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LYAPUNOV EQUATIONS, ENERGY FUNCTIONALS, AND MODEL ORDER REDUCTION OF BILINEAR AND STOCHASTIC SYSTEMS*

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Abstract. We discuss the relation of a certain type of generalized Lyapunov equations to Gramians of stochastic and bilinear systems together with the corresponding energy functionals. While Gramians and energy functionals of stochastic linear systems show a strong correspondence to the analogous objects for deterministic linear systems, the relation of Gramians and energy functionals for bilinear systems is less obvious. We discuss results from the literature for the latter problem and provide new characterizations of input and output energies of bilinear systems in terms of algebraic Gramians satisfying generalized Lyapunov equations. In any of the considered cases, the definition of algebraic Gramians allows us to compute balancing transformations and implies model reduction methods analogous to balanced truncation for linear deterministic systems. We illustrate the performance of these model reduction methods by showing numerical experiments for different bilinear systems.

Key words. Lyapunov equations, Gramians, energy functionals, balanced truncation, model order reduction, bilinear systems, stochastic systems

AMS subject classifications. 93A15, 93A30, 93C10, 93E20, 93B40

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1. Introduction.

Model order reduction by balanced truncation is a standard method, which has been introduced by Moore in [40] for linear deterministic control systems of the form

$$\dot{x} = Ax + Bu\,, \quad y = Cx\,, \tag{1.1}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$ and $x(t) \in \mathbb{R}^n$, $y(t) \in \mathbb{R}^p$, $u(t) \in \mathbb{R}^m$ are the state, output, and input of the system, respectively. It preserves stability and provides guaranteed error estimates. The main obstacle in its realization is the computation of controllability and observability Gramians as solutions of the dual Lyapunov equations

$$AP + PA^T = -BB^T, \quad A^TQ + QA = -C^TC\,. \tag{1.2}$$

Although this requires a higher effort than, e.g., methods, based on Krylov subspace approximations, there are algorithms which allow balanced truncation for sparse systems of dimensions $O(10^5)$ and more; see, e.g., [46, 5, 36, 29].

The appealing features of balanced truncation have motivated similar approaches for other system classes. In a series of papers, Scherpen and others (see, e.g., [50, 52, 51, 24, 27, 23, 59]) have developed a theory of balancing for nonlinear systems. The notion of Gramians is replaced by controllability and observability energy functionals. While on a conceptional base this generalization is quite attractive, often it is hardly practicable from the computational point of view, since the energy functionals are

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obtained as solutions of nonlinear Hamilton–Jacobi equations, which are very expensive for large dimensions. Recently in [33], there have been attempts to reduce the complexity of the optimality equations by POD methods, but nevertheless the scope of the approach seems to be limited. To overcome this drawback, other generalizations of Gramians have been considered, especially for bilinear systems in the context of model order reduction; cf. [1, 25, 26, 63, 64, 11]. These generalized Gramians are solutions of generalized Lyapunov equations of the forms

$$\begin{aligned} AP + PA^T + \sum_{j=1}^{m} A_j P A_j^T &= -BB^T, \\ A^T Q + QA + \sum_{j=1}^{m} A_j^T Q A_j &= -C^T C\,, \end{aligned} \tag{1.3}$$

where A, B, C are as in (1.1), (1.2), and $A_j \in \mathbb{R}^{n\times n}$ for $j = 1, \ldots, m$. If $A_j = 0$ for all j, then the linear matrix equations in (1.3) boil down to (1.2). Therefore, we call them generalized Lyapunov equations, but they should not be confused with other types of generalized Lyapunov equations such as

$$APE^T + EPA^T = -BB^T$$

arising in the context of generalized state-space systems [45]. The Gramians defined by (1.3) have already been considered in [49, 13] to characterize controllability and observability of bilinear systems

$$\dot{x} = Ax + \sum_{j=1}^{m} A_j u_j x + Bu\,, \quad y = Cx\,. \tag{1.4}$$

A first attempt to give an energy-based interpretation of these algebraic Gramians apparently was made by Gray and Mesko in [25]. Their results look quite promising and have been taken up recently, e.g., in [34, 17, 11, 28, 16]. Unfortunately, however, the characterization of energy functionals given in [25] does not hold in the stated generality. This issue, together with the nonuniqueness of singular value functions and balancing of nonlinear system, has been addressed by Gray and Scherpen in [24]. Their analysis is quite subtle and applies to general nonlinear systems. However, it does not discuss the special role of the algebraic Gramians of bilinear systems from [49, 13], which is of particular interest to us from the computational point of view. Moreover, the implications of [24] are not fully accounted for in subsequent papers on bilinear systems, e.g., [11].

Hence, for the special case of bilinear systems, we try to clarify conditions under which the algebraic Gramians give quantifiable information on reachability and observability properties of the state vectors. In section 3, we first suggest a new approach to characterize unreachable and unobservable states (Theorem 3.1) via the Gramians and then give a simple example to illustrate how an integrability condition contradicts the characterization of energy functionals in [25]. We also discuss some patches, which, however, do not give satisfactory error estimates for truncation errors.

Since we mainly aim at practical methods for model order reduction applicable to large-scale problems, we review solvability conditions for the generalized Lyapunov equations and, in section 4, provide numerical examples to support the significance of the generalized Gramians in (1.3) for model order reduction of a bilinear system (1.4)—at least in special cases.

On the other hand, it is a well-known fact that generalized Lyapunov equations of the forms (1.3) are naturally associated to stochastic linear control systems; see, e.g., [32, 31, 15]. Therefore it is not surprising that P and Q can be interpreted as Gramians of stochastic systems and that the method of balanced truncation can immediately be carried over to this class of systems. Although, of course, work has been done in this direction, e.g., in [42, 62, 59], to our knowledge this connection between bilinear and stochastic Gramians has not really been documented in the literature so far; it is thus another goal of this paper (pursued in section 2) to fill this gap and to open up the field for further research.

2. Gramians and energy functionals of linear systems. The representation of input and output energies for deterministic linear control systems as quadratic forms involving the Gramians is a classical result. Factorizations of the Gramians are used to compute balanced realizations which can be reduced by truncation. This method has first been described for time-invariant systems in [40] and for time-varying systems in [54, 60]. Our adaptation to stochastic systems is quite analogous. To clarify the idea and the notation as well as for later reference in the discussion of bilinear systems, we will briefly recapitulate some basic results for time-varying systems.

2.1. Time-varying deterministic linear equations. Let us consider a linear control system

$$\dot{x} = A(t)x + B(t)u\ , \quad y = C(t)x \tag{2.1}$$

with coefficient matrices $A(t) \in \mathbb{R}^{n\times n}$, $B(t) \in \mathbb{R}^{n\times m}$, and $C(t) \in \mathbb{R}^{p\times n}$ being measurable functions of t. Here $u \in \mathbb{R}^m$ and $y \in \mathbb{R}^p$ are called input and output vectors, while $x \in \mathbb{R}^n$ is the state vector. For a given measurable input function $u : \mathbb{R} \to \mathbb{R}^m$ and an initial vector $x_0 \in \mathbb{R}^n$, let $x(t, x_0, u)$ denote the solution of (2.1) with input u and $x(0, x_0, u) = x_0$; the corresponding output will be denoted by $y(t, x_0, u)$. For the fundamental solution of the homogeneous system $\dot{x} = A(t)x$, we write $\Phi(t, \tau)$.

Assuming that the homogeneous system $\dot{x} = A(t)x$ is exponentially asymptotically stable, we can define the controllability and observability Gramian by

$$\begin{aligned} P &= \int_{-\infty}^{0} \Phi(0,\tau)B(\tau)B(\tau)^T\Phi(0,\tau)^T\,d\tau\ , \\ Q &= \int_{0}^{\infty} \Phi(t,0)^T C(t)^T C(t)\Phi(t,0)\,dt\ . \end{aligned} \tag{2.2}$$

Furthermore, for $x_0 \in \mathbb{R}^n$, we define the input and output energy functionals as

$$\begin{aligned} E_c(x_0) &= \inf_{\substack{u\in L^2]-\infty,0] \\ x(-\infty,x_0,u)=0}} \int_{-\infty}^{0} \|u(t)\|^2\,dt\ , \\ E_o(x_0) &= \int_{0}^{\infty} \|y(t,x_0,0)\|^2\,dt\ . \end{aligned}$$

Note that $E_c(x_0) = \infty$ if x_0 cannot be reached from 0 over the time-interval $]-\infty, 0]$. It is easy to see that this is equivalent to $x_0 \notin \operatorname{Im} P$. The following result is well known. We present a proof both to motivate similar arguments for other systems and to discuss some issues of forward and backward solutions (see Remark 2.2), which are important for the stochastic case. Some details of the argument will also play a role in the bilinear setup.

THEOREM 2.1. *Consider the time-varying system* (2.1) *and the Gramians* P *and* Q *defined by* (2.2). *If* $x_0 \in \operatorname{Im} P$, *then*

$$E_c(x_0) = x_0^T P^\sharp x_0 \ ,$$

where $P^\sharp$ *denotes the Moore–Penrose inverse.*

For $x_0 \in \mathbb{R}^n$ *we have*

$$E_o(x_0) = x_0^T Q x_0 \ .$$

Proof. For fixed x_0 we define $u :]-\infty, 0] \to \mathbb{R}^m$ by

$$u(t) = B(t)^T \Phi(0,t)^T P^\sharp x_0 \ . \tag{2.3}$$

Then

$$x(t, x_0, u) = \int_{-\infty}^{t} \Phi(t,\tau) B(\tau) u(\tau)\, d\tau$$

is well defined by the exponential stability of the homogeneous equation and satisfies (2.1) as well as the boundary conditions $\lim_{t\to\infty} x(t, x_0, u) = 0$ and

$$\begin{aligned} x(0, x_0, u) &= \int_{-\infty}^{0} \Phi(0,\tau) B(\tau) u(\tau)\, d\tau = \int_{-\infty}^{0} \Phi(0,\tau) B(\tau) B(\tau)^T \Phi(0,\tau)^T\, d\tau P^\sharp x_0 \\ &= P P^\sharp x_0 = x_0 \ . \end{aligned}$$

Among all $\tilde{u}$ with $x(\infty, x_0, \tilde{u}) = 0$ the given control has minimal L^2-norm. To show this, let us assume that $\tilde{u} = u + \hat{u}$ is another solution to the control problem. Then

$$x_0 = \int_{-\infty}^{0} \Phi(0,\tau) B(\tau)(u(\tau) + \hat{u}(\tau))\, d\tau \ , \quad \text{whence} \quad \int_{-\infty}^{0} \Phi(0,\tau) B(\tau) \hat{u}(\tau)\, d\tau = 0 \ .$$

This implies $\int_{-\infty}^{0} u(t)^T \hat{u}(t)\, dt = 0$, so that

$$\|\tilde{u}\|_{L^2}^2 = \|u + \hat{u}\|_{L^2}^2 = \|u\|_{L^2}^2 + \|\hat{u}\|_{L^2}^2 \geq \|u\|_{L^2}^2 \ .$$

Since

$$\begin{aligned} \|u\|_{L^2}^2 &= \int_{-\infty}^{0} \|u(t)\|^2\, dt = \int_{-\infty}^{0} x_0^T P^\sharp \Phi(0,t) B(t) B(t)^T \Phi(0,t)^T P^\sharp x_0\, dt \\ &= x_0^T P^\sharp P P^\sharp x_0 = x_0^T P^\sharp x_0 \ , \end{aligned}$$

the proof of the first assertion is complete.

To prove the second, assume that the system starts in state x_0 and is not controlled. Then the corresponding output is $y(t) = C(t)\Phi(t,0)x_0$. The output energy is the L^2-norm of y,

$$\begin{aligned} E_o(x_0) = \|y\|_{L^2}^2 &= \int_0^\infty y(t)^T y(t)\, dt = x_0^T \left(\int_0^\infty \Phi(t,0)^T C(t)^T C(t) \Phi(t,0)\, dt \right) x_0 \\ &= x_0^T Q x_0 \ , \end{aligned}$$

which we had to show. ∎

Remark 2.2.

(i) If a state x_0 minimizes the quadratic form $x_0^T P x_0$, then either it is in $\ker P$ or it maximizes $x_0^T P^\sharp x_0$ among all $x_0 \in \operatorname{Im} P$. Hence a state is hard to reach if $x_0^T P x_0$ is small. Similarly, we can say that a state is hard to observe if $x_0^T Q x_0$ is small.

(ii) We will need later for the controllability Gramian to be interpreted as the observability Gramian of the dual system. Note that $\Phi(0,t) = \Phi(t,0)^{-1}$, whence

$$\frac{d}{dt}\Phi(0,t) = -\Phi(0,t)A(t)\Phi(t,0)\Phi(0,t) = -\Phi(0,t)A(t)\ ,$$
$$\frac{d}{dt}\Phi(0,-t)^T = A(-t)^T\Phi(0,-t)^T\ ;$$

see, e.g., [55]. Therefore,

$$\begin{aligned} P &= \int_0^\infty \Phi(0,-\tau)B(-\tau)B(-\tau)^T\Phi(0,-\tau)^T\,d\tau \\ &= \int_0^\infty \tilde{\Phi}(\tau,0)^T B(-\tau)B(-\tau)^T\tilde{\Phi}(\tau,0)\,d\tau\ , \end{aligned}$$

where $\tilde{\Phi}$ is the fundamental solution of the equation $\dot{x} = A(-t)^T x$.

(iii) It is customary to define $E_c(x_0)$ as the minimal energy needed to steer from 0 to x_0 over the interval $]-\infty, 0]$. Alternatively, one can steer asymptotically from t_0 to x_0 over an interval $[t_0, t_0+T]$, where $t_0 \in \mathbb{R}$ and $T > 0$ are arbitrary, and set

$$E_c^{(t_0)}(x_0) = \inf_{\substack{u\in L^2[t_0,t_0+T],T>0 \\ x(t_0+T,t_0,u)=x_0}} \int_{t_0}^{t_0+T} \|u(t)\|^2\,dt\ .$$

For time-varying systems in general this yields a different value, $E_c^{(t_0)}(x_0) \neq E_c(x_0)$, but in the time-invariant case it is the same (which is well known and follows also as a special case from our discussion in the next subsection). An advantage is that we may also consider solutions for positive times, and these are also defined for stochastic systems.

(iv) For completeness, let us recall that in the time-invariant case, P and Q satisfy the Lyapunov equations $AP + PA^T = -BB^T$ and $QA + A^TQ = -C^TC$.

2.2. Stochastic linear differential equations. Consider a stochastic linear control system of Itô type (see, e.g., [3, 44])

$$\begin{aligned} dx &= Ax\,dt + \textstyle\sum_{j=1}^N A_j x\,dw_j + Bu\,dt\ , \\ y &= Cx\ . \end{aligned} \tag{2.4}$$

The $w_j = w_j(t)$ are independent zero mean real Wiener processes on a probability space $(\Omega, \mathcal{F}, \mu)$ with respect to an increasing family $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$ of σ-algebras $\mathcal{F}_t \subset \mathcal{F}$.

Let $L_w^2(\mathbb{R}_+, \mathbb{R}^q)$ denote the corresponding space of nonanticipating stochastic processes v with values in $\mathbb{R}^q$ and norm

$$\|v(\cdot)\|_{L_w^2}^2 := \mathcal{E}\left(\int_0^\infty \|v(t)\|^2 dt\right) < \infty\ ,$$

where $\mathcal{E}$ denotes expectation. We assume that the homogeneous equation $dx = Ax\,dt + \sum A_j x\,dw_j$ is mean-square-stable, i.e., $\mathcal{E}(\|x(t)\|^2) \overset{t\to\infty}{\longrightarrow} 0$, for all initial conditions $x(0) = x_0$. Its fundamental solution will be denoted by Φ, so that $x(t) = \Phi(t,0)x_0$. Since stochastic differential equations in general can be solved only forward in time (see, e.g., [44]), note that $\Phi(t,\tau)$ is defined only for $t \geq \tau$. By time-invariance, we have $\Phi(t,\tau) = \Phi(t-\tau,0)$. For simplicity, we write $\Phi(t) = \Phi(t,0)$, where $t \geq 0$. By the stability assumption, the generalized Lyapunov equations

$$\begin{aligned} AP + PA^T + \sum_{j=1}^{N} A_j P A_j^T &= -BB^T\,, \\ A^T Q + QA + \sum_{j=1}^{N} A_j^T Q A_j &= -C^T C \end{aligned} \tag{2.5}$$

have nonnegative definite solutions $P \geq 0$ and $Q \geq 0$, which can be written (cf. [15]) as

$$P = \mathcal{E}\left(\int_0^\infty \Phi(t)BB^T\Phi(t)^T\,dt\right) \quad \text{and} \quad Q = \mathcal{E}\left(\int_0^\infty \Phi(t)^T C^T C\Phi(t)\,dt\right)\,.$$

Let $x_0 \in \mathbb{R}^n$ be given. We determine the minimal energy of an input u, so that $\mathcal{E}(x(T,0,u)) = x_0$ for some $T > 0$. In other words, u steers the average state from 0 to x_0 over an arbitrary time-interval $[0,T]$. Similarly, we consider the output energy produced by x_0. Thus, we consider the energy functionals

$$\begin{aligned} E_c(x_0) &= \inf_{\substack{u\in L^2_w[0,T],\,T>0 \\ x(T,x_0,u)=0}} \mathcal{E}\left(\int_0^T \|u(t)\|^2\,dt\right)\,, \\ E_o(x_0) &= \mathcal{E}\left(\int_0^\infty \|y(t,x_0,0)\|^2\,dt\right)\,. \end{aligned}$$

Note that $E_c(x_0) = \infty$ if the average state x_0 cannot be reached from 0. It is easy to see that this is equivalent to $x_0 \notin \operatorname{Im} P$. We have the following analogue of Theorem 2.1.

THEOREM 2.3. *Consider the stochastic system* (2.4) *and the Gramians* P *and* Q *defined by* (2.5). *If* $x_0 \in \operatorname{Im} P$, *then*

$$E_c(x_0) = x_0^T P^\sharp x_0\,.$$

For $x_0 \in \mathbb{R}^n$ *we have*

$$E_o(x_0) = x_0^T Q x_0\,.$$

Proof. For $T > 0$, we set $P_T = \mathcal{E}(\int_0^T \Phi(t)BB^T\Phi(t)^T\,dt)$, and for fixed $x_0 \in \operatorname{Im} P_T$, we define $u : [0,T] \to \mathbb{R}^m$ via $u(t) = B^T\Phi(T-t)^T P_T^\sharp x_0$. By the "variation of constants" formula for stochastic linear differential equations (see, e.g., [3, 15]) we have

$$\begin{aligned} \mathcal{E}\left(x(T,0,u)\right) &= \mathcal{E}\left(\int_0^T \Phi(T-t)Bu(t)\,dt\right) = \int_0^T \Phi(T-t)BB^T\Phi(T-t)^T P_T^\sharp x_0\,dt \\ &= \mathcal{E}\left(\int_0^T \Phi(\tau)BB^T\Phi(\tau)^T\,d\tau\right) P_T^\sharp x_0 = P_T P_T^\sharp x_0 = x_0\,. \end{aligned}$$

Moreover, u is the unique control with $\mathcal{E}(x(T,0,u)) = x_0$ and minimal $L^2_w[0,T]$-norm

$$\begin{aligned}\|u\|^2_{L^2_w} &= \mathcal{E}\left(\int_{-\infty}^{0} \|u(t)\|^2\,dt\right) = \mathcal{E}\left(\int_{-\infty}^{0} x_0^T P^\sharp \Phi(-t) B B^T \Phi(-t)^T P_T^\sharp x_0\,dt\right)\\ &= x_0^T P_T^\sharp P_T P_T^\sharp x_0 = x_0^T P_T^\sharp x_0\ .\end{aligned}$$

To prove minimality, we assume that $\tilde{u} = u + \hat{u}$ is another solution to the control problem. Then

$$x_0 = \mathcal{E}\left(\int_0^T \Phi(T-t)B(u(t)+\hat{u}(t))\,dt\right)\ ,\quad \text{whence}\quad \mathcal{E}\left(\int_0^T \Phi(T-t)B\hat{u}(t)\,dt\right) = 0\ .$$

This implies $\mathcal{E}(\int_0^T u(t)^T\hat{u}(t)\,dt) = 0$, so that

$$\|\tilde{u}\|^2_{L^2_w[0,T]} = \|u+\hat{u}\|^2_{L^2_w[0,T]} = \|u\|^2_{L^2_w[0,T]} + \|\hat{u}\|^2_{L^2_w[0,T]} \geq \|u\|^2_{L^2_w[0,T]}\ .$$

Hence $E_c(x_0) = \inf_{T>0} x_0^T P_T^\sharp x_0$. From the definitions it is clear that P_T is monotonically increasing and $\lim_{T\to\infty} P_T = P$. Hence $P_T^\sharp$ is decreasing, and the infimum is given by $x_0^T P^\sharp x_0$.

On the other hand, if the system starts in state x_0 and is not controlled, then the corresponding output is $y(t) = C\Phi(t)x_0$. The output energy is the L^2_w-norm of y,

$$\begin{aligned}E_o(x_0) = \|y\|^2_{L^2_w} &= \mathcal{E}\left(\int_0^\infty y(t)^T y(t)\,dt\right)\\ &= x_0^T\,\mathcal{E}\left(\int_0^\infty \Phi(t)^T C^T C \Phi(t)\,dt\right)x_0 = x_0^T Q x_0\ ,\end{aligned}$$

which concludes the proof. ∎

Remark 2.4.

(a) Likewise, the matrices P and Q can be interpreted as state and output covariances of a white-noise driven linear system; cf. [59]. We come back to this idea for bilinear systems in subsection 3.5.

(b) Balancing of systems with stochasticity is also discussed, e.g., in [43, 42], but not in the case of differential equations with multiplicative noise and not on the basis of the stochastic Lyapunov equations (2.5).

2.3. Stochastic linear discrete-time systems. Let us now consider a stochastic linear discrete-time control system (see, e.g., [9, 20])

$$\begin{aligned}x_{k+1} &= Ax_k + A_0 x_k w_k + Bu_k,\\ y_k &= Cx_k\ .\end{aligned}\tag{2.6}$$

Here (w_k) denotes a scalar stochastic process with zero mean and variance 1. Let $\ell^2_w(\mathbb{N},\mathbb{R}^q)$ denote the corresponding space of nonanticipating stochastic processes v with values in $\mathbb{R}^q$ and norm

$$\|v(\cdot)\|^2_{\ell^2_w} := \mathcal{E}\left(\sum_{j=0}^{\infty}\|v_j\|^2\right) < \infty\ .$$

As in the previous subsection, we could also introduce vector-valued Wiener processes and write

$$x_{k+1} = Ax_k + \sum_{j=1}^{N} A_j x_k w_k^{(j)} + Bu_k \ ,$$

but this only complicates the notation without leading to new insight.

We assume that the homogeneous equation $x_{k+1} = Ax_k + A_0 x w_j$ is mean-square-stable, which means that for all initial conditions x_0, we have $\mathcal{E}(\|x_k\|^2) \overset{k\to\infty}{\longrightarrow} 0$. Its fundamental solution is

$$\Phi_k = \prod_{j=0}^{k-1} (A + w_j A_0) \ ,$$

where multiplication is always from the left, i.e., $\Phi_{k+1} = (A + w_k A_0)\Phi_k$. Under the stability assumption, the generalized discrete-time Lyapunov equations

$$\begin{aligned} APA^T + A_0 P A_0^T - P &= -BB^T, \\ A^T Q A + A_0^T Q A_0 - Q &= -C^T C \end{aligned} \tag{2.7}$$

have nonnegative definite solutions $P \geq 0$ and $Q \geq 0$, which can be written (cf. [41]) as

$$P = \mathcal{E}\left(\sum_{j=0}^{\infty} \Phi_j B B^T \Phi_j^T\right) \quad \text{and} \quad Q = \mathcal{E}\left(\sum_{j=0}^{\infty} \Phi_j^T C^T C \Phi_j\right) \ .$$

Note that (1.3) is a special form of (2.7) if we set $A = \frac{1}{2} I$ and $N = 1$.

Let $x_0 \in \mathbb{R}^n$ be given. In analogy to the continuous-time situation, we define

$$E_c(x_0) = \inf_{\substack{u \in \ell_w^2[0,N],\, N>0 \\ x(N,x_0,u)=0}} \mathcal{E}\left(\sum_{j=0}^{N} \|u_j\|^2\right) \ ,$$

$$E_o(x_0) = \mathcal{E}\left(\sum_{j=0}^{\infty} \|y_j\|^2\right) \ .$$

Again, $E_c(x_0) = \infty$ if the average state x_0 cannot be reached from 0, which is equivalent to $x_0 \notin \operatorname{Im} P$. We have the following analogue of Theorem 2.3.

THEOREM 2.5. *Consider system* (2.6) *and the Gramians P and Q given by* (2.7). *If $x_0 \in \operatorname{Im} P$, then $E_c(x_0) = x_0^T P^\sharp x_0$, and for all $x_0 \in \mathbb{R}^n$ we have $E_o(x_0) = x_0^T Q x_0$.*

Proof. For $N > 0$ we set $P_N = \mathcal{E}\big(\sum_{j=0}^{N-1} \Phi_j B B^T \Phi_j^T\big)$, and for $x_0 \in \operatorname{Im} P$, we define a sequence $u_0, \ldots, u_{N-1}$ via $u_j = B^T \Phi_{(N-1)-j}^T P_N^\sharp x_0$. Then

$$\begin{aligned} \mathcal{E}\left(x(N,0,u)\right) &= \mathcal{E}\left(\sum_{j=0}^{N-1} \Phi_{(N-1)-j} B u_j\right) = \mathcal{E}\left(\sum_{j=0}^{N-1} \Phi_{(N-1)-j} B B^T \Phi_{(N-1)-j}^T P_N^\sharp x_0\right) \\ &= \mathcal{E}\left(\sum_{i=0}^{N-1} \Phi_i B B^T \Phi_i^T\right) P_N^\sharp x_0 = P_N P_N^\sharp x_0 = x_0 \ . \end{aligned}$$

Moreover, u is the unique control with $\mathcal{E}\left(x(N,0,u)\right) = x_0$ and minimal ℓ_w^2-norm

$$\|u\|_{\ell_w^2}^2 = \mathcal{E}\left(\sum_{j=0}^{N-1} \|u_j\|^2\right) = \mathcal{E}\left(\sum_{i=0}^{N-1} x_0^T P_N^\sharp \Phi_i B B^T \Phi_i^T P_N^\sharp x_0\right) = x_0^T P^\sharp P P^\sharp x_0$$
$$= x_0^T P^\sharp x_0 \ .$$

To prove minimality, we assume that $\tilde{u} = u + \hat{u}$ is another solution to the control problem. Then

$$x_0 = \mathcal{E}\left(\sum_{j=0}^{N-1} \Phi_{(N-1)-j} B(u_j + \hat{u}_j)\right) \ , \quad \text{whence} \quad \mathcal{E}\left(\sum_{j=0}^{N-1} \Phi_{(N-1)-j} B \hat{u}_j\right) = 0 \ .$$

This implies $\mathcal{E}\big(\sum_{j=0}^{N-1} u_j^T \hat{u}_j\big) = 0$, so that

$$\|\tilde{u}\|_{\ell_w^2}^2 = \|u + \hat{u}\|_{\ell_w^2}^2 = \|u\|_{\ell_w^2}^2 + \|\hat{u}\|_{\ell_w^2}^2 \geq \|u\|_{\ell_w^2}^2 \ .$$

As in the previous subsection, we have $E_c(x_0) = \inf_{N>0} x_0^T P_N^\sharp x_0 = x_0^T P^\sharp x_0$.

On the other hand, if the system starts in state x_0 and is not controlled, then the corresponding output is $y_j = C\Phi_j x_0$. The output energy is the ℓ_w^2-norm of y,

$$E_o(x_0) = \|y\|_{\ell_w^2}^2 = \mathcal{E}\left(\sum_{j=0}^{\infty} y_j^T y_j\right) = x_0^T \mathcal{E}\left(\sum_{0}^{\infty} \Phi_j^T C^T C \Phi_j\right) x_0 = x_0^T Q x_0 \ ,$$

which concludes the proof. ∎

Remark 2.6. The discrete-time stochastic Gramians (2.7) are also discussed, e.g., in [64, 62] in the context of model order reduction. But we have no reference for the specific energy interpretation given in this section.

In summary, in this section we have shown that the solutions P and Q of the Lyapunov equations (1.3) exactly represent the energy functionals for different classes of autonomous linear systems. We could also have added deterministic discrete-time systems as a special case of stochastic discrete-time systems (2.6) with $A_0 = 0$. Time-varying stochastic versions can be obtained along the lines of subsection 2.1. Thus, given P and Q the method of model reduction by balanced truncation can be applied to the corresponding system in an obvious way; we skip the details here but will discuss the same idea for bilinear systems in Remark 3.2.

3. Gramians and energy functionals for bilinear systems. Let us now consider a bilinear control system of the form

$$\dot{x} = Ax + \sum_{j=1}^{m} N_j x u_j + Bu, \tag{3.1}$$

$$y = Cx, \tag{3.2}$$

with $A, N_j \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$, $x(t) \in \mathbb{R}^n$, $u(t) = [u_1(t), \ldots, u_m(t)]^T \in \mathbb{R}^m$, $y(t) \in \mathbb{R}^p$.

The system is locally controllable if the pair (A, B) is controllable, and locally observable if the pair (A, C) is observable. It is asymptotically stable for all $u \in L^2[0, \infty[$

if A is stable. Assuming these properties, we can consider the energy functionals

$$E_c(x_0) = \min_{\substack{u\in L^2]-\infty,0] \\ x(-\infty,x_0,u)=0}} \|u\|^2_{L^2(]-\infty,0])}, \tag{3.3}$$

$$E_o(x_0) = E_o^\alpha(x_0) = \max_{\substack{u\in L^2[0,\infty[\\ \|u\|_{L^2}\le\alpha}} \|y(\cdot,x_0,u)\|^2_{L^2([0,\infty[)}\ , \tag{3.4}$$

where $\alpha > 0$ is a fixed small parameter. While in the definition of E_c we naturally consider the solution x of (3.1) with the given boundary conditions and minimize over all $u \in L^2\,]-\infty,0]$, there is some ambiguity concerning the roles of y and α in the definition of E_o. For reasons of duality (as we will see later), we prefer to consider y as the output of the following *homogeneous* system:

$$\begin{aligned} \dot{x} &= Ax + \sum_{j=1}^m N_j x u_j, \\ y &= Cx \end{aligned} \tag{3.5}$$

instead of the *inhomogeneous* system (3.1), (3.2). Both versions are considered by Gray and Mesko in [25], while, e.g., Scherpen in [50] defines E_o for general nonlinear systems with zero control input, i.e., with $\alpha = 0$.

Based on further duality considerations, we will suggest an alternative definition of E_o, which does not involve additional parameters. If the controllability condition or the observability condition is not satisfied, then we may have $E_c(x_0) = \infty$ or $E_o(x_0) = 0$ for arbitrarily small x_0.

The aim is to compare E_c and E_o with the quadratic forms defined by the algebraic Gramians $P \ge 0$ and $Q \ge 0$ from the generalized Lyapunov equations

$$\begin{aligned} AP + PA^T + \sum_{j=1}^m N_j P N_j^T &= -BB^T\ , \\ A^TQ + QA + \sum_{j=1}^m N_j^T Q N_j &= -C^TC\ . \end{aligned} \tag{3.6}$$

In section 3.4 we will review necessary and sufficient conditions for these Gramians to exist. Let us now show how the definiteness of the Gramians is related to reachability and observability of the bilinear system.

THEOREM 3.1.

(a) *Consider the bilinear system* (3.1) *and assume that* P *defined by* (3.6) *is nonnegative definite. Then* $x(t,0,u) \in \operatorname{Im} P$ *for all* $t \ge 0$ *and all input functions* u*, i.e.,* $E_c(x_0) = \infty$ *for all* $x_0 \notin \operatorname{Im} P$*.*

(b) *Consider the homogeneous bilinear system* (3.5) *and assume that* Q *defined by* (3.6) *is nonnegative definite. If* $x_0 \in \operatorname{Ker} Q$*, then* $y(t,x_0,0) = 0$ *for all* $t \ge 0$*, i.e.,* $E_o(x_0) = 0$*.*

Proof. (a) If $v \in \operatorname{Ker} P$, then by the defining equation for P we have

$$0 = v^T\left(AP + PA^T + \sum_{j=1}^m N_j P N_j^T + BB^T\right)v = v^T\left(\sum_{j=1}^m N_j P N_j^T + BB^T\right)v\ ,$$

which, by the nonnegativity of P, implies $B^Tv = 0$ and $PN_j^Tv = 0$ for $j = 1,\dots,m$. Hence $N_j^T \operatorname{Ker} P \subset \operatorname{Ker} P \subset \operatorname{Ker} B^T$. Again by (3.6), this implies $PA^Tv = 0$, i.e., $A^T \operatorname{Ker} P \subset \operatorname{Ker} P$. Let $x(t)$ denote an arbitrary solution of (3.1).

If $x(t) \in \operatorname{Im} P = (\operatorname{Ker} P)^\perp$ for some t, then

$$\dot{x}(t)^T v = x(t)^T \underbrace{A^T v}_{\in \operatorname{Ker} P} + \sum_{j=1}^{m} u_j(t) x(t)^T \underbrace{N_j^T v}_{\in \operatorname{Ker} P} + u(t)^T \underbrace{B^T v}_{=0} = 0 \,,$$

i.e., $\dot{x}(t) \perp v$ for all $v \in \operatorname{Ker} P$. Thus $\dot{x}(t) \in \operatorname{Im} P$ if $x(t) \in \operatorname{Im} P$, which means that $\operatorname{Im} P$ is invariant under the dynamics. Hence $x(0) = 0 \in \operatorname{Im} P$ implies $x(t) \in \operatorname{Im} P$ for all $t \geq 0$.

(b) For $x_0 \in \operatorname{Ker} Q$ we can argue as above to show that $N_j \operatorname{Ker} Q \subset \operatorname{Ker} Q \subset \operatorname{Ker} C$ and $A \operatorname{Ker} Q \subset \operatorname{Ker} Q$. Hence $x(t) \in \operatorname{Ker} Q$ implies $\dot{x}(t) \in \operatorname{Ker} Q$, so that $\operatorname{Ker} Q$ is invariant under the dynamics. If $x_0 \in \operatorname{Ker} Q$, then $x(t, x_0, u) \in \operatorname{Ker} Q$ for all $t \geq 0$, implying $y(t, x_0, u) = Cx(t, x_0, u) = 0$. ∎

Remark 3.2. It follows that states in $\operatorname{Ker} P$ or $\operatorname{Ker} Q$ are irrelevant for the transfer behavior of the system and can be eliminated. The idea of balanced truncation is to dispense also with those states which are *almost* in $\operatorname{Ker} P$ or $\operatorname{Ker} Q$, i.e., which belong to small singular values of P and Q. In analogy to the linear case (see, e.g., [2]), one can use factorizations $P = LL^T$ and $L^T Q L = U \Sigma^2 U^T$ to compute a transformation matrix $T = LU\Sigma^{-1/2}$ and an equivalent system

$$\dot{x} = \tilde{A}x + \sum_{j=1}^{m} \tilde{N}_j x u_j \,, \quad y = \tilde{C}x \,,$$

with

$$\tilde{A} = T^{-1} A T \,, \quad \tilde{N}_j = T^{-1} N_j T \,, \quad \tilde{B} = T^{-1} B \,, \quad \tilde{C} = CT \,.$$

Then (cf. [1]) the corresponding Gramians are *balanced*, i.e., equal and diagonal,

$$\tilde{P} = \tilde{Q} = \begin{bmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_n \end{bmatrix} \quad \text{with} \quad \sigma_1 \geq \cdots \geq \sigma_n > 0 \,. \tag{3.7}$$

Hence, we may assume without loss of generality that P and Q are balanced, when it is convenient. As in the linear case, we call the numbers $\sigma_1, \ldots, \sigma_n$ the *generalized Hankel singular values* of the bilinear system. If, e.g., $\sigma_{r+1}, \ldots, \sigma_n$ are particularly small for some $r \in \mathbb{N}$, then locally the states in the subspace spanned by the canonical unit vectors $e_{r+1}, \ldots, e_n$ are both hard to reach and hard to observe, and thus negligible for the transfer behavior. Partitions $T = [T_1, T_2]$ and $T^{-1} = \left[\begin{smallmatrix} S_1 \\ S_2 \end{smallmatrix} \right]$ with the projection matrices $T_1 \in \mathbb{R}^{n \times r}$ and $S_1 \in \mathbb{R}^{r \times n}$ then define the truncated system matrices $\tilde{A}^{(r)} = S_1 A T_1$, $\tilde{N}_j^{(r)} = S_1 N_j T_1$, $\tilde{B}^{(r)} = S_1 B$, and $\tilde{C}^{(r)} = C T_1$, which give a good local approximation of the original system. It is, however, not obvious how to measure the error caused by such a procedure.

3.1. Some comments on recent results. In [25] (see also [11]), it is stated that locally, i.e., for $x \neq 0$ with $\|x\|$ sufficiently small, the gradients of E_c and E_o are given by

$$\nabla E_c(x) = \tilde{P}(x)^{-1} x \quad \text{and} \quad \nabla E_o(x) = \tilde{Q}(x) x \tag{3.8}$$

with

$$A\tilde{P}(x) + \tilde{P}(x)A^T = -\sum_{j=1}^{m}(N_jx + b_j)(N_jx + b_j)^T, \tag{3.9}$$

$$A^T\tilde{Q}(x) + \tilde{Q}(x)A = -\sum_{j=1}^{m}\tilde{Q}(x)N_jxx^TN_j^T\tilde{Q}(x) - C^TC\ . \tag{3.10}$$

From this the authors derive the inequalities

$$E_c(x_0) > x_0^TP^{-1}x_0 \quad \text{and} \quad E_o(x_0) < x_0^TQx_0\ , \tag{3.11}$$

under the assumption that at least one of the N_j is nonsingular.

The statements (3.8), and thus the reasoning of [25, 11] leading to (3.11), however, require a further integrability condition. Note that $\tilde{P}(x)$ and $\tilde{Q}(x)$ are the Gramians of the linearization of (3.1) at $x \in \mathbb{R}^n$ and $u = 0$. The functionals $x^T\tilde{P}(x)^{-1}x$ thus describe the minimal local cost associated with moving in direction x. Equation (3.8) implies that the cost on a whole neighborhood of zero can be obtained from these local forms. Verriest in [59, sect. 5] discusses in detail that an integrability condition generally will be an obstruction to this. The following explicit example illustrates that the field $\tilde{P}(x)^{-1}x$ need not be integrable.

Example 3.3. Let us concentrate on the controllability Gramian. We introduce a locally controllable system for which the vector field $x \mapsto \tilde{P}(x)^{-1}x$ defined by (3.9) is not integrable, so that (3.8) cannot hold. To see this we consider the second derivative in direction $h \in \mathbb{R}^n$. Assuming (3.8) and (3.9), we have

$$\frac{\partial}{\partial x}\nabla E_c(x)(h) = \tilde{P}(x)^{-1}h - \tilde{P}(x)^{-1}\tilde{P}'_x(h)\tilde{P}(x)^{-1}x\ ,$$

with

$$A\tilde{P}'_x(h) + \tilde{P}'_x(h)A^T = -\sum_{j=1}^{m}N_jh(N_jx + b_j)^T + (N_jx + b_j)h^TN_j^T\ .$$

We now check the integrability condition for the field $F(x) := \tilde{P}(x)^{-1}x$. Its derivative is

$$h \mapsto F'(x)h = \tilde{P}(x)^{-1}h - \tilde{P}(x)^{-1}\tilde{P}'_x(h)\tilde{P}(x)^{-1}x\ .$$

The matrix $F'(x)$ must be symmetric for the field to be integrable (see, e.g., [39, sect. 18.2]). As a solution of a Lyapunov equation with symmetric right-hand side, $\tilde{P}(x)$ is symmetric. Hence we concentrate on the matrix representation of the second term, which we rewrite in terms of Kronecker products. We have

$$\operatorname{vec}\left(\tilde{P}(x)^{-1}\tilde{P}'_x(h)\tilde{P}(x)^{-1}x\right) = (x^T \otimes I)\left(\tilde{P}(x) \otimes \tilde{P}(x)\right)^{-1}\operatorname{vec}\tilde{P}'_x(h)\ ,$$

where

$$\operatorname{vec}\tilde{P}'_x(h) = (A \otimes I + I \otimes A)^{-1}\sum_{j=1}^{m}\left((N_jx + b_j) \otimes N_j + N_j \otimes (N_jx + b_j)\right)h\ .$$

Thus we have to check the symmetry of the $n \times n$ matrices F_j given by

$$(x^T \otimes I)\left(\tilde{P}(x) \otimes \tilde{P}(x)\right)^{-1} (A \otimes I + I \otimes A)^{-1} \left((N_j x + b_j) \otimes N_j + N_j \otimes (N_j x + b_j)\right).$$

In general, this condition is not fulfilled. Let us consider a simple explicit example:

$$A = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}, \quad N = \nu \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad x = \xi \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Here ν and ξ are parameters which can be chosen small, but which turn out to be irrelevant for the computation. We can solve explicitly for $\tilde{P}(x)^{-1}$:

$$A\tilde{P}(x) + \tilde{P}(x)A^T = -(Nx+b)(Nx+b)^T = -(1+\nu\xi)^2 \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$
$$\Rightarrow \tilde{P}(x) = (1+\nu\xi)^2 \begin{bmatrix} 1/2 & 1/3 \\ 1/3 & 1/4 \end{bmatrix}, \quad \tilde{P}(x)^{-1} = \frac{6}{(1+\nu\xi)^2} \begin{bmatrix} 3 & -4 \\ -4 & 6 \end{bmatrix}.$$

Hence the matrix $F_j = F$ now takes the form

$$\frac{36\nu\xi}{(1+\nu\xi)^3} \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}^T \begin{bmatrix} 9 & -12 & -12 & 16 \\ -12 & 18 & 16 & -24 \\ -12 & 16 & 18 & -24 \\ 16 & -24 & -24 & 36 \end{bmatrix} \begin{bmatrix} -\frac{1}{2} & & & \\ & \frac{1}{3} & & \\ & & \frac{1}{3} & \\ & & & \frac{1}{4} \end{bmatrix} \begin{bmatrix} 0 & 2 \\ 1 & 1 \\ 1 & 1 \\ 2 & 0 \end{bmatrix}$$
$$= \frac{12\nu\xi}{(1+\nu\xi)^3} \begin{bmatrix} 2 & -1 \\ -4 & 2 \end{bmatrix},$$

which is not symmetric. We can argue similarly for the observability Gramian and conclude that the corresponding assertions in [25, 11] are not true in general.

Remark 3.4. The critical step in [25] and [11] is the following. In the Hamilton–Jacobi equation

$$\frac{\partial E_c}{\partial x} Ax + x^T A^T \frac{\partial E_c}{\partial x}^T + \frac{\partial E_c}{\partial x} \sum_{j=1}^{m} (N_j x + b_j)(N_j x + b_j) \frac{\partial E_c}{\partial x}^T = 0,$$

the authors replace $\frac{\partial E_c}{\partial x}$ by the ansatz $x^T \tilde{P}(x)^{-1}$. This gives

$$x^T \tilde{P}(x)^{-1} Ax + x^T A^T \tilde{P}(x)^{-1} x + x^T \tilde{P}(x)^{-1} \sum_{j=1}^{m} (N_j x + b_j)(N_j x + b_j) \tilde{P}(x)^{-1} x = 0$$

for all $x \in \mathbb{R}^n$, from which, however, in general it cannot be concluded that

$$M(x) = \tilde{P}(x)^{-1} A + A^T \tilde{P}(x)^{-1} + \tilde{P}(x)^{-1} \sum_{j=1}^{m} (N_j x + b_j)(N_j x + b_j) \tilde{P}(x)^{-1} = 0.$$

In [24] this issue is treated properly. The authors call $M(x)$ a *null matrix function* if it satisfies $M(0) = 0$ and $x^T M(x) x = 0$ on an open neighborhood of V. Then they analyze the nonunique singular value decompositions of different null matrix functions. This analysis, however, does not give an explicit comparison of the energy functional E_c with the algebraic Gramian P.

3.2. Some patches. We wish to compare the energy functionals and the quadratic form given by the algebraic Gramians and follow another line of reasoning in [25] (see also [24, 22] for related fixes). It suffices to concentrate on the control energy $E_c(x_0)$ and to consider the system

$$\dot{x} = \left(A + \sum_{j=1}^{m} N_j u_j(t) \right) x + Bu \ .$$

Assume that P is the unique positive definite solution of a generalized Lyapunov equation

$$AP + PA^T + \sum_{j=1}^{m} \tilde{N}_j P \tilde{N}_j^T = -BB^T \ ,$$

where typically, but not necessarily, $\tilde{N}_j = N_j$. For simplicity let P be balanced as in (3.7). For a fixed $x_0 \in \mathbb{R}^n$ let $u = u_{x_0} :]-\infty, 0] \to \mathbb{R}^m$ denote the minimizing control in the definition of $E_c(x_0)$ (see, e.g., [50]). With this function we consider the time-varying homogeneous linear differential equation

$$\dot{\varphi} = \left(A + \sum_{j=1}^{m} N_j u_j(t) \right) \varphi =: A_u(t)\varphi(t)$$

and its fundamental solution $\Phi_u(t, \tau)$. The controllability Gramian of the time-varying control system $\dot{x} = A_u(t)x + Bu$ is then given by

$$P_u = \int_{-\infty}^{0} \Phi_u(0, \tau) BB^T \Phi_u(0, \tau)^T \, d\tau \ .$$

Since u also steers the state of the time-varying system from 0 to x_0, we have

$$\|u\|_{L^2}^2 \geq x_0^T P_u^\sharp x_0 \ .$$

By Remark 2.2(iii), we can write P_u also as an observability Gramian

$$P_u = \int_0^\infty \Psi_u(t, 0)^T BB^T \Psi_u(t, 0) \, dt \ ,$$

where Ψ_u is the fundamental solution of the dual system

$$\dot{\Psi}_u = \left(A^T + \sum_{j=1}^{m} N_j^T u_j(-t) \right) \Psi_u \ , \quad \Psi_u(t, t) = I \ .$$

With $x_u(t) = \Psi_u(t, 0)x_0$, we have

$$\begin{aligned}
x_0^T P x_0 &= -\int_0^\infty \frac{d}{dt} \Big(x_u(t)^T P x_u(t) \Big) \, dt \\
&= -\int_0^\infty x_u(t)^T \left(\left(A + \sum_{j=1}^{m} N_j u_j(-t) \right) P + P \left(A^T + \sum_{j=1}^{m} N_j^T u_j(-t) \right) \right) x_u(t) \, dt \\
&= -\int_0^\infty x_u(t)^T \left(\left(AP + PA^T + \sum_{j=1}^{m} \tilde{N}_j P \tilde{N}_j^T \right) \right) x_u(t) \, dt \\
&\quad + \int_0^\infty x_u(t)^T \sum_{j=1}^{m} \Big(\tilde{N}_j P \tilde{N}_j^T - N_j u_j(-t) P - P N_j^T u_j(-t) \Big) x_u(t) \, dt \ .
\end{aligned}$$

Here

$$-\int_0^\infty x_u(t)^T\left(AP+PA^T+\sum_{j=1}^m \tilde{N}_jP\tilde{N}_j^T\right)x_u(t)\,dt=\int_0^\infty x_u(t)^TBB^Tx_u(t)\,dt$$
$$=x_0^TP_ux_0\ .$$

Hence, if

$$\int_0^\infty x_u(t)^T\sum_{j=1}^m\Big(\tilde{N}_jP\tilde{N}_j^T-N_ju_j(-t)P-PN_j^Tu_j(-t)\Big)x_u(t)\,dt\geq 0\ , \tag{3.12}$$

then $x_0^TPx_0\geq x_0^TP_ux_0$ and $x_0^TP^{-1}x_0\leq x_0P_u^{-1}x_0=E_c$ for canonical unit vectors $x_0=e_j$ (cf. [30, Thm. 7.7.8]). Note that (3.12) obviously holds, if $\sum_{j=1}^m\tilde{N}_jPN_j^T>0$ (i.e., $P>0$) and $\sup_{t\in\mathbb{R}_-}\|u_{x_0}(t)\|$ is sufficiently small.

For small $\|x_0\|$, it follows from the continuous dependence of the solution $x(t,x_0,u)$ on the data that the optimal control $u=u_{x_0}$ for the bilinear system is close to the optimal control $u_{x_0}^{\mathrm{lin}}$ of the linearized system (compare (2.3)). Thus $\sup_{t\in\mathbb{R}_-}\|u_{x_0}(t)\|$ will be arbitrarily small if we restrict our attention to sufficiently small $\|x_0\|$. This reestablishes an essential part of the first assertion in (3.11), which we summarize as follows.

PROPOSITION 3.5. *Consider the bilinear system* (3.1) *and assume that the Gramian* P *defined by* (3.6) *is positive definite and diagonal, i.e.,* $P=\mathrm{diag}(\sigma_1,\dots,\sigma_n)$ *with* $\sigma_j>0$. *Then there exists an* $\varepsilon>0$, *so that for all canonical unit vectors* e_j *the inequality* $E_c(\varepsilon e_j)>\varepsilon^2e_j^TP^{-1}e_j=\frac{\varepsilon^2}{\sigma_j}$ *holds.*

Remark 3.6.

(a) In [25] an analogue of the inequality (3.12) is taken as an assumption at some stage in the discussion of the output energy and the observability Gramian.

(b) It is surprising and, in fact, dissatisfying that the N_j do not really play a role in our reasoning here. The same is true for the arguments in [25, 11]. In special examples, it could be that inequality (3.12) is more likely to hold for larger u_j if $\tilde{N}_j=N_j$, but there is no evidence for that.

(c) If we set $\tilde{N}_j=N_j$, and if none of these matrices has full rank, then (3.12) need not hold. Another approach might be based on considering the generalized Lyapunov equation with a shifted matrix A. This idea in different contexts can be found, e.g., in [7, 14] or [15, sects. 1.6 and 2.3.2]. Let

$$(A+\kappa I)P+P(A+\kappa I)^T+\sum_{j=1}^m N_jPN_j^T+BB^T=0\ ,$$

where $\kappa>0$ is small enough, so that $\sigma(A+\kappa I)\subset\mathbb{C}_-$. Then the previous computation yields

$$x_0^TPx_0-x_0^TP_ux_0$$
$$=\int_0^\infty x_u(t)^T\left(\sum_{j=1}^m\big(N_jPN_j^T-N_ju_j(-t)P-PN_j^Tu_j(-t)\big)+2\kappa P\right)x_u(t)\,dt\ ,$$

where now we have

$$\sum_{j=1}^m\Big(N_jPN_j^T-N_ju_j(-t)P-PN_j^Tu_j(-t)\Big)+2\kappa P\geq 0\ ,$$

as long as the $|u_j|$ are sufficiently small.

3.3. Dual definition and characterization of output energy. As mentioned before, the presence of the parameter α in the definition of the output energy is awkward. Here we wish to give an alternative definition, which—in analogy to Remark 2.2(ii)—is based on the input energy of a dual control problem. If we regard the homogeneous system (3.5) for fixed u as a time-varying linear system, then the output energy for a given x_0 is equal to the minimal energy needed to steer the dual system from x_0 to 0. The dual system, however, can be interpreted again as a bilinear system, which allows us to fix an adequate u. This is formalized in the following definition.

We can assume that $m = p$, since adding zero columns to B or zero rows to C changes neither the energies nor the Gramians.

DEFINITION 3.7. *Together with* (3.5)*, where* $y(t) \in \mathbb{R}^m$*, consider the antistable, locally controllable dual system*

$$\dot{\xi} = -A^T\xi - \sum_{j=1}^{m} N_j^T \xi u_j + C^T u \ . \tag{3.13}$$

For small $x_0 \in \mathbb{R}^n$ *let* $u = u_{x_0}$ *denote the control of minimal* L^2*-norm, so that* $\lim_{t\to\infty} \xi(t, x_0, u) = 0$.

With this input consider the output $y(\cdot, x_0, u_{x_0})$ *of* (3.5) *and define the output energy*

$$E_o(x_0) = \|y(\cdot, x_0, u_{x_0})\|^2_{L^2[0,\infty[}. \tag{3.14}$$

Note that, in contrast to maximizing over a class of bounded inputs in the definition (3.4), here we choose a special input u_{x_0} for each x_0. This input maximizes the output energy in the following sense. Let u be an arbitrary L^2-input to both the primal system (3.5) and the dual system (3.13). If x and ξ denote the corresponding solutions with initial value x_0, then we have

$$\begin{aligned}\frac{d}{dt}\xi^T x = \dot{\xi}^T x + \xi^T \dot{x} &= \left(-\xi^T A - \sum_{j=1}^{m} u_j \xi^T N_j + u^T C \right) x + \xi^T \left(Ax + \sum_{j=1}^{m} u_j N_j x \right) \\ &= u^T y \ .\end{aligned}$$

If $\xi(t, x_0, u)$ is bounded, then $\xi(t)^T x(t) \to 0$ as $t \to \infty$ and

$$\|x_0\|^2 = \left| \int_0^\infty \frac{d}{dt}\xi^T x \, dt \right| = \left| \int_0^\infty u(t)^T y(t, x_0, u) \, dt \right| \le \|u\|_{L^2} \|y(\cdot, x_0, u)\|_{L^2} \ ,$$

and

$$\|y(\cdot, x_0, u)\|_{L^2} \ge \frac{\|x_0\|^2}{\|u\|_{L^2}} \ .$$

The right-hand side is maximized if $u = u_{x_0}$.

Applying Proposition 3.5 to the dual system (3.13), we obtain the dual result.

PROPOSITION 3.8. *Consider the bilinear system* (3.2) *and assume that the Gramian* Q *defined by* (3.6) *is positive definite and diagonal, i.e.,* $Q = \operatorname{diag}(\sigma_1, \ldots, \sigma_n)$ *with* $\sigma_j > 0$*. Then there exists an* $\varepsilon > 0$*, so that for all canonical unit vectors* e_j *the inequality* $E_o(\varepsilon e_j) < \varepsilon^2 e_j^T Q e_j = \varepsilon^2 \sigma_j$ *holds.*

3.4. Solvability of generalized Lyapunov equations. In the previous sections we have encountered dual pairs of matrix equations of the forms (1.3). To characterize their solvability, it suffices to concentrate on the equation

$$AP + PA^T + \sum_{j=1}^{m} A_j P A_j^T = -BB^T \,, \tag{3.15}$$

where P is required to be nonnegative definite. We cite a general theorem (cf. [53, 15, 16]), which gives necessary and sufficient criteria for the existence of a positive definite solution $P > 0$. (In particular see [15, Thm. 3.6.1] for further equivalent conditions. Note that parts of this theorem are contained also in [61, 32].) To apply the result to (3.15), we have to note that the linear matrix operator

$$\Pi(X) \mapsto \sum_{j=1}^{m} A_j X A_j^T$$

is nonnegative in the sense that $\Pi(X) \geq 0$, whenever $X \geq 0$. Moreover, we write $\sigma(T) \subset \mathbb{C}$ for the spectrum of a linear operator T and $\rho(T) = \max\{|\lambda| \mid \lambda \in \sigma(T)\}$ for the spectral radius.

THEOREM 3.9. *Let $A \in \mathbb{R}^{n\times n}$, and consider linear operators $\mathcal{L}_A, \Pi : \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$, where $\mathcal{L}_A$ is defined by $\mathcal{L}_A(X) = AX + XA^T$, and Π is nonnegative. The following are equivalent:*

(a) *For all $Y > 0$: $\exists X > 0$: $\mathcal{L}_A(X) + \Pi(X) = -Y$;*
(b) *$\exists Y > 0$: $\exists X > 0$: $\mathcal{L}_A(X) + \Pi(X) = -Y$;*
(c) *$\exists Y \geq 0$ with (A, Y) controllable: $\exists X > 0$: $\mathcal{L}_A(X) + \Pi(X) = -Y$;*
(d) *$\sigma(\mathcal{L}_A + \Pi) \subset \mathbb{C}_-$;*
(e) *$\sigma(\mathcal{L}_A) \subset \mathbb{C}_-$ and $\rho(\mathcal{L}_A^{-1}\Pi) < 1$.*

In particular we note that (3.15) possesses a positive definite solution P if A is stable, (A, B) is controllable, and the norms of the A_j are sufficiently small. If the pair (A, B) is not controllable, then P is not necessarily definite. This is consistent with Theorem 3.1.

The algebraic Gramians may not exist if the A_j are too large. For the stochastic systems discussed in section 2, this is equivalent to the system being unstable, so that the energy functionals are not well defined either.

In the context of locally stable bilinear systems, however, we may always rescale the input variable u so that the algebraic Gramians exist (see also [11]). More precisely, we replace the bilinear system (3.1) by the equivalent one

$$\begin{aligned} \dot{x} &= Ax + \sum_{j=1}^{m} (\gamma N_j)\, x\, \frac{u_j}{\gamma} + (\gamma B)\frac{u}{\gamma} = Ax + \sum_{j=1}^{m} \tilde{N}_j x \tilde{u}_j + \tilde{B}\tilde{u}, \\ y &= Cx \,. \end{aligned}$$

Then equations (3.6) for the algebraic Gramians have to be replaced by

$$\begin{aligned} AP + PA^T + \gamma^2 \sum_{j=1}^{m} N_j P N_j^T &= -BB^T \,, \\ A^T Q + QA + \gamma^2 \sum_{j=1}^{m} N_j^T Q N_j &= -C^T C \,. \end{aligned} \tag{3.16}$$

Choosing $\gamma > 0$ small enough guarantees the existence of the algebraic Gramians at the price of possibly decreasing the region where the energy estimates hold (i.e., decreasing ε in Propositions 3.5 and 3.8). In particular, as we will also see in the following numerical examples, the smaller γ is chosen, the faster the decay of the generalized Hankel singular values. Note, however, that in exact arithmetic the kernel of P is independent of $\gamma > 0$.

3.5. Covariance approximation. To conclude this section we give another interpretation of the Gramians (3.16) as residual covariances; for linear systems this interpretation is suggested in [59]. Assume that (3.1) is driven by white noise, i.e., u is a vector of independent white-noise processes of given spectral density γ. Then we interpret the system as a linear Itô-type stochastic differential equation

$$dx = Ax\,dt + \sum_{j=1}^{m} N_j x\,dw_j + B\,dw, \tag{3.17}$$

$$y = Cx\ . \tag{3.18}$$

The covariance matrix $P(t) = E(xx^T)$ satisfies the deterministic differential equation

$$\dot{P}(t) = AP(t) + P(t)A^T + \gamma^2 \sum_{j=1}^{m} N_j P(t) N_j^T + BB^T\ .$$

If the system (3.17) is mean-square-stable, then $P(t)$ converges to a limiting covariance $P \geq 0$ satisfying (3.16). For the output we have

$$\frac{d}{dt}\,E(y(t)^T y(t)) = \frac{d}{dt} E(x(t)^T C^T C x(t)) = \langle \dot{P}(t), C^T C\rangle\ ,$$

where $\langle X, Y\rangle = \operatorname{trace} XY$ for symmetric matrices X and Y. Hence

$$E(y(t)^T y(t)) = \langle P(t), C^T C\rangle \overset{t\to\infty}{\to} \langle P, C^T C\rangle = \langle BB^T, Q\rangle\ .$$

Summarizing this section, we have shown that the kernels of the algebraic Gramians P and Q of bilinear systems are unreachable and unobservable, respectively. But we have observed some difficulties with the energy interpretation. In contrast to the linear case, we can only expect the Gramians to provide a bound for the energies. Moreover, this estimate seems to hold only locally. Nevertheless we expect that small Hankel singular values correspond to states, which are both hard to reach and hard to observe. We will provide some numerical evidence for this in the next section. In any case, the existence of the algebraic Gramians of bilinear systems allows us to apply balanced truncation to bilinear systems by using a truncated version of the contragradient transformation represented by $T \in \mathbb{R}^{n\times n}$ nonsingular, which balances P vs. Q via

$$TPT^T = T^{-T} Q T^{-1} = \operatorname{diag}(\sigma_1, \ldots, \sigma_n)\ ,$$

where the σ_j are the generalized Hankel singular values. The implementation of the corresponding model reduction method using the *square-root* [35, 56] or *balancing-free square-root* [58] versions of balanced truncation is straightforward: after solving the generalized Lyapunov equations (3.6) for the Cholesky factors of P, Q, the only difference is that the truncation operators have to be applied to the matrices N_j, analogous to their application to A. In the following section, we report results achieved by this approach.

4. Numerical examples. Numerically, the solution of the generalized Lyapunov equations (3.16) is a challenge even for medium-sized systems. A naive direct approach has complexity $\mathcal{O}(n^6)$, which is impracticable. Iterative methods can be based on the contraction property in Theorem 3.9(e). In [16] a preconditioned Krylov subspace method has been described, which converges reasonably fast in many cases and has complexity $\mathcal{O}(n^3)$. This allows us to deal with dimensions n up to a few thousand (let us call this medium-sized) on a standard computer, but in general not more. In the following we consider several medium-sized numerical examples and compare the approximation property of balanced truncation with that of a Krylov subspace projection method, developed in [4], which we briefly summarize below. (Other Krylov subspace methods for bilinear systems can be found, e.g., in [47, 38, 21, 12, 8].) Since it is not obvious to define a transfer norm for nonlinear systems, we compare just the outputs of the original system and the reduced systems for a given input function. As in the linear case, we observe that for a fixed $r < n$ the worst case approximation error of a reduced system of order r seems to be particularly small if we use balanced truncation. We interpret this as a strong indication that balanced truncation yields good error bounds also for bilinear systems.

We also try a two-step reduction procedure, where in the first step a system is reduced by a Krylov subspace projection, and the smaller system is reduced further in the second step by balanced truncation. This method can be used also for a large-scale system if the first step reduces it to medium-sized. We expect that the two-step reduction can give better error bounds than just a Krylov subspace projection.

Krylov subspace projection. In [4], Bai and Skoogh describe a Krylov subspace projection method for the order reduction of SISO bilinear systems (3.1), (3.2). Given parameters $q_1, p_2, q_2 \in \mathbb{N}$, the method produces a reduced system of order $r = q_1 + p_2 q_2$, which matches the first and second moments (cf. [19])

$$\begin{aligned} m(\ell_1) &= -CA^{-\ell_1}B && \text{for } \ell_1 = 1, 2, \ldots, q_1 \quad \text{and} \\ m(\ell_1, \ell_2) &= CA^{-\ell_2}NA^{-\ell_1}B && \text{for } \ell_1 = 1, 2, \ldots, p_2, \quad \ell_2 = 1, 2, \ldots, q_2\,. \end{aligned} \tag{4.1}$$

It is easy to generalize the algorithm for the MIMO case.

To sum up, we compare three methods:

BT. Balanced truncation as in Remark 3.2, where P and Q solve (3.16) with a given $\gamma \geq 0$. In particular, if $\gamma = 0$, the transformation is based only on the linear part of the system.

Krylov. Krylov subspace projection with moment matching for given parameters q_1, p_2, q_2.

Krylov & BT. Two-step reduction, where the first step is *Krylov* and the second is *BT*.

A nonlinear RC circuit. In [4, Example 2] a large system obtained by Carleman bilinearization of an RC circuit with nonlinear resistor is considered. See that paper for further details on the derivation of the model and the structure of the matrices; the same system has also been discussed, e.g., in [10, 18, 11, 12]. If the dimension of the nonlinear system is N_0, then the bilinear system has dimension $n = N_0^2 + N_0$. For $N_0 = 200$ the authors of [4] compute a reduced model of order $r = 21$ with $q_1 = 20$, $p_2 = q_2 = 1$.

Since $n = 40{,}200$ is too large for our generalized Lyapunov solver, we first consider the case $N_0 = 50$, which still gives a medium-sized bilinear system with $n = 2550$. This system is reduced to order $r = 21$ by the different methods. Our first plots in

Figure 4.1 show the advantage of BT with respect to the relative error. Even the mixed method, which first reduces the system to order 76 by Krylov and then to 21 by BT, is superior to Krylov alone in this example.

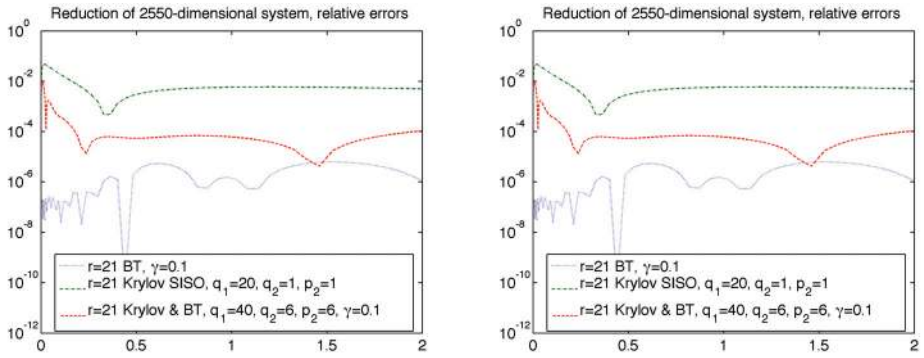


FIG. 4.1. *For the input function* $u(t) = e^{-t}$ *(left) and* $u(t) = \cos((2\pi t)/10) + 1)/2$ *(right) the outputs of the full and the reduced systems are compared. The plots show the relative errors over time.*

The choice of parameters, of course, is important. Let us have a look at the role of r and γ, since these are essential in BT. We solve the generalized Lyapunov equations (3.16) for $\gamma = 0$, $\gamma = 0.2$, $\gamma = 0.5$, and $\gamma = 1$ and compute the generalized Hankel singular values. From these we can read off a local error estimate if we truncate at $r = 21$. Moreover, we can determine the numerical rank r_γ of the balanced Gramians and truncate for this value as well. The results in Figure 4.2 clearly indicate that one should use $\gamma > 0$; i.e., for the approximation it is unfavorable to consider only the linearization. On the other hand, increasing γ does not improve the approximation in this special example. In fact, if we fix $r = 21$, then the best results are obtained for $\gamma = 0.2$, while the results are quite similar if we choose $r = r_\gamma$.

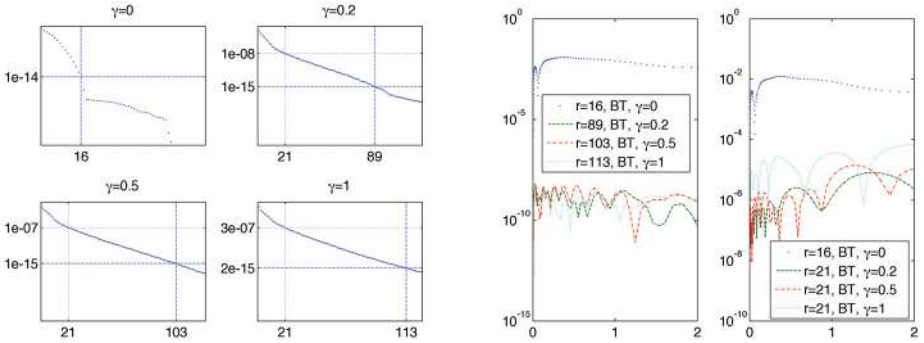


FIG. 4.2. *The plots on the left show the largest generalized Hankel values for different* γ*. The dotted line shows the truncation error for* $r = 21$*, and the dashed line shows the numerical rank. For* $\gamma = 0$ *the numerical rank is less than* 21*. On the right we see the relative errors at the output (with* $u = e^{-t}$*) for the corresponding* r *and* γ*.*

A heat transfer model. As another example we introduce a bilinear controlled heat transfer system. On the unit square $\Omega = [0,1] \times [0,1]$, we consider the heat equation $x_t = \Delta x$ with the mixed Dirichlet and Robin boundary conditions

$$\begin{aligned} n \cdot \nabla x &= u_1(x-1) \text{ on } \Gamma_1 := \{0\} \times]0,1[\,, \\ n \cdot \nabla x &= u_2(x-1) \text{ on } \Gamma_2 :=]0,1[\times \{0\}\,, \\ x &= 0 \text{ on } \Gamma_3 := \{1\} \times [0,1] \text{ and } \Gamma_4 = [0,1] \times \{1\}\,. \end{aligned}$$

Here the heat transfer coefficients u_1 and u_2 on the left and the lower boundaries Γ_1 and Γ_2 are the input variables. They can be interpreted, e.g., as spraying-intensities of a cooling-fluid acting on these boundaries (cf. [48, 57, 6, 37]), where in [6], linear and bilinear control systems are derived using different choices of controls. The temperature of the fluid is normalized to the value 1, and the heat flow over the boundary is proportional to the difference of temperatures $x - 1$ on the boundary. Note that the inputs u_j enter these conditions bilinearly.

By a finite difference discretization of the Poisson equation on an equidistant $k \times k$-mesh (meshsize $h = \frac{1}{k+1}$) with nodes x_{ij}, we obtain the well-known Poisson matrix

$$I \otimes T_k + T_k \otimes I\,, \quad \text{where} \quad T_k = \begin{bmatrix} -2 & 1 & & \\ 1 & -2 & \ddots & \\ & \ddots & \ddots & 1 \\ & & 1 & -2 \end{bmatrix} \in \mathbb{R}^{k \times k}\,.$$

Together with the boundary conditions, this leads to the bilinear system

$$\dot{x} = Ax + u_1 N_1 x + u_2 N_2 x + Bu$$

for $x = \operatorname{vec}(x_{ij})$, where

$$A = \frac{1}{h^2}\left(I \otimes T_k + T_k \otimes I + E_1 \otimes I + I \otimes E_k\right)\,, \quad E_j = e_j e_j^T\,.$$

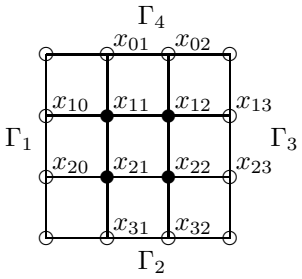
The coefficient matrices N_j and the columns b_j of B corresponding to the left and lower boundaries are given by

$$N_1 = \frac{1}{h} E_1 \otimes I\,,\ N_2 = \frac{1}{h} I \otimes E_k\,,\ b_1 = \frac{1}{h} e_1 \otimes e\,,\ b_2 = \frac{1}{h} e \otimes e_k\,,\ e = [1, \ldots, 1]^T \in \mathbb{R}^k\,.$$

As an output, we consider the average temperature

$$y = Cx = \frac{1}{k^2} \sum_{i,j=1}^{k} x_{ij} = \frac{1}{k^2} (e \otimes e)^T x\,.$$

Example 4.1. We illustrate these definitions for the simple 2×2 mesh (i.e., $k = 2$, $h = 1/3$) shown on the right. The state vector $x = [x_{11}, x_{21}, x_{12}, x_{22}]^T$ contains the temperatures at the inner points and the Laplacian is approximated via



$$\Delta x_{ij} \approx -\frac{1}{h^2}\left(4x_{ij} - x_{i+1,j} - x_{i,j+1} - x_{i-1,j} - x_{i,j-1}\right) .$$

For the boundary points a discretization of the Robin condition

$$n \cdot \nabla x = u(x - 1)$$

gives, e.g., the approximations

$$x_{10} \approx x_{11} - hu_1(x_{11} - 1) , \quad x_{20} \approx x_{21} - hu_1(x_{21} - 1) , \quad x_{31} \approx x_{21} - hu_2(x_{21} - 1) .$$

Altogether this leads to the bilinear system

$$\begin{aligned}
\dot{x} &= 9 \begin{bmatrix} -3 & 1 & 1 & 0 \\ 1 & -2 & 0 & 1 \\ 1 & 0 & -4 & 1 \\ 0 & 1 & 1 & -3 \end{bmatrix} x \; + \; 3 \left(\begin{bmatrix} x_{11} - 1 \\ x_{21} - 1 \\ 0 \\ 0 \end{bmatrix} u_1 + \begin{bmatrix} 0 \\ x_{21} - 1 \\ 0 \\ x_{22} - 1 \end{bmatrix} u_2 \right) \\
&= Ax + (N_1 x + b_1)u_1 + (N_2 x + b_2)u_2 , \\
y &= \frac{1}{4} \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} x = Cx ,
\end{aligned}$$

where the matrices A, N_1, N_2, C and the vectors b_1, b_2 are as above.

Now let $k = 50$, so that the discretized bilinear system has order $n = k^2 = 2500$, two inputs, and one output. As in the previous example, we first compare BT, Krylov, and Krylov & BT (see Figure 4.3). Then, in Figure 4.4 we compare the results of BT for different γ.

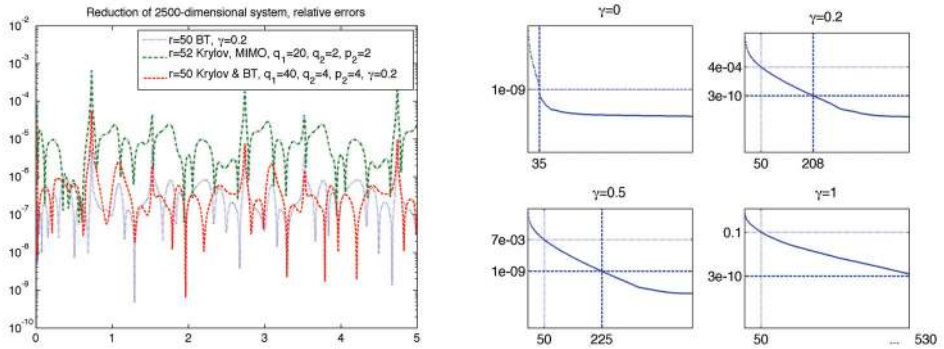


FIG. 4.3. *Relative output errors for the input functions* $u_j(t) = \cos(j\pi t)$ *and Hankel singular values for different* γ.

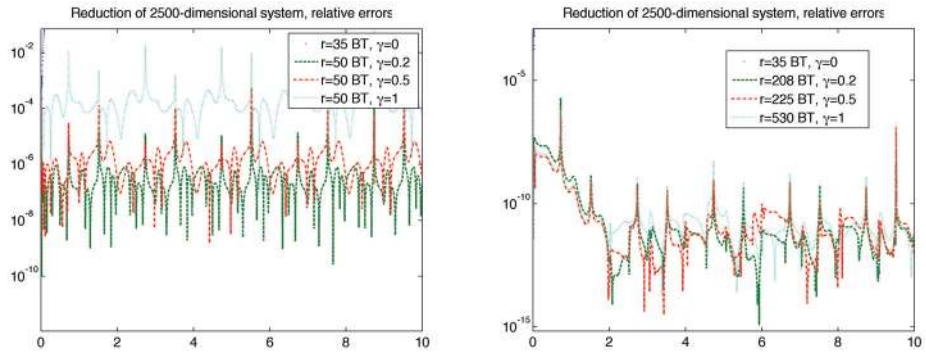


FIG. 4.4. *Relative output error for different* γ. *For* $\gamma = 0$ *the solution is unstable. For fixed* $r = 50$ *we see that the error seems to increase if* $\gamma \geq 0.2$ *increases. If we choose* $r = r_\gamma$, *i.e., the numerical rank of the balanced Gramians, then the errors are almost equal.*

5. Conclusions. We have discussed the relation of a certain type of generalized Lyapunov equations to Gramians of stochastic and bilinear systems together with the corresponding energy functionals. The Gramians of continuous- and discrete-time stochastic linear systems allow the same energy interpretations as in the case of deterministic linear systems. The relation of algebraic Gramians solving generalized Lyapunov equations and energy functionals for bilinear systems is less clear. We have discussed results from the literature for the latter problem and point out some inaccuracies in the energy interpretations used so far. In order to provide some motivation for using the algebraic Gramians of bilinear systems for model reduction, we have derived new characterizations of input and output energies of bilinear systems. In any of the considered cases, the definition of the algebraic Gramians allows us to compute balancing transformations, which in turn implies model reduction methods analogous to balanced truncation for linear deterministic systems. We have illustrated the performance of these model reduction methods by showing numerical experiments for different bilinear systems. The results demonstrate that model reduction for bilinear systems based on balanced truncation often appears to be superior to Krylov subspace methods. In order to make these methods more efficient and reliable, it would be necessary to derive numerical methods for the generalized Lyapunov equations that would enable us to solve such equations for dimensions $n > 10,000$, as well as error bounds for the reduced order models.

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