, E, \lambda)$ of this integral equation, the Lyapounov exponent γ would be easy to obtain by writing

$$\gamma = \int \mathrm{d}R\, P(R, E, \lambda) \log R . \tag{19}$$

In the following, we shall restrict ourselves to real energies. For real energies E, all the R_n are real. Since for positive R, one has $\log R = \log |R|$ and for negative R we choose $\log R = \log |R| + i\pi$, the real part $\mathrm{Re}\, \gamma$ and the imaginary part $\mathrm{Im}\, \gamma$ of γ are given by

$$\mathrm{Re}\, \gamma = \int_{-\infty}^{+\infty} \log |R|\, P(R, E, \lambda)\, \mathrm{d}R \tag{20}$$

$$\operatorname{Im} \gamma = i\pi \int_{-\infty}^{0} P(R, E, \lambda) \, \mathrm{d}R . \tag{21}$$

From (21), one sees that the density of states $\tilde{\rho}(E)$ is just

$$\tilde{\rho}(E) = - \frac{\mathrm{d}}{\mathrm{d}E} \left[\int_{-\infty}^{0} P(R, E, \lambda) \, \mathrm{d}R \right] = - \frac{1}{i\pi} \frac{\mathrm{d} \operatorname{Im} \gamma}{\mathrm{d}E} \tag{22}$$

since $\operatorname{Im} \gamma$ counts the number of nodes of the wave function.

One does not know how to solve (18) for an arbitrary distribution $\rho(V)$. What makes the calculations possible in the limit $E \to 2$ and $\lambda \to 0$ is that $P(R, E, \lambda)$ takes a scaling form

$$P(R, E, \lambda) \simeq \lambda^{-2/3} \, Q\left(\frac{R-1}{\lambda^{2/3}}, \frac{E-2}{\lambda^{4/3}} \right) \tag{23}$$

i.e. the function P which is a function of 3 variables becomes a function of 2 variables only.

One could have guessed this form because in section 2 we saw that when $(E-2)/\lambda^{4/3}$ becomes finite, several terms of the expansion (13) start to contribute and one has $\log R \sim \log A \sim \lambda^{2/3}$ for this range of values of E.

However the best justification of (23) is that by looking for a solution of the form (23), we can solve equation (18) to leading order in λ. To see that let us make the following change of variables

$$E = 2 + \lambda^{4/3} \, x \tag{24}$$

$$R = 1 + \lambda^{2/3} \, t \tag{25}$$

and let us define $H(x, t, \lambda)$ by

$$H(t, x, \lambda) = \lambda^{2/3} \, P(1 + \lambda^{2/3} \, t, 2 + \lambda^{4/3} \, x, \lambda) . \tag{26}$$

The integral equation (18) becomes

$$H(t, x, \lambda) = \int \rho(V) \, \mathrm{d}V (1 - \lambda^{2/3} \, t - \lambda V + \lambda^{4/3} \, x)^{-2} \, H\left(\frac{t + \lambda^{1/3} \, V - \lambda^{2/3} \, x}{1 - \lambda^{2/3} \, t - \lambda V + \lambda^{4/3} \, x}, x, \lambda \right) . \tag{27}$$

If we expand the right hand side of (27) in powers of λ, we get :

$$\begin{aligned} H = \int \rho(V) \, \mathrm{d}V \Big\{ & H + \lambda^{1/3} \, VH' + \lambda^{2/3} \left[2 \, tH + (t^2 - x) \, H' + \frac{V^2}{2} H'' \right] + \\ & + \lambda \left[2 \, VH + 4 \, VtH' + V(t^2 - x) \, H'' + \frac{V^3}{6} H''' \right] + \lambda^{4/3} \Big[(3 \, t^2 - 2 \, x) \, H + (3 \, t^3 - 4 \, xt + 3 \, V^2) \, H' \\ & + \left(3 \, V^2 \, t + \frac{(t^2 - x)^2}{2} \right) H'' + \frac{t^2 - x}{2} V^2 \, H''' + \frac{V^4}{24} H'''' \Big] + 0(\lambda^{5/3}) \Big\} \end{aligned} \tag{28}$$

where H, H', H'', H''', H'''' mean respectively $H(t, x, \lambda)$, $\frac{\partial}{\partial t} H(t, x, \lambda)$, $\frac{\partial^2}{\partial t^2} H(t, x, \lambda)$, etc...

If is easy to perform in (28) the average over V and one gets using the fact that $\int \rho(V) \, V \, \mathrm{d}V = 0$

$$H = H + \lambda^{2/3} \left[2 \, tH + (t^2 - x) \, H' + \frac{\langle V^2 \rangle}{2} H'' \right] + \lambda \left[\frac{\langle V^3 \rangle}{6} H''' \right] + 0(\lambda^{4/3}) . \tag{29}$$

One expects that the solution of (29) can be expanded in the following way :

$$H(t, x, \lambda) = H_0(t, x) + \lambda^{1/3} \, H_1(t, x) + \lambda^{2/3} \, H_2(t, x) + \cdots . \tag{30}$$

One sees that if we keep the leading order in λ of equation (29) (i.e. the order $\lambda^{2/3}$), the function H_0 has to obey the following differential equation

$$2 \, tH_0 + (t^2 - x) \, H_0' + \frac{\langle V^2 \rangle}{2} H_0'' = 0 . \tag{31}$$

The general solution of the differential equation (31) is easy to obtain by noticing that (31) can be rewritten as

$$\frac{\mathrm{d}}{\mathrm{d}t} \left[(t^2 - x) \, H_0 + \frac{\langle V^2 \rangle}{2} H_0' \right] = 0 \tag{32}$$

and the general solution of (32) is

$$H_0(t) = C \exp\left\{ -\frac{2}{\langle V^2 \rangle}\left(\frac{1}{3} t^{3} - xt\right)\right\} \int_{-\infty}^{t} \exp\left\{\frac{2}{\langle V^2 \rangle}\left(\frac{1}{3} t'^3 - xt'\right)\right\} \mathrm{d}t' + \\ + C_1 \exp - \frac{2}{\langle V^2 \rangle}\left(\frac{1}{3} t^3 - xt\right). \tag{33}$$

Since $H_0(t)$ is a probability distribution, it should be integrable and therefore the constant C_1 has to vanish

$$C_1 = 0 . \tag{34}$$

We can now find the expression of Re γ using (20), (25), (26) and (33) :

$$\operatorname{Re} \gamma \simeq \left[\int_{-\infty}^{+\infty} \log | 1 + \lambda^{2/3} t | H_0(t) \, \mathrm{d}t\right] \Big/ \left[\int_{-\infty}^{+\infty} H_0(t) \, \mathrm{d}t\right] \simeq$$

$$\simeq \lambda^{2/3} \frac{\displaystyle\int_{-\infty}^{+\infty} t \, \mathrm{d}t \int_{-\infty}^{t} \mathrm{d}t' \exp\left[\frac{2}{\langle V^2 \rangle}\left(\frac{1}{3} t'^3 - xt' - \frac{1}{3} t^3 + xt\right)\right]}{\displaystyle\int_{-\infty}^{+\infty} \mathrm{d}t \int_{-\infty}^{t} \mathrm{d}t' \exp\left[\frac{2}{\langle V^2 \rangle}\left(\frac{1}{3} t'^3 - xt' - \frac{1}{3} t^3 + xt\right)\right]} \tag{35}$$

which becomes after simplification

$$\operatorname{Re} \gamma = \lambda^{2/3} \langle V^2 \rangle^{1/3} \frac{1}{2} \frac{\displaystyle\int_0^{\infty} t^{1/2} \, \mathrm{d}t \exp\left(-\frac{1}{6} t^3 + 2 Xt\right)}{\displaystyle\int_0^{\infty} t^{-1/2} \, \mathrm{d}t \exp\left(-\frac{1}{6} t^3 + 2 Xt\right)} \tag{36}$$

where X is defined by

$$X = x/\langle V^2 \rangle^{2/3} = \frac{E - 2}{\lambda^{4/3} \langle V^2 \rangle^{2/3}} . \tag{37}$$

Similarly one finds for Im γ :

$$\operatorname{Im} \gamma = i\pi \left[\int_{-\infty}^{-\lambda^{-2/3}} H_0(t) \, \mathrm{d}t\right] \Big/ \left[\int_{-\infty}^{+\infty} H_0(t) \, \mathrm{d}t\right] \tag{38}$$

which becomes after simplification

$$\operatorname{Im} \gamma \simeq i\pi \frac{\lambda^{2/3} \langle V^2 \rangle^{1/3}}{\sqrt{2}\, \pi} \frac{1}{\displaystyle\int_0^{\infty} t^{-1/2} \, \mathrm{d}t \exp\left(-\frac{1}{6} t^3 + 2 Xt\right)} . \tag{39}$$

Formulae (36) and (39) give us the Lyapounov exponent γ in the neighbourhood of the band edge $E = 2$. The real part Re γ is just the inverse localization length whereas the density of states $\tilde{\rho}(E)$ is given by (22) :

$$\tilde{\rho}(E) = \frac{\lambda^{-2/3} \langle V^2 \rangle^{-1/3}}{\sqrt{\pi}} \frac{\sqrt{2} \displaystyle\int_0^{\infty} t^{1/2} \, \mathrm{d}t \exp\left(-\frac{1}{6} t^3 + 2 Xt\right)}{\left[\displaystyle\int_0^{\infty} t^{-1/2} \, \mathrm{d}t \exp\left(-\frac{1}{6} t^3 + 2 Xt\right)\right]^2} . \tag{40}$$

If we choose $E = 2$, i.e. $X = 0$, the integrals in (36) and (39) can be expressed in terms of Γ functions :

For $X = 0$,
$$\mathrm{Re}\,\gamma = (\lambda^2 \langle V^2 \rangle)^{1/3} \frac{(6)^{1/3}\sqrt{\pi}}{2\,\Gamma\left(\frac{1}{6}\right)} = 0.289\,3\ldots (\lambda^2 \langle V^2 \rangle)^{1/3} \tag{41}$$

$$\mathrm{Im}\,\gamma/i\pi = (\lambda^2 \langle V^2 \rangle)^{1/3} \frac{3}{\sqrt{2}\,\pi(6)^{1/6}\,\Gamma\left(\frac{1}{6}\right)} = 0.159\,5\ldots (\lambda^2 \langle V^2 \rangle)^{1/3}\,. \tag{42}$$

One should notice that expressions (36) and (40) are very similar to those found in the continuous case [14, 5, 17].

For $X \to +\infty$, one can estimate (36) by the saddle point method and one recovers (16). Similarly, for $X \to -\infty$, the combination of (36), (39) and of the steepest descent method gives (16).

4. Expansion near an energy $E = 2\cos\pi\alpha$ with α rational.

Let us now describe a method of deriving the weak disorder expansion of γ which should work at all the energies $E = 2\cos\pi\alpha$

$$E = 2\cos\pi\alpha \tag{43}$$

with α rational.

As in section 3, our starting point is the integral equation (18) and we shall use (20) and (21) to calculate γ.

Like Kappus and Wegner [15], we make the following change of variables

$$R = \frac{\sin(\varphi + \pi\alpha)}{\sin\varphi} \tag{44}$$

and we define $G(\varphi)$ by

$$G(\varphi) = P(R, E, \lambda)\frac{\mathrm{d}R}{\mathrm{d}\varphi}. \tag{45}$$

When R goes from $-\infty$ to $+\infty$, φ goes from 0 to π. The integral equation (18) becomes an integral equation for $G(\varphi)$

$$G(\varphi) = \int \rho(V)\,\mathrm{d}V\,G(\varphi')\frac{\partial\varphi'}{\partial\varphi} \tag{46}$$

where φ' is a function of φ, E and V given by

$$\varphi' = \varphi - \pi\alpha + \frac{1}{2i}\log\left[\frac{\sin\pi\alpha + \lambda V\,\mathrm{e}^{-i\varphi}\sin\varphi}{\sin\pi\alpha + \lambda V\,\mathrm{e}^{i\varphi}\sin\varphi}\right]. \tag{47}$$

Since (47) is equivalent to

$$R = \frac{\sin(\varphi + \pi\alpha)}{\sin\varphi} = 2\cos\pi\alpha - \lambda V - \frac{\sin\varphi'}{\sin(\varphi' + \pi\alpha)} = E - \lambda V - \frac{1}{R'}. \tag{48}$$

From formula (47), one can check that

$$\frac{\partial\varphi'}{\partial\lambda} = -\frac{V\sin^2\varphi}{\sin\pi\alpha}\frac{\partial\varphi'}{\partial\varphi} \tag{49}$$

and using this identity, one can show that for any function G, one has

$$\frac{\partial}{\partial\lambda}\left[G(\varphi')\frac{\partial\varphi'}{\partial\varphi}\right] = -\frac{V}{\sin\pi\alpha}\frac{\partial}{\partial\varphi}\left[\sin^2\varphi\,G(\varphi')\frac{\partial\varphi'}{\partial\varphi}\right]. \tag{50}$$

For $\lambda = 0$, one has $\varphi' = \varphi - \pi\alpha$, and therefore

For $\lambda = 0$,
$$G(\varphi')\frac{\partial\varphi'}{\partial\varphi} = G(\varphi - \pi\alpha)\,. \tag{51}$$

From (50) and (51), it follows that

$$G(\varphi')\frac{\partial\varphi'}{\partial\varphi} = \exp\left(-\frac{\lambda V}{\sin\pi\alpha}\frac{\partial}{\partial\varphi}\sin^2\varphi\right)G(\varphi-\pi\alpha) \tag{52}$$

$$= \sum_{p=0}^{\infty}\frac{(-)^p}{p\,!}\left(\frac{\lambda V}{\sin\pi\alpha}\right)^p\left(\frac{\partial}{\partial\varphi}\sin^2\varphi\right)^p G(\varphi-\pi\alpha)\,. \tag{53}$$

The integral equation (46) can therefore be rewritten as

$$G(\varphi) = \left\langle \exp\left(-\frac{\lambda V}{\sin\pi\alpha}\frac{\partial}{\partial\varphi}\sin^2\varphi\right)\right\rangle G(\varphi-\pi\alpha)\,. \tag{54}$$

Our task is to find the solution $G(\varphi)$ of (54) which is a periodic function of φ :

$$G(\varphi+\pi) = G(\varphi)\,. \tag{55}$$

Since equation (54) is completely equivalent to the integral equation (18), we have no hope to solve it in general. However, one can expand (54) in powers of λ and look for a solution $G(\varphi)$ that we expand also in λ

$$G(\varphi) = G_0(\varphi) + \lambda G_1(\varphi) + \lambda^2 G_2(\varphi) + \lambda^3 G_3(\varphi)\,. \tag{56}$$

Our method consists in finding the solution $G(\varphi)$ of (54) perturbatively in λ.

When we expand equation (54) up to a given power of λ, the main problem is that we get a differential equation which is non local since it relates the function G at the points φ and $\varphi - \pi\alpha$. The simplification which occurs for α rational

$$\alpha = \frac{r}{s} \tag{57}$$

is that one can iterate (54) s times and get

$$G(\varphi) = \prod_{p=1}^{s}\left[\left\langle \exp\left(-\frac{\lambda V}{\sin\pi\alpha}\frac{\partial}{\partial\varphi}\sin^2(\varphi+p\pi\alpha)\right)\right\rangle\right]G(\varphi)\,. \tag{58}$$

So for α rational, one can obtain a local equation.

One may be interested by a whole neighbourhood of an energy $2\cos\pi\alpha$ with α rational. If one consider an energy E' of the form

$$E' = E + \lambda^2 x = 2\cos\pi\alpha + \lambda^2 x \tag{59}$$

by definition of x, then the equation (54) is replaced by

$$G(\varphi) = \left\langle \exp\left(\frac{\lambda^2 x - \lambda V}{\sin\pi\alpha}\frac{\partial}{\partial\varphi}\sin^2\varphi\right)\right\rangle G(\varphi-\pi\alpha)\,. \tag{60}$$

In the appendix we give a useful expression of the expansion of (60) up to the power λ^4.

Once G is known up to a given power of λ, one can obtain the Lyapounov exponent formula by

$$\mathrm{Re}\,\gamma = \left[\int_0^{\pi}\log\left|\frac{\sin(\varphi+\pi\alpha)}{\sin\varphi}\right|G(\varphi)\,\mathrm{d}\varphi\right]\Big/\int_0^{\pi}G(\varphi)\,\mathrm{d}\varphi \tag{61}$$

and

$$\mathrm{Im}\,\gamma = i\pi\left[\int_{\pi(1-\alpha)}^{\pi}G(\varphi)\,\mathrm{d}\varphi\right]\Big/\int_0^{\pi}G(\varphi)\,\mathrm{d}\varphi \tag{62}$$

as one can see from (20), (21) and (44).

We shall see that (61) and (62) can be transformed to shorten the calculations. For example (61) can be rewritten as

$$\mathrm{Re}\,\gamma = \frac{\displaystyle\int_0^\pi \log(\sin\varphi)\,[G(\varphi - \pi\alpha) - G(\varphi)]\,\mathrm{d}\varphi}{\displaystyle\int_0^\pi G(\varphi)\,\mathrm{d}\varphi} \tag{63}$$

and since $G(\varphi - \pi\alpha) - G(\varphi)$ starts like λ^2, one needs to know $G(\varphi)$ up to order λ^{n-2} if one wants the expansion of $\mathrm{Re}\,\gamma$ up to order λ^n.

In the next sections, we shall consider explicitly the cases $\alpha = \frac{1}{2}$ and $\alpha = \frac{1}{3}$.

5. The band centre.

We shall now see how the method presented in the previous section can be applied to the case $\alpha = \frac{1}{2}$.

For a given energy E',

$$E' = \lambda^2 x \tag{64}$$

we are going to look for a solution of (60) of the form (56)

$$G(\varphi) = G_0(\varphi) + \lambda G_1(\varphi) + \lambda^2 G_2(\varphi) + \cdots.$$

Using the expression of (60) given in the appendix, we get a hierarchy of equations for G_0, G_1, G_2, ...

$$G_0(\varphi) = G_0\left(\varphi - \frac{\pi}{2}\right) \tag{65}$$

$$G_1(\varphi) = G_1\left(\varphi - \frac{\pi}{2}\right) \tag{66}$$

$$\begin{aligned} G_2(\varphi) - G_2\left(\varphi - \frac{\pi}{2}\right) = \frac{x}{2}\left[(1 - \cos 2\varphi)\frac{\partial}{\partial\varphi} + 2\sin 2\varphi\right] G_0\left(\varphi - \frac{\pi}{2}\right) + \\ + \frac{\langle V^2 \rangle}{16}\left[(3 - 4\cos 2\varphi + \cos 4\varphi)\frac{\partial^2}{\partial\varphi^2} + (12\sin 2\varphi - 6\sin 4\varphi)\frac{\partial}{\partial\varphi}\right. \\ \left. + (8\cos 2\varphi - 8\cos 4\varphi)\right] G_0\left(\varphi - \frac{\pi}{2}\right). \end{aligned} \tag{67}$$

One sees clearly that equation (65) or (66) are not sufficient to determine the functions G_0 and G_1. However since $G(\varphi)$ is a periodic function of period π, this means that $G_2\left(\varphi + \frac{\pi}{2}\right) = G_2\left(\varphi - \frac{\pi}{2}\right)$ and therefore equations (67) and (65) give

$$\begin{aligned} 0 &= G_2\left(\varphi + \frac{\pi}{2}\right) - G_2\left(\varphi - \frac{\pi}{2}\right) = G_2\left(\varphi + \frac{\pi}{2}\right) - G_2(\varphi) + G_2(\varphi) - G_2\left(\varphi - \frac{\pi}{2}\right) = \\ &= x\frac{\partial}{\partial\varphi} G_0(\varphi) + \frac{\langle V^2 \rangle}{8}\left[(3 + \cos 4\varphi)\frac{\partial^2}{\partial\varphi^2} G_0(\varphi) - 6\sin 4\varphi\frac{\partial}{\partial\varphi} G_0(\varphi) - 8\cos 4\varphi\, G_0(\varphi)\right] = 0 \end{aligned} \tag{68}$$

So (68) gives us a differential equation which will determine $G_0(\varphi)$. The idea followed to obtain (68) is exactly the same as the one which led to (60). Although (68) is a second order differential equation, the fact that G_0 is a periodic function (see (65)) determines G_0 uniquely. For example, when x is small, one can expand the general solution of (68). One finds for $x \ll 1$:

$$\begin{aligned} G_0(\varphi) = \frac{C}{(3 + \cos 4\varphi)^{1/2}}\left[1 + \frac{ix}{\langle V^2 \rangle\sqrt{2}} \log\left(\frac{e^{4i\varphi}(\sqrt{2} + 1) + \sqrt{2} - 1}{e^{4i\varphi}(\sqrt{2} - 1) + \sqrt{2} + 1}\right) + \right. \\ \left. + \frac{8\sqrt{\pi}\,\Gamma\left(\frac{3}{4}\right) x}{\Gamma\left(\frac{1}{4}\right)\langle V^2 \rangle}\int_0^\varphi (3 + \cos 4\varphi')^{-1/2}\,\mathrm{d}\varphi'\right] + \frac{C_1}{(3 + \cos 4\varphi)^{1/2}}\int_0^\varphi (3 + \cos 4\varphi')^{-1/2}\,\mathrm{d}\varphi' + 0(x^2). \end{aligned} \tag{69}$$

There are 2 arbitrary constants C and C_1 because (68) was a second order differential equation. However to satisfy the condition that $G(\varphi)$ is periodic, C_1 in (69) has to be zero

$$C_1 = 0 . \tag{70}$$

Similarly one can see easily that (68) determines G_0 uniquely for $x \gg 1$:

$$G_0(\varphi) = 1 + \frac{\langle V^2 \rangle}{4x} \sin 4\varphi - \frac{\langle V^2 \rangle^2}{32 x^2} (12 \cos 4\varphi + 3 \cos 8\varphi) + 0\left(\frac{1}{x^3}\right). \tag{71}$$

We were only able to find explicitly $G_0(\varphi)$ for $x \ll 1$ or $x \gg 1$. For finite x, one can solve numerically the differential equation (68).

Let us now obtain the expression of γ up to order λ^2 in terms of $G_0(\varphi)$. From (63), we see that

$$\operatorname{Re} \gamma = \frac{\displaystyle\int_0^\pi \log(\sin\varphi)\left(G\left(\varphi - \frac{\pi}{2}\right) - G(\varphi)\right) d\varphi}{\displaystyle\int_0^\pi G_0(\varphi)\, d\varphi}. \tag{72}$$

From (65), (66) and (67), one finds that

$$\operatorname{Re} \gamma = \lambda^2 \frac{\displaystyle\int_0^\pi \log(\sin\varphi)\left(G_2\left(\varphi - \frac{\pi}{2}\right) - G_2(\varphi)\right)}{\displaystyle\int_0^\pi G_0(\varphi)\, d\varphi} d\varphi + 0(\lambda^3) \tag{73}$$

which becomes after a short calculation (which uses (67))

$$\operatorname{Re} \gamma = \lambda^2 \frac{\langle V^2 \rangle}{8} \frac{\displaystyle\int_0^\pi (1 + \cos 4\varphi)\, G_0(\varphi)\, d\varphi}{\displaystyle\int_0^\pi G_0(\varphi)\, d\varphi} + 0(\lambda^3) \tag{74}$$

Using (62), we can obtain the imaginary part Im γ

$$\operatorname{Im} \gamma = \frac{i\pi \displaystyle\int_{\pi/2}^\pi G(\varphi)\, d\varphi}{\displaystyle\int_0^\pi G(\varphi)\, d\varphi} = \frac{i\pi}{2} + \frac{i\pi}{2} \frac{\displaystyle\int_{\pi/2}^\pi \left[G(\varphi) - G\left(\varphi - \frac{\pi}{2}\right)\right] d\varphi}{\displaystyle\int_0^\pi G(\varphi)\, d\varphi} \tag{75}$$

The using (65), (66) and (67), one finds that the expansion of Im γ up to the order λ^2 is just

$$\operatorname{Im} \gamma = \frac{i\pi}{2} + \frac{i\pi}{2} \lambda^2 \frac{\displaystyle\int_{\pi/2}^\pi \left[G_2(\varphi) - G_2\left(\varphi - \frac{\pi}{2}\right)\right] d\varphi}{\displaystyle\int_0^\pi G_0(\varphi)\, d\varphi}$$

$$\operatorname{Im} \gamma = \frac{i\pi}{2} - \frac{i\pi}{2} \lambda^2 \frac{x G_0(0) + \langle V^2 \rangle \frac{\partial G_0}{\partial \varphi}(0)/2}{\displaystyle\int_0^\pi G_0(\varphi)\, d\varphi} \tag{76}$$

For any value of x, one has to find first the periodic solution of the differential equation (68) and then Re γ and Im γ are given by (74) and (76).

For $x \ll 1$, we have in (69) the expression for $G_0(\varphi)$. In that case we get

$$\operatorname{Re}\gamma = \lambda^2 \left[\left(\frac{\Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{1}{4}\right)} \right)^2 \langle V^2 \rangle + 0(x) \right] \simeq 0.11424... \lambda^2 \langle V^2 \rangle \tag{77a}$$

$$\operatorname{Im}\gamma = \frac{i\pi}{2} \left[1 - \lambda^2 x 2\sqrt{2} \left(\frac{\Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{1}{4}\right)} \right)^2 \right] + 0(x^2)$$

$$\tilde{\rho}(E) = \sqrt{2} \left(\frac{\Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{1}{4}\right)} \right)^2 + 0(x) \simeq 0.161\,56\,... \tag{77c}$$

For $x \gg 1$, we get from (71)

$$\operatorname{Re}\gamma = \lambda^2 \langle V^2 \rangle \left[\frac{1}{8} - \frac{3}{128} \frac{\langle V^2 \rangle^2}{x^2} \right] + 0\left(\frac{1}{x^3}\right) \tag{78a}$$

$$\operatorname{Im}\gamma = \frac{i\pi}{2} \left[1 - \lambda^2 \left(\frac{x}{\pi} + \frac{1}{32\,\pi} \frac{\langle V^2 \rangle^2}{x} + 0\left(\frac{1}{x^2}\right) \right) \right] \tag{78b}$$

$$\tilde{\rho}(E) = \frac{1}{2\,\pi} \left(1 - \frac{1}{32} \frac{\langle V^2 \rangle^2}{x^2} \right) \tag{78c}$$

All our results (77) and (78) are in complete agreement with those of Kappus and Wegner [15] after an appropriate change of notation. As they did, we can compare these results with the order λ^2 of the expansion (13) (which is known to be incorrect in the limit $E \to 0$)

$$\operatorname{Re}\gamma = \frac{\lambda^2 \langle V^2 \rangle}{8}; \qquad \operatorname{Im}\gamma = \frac{i\pi}{2} \left[1 - \frac{\lambda^2 x}{\pi} \right]; \qquad \tilde{\rho}(E) = \frac{1}{2\,\pi}. \tag{79}$$

One shoult notice that (79) is just what one gets if in (74) and (76) we had replaced G_0 by a constant, i.e. we had believed that the solution G_0 of (65) is a constant and not a periodic function. In principle one should be able to calculate $G_1(\varphi)$, $G_2(\varphi)$... and to obtain higher orders in the λ expansions of γ.

6. The energy $E = 1$.

We want now to apply the method described in section 4 to the case $\alpha = \frac{1}{3}$. We have again to find perturbatively in λ the solution $G(\varphi)$ of (60) :

$$G(\varphi) = G_0(\varphi) + \lambda G_1(\varphi) + \lambda^2 G_2(\varphi) + \cdots$$

for an energy E'

$$E' = 1 + \lambda^2 x = 2 \cos\left(\frac{\pi}{3}\right) + \lambda^2 x. \tag{80}$$

As in section 5, the equation (60) gives us a hierarchy of equations for G_0, G_1, G_2, ... when we equate the two sides of the equation order by order in λ

$$G_0(\varphi) = G_0\left(\varphi - \frac{\pi}{3}\right) \tag{81}$$

$$G_1(\varphi) = G_1\left(\varphi - \frac{\pi}{3}\right). \tag{82}$$

The next order (order λ^2) determines the function G_0 and gives also an equation for $G_2(\varphi)$.

It implies that the second derivative of G_0 should vanish and therefore that G_0 is constant because of (81)

$$G_0(\varphi) = 1 \tag{83}$$

and then G_2 has to satisfy

$$G_2(\varphi) - G_2\left(\varphi - \frac{\pi}{3}\right) = \frac{x}{\sqrt{3}} 2 \sin 2\,\varphi + \frac{\langle V^2 \rangle}{12} (8 \cos 2\,\varphi - 8 \cos 4\,\varphi) \tag{84}$$

The next order (λ^3) gives an equation for $G_1(\varphi)$

$$\left(\frac{x}{\sqrt{3}}\frac{\partial}{\partial\varphi} + \frac{\langle V^2\rangle}{4}\frac{\partial^2}{\partial\varphi^2}\right)G_1\left(\varphi - \frac{\pi}{3}\right) + \frac{2\langle V^3\rangle}{3\sqrt{3}}\sin 6\varphi = 0\,. \quad (85)$$

These equations can be easily solved:

$$G_1(\varphi) = \frac{\langle V^3\rangle}{81\langle V^2\rangle^2 + 12x^2} \times \left[2\sqrt{3}\langle V^2\rangle\sin 6\varphi + \frac{4}{3}x\cos 6\varphi\right] \quad (86)$$

and

$$G_2(\varphi) = \frac{2\langle V^2\rangle}{3\sqrt{3}}\left[\cos\left(2\varphi - \frac{\pi}{6}\right) - \cos\left(4\varphi + \frac{\pi}{6}\right)\right] + \frac{2x}{3}\sin\left(2\varphi - \frac{\pi}{6}\right) + W(\varphi) \quad (87)$$

where $W(\varphi)$ is a periodic function of period $\frac{\pi}{3}$: $W(\varphi) = W\left(\varphi + \frac{\pi}{3}\right)$ which cannot be determined from (84) but should be determined from further equations in the hierarchy. We shall not determine it because it will not be used later.

Let us now calculate the real part and the imaginary part of γ. Re γ can be written as

$$\operatorname{Re}\gamma = \frac{\int_0^\pi \log(\sin\varphi)\left(G\left(\varphi - \frac{\pi}{3}\right) - G(\varphi)\right)\mathrm{d}\varphi}{\int_0^\pi G(\varphi)\,\mathrm{d}\varphi} \quad (88)$$

which can be written up to order λ^4 using the expression given in the appendix and a few integrations by parts

$$\begin{aligned}\left[\int_0^\pi G(\varphi)\,\mathrm{d}\varphi\right]\operatorname{Re}\gamma = {} & \frac{\lambda^2 x}{\sqrt{3}}\int_0^\pi \sin 2\varphi\, G\left(\varphi - \frac{\pi}{3}\right)\mathrm{d}\varphi \\ & - \frac{\lambda^4 x^2 + \lambda^2\langle V^2\rangle}{6}\int_0^\pi(-1 + 2\cos 2\varphi - \cos 4\varphi)\,G\left(\varphi - \frac{\pi}{3}\right)\mathrm{d}\varphi \\ & + \frac{3\lambda^4 x\langle V^2\rangle - \lambda^3\langle V^3\rangle}{9\sqrt{3}}\int_0^\pi(-3\sin 2\varphi + 3\sin 4\varphi - \sin 6\varphi)\,G\left(\varphi - \frac{\pi}{3}\right)\mathrm{d}\varphi \\ & - \frac{\lambda^4\langle V^4\rangle}{108}\int_0^\pi(3 - 12\cos 2\varphi + 8\cos 4\varphi - 12\cos 6\varphi + 3\cos 8\varphi)\,G\left(\varphi - \frac{\pi}{3}\right)\mathrm{d}\varphi\,.\end{aligned}$$

Using the expressions (83), (86) and (87), we find

$$\operatorname{Re}\gamma = \frac{\lambda^2\langle V^2\rangle}{6} + \lambda^4\left[x\frac{\langle V^2\rangle}{9} + \frac{3\langle V^2\rangle^2 - \langle V^4\rangle}{36} + \frac{\langle V^3\rangle^2\langle V^2\rangle}{9(81\langle V^2\rangle^2 + 12x^2)}\right] + 0(\lambda^5)\,. \quad (89)$$

Similarly by writting Im γ in the following way

$$\begin{aligned}\operatorname{Im}\gamma &= \frac{i\pi}{3}\left[\int_{\frac{2\pi}{3}}^{\pi} G(\varphi)\,\mathrm{d}\varphi\right]\Big/\left[\int_0^\pi G(\varphi)\,\mathrm{d}\varphi\right] \\ &= \frac{i\pi}{3}\left[1 + \frac{2\int_{2\pi/3}^{\pi}\left(G(\varphi) - G\left(\varphi - \frac{\pi}{3}\right)\right)\mathrm{d}\varphi + \int_{\pi/3}^{2\pi/3}\left(G(\varphi) - G\left(\varphi - \frac{\pi}{3}\right)\right)\mathrm{d}\varphi}{\int_0^\pi G(\varphi)\,\mathrm{d}\varphi}\right]\end{aligned}$$

and by using the expression given in the appendix, one gets:

$$\operatorname{Im}\gamma = \frac{i\pi}{3}\left[1 - \frac{\lambda^2 x\sqrt{3}}{\pi} - \frac{\lambda^3\langle V^2\rangle^2\langle V^3\rangle 9\sqrt{3}}{\pi(81\langle V^2\rangle^2 + 12x^2)}\right] + 0(\lambda^4)\,. \quad (90)$$

Formulae (89) and (90) give our final results for the neighbourhood of the energy $E = 1$. We see in (89) the presence of a term which contains $\langle V^3 \rangle$ whereas in the expansion (13) no term contains $\langle V^3 \rangle$ at the order λ^4. This term is an anomaly of the same nature as the one discussed in section 5.

We see also in (90) that the term which contains $\langle V^3 \rangle$ depends on $\langle V^2 \rangle$ whereas such a term does not appear in (13) at order λ^3.

In this section, we have seen that in the neighbourhood of $E = 1$, one can find an anomaly very similar to the one which occurs in the neighbourhood of $E = 0$. Such an anomaly at $E = 1$ has been noticed in numerical work by Pichard [16] and the analytic work of Lambert [18].

As in the section 5, we notice that the anomaly is due to the fact that $G_1(\varphi)$ is a periodic function of period $\frac{\pi}{3}$. If we had believed from (82) that $G_1(\varphi)$ was a constant, then, we would not have found the anomaly.

7. Conclusion.

In this paper we have described several kinds of weak disorder expansions of the Lyapounov exponent γ : the expansion of section 2 is valid outside the spectrum of the pure system, the expansion of section 3 is valid in the neighbourhood of the band edge and the expansion of section 4 should be valid in the neighbourhood of the energies of the form $E = 2 \cos \pi\alpha$ with α rational.

In section 5 and 6 we have applied the method described in section 4 to the cases $\alpha = \frac{1}{2}$ and $\alpha = \frac{1}{3}$. We think that it is interesting to notice that the band centre anomaly $\left(\alpha = \frac{1}{2}\right)$ has a counterpart for $\alpha = \frac{1}{3}$. We effect should occur for all rational $\alpha = r/s$ although the power of λ at which it can be seen will increase with s [18].

We believe that the origin of these anomalies is the fact that the function $G(\varphi)$ contains a periodic function of period $\pi\alpha$. It would be interesting to generalise the results for $\alpha = \frac{1}{2}$ and $\frac{1}{3}$ to other rationals. In doing so, we think that the method presented in section 4 constitutes a good starting point.

Also we think that is should be interesting to extend the results presented here to quasiperiodic situations.

Acknowledgments.

This work was supported by a Research Fellowship from the Royal Society. It was mostly motivated by discussions with J. Avron and J. Lacroix and by the numerical work of G. Bennetin. We would like to than them for their help as well as D. Pesme, J. L. Pichard and B. Souillard for giving us related references.

Appendix.

We give an expression of the expansion of (60) up to the power λ^4

$$
\begin{aligned}
G(\varphi) = \Bigg\{ 1 &+ \frac{\lambda^2 x}{2 \sin \pi\alpha}\left[(1 - \cos 2\varphi)\frac{\partial}{\partial\varphi} + 2 \sin 2\varphi\right] + \\
&+ \frac{\lambda^2 \langle V^2 \rangle + \lambda^4 x^2}{16 \sin^2 \pi\alpha}\Bigg[(3 - 4\cos 2\varphi + \cos 4\varphi)\frac{\partial^2}{\partial\varphi^2} + (12 \sin 2\varphi - 6 \sin 4\varphi)\frac{\partial}{\partial\varphi} + \\
&\qquad + (8 \cos 2\varphi - 8 \cos 4\varphi)\Bigg] \\
&- \frac{\lambda^3 \langle V^3 \rangle - 3\lambda^4 x \langle V^2 \rangle}{192 \sin^3 \pi\alpha}\Bigg[(10 - 15 \cos 2\varphi + 6 \cos 4\varphi - \cos 6\varphi)\frac{\partial^3}{\partial\varphi^3} + \\
&\qquad + (60 \sin 2\varphi - 48 \sin 4\varphi + 12 \sin 6\varphi)\frac{\partial^2}{\partial\varphi^2} \\
&\qquad + (-8 + 84 \cos 2\varphi - 120 \cos 4\varphi + 44 \cos 6\varphi)\frac{\partial}{\partial\varphi} \\
&\qquad + (-48 \sin 2\varphi + 96 \sin 4\varphi - 48 \sin 6\varphi)\Bigg]
\end{aligned}
$$

$$+ \frac{\lambda^4 \langle V^4 \rangle}{3\,072(\sin \pi\alpha)^4} \left[(35 - 56 \cos 2\varphi + 28 \cos 4\varphi - 8 \cos 6\varphi + \cos 8\varphi) \frac{\partial^4}{\partial\varphi^4} \right.$$

$$+ (280 \sin 2\varphi - 280 \sin 4\varphi + 120 \sin 6\varphi - 20 \sin 8\varphi) \frac{\partial^3}{\partial\varphi^3}$$

$$+ (-100 + 640 \cos 2\varphi - 1\,040 \cos 4\varphi + 640 \cos 6\varphi$$

$$- 140 \cos 8\varphi) \frac{\partial^2}{\partial\varphi^2} + (-800 \sin 2\varphi + 1\,760 \sin 4\varphi$$

$$- 1\,440 \sin 6\varphi + 400 \sin 8\varphi) \frac{\partial}{\partial\varphi} + (-384 \cos 2\varphi$$

$$\left. \left. + 1\,152 \cos 4\varphi - 1\,152 \cos 6\varphi + 384 \cos 8\varphi) \right] \right\} G(\varphi - \pi\alpha) .$$

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