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Brief paper

# Lyapunov tools for predictor feedbacks for delay systems: Inverse optimality and robustness to delay mismatch[☆]

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ABSTRACT

We consider LTI finite-dimensional, completely controllable, but possibly open-loop unstable, plants, with arbitrarily long actuator delay, and the corresponding predictor-based feedback for delay compensation. We study the problem of inverse-optimal re-design of the predictor-based feedback law. We obtain a simple modification of the basic predictor-based controller, which employs a low-pass filter, and has been proposed previously by Mondie and Michiels for achieving robustness to discretization of the integral term in the predictor feedback law. The key element in our work is the employment of an infinite-dimensional "backstepping" transformation, and the resulting Lyapunov function, for the infinite dimensional systems consisting of the state of the ODE plant and the delay state. The Lyapunov function allows us to quantify the Lyapunov stability properties under the modified feedback, the inverse optimality of the feedback, and its disturbance attenuation properties. For the basic predictor feedback, the availability of the Lyapunov function also allows us to prove robustness to small delay mismatch (in both positive and negative directions).



## 1. Introduction

We consider control systems of the form

$$\dot{X}(t) = AX(t) + BU(t-D), \quad (1)$$

where $X \in \mathbb{R}^n$, $(A, B)$ is a completely controllable pair, and the scalar-valued input signal $U(t)$ is delayed by $D$ units of time. We allow $A$ to be unstable and the delay $D$ to be arbitrarily large. In Artstein (1982), Kwon and Pearson (1980) and Manitius and Olbrot (1979) the following controller was developed which achieves asymptotic stabilization for any $D > 0$:

$$U(t) = K\left[e^{AD}X(t) + \int_{t-D}^{t} e^{A(t-\theta)}BU(\theta)d\theta\right]. \quad (2)$$

and which is viewed as a "delay-compensated" version of the 'nominal controller'[1]

$$U(t) = KX(t), \quad (3)$$

where the expression $e^{AD}X(t) + \int_{t-D}^{t} e^{A(t-\theta)}BU(\theta)d\theta$ in (2) should be understood as a $D$-seconds ahead *predictor* of $X(t)$, starting from $X(t)$ as an initial condition, and driven by the control history over the $D$-second window (the effect of using the predictor is that, after a transient lasting for $D$ seconds, the system's closed-loop response is perfect, as if there were no actuator delay). The controller (2) is infinite-dimensional due to its use of the history of the input $U(t)$ over the last $D$ time units. The"predictor-based controller" (2) is known in the literature also as a "finite-spectrum assignment", "Smith predictor (Smith, 1959) for unstable systems", and "reduction-based controller". The properties of this controller have been widely studied (Gu & Niculescu, 2003; Mondie & Michiels, 2003) and it has been extended to the parameter-adaptive case (Evesque, Annaswamy, Niculescu, & Dowling, 2003; Niculescu & Annaswamy, 2003).

In Krstic and Smyshlyaev (2007), using the backstepping method for PDEs, we have constructed the Lyapunov function for the closed-loop systems (1) and (2), which is in the form

$$V(t) = X(t)^{\mathrm{T}}PX(t) + \frac{a}{2}\int_{t-D}^{t}(1+\theta+D-t)\,W(\theta)^2d\theta, \quad (4)$$

where $P$ is the solution of the Lyapunov equation

$$P(A+BK) + (A+BK)^{\mathrm{T}}P = -Q, \quad (5)$$

$P$ and $Q$ are positive definite and symmetric, the constant $a > 0$ is sufficiently large, and $W(\theta)$ is defined as

$$W(\theta) = U(\theta) - K\left[\int_{t-D}^{\theta} e^{A(\theta-\sigma)}BU(\sigma)d\sigma + e^{A(\theta+D-t)}X(t)\right], \quad (6)$$

with $-D \le t - D \le \theta \le t$.

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[1] The nominal control gain $K$ may be designed by a LQR/Riccati approach, pole placement, or some other method that makes $A + BK$ Hurwitz.

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In this note we highlight some of the benefits of constructing the transformation (6) and of the Lyapunov function (4). The first benefit is the ability to derive an inverse-optimal controller, which incorporates a penalty, not only on the ODE state $X(t)$ and the input $U(t)$, but also on the delay state. Inverse optimality, as an objective in designing controllers for delay systems, was pursued by Jankovic (2001, 2003). The inverse-optimal feedback that we design in the note is of the form (where, for brevity and conceptual clarity, we mix the frequency and time domains, i.e., the lag transfer function on the right should be understood as an operator):

$$U(t) = \frac{c}{s+c}\left\{K\left[\mathrm{e}^{AD}X(t) + \int_{t-D}^{t}\mathrm{e}^{A(t-\theta)}BU(\theta)\mathrm{d}\theta\right]\right\}, \tag{7}$$

where $c > 0$ is sufficiently large, i.e., the inverse-optimal feedback is of the form of a low-pass filtered version of (2).

As it turns out, the low pass modification, proposed here for inverse optimality, has already been proposed in Mondie and Michiels (2003) as a tool for helping robustness in the discretization of the integral term in (2). This low-pass filtering is not required for robustness in discretization, as shown in Zhong (2006, Chapter 11), and Zhong (2006) and Zhong and Mirkin (2002), but it is helpful.

The second benefit of constructing the transformation (6) and of the Lyapunov function (4) is that one can prove robustness of the exponential stability of the predictor feedback to a *small* mismatch in the actuator delay, both in the positive and in the negative direction.

In Section 2 we establish inverse optimality of the feedback law (7) and its stabilization property for sufficiently large $c$. In Section 3 we consider the plant (1) in the presence of an additive disturbance and establish the inverse optimality of the feedback (7) in the sense of solving a meaningful differential game problem and we quantify its $L_\infty$ disturbance attenuation property. Finally, for the basic predictor feedback, in Section 4 we use our Lyapunov function to prove robustness to small delay mismatches.

## 2. Inverse optimal re-design

In the formulation of the inverse optimality problem we will consider $\dot{U}(t)$ as the input to the system, however $U(t)$ is still the actuated variable. Hence, our inverse optimal design will be implementable after integration in time, i.e., as dynamic feedback. Treating $\dot{U}(t)$ as an input is the same as adding an integrator, which has been seen to be beneficial in the control design for delay systems in Jankovic (2001).

**Theorem 1.** *There exists $c^*$ such that the feedback system (1) and (7) is exponentially stable in the sense of the norm*

$$N(t) = \left(|X(t)|^2 + \int_{t-D}^{t} U(\theta)^2\mathrm{d}\theta + U(t)^2\right)^{1/2} \tag{8}$$

*for all $c > c^*$. Furthermore, there exists $c^{**} > c^*$ such that, for any $c \geq c^{**}$, the feedback (7) minimizes the cost functional*

$$J = \int_0^\infty \left(\mathcal{L}(t) + \dot{U}(t)^2\right)\mathrm{d}t, \tag{9}$$

*where $\mathcal{L}$ is a functional of $(X(t), U(\theta))$, $\theta \in [t-D, t]$, and such that*

$$\mathcal{L}(t) \geq \mu N(t)^2 \tag{10}$$

*for some $\mu(c) > 0$ with a property that $\mu(c) \to \infty$ as $c \to \infty$.*

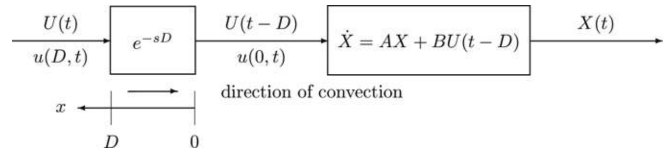


**Fig. 1.** Linear system $\dot{X} = AX + BU(t-D)$ with actuator delay $D$.

**Proof.** We start by writing (1) as the ODE-PDE system

$$\dot{X} = AX + Bu(0,t). \tag{11}$$
$$u_t(x,t) = u_x(x,t) \tag{12}$$
$$u(D,t) = U(t), \tag{13}$$

where $u(x,t) = U(t+x-D)$ and therefore the output $u(0,t) = U(t-D)$ gives the delayed input (see Fig. 1).

Consider the infinite-dimensional backstepping transformation of the delay state (Krstic & Smyshlyaev, 2007)

$$w(x,t) = u(x,t) - \left[\int_0^x K\mathrm{e}^{A(x-y)}Bu(y,t)\mathrm{d}y + K\mathrm{e}^{Ax}X(t)\right]. \tag{14}$$

It is readily verified that

$$\dot{X} = (A+BK)X + Bw(0,t) \tag{15}$$
$$w_t(x,t) = w_x(x,t). \tag{16}$$

Let us now consider $w(D,t)$. It is easily seen that

$$w_t(D,t) = u_t(D,t) - K\left[Bu(D,t) + \int_0^D \mathrm{e}^{A(D-y)}Bu(y,t)\mathrm{d}y + Ae^{AD}X(t)\right]. \tag{17}$$

Note that $u_t(D,t) = \dot{U}(t)$, which is designated as the control input penalized in (9). The inverse of (14) can be derived as[2]

$$u(x,t) = w(x,t) + \int_0^x K\mathrm{e}^{(A+BK)(x-y)}Bw(y,t)\mathrm{d}y + K\mathrm{e}^{(A+BK)x}X(t). \tag{18}$$

Plugging (18) into (17), after a lengthy calculation that involves a change of the order of integration in a double integral, we get

$$w_t(D,t) = u_t(D,t) - KBw(D,t) - K(A+BK) \times \left[\int_0^D M(y)Bw(y,t)\mathrm{d}y + M(0)X(t)\right], \tag{19}$$

where

$$M(y) = \int_y^D \mathrm{e}^{A(D-\sigma)}BK\mathrm{e}^{(A+BK)(\sigma-y)}d\sigma + \mathrm{e}^{A(D-y)} = \mathrm{e}^{(A+BK)(D-y)} \tag{20}$$

is a matrix-valued function defined for $y \in [0, D]$. Note that $N : [0,D] \to \mathbb{R}^{n\times n}$ is in both $L_\infty[0,D]$ and in $L_2[0,D]$.

Consider now a Lyapunov function

$$V(t) = X(t)^{\mathrm{T}}PX(t) + \frac{a}{2}\int_0^D (1+x)w(x,t)^2\mathrm{d}x + \frac{1}{2}w(D,t)^2, \tag{21}$$

[2] The fact that (18) is the inverse of (14) can be seen in various ways, including a direct substitution and manipulation of integrals, as well as by using a Laplace transform in $x$ and employing the identity $(\sigma I - A - BK)^{-1}\left(I - BK(\sigma I - A)^{-1}\right) = (\sigma I - A)^{-1}$, where $\sigma$ is the argument of the Laplace transform in $x$.

where $P > 0$ is defined in (5) and the parameter $a > 0$ is to be chosen later. We have

$$\begin{aligned}\dot V &= X^{\mathrm T}((A+BK)^{\mathrm T}P + P(A+BK))X + 2X^{\mathrm T}PBw(0,t)\\ &\quad + \frac{a}{2}\int_0^D (1+x)w(x,t)w_x(x,t)\mathrm{d}x + w(D,t)w_t(D,t)\\ &= -X^{\mathrm T}QX + 2X^{\mathrm T}PBw(0,t) + \frac{a}{2}(1+D)w(D,t)^2 - \frac{a}{2}w(0,t)^2\\ &\quad - \frac{a}{2}\int_0^D w(x,t)^2\,\mathrm{d}x + w(D,t)w_t(D,t) \qquad (22)\\ &\le -X^{\mathrm T}QX + \frac{2}{a}\|X^{\mathrm T}PB\|^2 - \frac{a}{2}\int_0^D w(x,t)^2\,\mathrm{d}x\\ &\quad + w(D,t)\left(w_t(D,t) + \frac{a(1+D)}{2}w(D,t)\right), \qquad (23)\end{aligned}$$

and finally,

$$\begin{aligned}\dot V &\le -\frac{1}{2}X^{\mathrm T}QX - \frac{a}{2}\int_0^D w(x,t)^2\,\mathrm{d}x\\ &\quad + w(D,t)\left(w_t(D,t) + \frac{a(1+D)}{2}w(D,t)\right), \qquad (24)\end{aligned}$$

where we have chosen

$$a = 4\frac{\lambda_{\max}(PBB^{\mathrm T}P)}{\lambda_{\min}(Q)}, \qquad (25)$$

where $\lambda_{\min}$ and $\lambda_{\max}$ are minimum and maximum eigenvalues of the corresponding matrices. Now we consider (24) along with (19). With a completion of squares, we obtain

$$\begin{aligned}\dot V &\le -\frac{1}{4}X^{\mathrm T}QX - \frac{a}{4}\int_0^D w(x,t)^2\,\mathrm{d}x\\ &\quad + \frac{|K(A+BK)M(0)|^2}{\lambda_{\min}(Q)}w(D,t)^2 + \frac{\|K(A+BK)MB\|^2}{a}w(D,t)^2\\ &\quad + \left(\frac{a(1+D)}{2} - KB\right)w(D,t)^2 + w(D,t)u_t(D,t). \qquad (26)\end{aligned}$$

(We suppress the details of this step in the calculation but provide the details on the part that may be the hardest to see: $-w(D,t)\langle K(A+BK)MB, w(t)\rangle \le |w(D)|\|K(A+BK)MB\|\|w(t)\| \le \frac{a}{4}\|w(t)\|^2 + \frac{\|K(A+BK)MB\|^2}{a}w(D,t)^2$, where the first inequality is the Cauchy–Schwartz and the second is Young's, the notation $\langle\cdot,\cdot\rangle$ denotes the inner product in the spatial variable $y$ which both $M(y)$ and $w(y,t)$ depend on, and $\|\cdot\|$ denotes the $L_2$ norm in $y$.)

Then, choosing

$$u_t(D,t) = -cw(D,t), \qquad (27)$$

we arrive at

$$\dot V \le -\frac{1}{4}X^{\mathrm T}QX - \frac{a}{4}\int_0^D w(x,t)^2\,\mathrm{d}x - (c-c^*)w(D,t)^2, \qquad (28)$$

where

$$\begin{aligned}c^* &= \frac{a(1+D)}{2} - KB + \frac{|K(A+BK)M(0)|^2}{\lambda_{\min}(Q)}\\ &\quad + \frac{\|K(A+BK)MB\|^2}{a}. \qquad (29)\end{aligned}$$

Using (14) for $x = D$ and the fact that $u(D,t) = U(t)$, from (27) we get (7). Hence, from (28), the first statement of the theorem is proved if we can show that there exist positive numbers $\alpha_1$ and $\alpha_2$ such that

$$\alpha_1 N^2 \le V \le \alpha_2 N^2, \qquad (30)$$

where

$$N(t)^2 = |X(t)|^2 + \int_0^D u(x,t)^2\,\mathrm{d}x + u(D,t)^2. \qquad (31)$$

This is straightforward to establish by using (14), (18) and (21), and employing the Cauchy–Schwartz inequality and other calculations, following a pattern of a similar computation in Smyshlyaev and Krstic (2004). Thus, the first part of the theorem is proved.

The second part of the theorem is established in a manner very similar to the lengthy proof of Theorem 6 in Smyshlyaev and Krstic (2004), which is based on the idea of the proof of Theorem 2.8 in Krstic and Deng (1998). We choose $c^{**} = 4c^*$ and

$$\begin{aligned}\mathcal{L}(t) &= -2c\dot V\big|_{(22)\text{ with }(19)\text{ and }(27),\text{and } c=2c^*} + c(c-4c^*)w(D,t)^2\\ &\ge c\left(\frac{1}{2}X^{\mathrm T}QX + \frac{a}{2}\int_0^D w(x,t)^2\,\mathrm{d}x + (c-2c^*)w(D,t)^2\right). \qquad (32)\end{aligned}$$

We have that $\mathcal{L}(t) \ge \mu N(t)^2$ for the same reason that (30) holds. This completes the proof of inverse optimality. □

**Remark 1.** We have established the stability robustness to varying the parameter $c$ from some large value $c^*$ to $\infty$, recovering in the limit, the basic, unfiltered predictor-based feedback (2). This robustness property might be intuitively expected from a singular perturbation idea, though an off-the-shelf theorem for establishing this property would be highly unlikely to be found in the literature, due to the infinite dimensionality and the special hybrid (ODE-PDE-ODE) structure of the system at hand.

**Remark 2.** The feedback (2) is not inverse optimal, however the feedback (7) is, for any $c \in [c^{**},\infty)$. Its optimality holds for a relevant cost functional, which is underbounded by the temporal $L_2[0,\infty)$ norm of the ODE state $X(t)$, the norm of the control $U(t)$, as well as the norm of its derivative $\dot U(t)$ (in addition to $\int_{-D}^0 U(\theta)^2\mathrm{d}\theta$ which is fixed because feedback has no influence on it). The controller (7) is stabilizing for $c = \infty$, namely, in its nominal form (2), however, since $\mu(\infty) = \infty$, it is not optimal with respect to a cost functional that includes a penalty on $\dot U(t)$.

**Remark 3.** Having obtained inverse optimality, one would be tempted to conclude that the controller (7) has an infinite gain margin and a phase margin of 60°. This is unfortunately not true, at least not in the sense of multiplicative (frequency domain) perturbations of the feedback law. These properties can be claimed only for the feedback law (27), i.e., $\dot U(t) = -cW(t)$. The meaning of the phase margin is that the feedback $\dot U(t) = -c(1+P(s))\{W(t)\}$ is also stabilizing for any $P(s)$ that is strictly positive real. For example, the feedback of the form (7) but with $\frac{c}{s+c}$ replaced by $\frac{c(s+\nu+\omega)}{s^2+(c+\omega)s+c(\nu+\omega)}$, which may be a lightly damped transfer function for some $\nu,\omega$, is stabilizing for all $\nu$ and $\omega$ and for $c > c^*$. This result is not obvious but can be obtained by mimicking the proof of Theorem 2.17 from (Krstic & Deng, 1998).

**Remark 4.** Here it is relevant to recall an important and elegant result on *direct* optimal control in the presence of actuator delay (Tadmor, 2000) (see also Zhong (2006, Chapter 7)). In general, for infinite dimensional systems, direct optimal control formulations lead to operator Riccati equations, which are infinite-dimensional nonlinear algebraic problems that can be only approached numerically, i.e., they cannot be simplified to finite-dimensional problems. The class of delay systems is an exception to this rule. For the class of systems (1) it was shown in Tadmor (2000) that the predictor-based 'nominally-optimal' feedback law

$$U(t) = -B^{\mathrm T}\Pi\left[\mathrm{e}^{AD}X(t) + \int_{t-D}^t \mathrm{e}^{A(t-\theta)}BU(\theta)\mathrm{d}\theta\right], \qquad (33)$$

where $\Pi$ is a positive definite and symmetric $n \times n$ solution to the *matrix* Riccati equation

$$\Pi A + A^{\mathrm{T}}\Pi - \Pi BB^{\mathrm{T}}\Pi + Q = 0 \tag{34}$$

(for a a positive definite and symmetric matrix $Q$), is actually the minimizer of the cost functional

$$J = \int_0^\infty \left(X(t)^{\mathrm{T}}QX(t) + U(t)^2\right)\mathrm{d}t. \tag{35}$$

This is a striking and subtle result, since the control $U(t)$ is penalized in (35) as both the control input and as the infinite-dimensional state of the actuator. Our inverse optimality result, whose cost functional (9) is such that $J \geq \int_0^\infty \left(\mu|X(t)|^2 + \mu U(t)^2 + \dot{U}(t)^2\right)\mathrm{d}t$, is far less general and its only advantage is that the optimal value function (21) is actually a legitimate Lyapunov function that can be used for proving exponential stability. In contrast, the optimal value function in Tadmor (2000) is given by

$$V\left(X(0), U([-D, 0])\right) = X(D)^{\mathrm{T}}\Pi X(D) + \int_0^D X(t)^{\mathrm{T}}QX(t)\mathrm{d}t \tag{36}$$

where

$$X(t) = \mathrm{e}^{At}X(0) + \int_{-D}^{t-D} \mathrm{e}^{A(t-D-\theta)}BU(\theta)\mathrm{d}\theta. \tag{37}$$

It is clear that (36) is positive *semi*-definite, but in general it is not clear (nor claimed in Tadmor (2000)) that it is *positive definite* in $(X(0), U([-D, 0]))$, i.e., that it is lower bounded in terms of $|X(0)|^2 + \int_{-D}^0 U(\theta)^2\mathrm{d}\theta$, hence, it may not be a valid Lyapunov function. So, for the controller (33), the Lyapunov function introduced in our work (Krstic & Smyshlyaev, 2007), that is, the Lyapunov function defined by (4) and (6) with $K = -B^{\mathrm{T}}\Pi$, is the first Lyapunov function made available for proving exponential stability. Note that in Tadmor (2000) exponential stability in a strict Lyapunov sense, namely a characterization that involves a dependence on the norm of the infinite dimensional state for $t \geq 0$, is neither stated nor quantified. Only 'exponential decay to zero' (in time) is claimed and argued qualitatively.

## 3. Disturbance attenuation

Consider the following system

$$\dot{X}(t) = AX(t) + BU(t - D) + Gd(t), \tag{38}$$

where $d(t)$ is an unmeasurable disturbance signal and $G$ is a vector. In this section, the availability of the Lyapunov function (21) lets us establish the disturbance attenuation properties of the controller (7), which we pursue in a differential game setting.

**Theorem 2.** *There exists $c^*$ such that, for all $c > c^*$, the feedback system (7) and (38) is $L_\infty$-stable, i.e., there exist positive constants $\beta_1, \beta_2, \gamma_1$, such that*

$$N(t) \leq \beta_1 \mathrm{e}^{-\beta_2 t}N(0) + \gamma_1 \sup_{\tau\in[0,t]} |d(\tau)|. \tag{39}$$

*Furthermore, there exists $c^{**} > c^*$ such that, for any $c \geq c^{**}$, the feedback (7) minimizes the cost functional*

$$J = \sup_{d\in\mathcal{D}} \lim_{t\to\infty} \left[2cV(t) + \int_0^t \left(\mathcal{L}(\tau) + \dot{U}(\tau)^2 - c\gamma_2 d(\tau)^2\right)\mathrm{d}\tau\right], \tag{40}$$

*for any*

$$\gamma_2 \geq \gamma_2^{**} = 8\frac{\lambda_{\max}(PBB^{\mathrm{T}}P)}{\lambda_{\min}(Q)}, \tag{41}$$

*where $\mathcal{L}$ is a functional of $(X(t), U(\theta)), \theta \in [t - D, t]$, and such that (10) holds for some $\mu(c, \gamma_2) > 0$ with a property that $\mu(c, \gamma_2) \to \infty$ as $c \to \infty$, and $\mathcal{D}$ is the set of linear scalar-valued functions of $X$.*

**Proof.** First, with a slight modification of the calculations leading to (28) we get that

$$\dot{V} \leq -\frac{1}{8}X^{\mathrm{T}}QX - \frac{a}{4}\int_0^D w(x, t)^2\,\mathrm{d}x - (c - c^*)w(D, t)^2 + \gamma_2^{**}d^2. \tag{42}$$

From here, a straightforward, though lengthy, calculation gives the $L_\infty$ stability result.

The proof of inverse optimality is obtained by specializing the proof of Theorem 2.8 in Krstic and Deng (1998) to the present case. The function $\mathcal{L}(t)$ is defined as

$$\mathcal{L}(t) = -2c\Omega(t) + 8c|PG|^2\frac{\gamma_2 - \gamma_2^{**}}{\gamma_2\gamma_2^{**}}|X(t)|^2 + c(c - 4c^*)w(D, t)^2, \tag{43}$$

where $\Omega(t)$ is defined as

$$\begin{aligned}\Omega(t) = {} & -X(t)^{\mathrm{T}}QX(t) + 2X(t)^{\mathrm{T}}PBw(0, t) + \frac{1}{\gamma_2^{**}}X(t)^{\mathrm{T}}PGG^{\mathrm{T}}PX(t) \\ & + \frac{a}{2}(1 + D)w(D, t)^2 - \frac{a}{2}w(0, t)^2 - \frac{a}{2}\int_0^D w(x, t)^2\,\mathrm{d}x \\ & - (2c^* + KB)w(D, t)^2 - K(A + BK) \\ & \times \left[\int_0^D M(y)Bw(y, t)\mathrm{d}y + M(0)X(t)\right]w(D, t).\end{aligned} \tag{44}$$

It is easy to see that $\Omega(t) \leq -\frac{1}{8}X^{\mathrm{T}}QX - \frac{a}{4}\int_0^D w(x, t)^2\,\mathrm{d}x - 2c^*w(D, t)^2$. Therefore,

$$\mathcal{L}(t) \geq c\left(\frac{\gamma_2 - \gamma_2^{**}/2}{\gamma_2}\lambda_{\min}(Q)|X(t)|^2 + \frac{a}{2}\int_0^D w(x, t)^2\,\mathrm{d}x + (c - 2c^*)w(D, t)^2\right), \tag{45}$$

which is lower-bounded by $\mu N(t)^2$ as in the proof of Theorem 1.

To complete the proof of inverse optimality, one can then show, by direct verification, that the cost of the two-player $(\dot{U}, d)$ differential game (40), along the solutions of the system, is

$$J = 2cV(0) + \int_0^\infty \left(u_t(D, t) - u_t^*(D, t)\right)^2\mathrm{d}t + c\gamma_2 \sup_{d\in\mathcal{D}}\left\{-\int_0^\infty \left(d(t) - d^*(t)\right)^2\mathrm{d}t\right\}, \tag{46}$$

where $u_t^*(D, t) = -cw(D, t)$ represents the optimal control as in (27), and $d^*(t)$ represents the "worst case disturbance"

$$d^*(t) = \frac{2}{\gamma_2}G^{\mathrm{T}}PX(t). \tag{47}$$

The choice $d(t) = d^*(t)$ achieves the supremum in the last term in (46), whereas the choice $u_t(D, t) = u_t^*(D, t)$, i.e., the choice given by (7), minimizes $J$. This completes the proof. □

**Remark 5.** Similar to the last point in Remark 2, the nominal predictor feedback (2), though not inverse optimal, is $L_\infty$ stabilizing. This is seen with a different Lyapunov function, $V(t) = X(t)^{\mathrm{T}}PX(t) + \frac{a}{2}\int_0^D(1 + x)w(x, t)^2\,\mathrm{d}x$, which yields $\mathrm{d}V(t)/\mathrm{d}t \leq -\frac{1}{4}X^{\mathrm{T}}QX - \frac{a}{2}\int_0^D w(x, t)^2\,\mathrm{d}x + \frac{\gamma_2^{**}}{2}d^2$.

## 4. Robustness to delay mismatch

Predictor-based feedbacks are known to be sensitive to errors in the knowledge of the value of actuator delay. This problem is discussed in Gu and Niculescu (2003), Michiels and Niculescu (2003) and Mondie and Michiels (2003) and other references. Despite the sensitivity, the predictor feedbacks are an 'irreplaceable and widely used tool' (Richard, 2003).

The existing studies of robustness to delay mismatch are frequency domain studies. We are not aware of robustness analyses performed using Lyapunov techniques. The result in Teel (1998) answers a similar question for ODE plants, however it does not apply to the present case where the nominal case (without delay mismatch) is infinite dimensional and the feedback law is also infinite dimensional.

We consider the feedback system

$$\dot{X} = AX + BU(t - D_0 - \Delta D), \tag{48}$$

$$U(t) = K\left[\mathrm{e}^{AD_0}X(t) + \int_{t-D_0}^{t} \mathrm{e}^{A(t-\theta)}BU(\theta)\mathrm{d}\theta\right]. \tag{49}$$

The reader should note that the actual actuator delay has a mismatch of $\Delta D$, which can be either positive or negative, relative to the assumed plant delay $D_0 > 0$, with the obvious necessary condition that $D_0 + \Delta D \geq 0$. Being in the possession of a Lyapunov function, we are able to prove the following result.

**Theorem 3.** *There exists $\delta > 0$ such that for all $\Delta D \in (-\delta, \delta)$ the system (48) and (49) is exponentially stable in the sense of the state norm*

$$N_2(t) = \left(|X(t)|^2 + \int_{t-\bar{D}}^{t} U(\theta)^2\mathrm{d}\theta\right)^{1/2}, \tag{50}$$

*where $\bar{D} = D_0 + \max\{0, \Delta D\}$.*

**Proof.** We use the same transport PDE formalism as in Theorem 1 and the transformations (14) and (18). First, we note that the feedback (49) is written as

$$u(D_0 + \Delta D, t) = K\left[\mathrm{e}^{AD_0}X(t) + \int_{\Delta D}^{D_0+\Delta D} \mathrm{e}^{A(D_0+\Delta D-y)}Bu(y,t)\mathrm{d}y\right], \tag{51}$$

which, using (14) for $x = D_0 + \Delta D$, gives us

$$w(D_0 + \Delta D, t) = K\mathrm{e}^{AD_0}\left[\left(I - \mathrm{e}^{A\Delta D}\right)X(t) - \int_{0}^{\Delta D} \mathrm{e}^{A(\Delta D-y)}Bu(y,t)\mathrm{d}y\right]. \tag{52}$$

Then, employing (18) under the integral, and performing certain calculations, we obtain

$$w(D_0 + \Delta D, t) = K\mathrm{e}^{AD_0} \times \left[\left(I - \mathrm{e}^{(A+BK)\Delta D}\right)X(t) - \int_{0}^{\Delta D} \mathrm{e}^{(A+BK)(\Delta D-y)}Bw(y,t)\mathrm{d}y\right]. \tag{53}$$

We then show that

$$w(D_0 + \Delta D, t)^2 \leq 2q_1|X|^2 + 2q_2\int_{\min\{0,\Delta D\}}^{\max\{0,\Delta D\}} w(x,t)^2\mathrm{d}x, \tag{54}$$

where the functions $q_1(\Delta D)$ and $q_2(\Delta D)$ are

$$q_1 = \left|K\mathrm{e}^{AD_0}\left(I - \mathrm{e}^{(A+BK)\Delta D}\right)\right|^2 \tag{55}$$

$$q_2 = \int_{\min\{0,\Delta D\}}^{\max\{0,\Delta D\}} \left(K\mathrm{e}^{AD_0}\mathrm{e}^{(A+BK)(\Delta D-y)}B\right)^2\mathrm{d}y. \tag{56}$$

Note that $q_1(0) = q_2(0) = 0$ and that $q_1$ and $q_2$ are both continuous functions of $\Delta D$ (note that the two integral terms in $q_2$ are both zero at zero, and continuous in $\Delta D$).

The cases $\Delta D > 0$ and $\Delta D < 0$ have to be considered separately. The case $\Delta D > 0$ is easier and the state of the system is $X(t), u(x,t), x \in [0, D_0+\Delta D]$, i.e., $X(t), U(\theta), \theta \in [t-D_0-\Delta D, t]$. The case $\Delta D < 0$ is more intricate, as the state of the system is $X(t), u(x,t), x \in [\Delta D, D_0 + \Delta D]$, i.e., $X(t), U(\theta), \theta \in [t - D_0, t]$.

*Case $\Delta D > 0$.* We take the Lyapunov function

$$V(t) = X(t)^{\mathrm{T}}PX(t) + \frac{a}{2}\int_0^{D_0+\Delta D} (1+x)w(x,t)^2\,\mathrm{d}x. \tag{57}$$

A calculation similar to that at the beginning of the proof of Theorem 1 gives

$$\dot{V} = -X^{\mathrm{T}}QX + 2X^{\mathrm{T}}PBw(0,t) + \frac{a}{2}(1+D)w(D_0+\Delta D,t)^2 - \frac{a}{2}w(0,t)^2 - \frac{a}{2}\int_0^{D_0+\Delta D} w(x,t)^2\,\mathrm{d}x \tag{58}$$

$$\leq -\left(\frac{\lambda_{\min}(Q)}{2} - a(1+D)q_1(\Delta D)\right)|X|^2 - a\left(\frac{1}{2} - (1+D)q_2(\Delta D)\right)\int_0^{D_0+\Delta D} w(x,t)^2\,\mathrm{d}x, \tag{59}$$

where $a$ is chosen as in (25), and where we have denoted $D = D_0 + \Delta D$ for brevity. This proves exponential stability of the origin of the $(X(t), w(x,t), x \in [0, D_0 + \Delta D])$ system. Exponential stability in the norm $N_2(t)$ is obtained using the standard procedures for over- and under-bounding $V(t)$ by a linear function of $N_2^2(t)$.

*Case $\Delta D < 0$.* In this case we use a different Lyapunov function,

$$V(t) = X(t)^{\mathrm{T}}PX(t) + \frac{a}{2}\int_0^{D_0+\Delta D} (1+x)w(x,t)^2\,\mathrm{d}x + \frac{1}{2}\int_{\Delta D}^{0}(D_0+x)w(x,t)^2\mathrm{d}x, \tag{60}$$

and obtain

$$\dot{V} \leq -\left(\frac{\lambda_{\min}(Q)}{2} - a(1+D)q_1(\Delta D)\right)|X|^2 - \left(\frac{a}{2} - \frac{D_0}{2} - \frac{2|PB|^2}{\lambda_{\min}(Q)}\right)w(0,t)^2 - \left(\frac{1}{2} - a(1+D)q_2(\Delta D)\right)\int_{\Delta D}^{0} w(x,t)^2\mathrm{d}x - \frac{D}{2}w(\Delta D,t)^2 - \frac{\max\{a,1\}}{4}\int_{\Delta D}^{D_0+\Delta D} w(x,t)^2\mathrm{d}x. \tag{61}$$

This quantity is made negative definite by first choosing $a > D_0 + \frac{4|PB|^2}{\lambda_{\min}(Q)}$, and then choosing a sufficiently small $\delta > 0$ as the largest value of $|\Delta D|$ so that $\frac{\lambda_{\min}(Q)}{2} > a(1+D)q_1(\Delta D)$ and $\frac{1}{2} > a(1+D)q_2(\Delta D)$. One thus gets exponential decay estimates in terms of $|X(t)|^2 + \int_{\Delta D}^{D_0+\Delta D} w(x,t)^2\mathrm{d}x$, and with some further work also in terms of $|X(t)|^2 + \int_{\Delta D}^{D_0+\Delta D} u(x,t)^2\mathrm{d}x$, i.e., in terms of $|X(t)|^2 + \int_{t-D_0}^{t} U(\theta)^2\mathrm{d}\theta$. □

**Remark 6.** The result of Theorem 3 is fairly subtle. The case when $\Delta D > 0$ is clear, the robustness to a "surplus" of actuator delay is a result that already holds for ODEs (Teel, 1998). The case $\Delta D < 0$ is more tricky. The controller, which overestimates the delay to be $D_0 > D_0+\Delta D$, introduces the delayed inputs from the time interval

$[t - D_0, t - D_0 - \Delta D]$ into the overall dynamic system, making its state consist of control inputs $U(\theta)$ from the entire interval $\theta \in [t - D_0, t]$, even though the actual actuator delay $D_0 + \Delta D$ is shorter. This peculiarity results in a more complicated analysis for $\Delta D < 0$, with different weights on the Krasovskii functionals for the different parts of the delay interval (with the lesser weight on the subinterval that represents the delay "mismatch"). The greater difficulty in proving the result for $\Delta D < 0$, leads us to conjecture that the predictor-based controllers *may* exhibit greater sensitivity[3] to delay mismatch in the cases where the delay is "over-estimated" (and thus "overcompensated") rather than when it is "underestimated". This means that, while there is no question that predictor-based delay compensation is indispensable for dealing with long actuator delays, and thus, that "some amount" of delay compensation is better than none, when faced with a delay of uncertain length—if our conjecture is true—"less" may be better than "more", i.e., it may be better to err on the side of caution and design for the lower end of the delay range expected.

The next result, for $\Delta D = -D_0 > 0$, shows that, even if the system has no actuator delay, it is robust to a small amount of predictor feedback.

**Corollary 1.** *There exists $\delta > 0$ such that for all $D_0 \in [0, \delta)$ the system*

$$\dot{X} = AX + BU(t), \tag{62}$$

$$U(t) = K\left[e^{AD_0}X(t) + \int_{t-D_0}^{t} e^{A(t-\theta)}BU(\theta)d\theta\right] \tag{63}$$

*is exponentially stable in the sense of the state norm $(|X(t)|^2 + \int_{t-D_0}^{t} U(\theta)^2 d\theta)^{1/2}$.*

**Proof.** The closed loop system is $\dot{X} = (A + BK)X + Bw(0, t)$, with $w_t = w_x$ evolving over $x \in [-D_0, 0]$ and $w(0, t)$ satisfying the relations (53)–(56) for $D_0 + \Delta D = 0$. The Lyapunov function $V = X^T PX + \frac{1}{2}\int_{-D_0}^{0}(D_0 + b + x)w(x)^2 dx$, where $b > 0$, satisfies $\dot{V} \leq -\left(\frac{\lambda_{\min}(Q)}{2} - \Omega q_1\right)|X|^2 - \left(\frac{1}{2} - \Omega q_2\right)\int_{-D_0}^{0} w(x)^2 dx - \frac{b}{2}w(-D_0)^2$, where $\Omega = \frac{4|PB|}{\lambda_{\min}(Q)} + D_0 + b$. Then $D_0$ can be chosen sufficiently small to make $q_1$ and $q_2$ arbitrarily small and achieve exponential stability. □

## 5. Conclusions

In this note we derived inverse optimality results for stabilization and disturbance attenuation with the low-pass filtered modification of the predictor-based feedback for actuator delay compensation. Having also established robustness to small delay mismatch, as the most critical form of robustness in the predictor-feedback problem (as well as robustness to 'bandwidth limitation,' in the form of a low-pass filter), other forms of robustness are worth studying next, using the Lyapunov functions, with the help of the backstepping transformation and its inverse, (14) and (18).

It is worth noting that, due to the constructive character of the proofs of the Theorems 1–3 and Corollary 1, all of the constants in their statements ($c^*, c^{**}, \beta_1, \beta_2, \gamma_1, \gamma_2^{**}, \delta$) can be given as explicit (albeit conservative) estimates.

The robustness result for delay mismatch in Theorem 2 is best appreciated if one is aware of the negative results on robustness of infinite dimensional systems with actuator delay. In Datko (1988) Datko revealed that exponentially stabilizing results for hyperbolic PDE systems (such as wave and beam equations) have zero robustness to delay in the feedback loop—an arbitrarily small $D > 0$ produces eigenvalues in the right half plane, no matter how "deeply" in the left half plane the closed-loop eigenvalues are for $D = 0$ (note that the addition of the delay $D > 0$ introduces more eigenvalues, i.e., this result contains no discontinuity in the dependence of the eigenvalues on $D$). Due to this result for hyperbolic PDEs, and given that the actuator delay in our problem is also a hyperbolic (though first order) PDE system, at the start of this research attempt we did not know, even at the intuitive level, if the predictor feedback would actually have a positive robustness margin to delay uncertainty.

[3] This is to be ascertained by a separate study, which may be hard to conduct analytically and may have to be mainly numerical, for select examples.

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