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Lyapunov Exponents, Periodic Orbits and Horseshoes for Mappings of Hilbert Spaces

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Abstract. We consider smooth (not necessarily invertible) maps of Hilbert spaces preserving ergodic Borel probability measures, and prove the existence of hyperbolic periodic orbits and horseshoes in the absence of zero Lyapunov exponents. These results extend Katok's work on diffeomorphisms of compact manifolds to infinite dimensions, with potential applications to some classes of periodically forced PDEs.

1. Introduction

For finite-dimensional dynamical systems, there is a fairly well-developed smooth ergodic theory. We focus here on the theory of *nonuniformly hyperbolic systems*, see e.g. [3,6,11–14]. The body of results contained in these and other papers[1] provides a firm foundation for understanding chaotic phenomena on a theoretical level. The present paper is a step in a program to extend these results to infinite dimensions, so they can be applied, among other things, to systems defined by evolutionary PDEs.

Central to nonuniform hyperbolic theory is the idea of *Lyapunov exponents*, which measure the infinitesimal rates at which nearby orbits diverge. Given a diffeomorphism of a finite-dimensional manifold, almost everywhere with respect to an invariant measure there is a decomposition of the tangent space into an expanding, a neutral and a contracting subspace corresponding to positive, zero and negative Lyapunov exponents. In infinite dimensions, this decomposition continues to make sense provided the system is asymptotically contracting in all but a finite number of directions. (This is not a requirement

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[1] Other topics of hyperbolic theory not discussed here include, e.g. Axiom A, piecewise hyperbolic and partially hyperbolic systems.

for uniform hyperbolicity.) The systems we consider will be assumed to have this property.

The purpose of the present paper is to generalize the results of Katok [3] to mappings of Hilbert spaces. Katok's results assert the following: Let f be a C^2 diffeomorphism of a compact Riemannian manifold, and let μ be an f-invariant Borel probability measure. Assume that (f, μ) has nonzero Lyapunov exponents and positive metric entropy. Then horseshoes are present, implying in particular an abundance of hyperbolic periodic points. Katok's results were proved for diffeomorphisms of compact manifolds. In this paper, we extend these results to mappings of Hilbert spaces without any assumptions on the invertibility of f or its Fréchet derivative Df_x. Our main results are stated as Theorems A–D in Sect. 2. Along the way we make a point of isolating and properly formulating for future use a number of basic facts which we extend to infinite dimensions.

In the paragraphs to follow, we will review previously known results, elaborate on the facts alluded to at the end of the last paragraph, and discuss what Theorems A–D will and will not tell us about systems defined by evolutionary PDEs.

1.1. Previously Known Results in Ergodic Theory of Infinite-Dimensional Systems

On the *infinitesimal* level, i.e. with regard to the asymptotic properties of Df_x^n, generalizations of Oseledets' Multiplicative Ergodic Theorem [11] to cocycles of linear maps of Hilbert and Banach spaces have been known for some time: a version of this result for compact operators of Hilbert spaces was proved in [15]; Banach space operators permitting nontrivial essential spectra were treated in [7,10,18]. These results are cited without proof in the present paper as Theorem 1 (see Sect. 3.1).

Turning to *local* results, i.e. dynamical properties in neighborhoods of typical orbits, the existence of local stable and unstable manifolds was proved for Hilbert and Banach space maps in e.g. [7,15]. These results also follow from Propositions 5 and 6 of the present paper and are stated as Corollary 7 in Sect. 5.1, but they are not the reason for our work in Sect. 5 nor do we claim priority for them.

On a more *nonlocal* level, we know of few results. Closer to the work discussed here are [18], which contains, among other things, an entropy inequality, and [9], which proves the existence of SRB measures in a special situation.

1.2. Techniques Borrowed from Finite-Dimensional Hyperbolic Theory

Our main results are stated in Sect. 2. We do not repeat them here, but would like instead to mention two sets of techniques used in the proofs of Theorems A–D that are of a foundational nature and are certain to be useful in future works.

1. *Lyapunov charts.* In nonuniform hyperbolic theory, it simplifies the estimates greatly to work in coordinates in which the values of Lyapunov exponents, which are by definition asymptotic quantities, are reflected

in single iterations of the map. In finite dimensions, such coordinates were introduced in [12] and are known as Lyapunov charts. These point-dependent changes of coordinates were used extensively in [3,6] and in a number of subsequent papers. Infinite-dimensional versions of Lyapunov charts had not been introduced before; their construction is carried out in Sect. 3.2.

2. *Exploiting uniform hyperbolicity on noninvariant sets.* In nonuniformly hyperbolic systems, there are, by definition, positive measure sets on which hyperbolic estimates are uniform. These sets are, in general, not invariant, and one's ability to effectively exploit the uniform hyperbolicity on such sets is key to success. Ideas of this type have been used extensively in virtually all papers in the subject in finite dimensions. In Sects. 5 and 6, we isolate and extend to infinite dimensions some of the relevant estimates.

Tempting as it may be at times, one must not pass from finite to infinite dimensions casually: even when the statements turn out not to be very different, many parts of the proofs often need to be reworked. Noninvertibility of the map, as manifested in the absence of inverse images for many phase points, infinitely large contractions, and the lack of local compactness in the phase space—these are some of the issues one has to contend with.

1.3. Application to Systems Defined by PDEs

While infinite-dimensional dynamical systems are interesting in their own right, the main applications we have in mind are to certain evolutionary PDEs, and the conditions in Sect. 2 are tailored to this application. Specifically, the setting of this paper is consistent with those of systems defined by dissipative parabolic PDEs, such as reaction-diffusion type equations including the (2D) Navier–Stokes equations. In a program to extend finite-dimensional hyperbolic theory to infinite dimensions, it is natural to begin with systems of this kind, for they have attractors that are finite dimensional in character (even though these attractors do not live in any finite-dimensional space). For our results to be applicable, we add a periodic forcing to the equations above. This is necessary because the main dynamical assumption in this paper, namely the absence of zero Lyapunov exponents, is violated by time-t maps of semiflows arising from time-independent equations.

The time-independent case is treated in a forthcoming paper [8], which builds upon the present work and proves results analogous to Theorems A–D for *semiflows on Hilbert spaces* under the assumption that the system has at most one zero Lyapunov exponent.

Having asserted that our results are potentially applicable to systems defined by PDEs of certain types, we must now clarify the nature of this application: Theorems A–D are *dynamical systems results.* As with most results from nonuniform hyperbolic theory, they are intended to help build a conceptual picture, to describe the qualitative behaviors of "typical" solutions once certain conditions are met. They offer no concrete information or estimate on any specific equation or specific solutions of any equation, as analytical results for PDEs often do. A case in point: Assuming the absence of zero Lyapunov exponents, Theorem D says that dynamical complexity (in the sense of entropy) implies the existence of infinitely many periodic solutions, and Theorem C compares the diversity of time evolutions to the flipping of a coin. Checking the no-zero-exponent and positive-entropy conditions for a specific invariant measure of a concrete PDE is difficult if not impossible, yet these results paint a qualitative picture—they contribute to an improved understanding on a theoretical level—for a very large class of equations.

2. Setting and Results

2.1. Setting

In this paper, $(\mathbb{H}, < \cdot, \cdot >)$ is a separable Hilbert space with norm $|\cdot|$. We consider a C^2 map $f : \mathbb{H} \to \mathbb{H}$, and let Df_x denote the Fréchet derivative of f at x. Let $A \subset \mathbb{H}$ be a compact subset. The following are assumed throughout:

(D1) $f(A) = A$, and f is one-to-one in a neighborhood of A;

(D2) For all $x \in A, Df_x$ is (i) injective, and (ii) compact;

(D3) μ is an ergodic f-invariant Borel probability measure on A. All of our results are in fact valid with (D2)(ii) replaced by:

(D2) (ii′) For all $x \in A$,

$$\kappa(x) := \lim_{n\to\infty} \frac{1}{n} \log \kappa_0(Df_x^n) < 0$$

where for an operator $T, \kappa_0(T)$ is the Kuratowski measure of noncompactness of T.

Recall that $\kappa_0(T)$ is defined to be the infimum of the set of numbers $r > 0$ where $T(B), B$ being the unit ball, can be covered by a finite number of balls of radius r. Since $\kappa_0(T_1 \circ T_2) \le \kappa_0(T_1)\kappa_0(T_2)$, the limit in the definition of $\kappa(x)$ is well defined by subadditivity.

2.2. Results

Under the conditions above, positive and zero Lyapunov exponents of (f, μ) are well defined, see Sect. 3.1.

Theorem A. *Assume (f, μ) has no Lyapunov exponents ≥ 0. Then μ is supported on a stable periodic orbit.*

In this paper, a *stable* periodic point is one that is linearly stable in a strict sense, meaning if $f^p(x) = x$, then the spectrum of Df_x^p is contained in $\{|z| < 1\}$. Likewise, by an *unstable* periodic point, we refer to one that is linearly unstable in a strict sense, meaning the spectrum of Df_x^p meets $\{|z| > 1\}$.

Theorem B. *If (f, μ) has no zero Lyapunov exponents, then one of the following holds:*

(a) *μ is supported on a single periodic orbit, stable or unstable; or*

(b) *μ is supported on the closure of a set of infinitely many unstable periodic orbits.*

Our next result gives conditions that imply the existence of a complex dynamical structure called a *horseshoe*.

Horseshoes in Infinite-Dimensional Spaces. Since f is generally not invertible, we think it is natural to have a notion of horseshoes that involves only forward time in addition to the usual definition in finite dimensions. Let $k \in \mathbb{Z}^+$. We say $\sigma : \Pi_0^\infty\{1, \dots, k\} \to \Pi_0^\infty\{1, \dots, k\}$ is a one-sided full shift on k symbols if for $\mathbf{a} = (a_0, a_1, a_2, \dots) \in \Pi_0^\infty\{1, \dots, k\}, \sigma(\mathbf{a}) = (a_1, a_2, \dots)$. The corresponding two-sided full shift on $\Pi_{-\infty}^\infty\{1, \dots, k\}$ is defined similarly. We also let $\mathcal{D}$ be the open unit disk in a separable Hilbert space, and let $\mathrm{Emb}^1(\mathcal{D}, \mathbb{H})$ be the space of C^1-embeddings of $\mathcal{D}$ into $\mathbb{H}$.

We say f has a *forward-invariant horseshoes with k symbols* if there is a continuous map $\Psi : \Pi_0^\infty\{1, \dots, k\} \to \mathrm{Emb}^1(\mathcal{D}, \mathbb{H})$ such that for each $\mathbf{a} \in \Pi_0^\infty\{1, \dots, k\}$,

(i) $\Psi(\mathbf{a})(\mathcal{D})$ is a stable manifold of finite codimension,
(ii) $f(\Psi(\mathbf{a})(\mathcal{D})) \subset \Psi(\sigma(\mathbf{a}))(\mathcal{D})$.

We sometimes refer to $\cup_{\mathbf{a}}\Psi(\mathbf{a})(\mathcal{D})$ as "the horseshoe".

We say f has a *bi-invariant horseshoe with k symbols* if there is a continuous embedding $\Psi : \Pi_{-\infty}^\infty\{1, \dots, k\} \to \mathbb{H}$ such that if $\Omega = \Psi(\Pi_{-\infty}^\infty\{1, \dots, k\})$, then

(i) $f|_\Omega$ is one-to-one and is conjugate to σ;
(ii) $f|_\Omega$ is uniformly hyperbolic.

We sometimes refer to the set Ω as "the horseshoe".

By the *uniformly hyperbolicity* of $f|\Omega$, we refer to the fact that there is a splitting of the tangent space of $x \in \Omega$ into $E^u(x) \oplus E^s(x)$ such that $E^u(x)$ and $E^s(x)$ vary continuously with $x, Df_x(E^u(x)) = E^u(fx), Df_x(E^s(x)) \subset E^s(fx)$, and there exist $N \in \mathbb{Z}^+$ and $\chi > 1$ such that for all $x \in \Omega$, $\|Df_x^N|_{E^s(x)}\| \le \chi^{-1}$ and $|Df_x^N(v)| \ge \chi|v|$ for all $v \in E^u(x)$.

Let $h_\mu(f)$ denote the metric entropy of f with respect to μ, and $h_{\mathrm{top}}(\cdot)$ the topological entropy of a map. Recall that if σ is the full shift on k symbols, then $h_{\mathrm{top}}(\sigma) = \log k$.

Theorem C. *Suppose $h_\mu(f) > 0$ and (f, μ) has no zero Lyapunov exponents. Then given $\varepsilon > 0$, there exist $m, n \in \mathbb{Z}^+$ with*

$$\frac{1}{n} \log m > h_\mu(f) - \varepsilon$$

such that the map f^n has both forward-invariant and bi-invariant horseshoes with m symbols. This implies in particular that $h_{\mathrm{top}}(f|_{\hat{\Omega}}) > h_\mu(f) - \varepsilon$ where Ω is the bi-invariant horseshoe for f^n and $\hat{\Omega} = \cup_{i=0}^{n-1} f^i(\Omega)$.

For $E \subset \mathbb{H}$, we use $|E|$ to denote the cardinality of E.

Theorem D. *Suppose $h_\mu(f) > 0$ and (f, μ) has no zero Lyapunov exponents. For $n \in \mathbb{Z}^+$, let $P_n(f) = \{x \in \mathbb{H} : f^n(x) = x\}$. Then*

$$\limsup_{n\to\infty} \frac{1}{n} \log |P_n(f)| \ge h_\mu(f).$$

Theorem D follows immediately from Theorem C together with the fact that $|P_n(\sigma)| = k^n$ where σ is the full shift on k symbols.

Remark. As discussed in the Introduction, for diffeomorphisms of compact (finite-dimensional) manifolds, the theorems above were first proved by Katok [3]. (The analog of Theorem C in [3] asserts only that $h_{\text{top}}(f^n|_\Omega) > 0$, but the conclusion of Theorem C is easily deduced from the arguments in that paper.)

Remarks on Applications to Systems Defined by PDEs. The setting above is consistent with those of systems defined by periodically driven nonlinear dissipative parabolic PDEs. Let $\{f^t, t \geq 0\}$ denote the family of time-t maps of such a system, i.e. $f^t(u_0) = u(t)$, where $u(t)$ is the solution with $u(0) = u_0$. Assuming the forcing has time-period T, the evolution of the system is captured by iterating f^T, which we take to be the mapping f in this paper. Choosing our function space appropriately, we may assume that f maps a Hilbert space $\mathbb{H}$ into itself and is C^r for $r \geq 2$. It is well known that many equations of the type above have absorbing balls and compact attracting sets; we assume the set A at the beginning of this section is the attractor or is contained in one. Injectivity of f (and of Df_x) is the backward uniqueness property; it and condition (D2)(ii) or (ii$'$) are typically satisfied for parabolic equations. These issues are discussed in e.g. [4,16,17]. With regard to our dynamical assumption of nonzero Lyapunov exponents, this is what causes us to consider systems that are periodically forced: PDEs with time-independent coefficients give rise to semiflows with zero Lyapunov exponents (see our forthcoming paper [8]), but there is no such constraint for time-T maps of periodically forced systems with forcing period T. Finally, periodic orbits of $f = f^T$ (the existence of which is asserted in Theorems A–D) correspond to periodic solutions of the original continuous-time system.

3. Lyapunov Exponents and Lyapunov Charts

3.1. The Multiplicative Ergodic Theorem (Mostly Review)

The multiplicative ergodic theorem (MET) for finite-dimensional maps or matrix-valued cocycles was first proved by Oseledets [11]. This result has since been generalized, with the matrices in Oseledets' theorem replaced by linear maps of Hilbert and Banach spaces, see [7,10,15,18]. We state below a version that will be used in this paper. It is a simplified version, in which one distinguishes only between Lyapunov exponents of different signs, i.e. positive, zero, or negative.

Theorem 1 (Version of MET used in this paper). *Let (f, μ) be as in Sect.* 2.1. *Then there is an invariant Borel subset $\Gamma \subset A$ with $\mu(\Gamma) = 1$ and a number $\lambda_0 > 0$ such that for every $x \in \Gamma$, there is a splitting of the tangent space $\mathbb{H}_x$ at x into*

$$\mathbb{H}_x = E^u(x) \oplus E^c(x) \oplus E^s(x)$$

(*some of these factors may be trivial*) *with the following properties:*

1. (a) *for* $\tau = u, c, s, x \mapsto E^\tau(x)$ *is Borel;*
   (b) $\dim E^\tau(x) < \infty$ *for* $\tau = u, c$*;*
   (c) $Df_x E^\tau(x) = E^\tau(fx)$ *for* $\tau = u, c$*, and* $Df_x E^s(x) \subset E^s(fx)$*.*
2. *For* $u \in E^\tau(x)$*,* $\tau = u, c$*, and* $n > 0$*, there is a unique* $v \in E^\tau(f^{-n}x)$*, denoted* $Df_x^{-n}u$*, such that* $Df^n_{f^{-n}x}v = u$*.*[2]
   (a) *For* $u \in E^u(x)\backslash\{0\}$*,*
   $$\lim_{n\to\pm\infty} \frac{1}{n} \log |Df_x^n u| \geq \lambda_0.$$
   (b) *For* $u \in E^c(x)\backslash\{0\}$*,*
   $$\lim_{n\to\pm\infty} \frac{1}{n} \log |Df_x^n u| = 0.$$
   (c) *For* $u \in E^s(x)\backslash\{0\}$*,*
   $$\limsup_{n\to\infty} \frac{1}{n} \log \|Df_x^n|_{E^s(x)}\| \leq -\lambda_0.$$
3. *The projections* $\pi_x^u, \pi_x^c, \pi_x^s$ *with respect to the splitting* $\mathbb{H}_x = E^u(x) \oplus E^c(x) \oplus E^s(x)$ *are Borel, and if for closed subspaces* $E, F \subset \mathbb{H}$*, we define*
   $$\measuredangle(E, F) = \inf\left\{\frac{|u \wedge v|}{|u||v|}\right\}_{u\in E\backslash\{0\}, v\in F\backslash\{0\}},$$
   then for $(E, F) = (E^u, E^c), (E^c, E^s), (E^u, E^c \oplus E^s)$ *and* $(E^u \oplus E^c, E^s)$*, we have*
   $$\lim_{n\to\pm\infty} \frac{1}{n} \log \measuredangle(E(f^n(x)), F(f^n(x))) = 0.$$

Clarification. The decomposition into $E^u \oplus E^c \oplus E^s$ as well as the finite dimensionality of E^u and E^c depends crucially on condition (D2)(ii) or (D2)(ii′) in Sect. 2.1 and on the invertibility of $f|_A : A \to A$. We elaborate on these important points:

1. Since μ is ergodic, there exists $\bar\kappa < 0$ such that $\kappa(x) = \bar\kappa$ μ-a.e. It is proved that for every $\varepsilon > 0$, there are at most finitely many Lyapunov exponents $\geq \bar\kappa + \varepsilon$, each with finite-dimensional associated subspaces. For more detail, see e.g. [7].

Lyapunov exponents $\leq \bar\kappa$ are undefined; all one can say is that there is a closed subspace on which the norm of Df_x^n grows at rate $\bar\kappa$. In infinite dimensions, $\bar\kappa$ can be $> -\infty$, and this subspace can, in general, be all of $\mathbb{H}$ (whereas in finite dimensions, $\bar\kappa = -\infty$, with the associated subspace being trivial in the case of diffeomorphisms).

In this paper, (D2)(ii) implies $\bar\kappa = -\infty$, and (D2)(ii′) implies $\bar\kappa < 0$. The latter is both necessary and sufficient for our purposes, namely to distinguish between positive, zero and negative Lyapunov exponents, to conclude that

[2] Throughout this paper, "u" is used both to denote the unstable direction, as in E^u, and as the generic name for a vector in $\mathbb{H}$. We apologize for the abuse of notation but do not think it will lead to confusion.

positive and zero exponents have at most finite multiplicities, and to have a well defined contracting subspace E^s.

2. Decompositions of the type $\mathbb{H}_x = E^u(x) \oplus E^c(x) \oplus E^s(x)$ relies on knowledge of backward orbits of f, without which one can get only a filtration of the form $E^s \subset E^{cs} \subset \mathbb{H}$. Invertibility for f on $\mathbb{H}$ is not required.

Thus, the condition in Theorem A is $E^u = E^c = \{0\}$, the condition in Theorem B is $E^c = \{0\}$, and so on.

3.2. Lyapunov Metrics

Lyapunov exponents are, by definition, asymptotic quantities. It simplifies the proofs greatly to work in coordinates in which these values are reflected in a single iteration of the map. In finite dimensions, Lyapunov metrics were introduced for that purpose. These metrics were first used in [12] and later in e.g. [3,6], see also the exposition in [19]. In this section, we carry out the corresponding constructions in Hilbert spaces. The adaptation is straightforward, but we include it for completeness, since the coordinate changes (or chart systems) constructed here will be used heavily in the rest of the paper.

Let δ_0 be such that $0 < \delta_0 < \frac{1}{100}\lambda_0$. This number denotes an accepted margin of error for the Lyapunov exponents and will be fixed throughout. There is another number, called δ, on which our chart system will depend: δ is a measure of the nonlinearity in charts and variation of chart sizes along orbits (for simplicity we group these two into a single constant). We will need this number to be small enough depending on the purpose at hand, and will specify conditions on δ each time a chart system is used.

Let $\lambda = \lambda_0 - 2\delta_0$. We begin with the following point-dependent changes of inner products. Recall that the (original) inner product and norm on $\mathbb{H}$ are denoted by $< \cdot, \cdot >$ and $|\cdot|$.

Lemma 2. *For μ-a.e. x, there is an inner product $< \cdot, \cdot >'_x$ on $\mathbb{H}_x$ with induced norm $|\cdot|'_x$ such that*

(i) $|Df_x u|'_{fx} \geq e^{\lambda}|u|'_x$ *for all* $u \in E^u(x)$.

(ii) $e^{-2\delta_0}|u|'_x \leq |Df_x u|'_{fx} \leq e^{2\delta_0}|u|'_x$ *for all* $u \in E^c(x)$.

(iii) $|Df_x u|'_{fx} \leq e^{-\lambda}|u|'_x$ *for all* $u \in E^s(x)$.

(iv) *Identifying $\mathbb{H}_x$ with $\mathbb{H}$, the function $x \mapsto < u, v >'_x$ is Borel for any fixed $u, v \in \mathbb{H}$.*

(v) *For all $p \in \mathbb{H}_x$,*

$$\frac{\sqrt{3}}{3}|p| \leq |p|'_x \leq K(x)|p|$$

for some Borel function K with

$$\lim_{n\to\pm\infty} \frac{1}{n}\log K(f^n x) = 0.$$

Proof. For $x \in \Gamma$, define $< \cdot, \cdot >'_x$ by

$$< u, v >'_x = \begin{cases} \sum_{n=-\infty}^{0} \frac{<Df_x^n u, Df_x^n v>}{e^{2n(\lambda_0 - 2\delta_0)}} & \text{for } u, v \in E^u(x), \\ \sum_{n=-\infty}^{+\infty} \frac{<Df_x^n u, Df_x^n v>}{e^{4|n|\delta_0}} & \text{for } u, v \in E^c(x), \\ \sum_{n=0}^{+\infty} \frac{<Df_x^n u, Df_x^n v>}{e^{2n(-\lambda_0 + 2\delta_0)}} & \text{for } u, v \in E^s(x), \\ 0 & \text{for } u \in E^{\tau_1}(x), v \in E^{\tau_2}(x), \tau_1 \neq \tau_2. \end{cases} \tag{1}$$

(i)–(iii) follow from straightforward computations using the definitions above, and (iv) follows Lemma 1. Part of (v) is also immediate: Let $p = u + w + v$ where $u \in E^u(x), w \in E^c(x)$ and $v \in E^s(x)$. Then

$$|p|^2 \le 3(|u|^2 + |w|^2 + |v|^2) \le 3\{(|u|'_x)^2 + (|w|'_x)^2 + (|v|'_x)^2\} = 3(|p|'_x)^2.$$

It remains to bound $|p|'_x$ above by a quantity related to $|p|$.

By 2(a)–(c) of Theorem 1, there is a Borel function $R(x) \ge 1$ with

$$\lim_{n \to \pm\infty} \frac{1}{n} \log R(f^n x) = 0$$

such that for u, w, v as above,

$$|Df_x^{-n} u| \le R(x) e^{-n(\lambda_0 - \delta_0)} |u| \quad \text{for } n \ge 0,$$
$$(R(x) e^{|n|\delta_0})^{-1} |w| \le |Df_x^n w| \le R(x) e^{|n|\delta_0} |w| \quad \text{for } n \in \mathbb{Z},$$
$$|Df_x^n v| \le R(x) e^{-n(\lambda_0 - \delta_0)} |v| \quad \text{for } n \ge 0.$$

Thus,

$$(|u|'_x)^2 = \sum_{n=-\infty}^{0} \frac{|Df_x^n u|^2}{e^{2n(\lambda_0 - 2\delta_0)}} \le \sum_{n=-\infty}^{0} \frac{(R(x) e^{n(\lambda_0 - \delta_0)} |u|)^2}{e^{2n(\lambda_0 - 2\delta_0)}} = \frac{R(x)^2}{1 - e^{-2\delta_0}} |u|^2,$$

with similar estimates for w and v. Also, we have

$$|u + w + v|^2 \ge \theta(x)^2 (|u|^2 + |w|^2 + |v|^2)$$

where $\theta(x) = \frac{1}{2} \measuredangle(E^u(x), E^{cs}(x)) \cdot \measuredangle(E^c(x), E^s(x))$. From these two sets of inequalities, we deduce that

$$(|u + w + v|'_x)^2 \le \frac{R(x)^2 (1 + e^{-2\delta_0})}{\theta(x)^2 (1 - e^{-2\delta_0})} |u + w + v|^2.$$

The function $K(x)$ defined by the inequality above inherits its subexponential growth property from R and θ. □

We introduce next a family of point-dependent coordinate changes $\{\Phi_x\}$ where for each x, Φ_x is an affine map taking a neighborhood of 0 in $\mathbb{H}$ to a neighborhood of x in $\mathbb{H}$. Noting that the dimensions of E^u and E^c and the codimension of E^s are constant μ-a.e., we fix orthogonal subspaces $\tilde{E}^u, \tilde{E}^c$ and $\tilde{E}^s$ of $\mathbb{H}$ such that $\dim \tilde{E}^u = \dim E^u$, $\dim \tilde{E}^c = \dim E^c$ and $\operatorname{codim} \tilde{E}^s = \operatorname{codim} E^s$. For a.e. x, we let $L_x : \mathbb{H}_x \to \mathbb{H}$ be such that

(i) $L_x(E^\tau(x)) = \tilde{E}^\tau$, $\tau = u, c, s$; and
(ii) $< L_x u, L_x v > = < u, v >'_x$ for all $u, v \in \mathbb{H}_x$.

Such a linear map exists and can be chosen to vary measurably with respect to x (see e.g. [2]). For $r > 0$, let $\tilde{B}(0,r) = \tilde{B}^u(0,r) \times \tilde{B}^c(0,r) \times \tilde{B}^s(0,r)$ where $\tilde{B}^\tau(0,r)$ is the ball of radius r centered at 0 in $\tilde{E}^\tau$. The coordinate patches $\{\Phi_x\}$ are then given by

$$\Phi_x : \tilde{B}(0, \delta l(x)^{-1}) \to \mathbb{H}, \quad \Phi_x(u) = \mathrm{Exp}_x(L_x^{-1}(u))$$

where $\mathrm{Exp}_x : \mathbb{H}_x \to \mathbb{H}$ is the exponential map (the usual identification of the tangent space $\mathbb{H}_x$ at x with $\{x\} + \mathbb{H}$), δ is the constant at the beginning of this subsection, and l is a function to be determined. Maps connecting charts along orbits are denoted by

$$\tilde{f}_x = \Phi_{fx}^{-1} \circ f \circ \Phi_x.$$

Since Φ_{fx}^{-1} is extendible to an affine map on all of $\mathbb{H}$, we sometimes view $\tilde{f}_x$ as $\tilde{f}_x : \tilde{B}(0, \delta l(x)^{-1}) \to \mathbb{H}$.

Properties of Φ_x and $\tilde{f}_x$ are summarized in Proposition 4. To be consistent with earlier notation, $D(\tilde{f}_x)_0$ means the derivative of $\tilde{f}_x$ evaluated at the point 0 in the chart, and so on. To control the nonlinearity in charts, we will need the following bound which follows easily from the conditions in Sect. 2:

Lemma 3. *There exist $M_2 > 0$ and $r_0 > 0$ such that $\|D^2 f_x\| < M_2$ for all $x \in \mathbb{H}$ with $dist(x, A) < r_0$.*

Proposition 4. *Given $\delta \in (0, \frac{\sqrt{3}}{3} r_0)$, there is a measurable function $l : \Gamma \to [1, +\infty)$ with $e^{-\delta} l(x) \le l(f(x)) \le e^{\delta} l(x)$ such that the following hold at μ-a.e. x:*

(a) *For all $y, y' \in B(0, \delta l(x)^{-1})$,*

$$l(x)^{-1}|y - y'| \le |\Phi_x(y) - \Phi_x(y')| \le \sqrt{3}|y - y'|.$$

(b) *$D(\tilde{f}_x)_0$ maps each $\tilde{E}^\tau$, $\tau = u, c, s$, into itself, with*

$$|D(\tilde{f}_x)_0 u| \ge e^{\lambda}|u|, \quad e^{-2\delta_0}|w| \le |D(\tilde{f}_x)_0 w| \le e^{2\delta_0}|w|,$$
$$\text{and} \quad |D(\tilde{f}_x)_0 v| \le e^{-\lambda}|v|$$

for $u \in \tilde{E}^u, w \in \tilde{E}^c$ and $v \in \tilde{E}^s$.

(c) *The following hold on $B(0, \delta l(x)^{-1})$:*
  (i) *$Lip(\tilde{f}_x - D(\tilde{f}_x)_0) < \delta$;*
  (ii) *$Lip(D\tilde{f}_x) \le l(x)$.*

Proof. From Lemma 2(v), it follows that

$$\frac{1}{K(x)}|v| \le |L_x^{-1} v| \le \sqrt{3}|v|, \quad v \in \mathbb{H}, \tag{2}$$

so (a) holds if $l(x) \ge K(x)$. (b) is nothing more than a rephrasing of Lemma 2, (i)–(iii), together with property (ii) of L_x.

Proceeding to (c), since $D(\tilde{f}_x)_y = D\Phi_{fx}^{-1} \circ Df_{\Phi_x(y)} \circ D\Phi_x$, we have

$$\begin{aligned}\|D(\tilde{f}_x)_y - D(\tilde{f}_x)_z\| &= \|D\Phi_{fx}^{-1} \cdot (Df_{\Phi_x(y)} - Df_{\Phi_x(z)}) \cdot D\Phi_x\| \\ &\leq \|L_{fx}\| \cdot M_2 |L_x^{-1}y - L_x^{-1}z| \cdot \|L_x^{-1}\| \\ &\leq 3M_2 K(fx)|y-z|,\end{aligned}$$

i.e. $\mathrm{Lip}(D\tilde{f}_x) < 3M_2K(fx)$. Here, M_2 is the constant in Lemma 3 , and in using this lemma we have taken for granted that $|\Phi_x(y) - x|$ and $|\Phi_x(z) - x| < r_0$ where r_0 is in Lemma 3. Thus, (c)(ii) holds if these conditions are valid and $l(x) \geq 3M_2K(fx)$.

Finally, to estimate (c)(i), we use

$$\mathrm{Lip}(\tilde{f}_x - D(\tilde{f}_x)_0) \leq \sup_{y \in \tilde{B}(0, l(x)^{-1}\delta)} \|D(\tilde{f}_x - D(\tilde{f}_x)_0)_y\|$$

and for $y \in \tilde{B}(0, l(x)^{-1}\delta)$, provided that $\mathrm{Lip}(D\tilde{f}_x) < l(x)$, we have

$$\|D(\tilde{f}_x - D(\tilde{f}_x)_0)_y\| \leq \|D(\tilde{f}_x)_y - D(\tilde{f}_x)_0\| \leq \mathrm{Lip}(D\tilde{f}_x) \cdot |y| \leq \delta.$$

Let $B(x, r_0)$ denote the ball of radius r_0 centered at x. Letting $\tilde{l}(x) = \max\{K(x), 3M_2K(fx), 1\}$ and noting that $\sqrt{3}\delta < r_0$, we have $\Phi_x(\tilde{B}(0, \delta\tilde{l}(x)^{-1})) \subset B(x, r_0)$. All of the conditions required of l are thus satisfied by $\tilde{l}$ – except for one: $\tilde{l}$ need not fluctuate slowly along orbits. To finish, observe that

$$\lim_{n \to \pm\infty} \frac{1}{n} \log \tilde{l}(f^n(x)) = 0$$

since K has this property (Lemma 2(v)). For such an $\tilde{l}$, it is a standard fact (see e.g. Sect. 4.3 of [1]) that there exists $l \geq \tilde{l}$ with $e^{-\delta}l(x) \leq l(f(x)) \leq e^{\delta}l(x)$. Since increasing $\tilde{l}$ cannot spoil any of the estimates, this is an acceptable function. □

Noninvariant sets with uniform estimates. For $l_0 > 0$, let $\Gamma_{l_0} = \{x \in \Gamma | l(x) \leq l_0\}$ where l is the function in Proposition 4. The sets Γ_l are generally not invariant, but since $\Gamma = \cup_{l \geq 1} \Gamma_l$ has full measure, $\mu(\Gamma_l) > 0$ for all large enough l. Notice that for each (fixed) l, we have uniform bounds for the domains of the charts $\tilde{B}(0, l(x)^{-1}\delta)$, $\|\Phi_x\|$, and $\mathrm{Lip}(D\tilde{f}_x)$ for all $x \in \Gamma_l$.

4. Proof of Theorem A

Let $\delta > 0$ be such that $e^{-\lambda} + \delta < e^{-\delta}$. We fix a chart system as in Sect. 3.2 using this δ, and begin with the following easy but important observation: In the setting of Theorem A, where $E^u = E^c = \{0\}$, we have at a.e. x that $\tilde{f}_x$ maps $\tilde{B}(0, \delta l(x)^{-1})$ into $\tilde{B}(0, \delta l(fx)^{-1})$ and is a contraction with $\mathrm{Lip}(\tilde{f}_x) < e^{-\lambda} + \delta$. This follows immediately from $l(fx) < e^{\delta}l(x)$ and property (c)(i) in Proposition 4.

Let Γ_l be as in the last paragraph of Sect. 3.2. We claim that to prove the theorem, it suffices to find l_0, x and n such that

(i) $x, f^n(x) \in \Gamma_{l_0}$, and x is in the support of μ (i.e. every open neighborhood of x has positive μ-measure);

(ii) x and $f^n(x)$ are sufficiently close that if we let

$$\tilde{F} = \Phi_x^{-1} \circ \Phi_{f^n x} \circ \tilde{f}_{f^{n-1}x} \circ \cdots \circ \tilde{f}_x,$$

then $\tilde{F}$ is defined on all of $\tilde{B}(0, \delta l_0^{-1})$ and maps $\tilde{B}(0, \delta l_0^{-1})$ into itself with $|\tilde{F}(0)| < \frac{1}{3}\delta l_0^{-1}$;

(iii) $\mathrm{Lip}(\tilde{F}) < \frac{1}{3}$.

We first finish the proof assuming (i)–(iii) can be arranged: From (ii) and (iii), it follows that $\tilde{F}$ has a unique fixed point $\tilde{z} \in \tilde{B}(0, \delta l_0^{-1})$. Clearly, $z = \Phi_x(\tilde{z})$ satisfies $f^n(z) = z$, and $\mu|_{\Phi_x(\tilde{B}(0,\delta l_0^{-1}))}$, which is nonzero by design, is necessarily concentrated at z. Since μ is ergodic, it follows that the entire measure is supported on the orbit of z, which is what the theorem asserts.

To justify (i)–(iii), we first fix l_0 with $\mu(\Gamma_{l_0}) > 0$. Next we choose $U \subset \Gamma_{l_0}$ such that $\mu(U) > 0$ but U is small enough that for all $y \in U, U \subset \Phi_y(\tilde{B}(0, \frac{1}{3}\delta l_0^{-1}))$; this is possible by Proposition 4(a). We then pick $x \in U$ with the property that its orbit returns to U infinitely often; this is feasible by the Poincaré Recurrence Theorem. Finally, let n be a large enough return time for x so that $\sqrt{3} l_0 (e^{-\lambda} + \delta)^n < \frac{1}{3}$. Then $f^n(x) \in U \subset \Phi_x(\tilde{B}(0, \frac{1}{3}\delta l_0^{-1}))$, implying $|\tilde{F}(0)| < \frac{1}{3}\delta l_0^{-1}$. (iii) follows from the fact that $\|\Phi_{f^n x}\| \leq \sqrt{3}$ and $\|(\Phi_x)^{-1}\| \leq l_0$. □

5. Stable and Unstable Manifolds

This section contains the main technical preparation for the proofs of Theorems B and C. The results needed are stable and unstable manifold theorems, of which many versions with different technical assumptions have been proved in the literature. In this section, we develop a version that will be very useful in much of nonuniform hyperbolic theory.

The following notation is used throughout: For linear spaces X and $Y, \mathcal{L}(X, Y)$ denotes the set of all bounded linear maps from X to Y.

5.1. Setting and Statement of Results

The setting and conclusions in Propositions 5, 6 and 8 are independent of the material in previous sections, though the setting is clearly motivated by chart maps $\{\tilde{f}_{f^i x}, i \in \mathbb{Z}\}$.

Setting. Let $\lambda_1 > 0$ be fixed, and let δ_1 and $\delta_2 > 0$ be as small as need be depending on λ_1. We assume there is a splitting of $\mathbb{H}$ into orthogonal subspaces $\mathbb{H} = E^u \oplus E^s$ with $\dim(E^u) < \infty$. For $i \in \mathbb{Z}$, let r_i be positive numbers such that $r_{i+1}e^{-\delta_1} < r_i < r_{i+1}e^{\delta_1}$ for all i, and let $B_i = B_i^u \times B_i^s$ where $B_i^\tau = B^\tau(0, r_i), \tau = u, s$. We consider a sequence of differentiable maps

$$g_i : B_i \to \mathbb{H}, \qquad i = \ldots, -1, 0, 1, 2, \ldots,$$

such that for each $i, g_i = \Lambda_i + G_i$ where Λ_i and G_i are as follows:

(I) $\Lambda_i \in \mathcal{L}(\mathbb{H}, \mathbb{H})$ and splits into $\Lambda_i = \Lambda_i^u \oplus \Lambda_i^s$ where $\Lambda_i^u \in \mathcal{L}(E^u, E^u), \Lambda_i^s \in \mathcal{L}(E^s, E^s)$, and $\|(\Lambda_i^u)^{-1}\|, \|\Lambda_i^s\| \leq e^{-\lambda_1}$;

(II) $|G_i(0)| < \delta_2 r_{i+1}$, and $\|DG_i(x)\| < \delta_2$ for all $x \in B_i$.

For slightly stronger results, we assume also

(III) there are positive numbers ℓ_i with $\ell_{i+1}e^{-\delta_1} < \ell_i < \ell_{i+1}e^{\delta_1}$ such that $\mathrm{Lip}(DG_i) < \ell_i$.

Orthogonal projections from $\mathbb{H}$ to E^u and E^s are denoted by π^u and π^s, respectively.

Proposition 5 (Local unstable manifolds). *Assume* (I) *and* (II), *and let δ_1 and δ_2 (depending only on λ_1) be sufficiently small. Then for each i there is a differentiable function $h_i^u : B_i^u \to B_i^s$ depending only on $\{g_j, j < i\}$, with*

(i) $|h_i^u(0)| < \frac{1}{2} r_i$ *and*

(ii) $\|Dh_i^u\| \leq \frac{1}{10}$

such that if $W_i^u = \mathrm{graph}(h_i^u)$, then

(a) $g_i(W_i^u) \supset W_{i+1}^u$;

(b) *for $x, y \in W_i^u$ such that $g_i x, g_i y \in B_{i+1}$,*

$$|\pi^u(g_i x) - \pi^u(g_i y)| > (e^{\lambda_1} - 2\delta_2)|\pi^u x - \pi^u y|.$$

If (III) *holds additionally, then $h_i^u \in C^{1+Lip}$ with $Lip(Dh_i^u) < const{\cdot}\ell_i$.*

Proposition 6 (Local stable manifolds). *Assume* (I) *and* (II), *and let δ_1 and δ_2 (depending only on λ_1) be sufficiently small. Then for each i there is a differentiable function $h_i^s : B_i^s \to B_i^u$ depending only on $\{g_j, j \geq i\}$, with*

(i) $|h_i^s(0)| < \frac{1}{2} r_i$ *and*

(ii) $\|Dh_i^s\| \leq \frac{1}{10}$

such that if $W_i^s = \mathrm{graph}(h_i^s)$, then

(a) $g_i W_i^s \subset W_{i+1}^s$;

(b) *for $x, y \in W_i^s, |\pi^s(g_i x) - \pi^s(g_i y)| < (e^{-\lambda_1} + 2\delta_2)|\pi^s x - \pi^s y|$.*

If (III) *holds additionally, then $h_i^s \in C^{1+Lip}$ with $Lip(Dh_i^s) < const{\cdot}\ell_i$.*

We remark that the $C^{1+\mathrm{Lip}}$ property of h_i^u and h_i^s in Propositions 5 and 6 can be replaced by $C^{1+\alpha}$ with the $\mathrm{Lip}(DG_i)$ condition in (III) replaced by one on the C^α-norm of DG_i. Notice also that δ_1 and δ_2 do not depend on r_i or ℓ_i.

The following result, which gives local stable and unstable manifolds μ-a.e. in the context of Sect. 2, is an immediate corollary of Propositions 5 and 6. Various versions of this result have been proved before, see e.g. [7,15].

Corollary 7. *In the setting of Sect. 2 with $E^c = \{0\}$, consider a chart system with $\delta \leq \min\{\delta_1, \delta_2\}$ where δ_1 and δ_2 are as in Propositions 5 and 6 with $\lambda_1 = \lambda$. Then the results above apply to the chart maps $\{\tilde{f}_{f^i x}, i \in \mathbb{Z}\}$ for μ-a.e. x, giving*

$$\tilde{W}_x^s = graph(\tilde{h}_x^s) \quad and \quad \tilde{W}_x^u = graph(\tilde{h}_x^u)$$

where $\tilde{h}^s_x : \tilde{B}^s(0, \delta l(x)^{-1}) \to \tilde{B}^u(0, \delta l(x)^{-1})$ *and* $\tilde{h}^u_x : \tilde{B}^u(0, \delta l(x)^{-1}) \to \tilde{B}^s(0, \delta l(x)^{-1})$ *satisfy*

$$\tilde{h}^s_x(0) = 0,\ (D\tilde{h}^s_x)_0 = 0 \quad \textit{and} \quad \tilde{h}^u_x(0) = 0,\ (D\tilde{h}^u_x)_0 = 0$$

and have the properties in Propositions 5 *and* 6*.*

The Φ_x-images of $\tilde{W}^s_x$ and $\tilde{W}^u_x$ are called the *local stable* and *unstable manifolds* at x.

Finally, we will also need the following result, which tells us how h^s_0 and h^u_0 vary in the C^1-topology with $\{g_i\}$ in the setting at the beginning of this subsection.

Proposition 8. *Let* λ_1, δ_1 *and* δ_2 *be as in Proposition* 6*, and let* r_0 *and* ℓ_0 *be fixed. Given* $\varepsilon > 0$*, there exists* $N = N(\varepsilon)$ *such that if* $\{g_i\}$ *and* $\{\hat{g}_i\}$ *are two sequences of maps satisfying Conditions* (I)–(III) *and* $g_i = \hat{g}_i$ *for all* $0 \le i \le N$*, then* $\|h^s_0 - \hat{h}^s_0\|_{C^1} < \varepsilon$ *where* h^s_0 *and* $\hat{h}^s_0$ *are as in Proposition* 6 *for* $\{g_i\}$ *and* $\{\hat{g}_i\}$*, respectively.*

Analogous results hold for h^u_0 *provided* $g_i = \hat{g}_i$ *for* $-N < i < 0$ *for sufficiently large* N*.*

5.2. Proof of Proposition 6

The proofs of Propositions 5 and 6 are quite similar to the corresponding proofs for a fixed map at a fixed point. We give only the stable manifolds proof, which illustrates how one deals with the noninvertibility of the maps. The proof of Proposition 5 proceeds similarly, and is simpler in that graph transforms for g_i and $Dg_i(x)$ for fixed x are defined (but not those for g_i^{-1} or $Dg_i^{-1}(x)$). See the remark following the statement of Lemma 9.

We have divided the proof of Proposition 6 into three main steps.

Step 1. *Proof of existence of a Lipschitz* h^s_i *with properties* (i)*,* (ii)*,* (a) *and* (b) *in Proposition* 6. (Our arguments here follow [5].) Define

$$\mathcal{W}_i = \left\{ w_i : B^s_i \to B^u_i \mid |w_i(0)| \le \frac{1}{2} r_i,\ \mathrm{Lip}(w_i) \le \frac{1}{10} \right\}.$$

Equipped with the C^0 norm, $\mathcal{W}_i$ is a complete metric space. We begin by defining what is effectively a graph transform by g_i^{-1}—in spite of the fact that g_i is not invertible.

Lemma 9. *Given any* $w_{i+1} \in \mathcal{W}_{i+1}$*, there is a unique* $w_i \in \mathcal{W}_i$ *such that*

$$g_i(\mathrm{graph}(w_i)) \subset \mathrm{graph}(w_{i+1}).$$

Remark on unstable manifolds case. If $\mathcal{V}_i$ is the analog of $\mathcal{W}_i$ with u and s interchanged, then given $v_i \in \mathcal{V}_i, v_{i+1}$ is simply the map whose graph is $g_i(\mathrm{graph}(v_i)) \cap B_{i+1}$.

Proof. Let $w_{i+1} \in \mathcal{W}_{i+1}$ be fixed throughout. For $w_i : B^s_i \to B^u_i$ to have the property in the lemma, it is sufficient that for every $\eta \in B^s_i$,

$$w_{i+1}(\Lambda^s_i \eta + \pi^s G_i(\eta, w_i(\eta))) = \Lambda^u_i w_i(\eta) + \pi^u G_i(\eta, w_i(\eta)),$$

equivalently,

$$w_i(\eta) = (\Lambda_i^u)^{-1}\left[w_{i+1}(\Lambda_i^s\eta + \pi^s G_i(\eta, w_i(\eta))) - \pi^u G_i(\eta, w_i(\eta))\right]. \tag{3}$$

For $w_i \in \mathcal{W}_i$, we let $\tilde{w}_{i+1}(w_i)$ be the mapping from B_i^s to E^u where $\tilde{w}_{i+1}(w_i)(\eta)$ is given by the right side of (3). The problem then becomes finding $w_i \in \mathcal{W}_i$ with $\tilde{w}_{i+1}(w_i) = w_i$. We do this in two steps.

(i) We show that $\tilde{w}_{i+1}(w_i) \in \mathcal{W}_i$ for every $w_i \in \mathcal{W}_i$. First,

$$\begin{aligned}
&|\tilde{w}_{i+1}(w_i)(0)| \\
&\quad \le e^{-\lambda_1}\cdot\left[|w_{i+1}(0)| + \frac{1}{10}|\pi^s G_i(0, w_i(0))| + |G_i(0,0)| + \mathrm{Lip}(G_i)\cdot|(0, w_i(0))|\right] \\
&\quad < e^{-\lambda_1}\cdot\left[\frac{1}{2}r_{i+1} + \frac{11}{10}\delta_2 r_{i+1} + \delta_2\cdot\frac{1}{2}r_i\right]
\end{aligned}$$

which is $< \frac{1}{2}r_i$ if δ_1 and δ_2 are small enough.

Next we estimate the Lipschitz constant of $\tilde{w}_{i+1}(w_i)$. Let $\eta, \xi \in B_i^s$. Using $\mathrm{Lip}(w_i)$, $\mathrm{Lip}(w_{i+1}) \le \frac{1}{10}$, we have

$$\begin{aligned}
&|(\tilde{w}_{i+1}(w_i))\,\eta - (\tilde{w}_{i+1}(w_i))\,\xi| \\
&\quad \le \|(\Lambda_i^u)^{-1}\|\cdot[\ |w_{i+1}(\Lambda_i^s\eta + \pi^s G_i(\eta, w_i(\eta))) - w_{i+1}(\Lambda_i^s\xi + \pi^s G_i(\xi, w_i(\xi)))| \\
&\qquad + \|\pi^u\|\cdot|G_i(\eta, w_i(\eta)) - G_i(\xi, w_i(\xi))|\ ] \\
&\quad \le e^{-\lambda_1}\left\{\frac{1}{10}|\Lambda_i^s(\eta-\xi)| + \left(1+\frac{1}{10}\right)\ \mathrm{Lip}(G_i)|(\eta, w_i(\eta)) - (\xi, w_i(\xi))|\right\} \\
&\quad < e^{-\lambda_1}\frac{1}{10}(e^{-\lambda_1} + 2\delta_2)|\eta - \xi|
\end{aligned}$$

which is $< \frac{1}{10}|\eta - \xi|$ with δ_2 sufficiently small.

The two estimates above imply that $|\,(\tilde{w}_{i+1}(w_i))\,\eta| < r_i$ for all $\eta \in B_i^s$, completing the proof of $\tilde{w}_{i+1}(w_i) \in \mathcal{W}_i$.

(ii) We prove $\tilde{w}_{i+1} : \mathcal{W}_i \to \mathcal{W}_i$ is a contraction. Let $w_i^1, w_i^2 \in \mathcal{W}_i$. Then

$$\begin{aligned}
&\left|(\tilde{w}_{i+1}(w_i^1))\eta - (\tilde{w}_{i+1}(w_i^2))\eta\right| \\
&\quad \le \|(\Lambda_i^u)^{-1}\|\cdot(\mathrm{Lip}(w_{i+1}) + 1)\cdot\left|G_i\left(\eta, w_i^1(\eta)\right) - G_i\left(\eta, w_i^2(\eta)\right)\right| \\
&\quad < e^{-\lambda_1}2\delta_2\left|w_i^1(\eta) - w_i^2(\eta)\right|.
\end{aligned}$$

The unique fixed point of $\tilde{w}_{i+1}$ is the w_i in the lemma. □

For each i, we let $\Gamma_i : \mathcal{W}_{i+1} \to \mathcal{W}_i$ be the mapping defined by $\Gamma_i(w_{i+1}) = w_i$ where w_i and w_{i+1} are related as in Lemma 9. For $k \in \mathbb{Z}^+$, let $\Gamma_i^k : \mathcal{W}_{i+k} \to \mathcal{W}_i$ denote the composition $\Gamma_i \circ \cdots \circ \Gamma_{i+k-1}$.

Lemma 10. *There is a unique sequence $w_i^*, i \in \mathbb{Z}$, such that for all $i, w_i^* \in \mathcal{W}_i$ and $\Gamma_i(w_{i+1}^*) = w_i^*$.*

Proof. Let i be fixed throughout. First we show Γ_i is a contraction: Let $w_i^1 = \Gamma_i(w_{i+1}^1)$ and $w_i^2 = \Gamma_i(w_{i+1}^2)$. Proceeding as above, we obtain for $\eta \in B_i^s$,

$$|w_i^1(\eta) - w_i^2(\eta)| \le e^{-\lambda_1}\cdot(\|w_{i+1}^1 - w_{i+1}^2\| + 3\delta_2|w_i^1(\eta) - w_i^2(\eta)|).$$

Thus,

$$\|w_i^1 - w_i^2\| \le c\|w_{i+1}^1 - w_{i+1}^2\| \quad \text{where} \quad c = \frac{e^{-\lambda_1}}{1 - 3\delta_2 e^{-\lambda_1}}.$$

Inductively, we obtain $\operatorname{diam}(\Gamma_i^k(\mathcal{W}_{i+k})) \le 2c^k r_{i+k}$ where

$$\operatorname{diam}(\Gamma_i^k(\mathcal{W}_{i+k})) = \sup_{w^1, w^2 \in \Gamma_i^k(\mathcal{W}_{i+k})} \|w^1 - w^2\|.$$

Since $r_{i+k} < r_i e^{k\delta_1}$ and $c < e^{-\delta_1}$, it follows that $\operatorname{diam}(\Gamma_i^k(\mathcal{W}_{i+k})) \to 0$ as $k \to \infty$. From the completeness of $\mathcal{W}_i$, we have that $\cap_{k>0}\overline{\Gamma_i^k(\mathcal{W}_{i+k})}$ (where $\overline{\Omega}$ denotes the closure of Ω) contains exactly one point. This is our w_i^*. □

It is an easy exercise to check that if we let $h_i^s = w_i^*$, then $\operatorname{Lip}(h_i^s) \le \frac{1}{10}$, and properties (a) and (b) in Proposition 6 hold.

Step 2. *Proof of differentiability of $h_i = h_i^s$.*

Fix $x \in B_i$ for the moment. Identifying the tangent spaces $\mathbb{H}_x$ and $\mathbb{H}_{g_i(x)}$ with $\mathbb{H} = E^u \oplus E^s$, we now define the surrogate for the graph transform by the linear map $(Dg_i)_x^{-1}$:

Lemma 11. *Given $x \in B_i$ and $s_{i+1} \in \mathcal{L}(E^s, E^u)$ with $||s_{i+1}|| \le \frac{1}{10}$, there is a unique $s_i \in \mathcal{L}(E^s, E^u)$ with $||s_i|| \le \frac{1}{10}$ such that*

$$(Dg_i)_x(\operatorname{graph}(s_i)) \subset \operatorname{graph}(s_{i+1}).$$

The setup is a special case of Lemma 9, with g_i taken to be linear (and globally defined on $\mathbb{H}_x$). We omit the proof; the only point that needs to be checked is that with $\tilde{s}_{i+1}$ defined analogously to $\tilde{w}_{i+1}, \tilde{s}_{i+1}(s_i)$ is also linear, and that is obvious from Eq. (3). Let $\tilde{\Gamma}_{i,x}$ denote the mapping given by $\tilde{\Gamma}_{i,x}(s_{i+1}) = s_i$.

Returning to the problem at hand, namely the regularity of h_i, we let

$$\mathcal{Z}_i = \left\{ \sigma_i \in \mathcal{B}(B_i^s, \mathcal{L}(E^s, E^u)) : ||\sigma_i|| \le \frac{1}{10} \right\}$$

where $\mathcal{B}(X, Y)$ is the set of bounded maps from X to Y, and $\|\cdot\|$ is the sup norm. For each $\sigma_i \in \mathcal{Z}_i$ and $\eta \in B_i^s$, we consider the graph of $\sigma_i(\eta)$ as a *trial tangent plane* for the graph of h_i at $x = (\eta, h_i(\eta))$. From Lemma 11, we obtain immediately the following:

Lemma 12. *Given any i and $\sigma_{i+1} \in \mathcal{Z}_{i+1}$, there is a unique $\sigma_i \in \mathcal{Z}_i$ such that for all $\eta \in B_i^s$, if $x = (\eta, h_i(\eta))$, then*

$$(Dg_i)_x(\operatorname{graph}(\sigma_i(\eta))) \subset \operatorname{graph}(\sigma_{i+1}(\pi^s(g_i(x)))).$$

We let $\tilde{\Gamma}_i : \mathcal{Z}_{i+1} \to \mathcal{Z}_i$ be given by $\tilde{\Gamma}_i(\sigma_{i+1}) = \sigma_i$ where σ_i and σ_{i+1} are as above. Equivalently, for $\eta \in B_i^s$, if $x = (\eta, h_i(\eta))$, then

$$\tilde{\Gamma}_i(\sigma_{i+1})(\eta) = \tilde{\Gamma}_{i,x}(\sigma_{i+1}(\pi^s g_i(x))).$$

We also record for later use (in the Proof of Lemma 12) the following: For $v \in E^s$,

$$(\sigma_i(\eta))v = (\Lambda_i^u)^{-1}[(\sigma_{i+1}(\pi^s g_i(x)))(\Lambda_i^s v + \pi^s(DG_i)_x(v,(\sigma_i(\eta))v))] \\ -(\Lambda_i^u)^{-1}[\pi^u(DG_i)_x(v,(\sigma_i(\eta))v)]. \tag{4}$$

Lemma 13. *There is a unique sequence $\sigma_i^*, i \in \mathbb{Z}$, such that for all $i, \sigma_i^* \in \mathcal{Z}_i$ and $\tilde{\Gamma}_i(\sigma_{i+1}^*) = \sigma_i^*$.*

Proof. As an immediate consequence of the contractive property of the individual $\tilde{\Gamma}_{i,x}$, we have

$$\|\tilde{\Gamma}_i(\sigma_i^1) - \tilde{\Gamma}_i(\sigma_i^2)\| \leq c'\|\sigma_i^1 - \sigma_i^2\|$$

for some $c' < 1$. From this we conclude that for each i,

$$\cap_{k>0}\overline{\tilde{\Gamma}_i^k(\mathcal{Z}_{i+k})} = \{\sigma_i^*\}.$$

□

It remains to show that $Dh_i = \sigma_i^*$. Let $\Delta h_i(\eta) = h_i(\eta + \Delta\eta) - h_i(\eta)$, and define

$$M_i(\sigma_i) = \sup_{\eta \in B_i^s}\left(\limsup_{|\Delta\eta|\to 0}\frac{|\Delta h_i(\eta) - \sigma_i(\eta)\Delta\eta|}{|\Delta\eta|}\right).$$

Lemma 14. *There is $c'' < 1$ such that for all i and for all $\sigma_{i+1} \in \mathcal{Z}_{i+1}$,*

$$M_i(\tilde{\Gamma}_i(\sigma_{i+1})) \leq c'' M_{i+1}(\sigma_{i+1}).$$

It follows that $M_i(\sigma_i^*) = 0$, i.e. $Dh_i = \sigma_i^*$.

Proof. Let $\sigma_i = \tilde{\Gamma}_i(\sigma_{i+1})$. Using $\mathrm{Lip}(h_i)$, $\mathrm{Lip}(h_{i+1}) < \frac{1}{10}$ and (4), we obtain after a straightforward computation that

$$|\Delta h_i(\eta) - \sigma_i(\eta)\Delta\eta| \leq e^{-\lambda}\left(e^{-\lambda} + \frac{11}{10}\delta_2\right)(1 + o(|\Delta\eta|))M_{i+1}(\sigma_{i+1})|\Delta\eta| \\ + \frac{11}{10}e^{-\lambda}\delta_2|\Delta h_i(\eta) - \sigma_i(\eta)\Delta\eta| + o(|\Delta\eta|).$$

This proves the inequality in the lemma. Together with the fact that $M_i(\sigma_i) \leq \frac{1}{5}$ for all $\sigma_i \in \mathcal{Z}_i$, it gives $M_i(\sigma_i^*) = 0$. □

Step 3. *Proof of Lipschitzness of Dh_i.*

Let $\eta_1, \eta_2 \in B_i^s$ with $x_j = (\eta_j, h_i(\eta_j))$. Assuming condition (III) in Sect. 5.1, we obtain, after a computation similar to previous ones, that

$$\|(Dh_i)_{\eta_1} - (Dh_i)_{\eta_2}\| \leq 2\ell_i|\eta_1 - \eta_2| + q\|(Dh_{i+1})_{\pi^s g_i(x_1)} - (Dh_{i+1})_{\pi^s g_i(x_2)}\|$$

where $q = (e^{-2\lambda_1} + \frac{11}{10}e^{-\lambda_1}\delta_2)/(1 - \frac{11}{10}e^{-\lambda_1}\delta_2) \approx e^{-2\lambda_1}$ assuming δ_2 is small enough. We also have the estimate

$$|\pi^s g_i(x_1) - \pi^s g_i(x_2)| \leq p|\eta_1 - \eta_2|$$

where $p = e^{-\lambda_1} + 2\delta_2$ (Proposition 6(b)). Combining, one shows inductively that

$$\|(Dh_i)_{\eta_1} - (Dh_i)_{\eta_2}\| \le \left(2\sum_{j=0}^{\infty}(pq)^j \ell_{i+j}\right)|\eta_1 - \eta_2| \ ,$$

which is $<$ const $\cdot\, \ell_i|\eta_1 - \eta_2|$ where const $= 2\sum(pqe^{\delta_1})^j$.

This completes the proof of Proposition 6.

5.3. Proof of Proposition 8

Let $\varepsilon > 0$ be given. For any two admissible sequences $\{g_i\}$ and $\{\hat{g}_i\}$, let h_i^s and $\hat{h}_i^s$ be the functions whose graphs are stable manifolds for $\{g_i\}$ and $\{\hat{g}_i\}$ respectively.

C^0-bound for $h_0^s - \hat{h}_0^s$: Let c be as in Lemma 10, and let N be such that $2r_0(ce^{\delta_1})^N < \varepsilon$. Then since $\|h_N^s - \hat{h}_N^s\|_{C^0} < 2r_N < 2r_0e^{N\delta_1}$ and $g_i = \hat{g}_i$ for $i = 0, 1, \ldots, N-1$, we have $\|h_0^s - \hat{h}_0^s\|_{C^0} < \varepsilon$.

C^0-bound for $Dh_0^s - D\hat{h}_0^s$: Here we assume $g_i = \hat{g}_i$ for $i = 0, 1, \ldots, N-1$ where $N = N_1 + N_2, N_1$ and N_2 to be specified at the end of the proof.

We first estimate $\|Dh_i^s - D\hat{h}_i^s\|_{C^0}$ for $0 \le i \le N_1 - 1$. Let $x_i = (\eta, h_i^s(\eta)), \hat{x}_i = (\eta, \hat{h}_i^s(\eta))$ for $\eta \in B^s(0, r_i)$. Recall that for $0 \le i \le N-1$ and any $v \in E^s$,

$$\begin{aligned}(Dh_i^s)_\eta v = (\Lambda_i^u)^{-1}[&(Dh_{i+1}^s)_{\pi^s g_i(x)}(\Lambda_i^s v + \pi^s(DG_i)_x(v, (Dh_i^s)_\eta v))\\ &-\pi^u(DG_i)_x(v, (Dh_i^s)_\eta v)],\end{aligned}$$

with $D\hat{h}_i^s$ satisfying an analogous equation. Let $I^s \in \mathcal{L}(E^s, E^s)$ denote the identity map. We then have

$$\|(Dh_i^s)_\eta - (D\hat{h}_i^s)_\eta\| \le e^{-\lambda_1}\{(a) + (b) + (c)\}$$

where

$$\begin{aligned}(a) = \|&(Dh_{i+1}^s)_{\pi^s g_i(x)}(\Lambda_i^s + \pi^s(DG_i)_x(I^s, (Dh_i^s)_\eta))\\ &-(Dh_{i+1}^s)_{\pi^s g_i(x)}(\Lambda_i^s + \pi^s(DG_i)_{\hat{x}}(I^s, (D\hat{h}_i^s)_\eta))\|,\\ (b) = \|&(Dh_{i+1}^s)_{\pi^s g_i(x)}(\Lambda_i^s + \pi^s(DG_i)_{\hat{x}}(I^s, (D\hat{h}_i^s)_\eta))\\ &-(D\hat{h}_{i+1}^s)_{\pi^s g_i(\hat{x})}(\Lambda_i^s + \pi^s(DG_i)_{\hat{x}}(I^s, (D\hat{h}_i^s)_\eta))\|\\ (c) = \|&\pi^u(DG_i)_x(I^s, (Dh_i^s)_\eta) - \pi^u(DG_i)_{\hat{x}}(I^s, (D\hat{h}_i^s)_\eta)\|.\end{aligned}$$

A computation similar to those in Sect. 5.2 gives

$$\begin{aligned}(a) + (c) &\le \frac{11}{10}\left(\delta_2\|(Dh_i^s)_\eta - (D\hat{h}_i^s)_\eta\| + \frac{11}{10}l_i|h_i^s(\eta) - \hat{h}_i(\eta)|\right),\\ (b) &\le \left(e^{-\lambda_1} + \frac{11}{10}\delta_2\right)(\|(Dh_{i+1}^s)_{\pi^s g_i(x)} - (D\hat{h}_{i+1}^s)_{\pi^s g_i(x)}\|\\ &\quad +\delta_2\mathrm{Lip}\ (D\hat{h}_{i+1}^s)|h_i^s(\eta) - \hat{h}_i^s(\eta)|).\end{aligned}$$

In (b) we used that $|\pi^s g_i(x) - \pi^s g_i(\hat{x})| \le \delta_2|h_i^s(\eta) - \hat{h}_i^s(\eta)|$.

Summarizing, we have

$$\|Dh_i^s - D\hat{h}_i^s\|_{C^0} \le c_1\|Dh_{i+1}^s - D\hat{h}_{i+1}^s\|_{C^0} + (c_2 l_i + c_3 l_{i+1})\|h_i^s - \hat{h}_i^s\|_{C^0}$$

where

$$c_1 = \frac{e^{-\lambda_1}(e^{-\lambda_1} + \frac{11}{10}\delta_2)}{1 - \frac{11}{10}\delta_2 e^{-\lambda_1}}, \tag{5}$$

and c_2 and c_3 are constants depending only on λ_1, δ_1 and δ_2.

Applying the formula above for successive i, we obtain

$$\begin{aligned}
&\|Dh_0^s - D\hat{h}_0^s\|_{C^0} \\
&\quad \le c_1^{N_1}\|Dh_{N_1}^s - D\hat{h}_{N_1}^s\|_{C^0} + \sum_{k=0}^{N_1-1} c_1^k(c_2 l_k + c_3 l_{k+1})\|h_k^s - \hat{h}_k^s\|_{C^0} \\
&\quad \le c_1^{N_1}\|Dh_{N_1}^s - D\hat{h}_{N_1}^s\|_{C^0} + \sum_{k=0}^{N_1-1} (c_1 e^{\delta_1})^k(c_2 l_0 + c_3 l_1)\|h_k^s - \hat{h}_k^s\|_{C^0} \\
&\quad \le \frac{1}{5}c_1^{N_1} + \frac{c_2 l_0 + c_3 l_1}{1 - c_1 e^{\delta_1}} \cdot (2r_0 e^{N\delta_1} \cdot c^{N_2}).
\end{aligned}$$

In the last inequality, we have used $c_1 e^{\delta_1} < 1$, which is true provided δ_1 and δ_2 are small enough. The quantity in parenthesis is an upper bound for $\|h_k^s - \hat{h}_k^s\|_{C^0}$ for all $k < N_1$.

Finally, we specify N_1 and N_2 as follows: First we choose N_1 large enough that $\frac{1}{5}c_1^{N_1} < \frac{1}{2}\varepsilon$. With N_1 fixed, we choose N_2 to ensure that the second term in the last displayed inequality is $< \frac{1}{2}\varepsilon$; this is made possible by the fact that $ce^{\delta_1} < 1$. □

6. Switching Charts

In the proof of Theorem A, we considered a sequence of chart maps in which we "switched charts" periodically from the one at $f^n(x)$ to the one at x where $f^n(x)$ and x are nearby points. The proofs of Theorem B and C will involve similar concatenations in a hyperbolic setting. In this section, we dispose of the more technical estimates.

6.1. Desired Technical Result

Returning to the setting of Sect. 2 and the notation of Sect. 3, we assume that

- $E^u, E^s \neq \{0\}$ and $E^c = \{0\}$;
- δ and a chart system $\{\Phi_x\}$ has been fixed; and
- l_0 with $\mu(\Gamma_{l_0}) > 0$ is chosen.

The result we need is the following:

Proposition 15. *Given $\varepsilon > 0$, there exists $\delta_3 > 0$ (depending on ε and the chart system above) such that the following holds for all $x, y \in \Gamma_{l_0}$ with $|x - y| < \delta_3$: Let x' be such that $f(x') = x$, and let*

$$g : \tilde{B}\left(0, \delta e^{-\delta} l_0^{-1}\right) \to \mathbb{H} \quad \textit{be given by} \quad g = \Phi_y^{-1} \circ \Phi_x \circ \tilde{f}_{x'}.$$

Then we have the following estimates:

(1) *If* $\Lambda = \Lambda^u \oplus \Lambda^s$ *where for* $\tau = u, s, \Lambda^\tau = \tilde{\pi}_y^\tau (Dg)_0|_{\tilde{E}^\tau} \in \mathcal{L}(\tilde{E}^\tau, \tilde{E}^\tau)$, *then* $\|\Lambda^s\|, \|(\Lambda^u)^{-1}\| \le (1+\varepsilon)e^{-\lambda}$.

(2) *If* $G = g - \Lambda$, *then*
  (i) $|G(0)| < \varepsilon$,
  (ii) $\|DG\| \le (1+\varepsilon)\delta$, *and*
  (iii) $Lip(DG) \le (1+\varepsilon)\ Lip(D\tilde{f}_{x'})$.

This proposition is deduced from the following: Let $J_{x,y} = \Phi_y^{-1} \circ \Phi_x$, viewing Φ_x and Φ_y as affine maps defined on all of $\mathbb{H}$. Then confusing (deliberately) $u \in \mathbb{H}_x$ with $u + x \in \mathbb{H}$, we get

$$\begin{aligned} J_{x,y} v &= (L_y \circ \mathrm{Exp}_y^{-1} \circ \mathrm{Exp}_x \circ L_x^{-1}) v \\ &= L_y(-y + x + L_x^{-1} v) \\ &= L_y L_x^{-1} v + L_y(x - y). \end{aligned}$$

That is to say, $J_{x,y}$ is an affine map with $J_{x,y}(0) = L_y(x-y)$ and $DJ_{x,y} = L_y L_x^{-1}$. Since $|L_y(x-y)| \le l_0 |x-y|$ for $x, y \in \Gamma_{l_0}$, we can arrange to have $|G(0)|$ as small as we wish by letting $|x-y| \to 0$.

Thus, it suffices to focus on the linear part of the map, namely $L_y L_x^{-1}$. Notice that $L_y L_x^{-1}$ is a linear isomorphism. In the next subsection, we will prove a result (Proposition 17) which says that it is very close to a linear isometry which carries $\tilde{E}^u$ to a subspace near $\tilde{E}^u$ and $\tilde{E}^s$ to a subspace near $\tilde{E}^s$.

6.2. Continuity of Splitting on Γ_{l_0}

The main ingredient behind the result we need is the continuity of the $E^u \oplus E^s$ splitting on Γ_{l_0}. Since this is a very basic fact which is likely to be useful elsewhere, we will prove it in a more general setting:

In this subsection, we assume the setting is as in Sect. 3.2, and that $\delta, \{\Phi_x\}$, *and* l_0 *have been fixed* (and we do not assume $E^c = \{0\}$). In what follows, tangent spaces are identified with $\mathbb{H}$, so it makes sense to write $u - v$ where $u \in \mathbb{H}_x$ and $v \in \mathbb{H}_y, x \ne y$.

Proposition 16. *For* $x \in \Gamma_{l_0}$, *the subspaces* $E^u(x), E^c(x)$ *and* $E^s(x)$ *vary continuously with* x, *as do the corresponding projections.*

Proof. First we prove the continuity of $x \mapsto E^s(x)$ on Γ_{l_0}. Let $x, y \in \Gamma_{l_0}$, and consider a unit vector $v \in E^s(y)$. We will estimate $|\pi_x^{uc} v|$ in terms of $|y - x|$ where π_x^{uc} is the projection onto $E^{uc}(x) = E^u(x) \oplus E^c(x)$. Using the fact that $\pi_{f^n(x)}^{cu} Df_x^n = Df_x^n \pi_x^{cu}$, we have

$$\begin{aligned} |\pi_x^{uc} v| &= |(Df_x^n)^{-1} \pi_{f^n(x)}^{cu} Df_x^n v| \\ &\le |(Df_x^n)^{-1} \pi_{f^n(x)}^{cu} Df_y^n v| + |(Df_x^n)^{-1} \pi_{f^n(x)}^{cu} (Df_x^n v - Df_y^n v)|. \quad (6) \end{aligned}$$

To estimate the quantities above, we use Proposition 4, remembering that $l(f^n x) \le l_0 e^{\delta n}$ and $\|\pi_{f^n(x)}^{cu}\| \le \sqrt{3} l(f^n x)$. The first term above is

$$\le (\sqrt{3} l_0 e^{-\delta n} e^{2\delta_0 n})(\sqrt{3} l_0 e^{\delta n})(\sqrt{3} l_0 e^{-\lambda n}) = 3\sqrt{3} l_0^3 e^{-n(\lambda - 2\delta_0 - 2\delta)},$$

while the second term is

$$\leq 3l_0^2 e^{n(2\delta_0+2\delta)}\|Df_x^n - Df_y^n\|.$$

For a given ε, we fix an n so that

$$3\sqrt{3}l_0^3 e^{-n(\lambda-2\delta_0-2\delta)} \leq \frac{1}{2}\varepsilon.$$

Since f is C^1, there exists Δ such that if $|x-y| \leq \Delta$, then

$$3l_0^2 e^{n(2\delta_0+2\delta)}\|Df_x^n - Df_y^n\| \leq \frac{1}{2}\varepsilon.$$

Note that n and Δ depend on l_0, δ_0, δ and ε only; they do not depend on x, y or v. This proves the continuity of $x \mapsto E^s(x)$ on Γ_{L_0}.

The continuity of E^{cs} is proved similarly: Let v be a unit vector in $E^{cs}(y)$. By an argument entirely parallel to that above, we get

$$|\pi_x^u v| \leq 3\sqrt{3}l_0^3 e^{-n(\lambda-2\delta_0-2\delta)} + 3l_0^2 e^{-n(\lambda-2\delta)}\|Df_x^n - Df_y^n\|,$$

and we finish as before.

To prove the continuity of E^u, we again consider $x, y \in \Gamma_{l_0}$ and a unit vector $v \in E^u(y)$, but estimate $|\pi_x^{cs} v|$ by iterating *backwards*, obtaining

$$\begin{aligned} |\pi_x^{cs} v| &= |\pi_x^{cs} Df_{f^{-n}(y)}^n Df_y^{-n} v| \\ &\leq |\pi_x^{cs} Df_{f^{-n}(x)}^n Df_y^{-n} v| + |\pi_x^{cs}(Df_{f^{-n}(y)}^n - Df_{f^{-n}(x)}^n) Df_y^{-n} v| \\ &= |Df_{f^{-n}(x)}^n \pi_{f^{-n}(x)}^{cs} Df_y^{-n} v| + |\pi_x^{cs}(Df_{f^{-n}(y)}^n - Df_{f^{-n}(x)}^n) Df_y^{-n} v| \\ &\leq 3\sqrt{3}l_0^3 e^{-n(\lambda-2\delta_0-2\delta)} + 3l_0^2 e^{-n\lambda}\|Df_{f^{-n}(y)}^n - Df_{f^{-n}(x)}^n\|. \end{aligned} \tag{7}$$

As before, given $\varepsilon > 0$, we fix n large enough that $3\sqrt{3}l_0^3 e^{-n(\lambda-2\delta_0-2\delta)} < \frac{1}{2}\varepsilon$. Since $x \mapsto f^{-n}(x)$ is continuous on A and f is C^1, there exists $\Delta > 0$ such that if $|x-y| \leq \Delta$, then $3l_0^2 e^{-n\lambda}\|Df_{f^{-n}(x)}^n - Df_{f^{-n}(y)}^n\| \leq \frac{1}{2}\varepsilon$. This proves the continuity of E^u.

The proof for E^{uc} is entirely analogous.

It remains to deduce the continuity of E^c from the above: Since $E^c(\cdot) = E^{uc}(\cdot) \cap E^{cs}(\cdot)$, we have, for a unit vector $v \in E^c(y)$,

$$|v - \pi_x^{us} v| \leq |\pi_x^s v| + |\pi_x^u v|,$$

which tends to 0 as $|y-x| \to 0$ by the continuity of E^{uc} and E^{cs}.

The assertions for the projections follow immediately. $\square$

For $x, y \in \Gamma_{l_0}$ and $\tau, \tau' = u, s, c$, we define $J_{x,y}^{\tau,\tau'}$ to be the linear map $J_{x,y}^{\tau,\tau'} = \tilde{\pi}_y^{\tau'}(L_y L_x^{-1})|_{\tilde{E}^\tau} \in \mathcal{L}(\tilde{E}^\tau, \tilde{E}^{\tau'})$.

Proposition 17. *For any $\varepsilon > 0$, there exists Δ such that the following hold for any $\tau, \tau' = u, s$ or c: If $|x-y| < \Delta$, then for $v \in \tilde{E}^\tau$,*

(i) $(1-\varepsilon)|v| < |J_{x,y}^{\tau,\tau} v| \leq (1+\varepsilon)|v|$;

(ii) $|J_{x,y}^{\tau,\tau'}| < \varepsilon|v|$ *when* $\tau' \neq \tau$.

We first prove a technical lemma:

Lemma 18. *For given $\varepsilon > 0$, there exists $\Delta > 0$ such that if $x, y \in \Gamma_{l_0}$ with $|x - y| \le \Delta$, then for any $z \in \mathbb{H}$ and $\tau = u, c, s$,*

$$\big||L_y \pi_y^\tau z| - |L_x \pi_x^\tau z|\big| \le \varepsilon |z|. \tag{8}$$

Proof. Consider first the case $\tau = s$, and let $|z| = 1$. Notice that

$$|L_x \pi_x^s z| = |\pi_x^s z|'_x = \left(\sum_{i=0}^{\infty} \frac{|Df_x^n \pi_x^s z|^2}{e^{-2n\lambda}}\right)^{\frac{1}{2}},$$

and for $x \in \Gamma_{l_0}, \|Df_x^n \pi_x^s\| \le 3l_0^2 e^{-n(\lambda+2\delta)}$. Let $\varepsilon > 0$ be given. Then for $x, y \in \Gamma_{l_0}$, there exists $N > 0$ such that

$$\sum_{i=N+1}^{\infty} \frac{\|Df_y^n \pi_y^s - Df_x^n \pi_x^s\|^2}{e^{-2n\lambda}} \le \sum_{i=N+1}^{\infty} \frac{36 l_0^4 e^{-2n(\lambda+2\delta)}}{e^{-2n\lambda}} \le \frac{1}{2}\varepsilon^2.$$

For x and y close enough, we have also that the sum from 0 to N is $\le \frac{1}{2}\varepsilon^2$, since $x \mapsto Df_x^n \pi_x^s$ is continuous on Γ_{l_0}. Thus

$$\begin{aligned}
\big||L_y \pi_y^s z| - |L_x \pi_x^s z|\big| &= \left|\left(\sum_{i=0}^{+\infty} \frac{|Df_y^n \pi_y^s z|^2}{e^{-2n\lambda}}\right)^{\frac{1}{2}} - \left(\sum_{i=0}^{+\infty} \frac{|Df_x^n \pi_x^s z|^2}{e^{-2n\lambda}}\right)^{\frac{1}{2}}\right| \\
&\le \left(\sum_{i=0}^{\infty} \frac{(|Df_y^n \pi_y^s z| - |Df_x^n \pi_x^s z|)^2}{e^{-2n\lambda}}\right)^{\frac{1}{2}} \\
&\le \left(\sum_{i=0}^{\infty} \frac{\|Df_y^n \pi_y^s - Df_x^n \pi_x^s\|^2}{e^{-2n\lambda}}\right)^{\frac{1}{2}} |z| \\
&\le \varepsilon |z|.
\end{aligned}$$

The case of $\tau = u$ is proved similarly using the fact that for any fixed $n \ge 1, x \mapsto Df_x^{-n} \pi_x^u$ is continuous on Γ_{l_0}. For $\tau = c$, we treat the positive and negative parts of the bi-infinite sum separately. $\square$

Proof of Proposition 17. Let $v \in \tilde{E}^\tau$, and let us suppress x, y in $J_{x,y}^{\tau,\tau}$. Since $J_{x,y}^{\tau,\tau} v = L_y \pi_y^\tau L_x^{-1} v$ and $L_x \pi_x^\tau L_x^{-1} v = v$, we have

$$\big|\,|J_{x,y}^{\tau,\tau} v| - |v|\,\big| = \big||L_y \pi_y^\tau (L_x^{-1} v)| - |L_x \pi_x^\tau (L_x^{-1} v)|\big|,$$

which by Lemma 18 and $\|L_x^{-1}\| \le \sqrt{3}$ can be made $< \varepsilon |v|$ by taking x and y sufficiently near each other. For $\tau \ne \tau'$, the bound for $|J_{x,y}^{\tau,\tau'} v|$ is proved similarly, except that here $L_x \pi_x^{\tau'} L_x^{-1} v = 0$. $\square$

7. Proofs of Theorems B and C

7.1. Proof of Theorem B

We assume for definiteness that $E^u, E^s \ne \{0\}$ and $E^c = \{0\}$, and that μ is not supported on a periodic orbit. Let x_0 be an arbitrary point in the support of μ, and let $\varepsilon_0 > 0$ be given. We will show that $B(x_0, \varepsilon_0)$, the ball of radius ε_0 centered at x_0, contains a periodic point.

Let $\lambda_1 = \frac{99}{100}\lambda$ where λ is as in Sect. 3.2, and let δ_1 and δ_2 be given by Propositions 5 and 6. We let $\delta < \frac{1}{2}\min\{\delta_1, \delta_2\}$, and fix a chart system $\{\Phi_x\}$ using this δ. We then pick l_0 so that $\mu(\Gamma_{l_0} \cap B(x_0, \frac{1}{2}\varepsilon_0)) > 0$ where the sets Γ_l are as in Sect. 3.2.

By an argument similar to that in the proof of Theorem A, we can find $x \in \Gamma_{l_0} \cap B(x_0, \frac{1}{2}\varepsilon_0)$ and $n \in \mathbb{Z}^+$ such that $f^n x \in \Gamma_{l_0} \cap B(x_0, \frac{1}{2}\varepsilon_0)$ and $|x - f^n x|$ is smaller than any prescribed number. Our plan is to (a) introduce a periodic sequence of maps $\{g_i\}$ which are mostly chart maps along the orbit segment from x to $f^n(x)$, (b) show that $\{g_i\}$ satisfies the conditions in Sect. 5.1, and (c) use the local stable and unstable manifolds given by Propositions 5 and 6 to produce a periodic point.

(a) The maps in question are, for $i \in \mathbb{Z}$,

$$g_i : \tilde{B}^u(0, r_i) \times \tilde{B}^s(0, r_i) \to \mathbb{H}, \quad g_{i+n} = g_i \quad \text{and} \quad r_{i+n} = r_i,$$

defined by

$$\begin{aligned} g_i &= \tilde{f}_{f^i x}, \quad i = 0, 1, \ldots, n-2, \\ g_{n-1} &= \Phi_x^{-1} \circ \Phi_{f^n x} \circ \tilde{f}_{f^{n-1}x}, \end{aligned}$$

and

$$r_i = \min\left\{ \frac{1}{2\sqrt{3}}\varepsilon_0,\ \delta l_0^{-1},\ \delta l(f^i x)^{-1} \right\}.$$

The purpose of the constant $\frac{1}{2\sqrt{3}}\varepsilon_0$ in the preceding line is to ensure that for every $z \in \tilde{B}^u(0, r_0) \times \tilde{B}^s(0, r_0)$, $\Phi_x(z) \in B(x, \frac{1}{2}\varepsilon_0) \subset B(x_0, \varepsilon_0)$, so this is where we will look for our candidate periodic point.

(b) To check that $\{g_i\}$ satisfies the conditions in Sect. 5.1, first we show that $\{r_i\}$ satisfies $r_i e^{-\delta} < r_{i+1} < r_i e^{\delta}$ for all i. Since the function $l(\cdot)$ has such a property along orbits, and this property is not spoiled by taking the minimum with a constant, we need only be concerned about the relation between r_{n-1} and $r_n = r_0$, where the switching of charts occurs. Here we have $l(x) \leq l_0$, so $r_n = \min\{\frac{1}{2\sqrt{3}}\varepsilon_0,\ \delta l_0^{-1}\}$, while $l(f^{n-1}x)^{-1} > e^{-\delta} l(f^n x)^{-1} \geq e^{-\delta} l_0^{-1}$, so $e^{-\delta} r_n \leq r_{n-1} \leq r_n$.

Next we check that conditions (I), (II) and (III) hold for g_i: For $i = 0, 1, \ldots, n-2$, these conditions are satisfied with $G_i(0) = 0$ and $\ell_i = l(f^i x)$. Again, the main concern is for g_{n-1}. This is where Proposition 15 is needed: Condition (I) is assured by item (1) in Proposition 15 if ε is small enough that $(1+\varepsilon)e^{-\lambda} < e^{-\frac{99}{100}\lambda}$. Condition (II) is given by item (2) if $\varepsilon < \delta^2 l_0^{-1} \leq \delta r_n$, and Condition (III) is satisfied if we take $\ell_i = (1 + 4\varepsilon) l(f^i(x))$.

(c) Proposition 6 then gives for each i, a local stable manifold $W_i^s \subset \tilde{B}^u(0, r_i) \times \tilde{B}^s(0, r_i)$. Since g_i contracts points on W_i^s (Proposition 6(b)) and $g^n(W_0^s) \subset W_0^s$ where $g^n \equiv g_{n-1} \circ \cdots g_1 \circ g_0$ (Proposition 6(a)), we obtain by the Contraction Mapping Theorem a fixed point $z \in W_0^s$ of g^n. (Alternately, we may take $W_0^s \cap W_0^u = \{z\}$.)

Finally, $g^n(z) = z$ implies that $f^n(\Phi_x(z)) = \Phi_x(z)$. That z is a hyperbolic fixed point of g^n of saddle type follows immediately from the estimates in Sect. 5. These hyperbolic properties are passed directly to $\Phi_x(z)$. As noted earlier, $\Phi_x(z) \in B(x_0, \varepsilon_0)$, completing the proof. $\square$

7.2. Proof of Theorem C

Preliminaries on entropy. Let $T : X \to X$ be a continuous map of a compact metric space with metric $d(\cdot, \cdot)$, and let ν be a T-invariant Borel probability measure on X. For $n \in \mathbb{Z}^+$, we define the d_n^T-metric on X by

$$d_n^T(x, y) = \max_{0 \le i < n} d(T^i(x), T^i(y)),$$

and for $\alpha, \beta > 0$, let $\mathcal{N}(n, \alpha; \beta)$ denote the minimum number of α-balls in the d_n^T-metric needed to cover a set of measure $\ge \beta$ in X. The following result, first proved in [3], is by now a standard fact:

Assume (T, ν) is ergodic. Then given $\beta \in (0, 1)$,

$$h_\nu(T) = \lim_{\alpha \to 0} \liminf_{n \to \infty} \frac{1}{n} \ln \mathcal{N}(n, \alpha; \beta). \tag{9}$$

A set $E \subset X$ is called (n, α)-*separated* if for every $x, y \in E$, $d_n^T(x, y) > \alpha$. We use $|E|$ to denote the cardinality of E. A version of the following lemma is proved in [3].

Lemma 19. *Assume (T, ν) is ergodic, and $h_\nu(T) > 0$. Given $\gamma > 0$ and $\beta \in (0, \frac{1}{2})$, there exists $\alpha_0 > 0$ such that the following holds for all $\alpha \le \alpha_0$: Let $S \subset X$ be any Borel subset with $\nu(S) \ge 2\beta$, and let $N > n_0$ be given. Then there exist $n \ge N$ and an $(n - n_0, \alpha)$-separated set E such that*

(a) $E, T^n(E) \subset S$,
(b) $\frac{1}{n} \ln |E| \ge h_\nu(T) - \gamma$.

Proof. We begin with the following general observation: For a Borel set $S \subset X$, let χ_S denote the indicator function of S, and define

$$S_k^\varepsilon = \left\{ x \in S : \left| \frac{1}{k} \sum_{i=0}^{k-1} \chi_S(T^i(x)) - \nu(S) \right| \le \frac{\varepsilon}{3} \nu(S) \right\}.$$

Then for ε small enough and k large enough depending on S and ε, we have

(i) $\nu(S_k^\varepsilon \cap S_{(1+\varepsilon)k}^\varepsilon) > \frac{1}{2}\nu(S)$, and
(ii) for each $x \in S_k^\varepsilon \cap S_{(1+\varepsilon)k}^\varepsilon$, there exists $m(x) \in (k, (1 + \varepsilon)k]$ such that $T^{m(x)}(x) \in S$.

Here $S_\rho^\varepsilon = S_{[\rho]}^\varepsilon$ where $[\rho]$ is the integer part of ρ. (i) above follows from the Birkhoff Ergodic Theorem, and (ii) follows from the definition of S_k^ε.

We now turn to the setting of the lemma. For a given $\gamma > 0$ and $\beta \in (0, \frac{1}{2})$, let α be small enough that the lim inf in (9) is $> h_\nu(T) - \frac{1}{3}\gamma$, and let S, N and n_0 be given. Other conditions on ε and k will be specified later. For now, we require that ε be small enough and $k \ge N$ large enough that (i) and (ii) above are satisfied. Let E' be a maximal $(k - n_0, \alpha)$-separated set in $S_k^\varepsilon \cap S_{(1+\varepsilon)k}^\varepsilon$. Since $S_k^\varepsilon \cap S_{(1+\varepsilon)k}^\varepsilon \subset \cup_{x \in E'} B_{d_{k-n_0}^T}(x, \alpha)$ where $B_{d_{k-n_0}^T}$ is the ball with respect to the $d^T_{k-n_0}$-metric, and $\nu(S^\varepsilon_k \cap S^\varepsilon_{(1+\varepsilon)k}) > \beta$ by (i), it follows that $|E'| \geq \mathcal{N}(k-n_0, \alpha; \beta)$. By (ii), every $x \in E'$ makes a return to S in the time interval $(k, (1+\varepsilon)k]$. Let $n \in (k, (1+\varepsilon)k]$ be such that at least $\frac{1}{\varepsilon k}$ of the points in E' return to S at time n. We claim that for this $n, E = \{x \in E' : T^n(x) \in S\}$ is the desired $(n-n_0, \alpha)$-separated set. Notice that

$$|E| \geq \frac{1}{\varepsilon k} \mathcal{N}(k-n_0, \alpha; \beta). \tag{10}$$

The conditions we needed to impose on ε and k are now clear: First, ε should be small enough that

$$\frac{1}{1+\varepsilon}\left(h_\nu(T) - \frac{2}{3}\gamma\right) > h_\nu(T) - \gamma. \tag{11}$$

Then k is chosen large enough to satisfy, in addition to the condition imposed earlier,

$$\frac{1}{k}\ln\left(\frac{1}{\varepsilon k}\mathcal{N}(k-n_0, \alpha; \beta)\right) > h_\nu(T) - \frac{2}{3}\gamma. \tag{12}$$

Assertion (b) in the lemma then follows from (10), (11) and (12), together with the fact that $n \leq (1+\varepsilon)k$. □

We now return to the setup and notation of Theorem C. We will proceed as in the proof of Theorem B, but instead of concatenating a fixed sequence of charts along an orbit segment of a single point, we concatenate charts following orbit segments starting from all possible points in a small (n, α)-separated set. Following the charts of two points that are (n, α)-separated will not guarantee that the resulting stable manifolds are disjoint, however: take, for example, x and y in the same stable manifold with $|x-y| > \alpha$. The next lemma is used to remedy the situation.

Assume that a chart system is fixed. We let $\tilde{f}^i_x = \tilde{f}_{f^{i-1}x} \circ \cdots \circ \tilde{f}_x$, and define $C_n(x) = \Phi_x(\tilde{C}_n(x))$ where

$$\tilde{C}_n(x) = \left\{y : \tilde{f}^i_x(y) \in \tilde{B}^u\left(0, \delta l(f^i x)^{-1}\right)\right) \times \tilde{B}^s\left(0, \frac{1}{2}\delta l(f^i x)^{-1}\right), 0 \leq i \leq n\right\}.$$

Lemma 20. *Given $\alpha > 0$, there exists $N_0 = N_0(\alpha)$ such that for all x and $n > 2N_0$,*

$$diam(f^k(C_n(x))) < \frac{1}{2}\alpha \quad for \quad all\ k \in [N_0, n-N_0].$$

Proof. Since real distance is ≤ 3 times distances in charts (Proposition 4), it suffices to show that $\tilde{f}^k_x(\tilde{C}_n)$ has diameter $< \frac{1}{6}\alpha$.

We foliate $\tilde{C}_0(x)$ with planes P having the same dimension as $\tilde{E}^u$ and parallel to $\tilde{E}^u$. From the proof of Proposition 5 (see the Remark following Lemma 9), we have that for each $P, P_1 := \tilde{f}_x(P) \cap \tilde{B}(0, \delta l(fx)^{-1})$ is the graph of a function from $\tilde{B}^u(0, \delta l(fx)^{-1})$ to $\tilde{B}^s(0, \frac{1}{2}\delta l(fx)^{-1})$ with slope $< \frac{1}{10}$. The same holds true for $P_2 := \tilde{f}_{fx}(P_1) \cap \tilde{B}(0, \delta l(f^2x)^{-1}), P_3, \ldots, P_n$. Moreover, Proposition 5(b) tells us that the diameter of $\tilde{f}^{-i}_{f^{n-i}x}(P_n)$ decreases with i faster than a fixed exponential rate.

For $z \in \tilde{C}_n$, let z' be the unique point of intersection between the P that contains z and $\tilde{W}^s_x$ where $\tilde{W}^s_x$ is the local stable manifold (in the chart of x) given by Corollary 7. Since $|\tilde{f}^i_x(z')|$ also decreases with i faster than a fixed exponential rate (Proposition 6(b)), and the boxes $\tilde{B}(\cdot, \delta l(\cdot)^{-1})$ are uniformly bounded in diameter, an N_0 with the desired property clearly exists. □

Proof of Theorem C. From the $h_\mu(f) > 0$ hypothesis, it follows that $E^u \neq \{0\}$ (Theorem A). We let $\lambda_1, \delta_1, \delta_2$ and δ be as in Sect. 7.1, fix a chart system Φ_x, an l_0 with $\mu(\Gamma_{l_0}) > 0$, and a set $U \subset \Gamma_{l_0}$ with $\mu(U) > 0$ small enough to permit the switching of charts for points in Γ_{l_0} as in the proof of Theorem B.

Capturing entropy: Let $\varepsilon > 0$ in the statement of Theorem C be given. With $\beta = \frac{1}{2}\mu(U)$, we let α be such that the lim inf in (9) is $> h_\mu(f) - \frac{1}{2}\varepsilon$. Let N_0 be a number given by Lemma 20 for this α, and let $S = f^{N_0}(U)$. With S here playing the role of S in Lemma 19, α as above, $\gamma = \varepsilon$ and $n_0 = 2N_0$, we let $E \subset S$ be given by Lemma 19, and let $\hat{E} = f^{-N_0}(E)$. We have thus found a finite set $\hat{E} = \{z_1, \ldots, z_m\}$ and an $n \in \mathbb{Z}^+$ with the properties that

(i) $\hat{E}, f^n(\hat{E}) \subset U \subset \Gamma_{l_0}$,
(ii) $\frac{1}{n}\log|\hat{E}| > h_\mu(f) - \varepsilon$, and
(iii) for all $x, y \in \hat{E}$, $|f^k(x) - f^k(y)| > \alpha$ for some $k \in [N_0, n - N_0]$.

Forward-invariant horseshoe for f^n: For each $\mathbf{a} = (a_j) \in \Pi_0^\infty\{1, \ldots, m\}$, we define $\{g_i, i \geq 0\}$ as follows: For $k = 0, 1, 2, \ldots$, let

$$g_{kn+i} = \tilde{f}_{f^i z_{a_k}} \quad \text{for} \quad i = 0, 1, \ldots, n-2, \tag{13}$$

$$g_{(k+1)n-1} = \Phi^{-1}_{z_{a_{k+1}}} \circ \Phi_{f^n z_{a_k}} \circ \tilde{f}_{f^{(n-1)} z_{a_k}}. \tag{14}$$

The domains are as in the proof of Theorem B (without the $\frac{1}{2\sqrt{3}}\varepsilon_0$ factor in the definition of r_i). This sequence g_i is admissible with regard to the conditions in Sect. 5 for the same reasons as before. For each $\mathbf{a} \in \Pi_0^\infty\{1, \ldots, m\}$, let $W_0^s = W_0^s(\mathbf{a})$ be the stable manifold given by Proposition 6, and let $\Phi_{z_{a_0}}(W_0^s(\mathbf{a})) = \Psi(\mathbf{a})(\mathcal{D})$ where $\mathcal{D}$ is the unit disk in the definition of horseshoes in Sect. 2.2. It then follows from the invariance of stable manifolds (Proposition 6) that $f^n(\Psi(\mathbf{a})(\mathcal{D})) \subset \Psi(\sigma(\mathbf{a}))(\mathcal{D})$ where σ is the shift map on $\Pi_0^\infty\{1, \ldots, m\}$.

To check that $\Psi(\mathbf{a})(\mathcal{D}) \cap \Psi(\mathbf{a}')(\mathcal{D}) = \emptyset$ for $\mathbf{a} \neq \mathbf{a}'$, we consider $\Psi(\sigma^i(\mathbf{a}))(\mathcal{D})$ and $\Psi(\sigma^i(\mathbf{a}'))(\mathcal{D})$ if $a_i \neq a_i'$. Since $\Psi(\mathbf{a})(\mathcal{D}) \subset C_n(z_j)$ for $\mathbf{a} = (a_i)$ with $a_0 = j$, it suffices to show $C_n(z_j) \cap C_n(z_k) = \emptyset$ for $j \neq k$. That is guaranteed by (iii) above together with Lemma 20. (Since U is very small compared to the domains of the charts at z_j, the slight discrepancy with the statement of Lemma 20 due to the changing of charts is easily absorbed.) Proposition 8 tells us that the family $\Psi(\mathbf{a})(\mathcal{D})$ varies continuously in the C^1 topology.

Bi-invariant horseshoe for f^n: We extend g_i to all $i \in \mathbb{Z}$ in the obvious way, and let $\Psi(\mathbf{a}) = \Phi_{z_{a_0}}(W_0^u(\mathbf{a}) \cap W_0^s(\mathbf{a}))$ where W_0^u is given by Proposition 5.

That $f^n(\Psi(\mathbf{a})) = \Psi(\sigma(\mathbf{a}))$ follows from the invariance of stable and unstable manifolds (Propositions 5 and 6), and the continuity of Ψ follows from Proposition 8. Letting $\Omega = \Psi(\Pi_{-\infty}^{\infty}\{1,\ldots,m\})$, we have proved that Ψ is at least a semi-conjugacy between $f|_\Omega$ and σ. To prove that Ψ is a conjugacy, i.e. that it is one-to-one, consider $\mathbf{a} = (a_i)$ and $\mathbf{a}' = (a_i')$ with $\mathbf{a} \neq \mathbf{a}'$. If $a_i \neq a_i'$ for some $i \geq 0$, the proof is as in the forward-invariant case. If $a_{-i} \neq a_{-i}'$ for some $i > 0$, then $\Psi(\sigma^{-i}(\mathbf{a})) \neq \Psi(\sigma^{-i}(\mathbf{a}'))$, and by the injectivity of f on a neighborhood of A (Condition (D1) in Sect. 2) we conclude that $\Psi(\mathbf{a}) = f^{in}(\Psi(\sigma^{-i}(\mathbf{a}))) \neq f^{in}(\Psi(\sigma^{-i}(\mathbf{a}'))) = \Psi(\mathbf{a}')$. Finally, $f^n|_\Omega$ is uniformly hyperbolic because the maps g_i are: the stable and unstable subspaces at $\Psi(\mathbf{a})$ are exactly the $D\Phi_{z_{a_0}}$-images of the subspaces tangent to $W_0^u(\mathbf{a})$ and $W_0^s(\mathbf{a})$ at $W_0^u(\mathbf{a}) \cap W_0^s(\mathbf{a})$. □

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