M - ideals in complex function spaces and algebras.

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Introduction.

The aim of this note is to give a characterization of the M-ideals of a complex function space $A \subseteq \mathcal{C}_{\mathbb{C}}(X)$.

The concept of an M-ideal was defined for real Banach spaces by Alfsen and Effros [AE], but it can be easily transferred to the complex case [Th. 1.3].

The main result is the following: Let J be a closed subspace of a complex function space A , then J is an M-ideal in A if and only if

$$J = \{a \in A \mid a \equiv 0 \text{ on } E\} ,$$

where $E \subseteq X$ is an A-convex set having the properties:

(i) $\mu \in M_1^+(\partial_A X)$, $\nu \in M_1^+(E)$, $\mu - \nu \in A^\perp \implies \mathrm{Supp}(\mu) \subseteq E$

(ii) $\mu \in A^\perp \cap M(\partial_A X) \implies \mu|_E \in A^\perp$

In case A is a uniform algebra these sets are precisely the p-sets (generalized peak sets).

Following the lines of [AE] we shall study M-ideals in A by means of the corresponding L-ideals in A^* , which in turn are studied by geometric and analytic properties of the closed unit ball K in A^* .

Although we have an isometric complex-linear representation of the given function space as the space of all complex-valued linear functions on K , it turns out that the smaller compact, convex set $Z = \mathrm{conv}(S_A \cup -iS_A)$, where S_A denotes the state space of A , will contain enough structure to determine the L-ideals. The set Z was first studied by Azimow in [Az]. Note also that the problems which always arize in the presence of complex orthogonal measures can to a certain extent be given a geometric treatment when we consider the compact, convex set Z [Prop. 2.4].

Another usefull tool in this context is the possibility of representing complex linear functionals by complex boundary measures of same norm, as was recently proved by Hustad in [Hu].

Specializing to uniform algebras we characterize the M-summands (see [AE, §5]), and we conclude by pointing out that the structure-topology of Alfsen and Effros [AE, §6] coincides with the symmetric facial topology studied by Ellis in [E].

This result yields a description of the structure space, Prim A (see [AE, §6]), in terms of concepts more familiar to function algebraists. Specifically, Prim A is (homeomorphic to) the Choquet-boundary of X endowed with the p-set topology.

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1. Preliminaries and notation.

Let W denote a real Banach space. Following [AE, §3] we define an <u>L-projection</u> e on W to be a linear map of W into itself such that,

i) $e^2 = e$

ii) $\|p\| = \|e(p)\| + \|p - e(p)\| \qquad \forall p \in W$

and we define the range of an L-projection to be an <u>L-ideal</u> in W.

To every L-ideal $N = eW$ there is associated a <u>complementary L-ideal</u> $N' = (I-e)W$, cf [AE, §3].

We say that a closed subspace J of a real Banach space V is an <u>M-ideal</u> if the polar of J is an L-ideal in $W = V^*$.

Also, we define a linear map e of V into itself to be an <u>M-projection</u> if

i) $e^2 = e$

ii) $\|v\| = \max\{\|e(v)\|, \|v - e(v)\|\} \quad \forall v \in V$

and we define a subspace of V to be an <u>M-summand</u> if it is the range of an M-projection. It follows from [AE, Cor.5.16] that M-summands are M-ideals.

<u>Lemma 1.2.</u> Let N be an L-ideal in a real Banach-space W, and let e be the corresponding L-projection. If T is an isometry of W onto itself, then TN is an L-ideal and the corresponding L-projection e_T is given by

(1.1) $$e_T = T\,e\,T^{-1}$$

Also

$$(TN)' = T(N')$$

<u>Proof</u>: Straightforward verification.

If V is a complex Banach space, then we shall denote by V_r the <u>subordinate real space</u>, having the same vectors but equipped with real scalars only. By an elementary theorem [P, §6] it follows that there is a natural isometry φ of $(V^*)_r$ onto $(V_r)^*$, defined by

(1.2) $$\varphi(p)(v) = \mathrm{Re}\; p(v) \qquad v \in V.$$

<u>Theorem 1.2.</u> (Effros) Let W be a complex Banach space with subordinate real space W_r. If N is an L-ideal in W_r then

N is a complex linear subspace of W.

Proof: It suffices to prove that $ip \in N$ for all $p \in N$. Let $p \in N$ and consider

$$q = p - e_T p$$

where T is the isometry $T(p) = ip \quad \forall p \in W$ and e_T is defined as in (1.1).

Then

$$q = e(p) - e_T e(p) = e(p - e_T(p)) \in N$$

since L-projections commute [AE, §3].

Also we shall have

$$iq = i(I - e_T)(p) \in i(T(N')) = N'$$

Thus

$$\sqrt{2}\,\|q\| = \|q + iq\| = \|q\| + \|iq\| = 2\|q\|$$

such that $q = 0$ and hence $ip \in N$.

Corollary 1.3. Let V be a complex Banach space with subordinate real space V_r. If J is an M-ideal in V_r, then J is a complex linear subspace of V.

Proof: By the bipolar theorem it suffices to show that the polar J^o of J in $W = V^*$ is a complex subspace of W. To this end, we first consider J as a real linear subspace of V_r, and we denote by J_r^o the polar of J in $(V_r)^*$. J_r^o is an L-ideal in $(V_r)^*$ since J is an M-ideal in V_r. If $\varphi : W_r \to (V_r)^*$ is the isometry defined in (1.2), then $\varphi^{-1}(J_r^o)$ is an L-ideal in W_r.

Moreover $J^o = \varphi^{-1}(J_r^o)$ since $\varphi^{-1}(J_r^o)$ is a complex linear subspace of W according to theorem 1.2.

The above results justify the use of the terms <u>L- and M-ideals</u> for <u>complex Banach spaces</u> to denote L- and M-ideals in the subordinate real spaces.

Let V be a complex Banach space, $W = V^*$, and K the closed unit ball of W. If N is a w*-colsed L-ideal in W with corresponding L-projection e, then it follows from [AE, Cor.4.2] that for a given $v \in V$ considered as a complex linear function in W one has:

(1.3) $$(v \circ e)(p) = \int_K (v \circ e)d\mu \quad \forall p \in K, \ \forall \mu \in M_p^+(K)$$

and

(1.4) $$(v \circ e)(p) = \int_{N \cap K} v \ d\mu \quad \forall p \in K, \ \forall \mu \in M_p^+(\partial_e K)$$

where $M_p^+(K)$ denotes the set of all probability measures on K with barycenter p, and $M_p^+(\partial_e K)$ the set of all measures in $M_p^+(K)$ which are maximal in Choquets ordering (boundary measures).

2. M-ideals in complex function spaces.

In this section X shall denote a compact Hausdorff space and A a closed, linear subspace of $\mathscr{C}_{\mathbb{C}}(X)$, which separates the points of X and contains the constant functions. The <u>state-space</u> of A i.e.

$$S_A = \{p \in A^* \mid p(\mathbb{1}) = \|p\| = 1\}$$

is a w*-closed face of the closed unit ball K of A^*, We shall assume that K is endowed with w*-topology.

Since A separates the points of X, we have a homeomorphic embedding Φ of X into S_A, defined by

(2.1) $$\Phi(x)(a) = a(x) \qquad \forall a \in A .$$

We use θa to denote the function on A^* defined by

(2.2) $$\theta a(p) = \mathrm{Re}\ p(a) \qquad \forall p \in A^* .$$

For convenience we shall use the same symbol θa to denote the restriction of this function to various compact, convex subsets of A^*.

An enlargement of S_A, which was introduced by Azimow, is the following set

(2.3) $$Z = \mathrm{conv}(S_A \cup -iS_A)$$

Appealing to [Az, Prop 1] the embedding $a \to \theta a$ is a bicontinuous real linear isomorphism of A onto the space $A(Z)$ of all real-valued w*-continuous affine functions on Z.

We shall denote by $M_1^+(S_A)$ resp. $M_1^+(Z)$ the w*-compact convex set of probability measures on S_A resp. Z. The set of extreme points of S_A resp. Z, K will be denoted by $\partial_e S_A$ resp. $\partial_e Z$, $\partial_e K$ and the Choquet boundary of X with respect to A is defined as the set

$$\partial_A X = \{x \in X \mid \Phi(x) \in \partial_e S_A\}$$

It follows from [P, p.38] that $\partial_e S_A \subseteq \Phi(X)$. Moreover,

$$\partial_e K = \{\lambda\Phi(x) \mid |\lambda| = 1 , x \in \partial_A X\}$$

cf [DS, p.441].

Also we agree to write $M_p^*(S_A)$ resp. $M_z^+(Z)$ for the w*-compact convex set of probability measures on S_A resp. Z which has barycenter $p \in S_A$ resp. $z \in Z$. By $M_p^+(\partial_e S_A)$ resp. $M_z^+(\partial_e Z)$ we denote the maximal representing measures for p resp. z (boundary measures).

A real measure μ on S_A resp. Z, K is said to be a boundary measure on S_A resp. Z, K if the total variation $|\mu|$ is a maximal element in the Choquet ordering, and we denote them by $M(\partial_e S_A)$ resp. $M(\partial_e Z)$, $M(\partial_e K)$.

Finally we denote by $M(\partial_A X)$ those complex measures μ on X for which the direct image measure $\Phi(|\mu|)$ on S_A is an element of $M(\partial_e S_A)$.

It is well-known (see e.g. [A, Prop.I.4.6]) that boundary measures are supported by the closure of the extreme boundary.

As mentioned we shall study M-ideals in A by considering the corresponding L-ideals in A^*. Let N be a w*-closed L-ideal in A^* with corresponding L-projection e.

Lemma 2.1. Let $p \in S_A$. Then

$$e(p) \in \operatorname{conv}(\{0\} \cup S_A)$$

Proof: Let $p \in S_A$ and decompose

$$p = q + r$$

where $q = e(p)$ and $r = (I-e)(p)$. If $q = 0$ or $r = 0$ there is nothing to prove.

Otherwise

$$p = \|q\|\left(\frac{q}{\|q\|}\right) + \|r\|\left(\frac{r}{\|r\|}\right)$$

is a convex combination of points in K . Since S_A is a face of K we obtain $\frac{q}{\|q\|} \in S_A$. Hence

$$e(p) = q \in \operatorname{conv}(\{0\} \cup S_A) .$$

Lemma 2.2. Let $p \in N \cap Z$ be of the form

$$p = \lambda p_1 + (1-\lambda)(-ip_2) ,$$

where $p_1, p_2 \in S_A$ and $0 < \lambda < 1$.

Then $p_1, p_2 \in N \cap Z$.

Proof: Let $p \in N \cap Z$ be of the form

$$p = \lambda p_1 + (1-\lambda)(-ip_2) , \quad p_i \in S_A \quad i = 1,2 \text{ and } 0<\lambda<1$$

Decompose p_i as

$$p_i = q_i + r_i ,$$

where $e(p_i) = q_i$ and $r_i = (I-e)(p_i)$ for $i = 1,2$.

Since $e(p) = p$ it follows that

$$0 = \lambda r_1 + (1-\lambda)(-ir_2) \tag{2.4}$$

Hence $q_i \neq 0$ for $i = 1,2$.

Now assume $r_1 \neq 0$; then

$$p_1 = \|q_1\|\left(\frac{q_1}{\|q_1\|}\right) + \|r_1\|\left(\frac{r_1}{\|r_1\|}\right)$$

is a convex combination, and we conclude that $\frac{r_1}{\|r_1\|} \in S_A$ which contradicts (2.4). Thus $r_i = 0$ and $p_i \in N$ for $i = 1,2$.

If Q is a closed face of a compact, convex set H , then the complementary face Q' is the union of all faces disjoint from Q . Q is said to be a split face of H if Q' is convex and each point in $K \setminus (Q \cup Q')$ can be expressed uniquely as convex combination of a point in Q and a point in Q' , cf [A,p.33].

We denote by $A_s(H)$ the smallest uniformly closed subspace of the space of all real valued bounded functions on H containing the bounded u.s.c. affine functions. According to [A, Th.II 6.12] and [An, Prop 3] we have that for a closed face Q of H the following statements are equivalent:

(i) Q is a split face

(ii) If $\mu \in M(\partial_e H)$ annihilates all continuous affine functions, then $\mu|_Q$ has the same property.

(iii) If $a \in A_s(Q)$ then a has an extension $\tilde{a} \in A_s(H)$ such that $\tilde{a} \equiv 0$ on Q' .

We remark that the functions in $A_s(H)$ satisfy the barycentric calculus.

Theorem 2.3. Let N be a w*-closed L-ideal of A^* and let $F = N \cap Z$. Then F is a split face of Z , and $F' = N' \cap Z$.

Proof: Applying lemma 2.2 twice it follows that F is a face of Z . Let $z \in F'$ and $\mu \in M_z^+(\partial_e Z)$, then $\mu(F) = 0$ [H,Lem. 2.11].

Moreover, the Milman theorem implies that $\partial_e Z \subseteq (S_A \cup -i\, S_A)$ and hence $\mathrm{Supp}(\mu) \subseteq (S_A \cup -i\, S_A)$.

Since these two sets are faces of K we may consider μ as a boundary measure on K .

According to (1.4) we also have

$$(\theta a \circ e)(z) = \int_F \theta a \, d\mu = 0 \qquad \forall a \in A ,$$

where e is the L-projection corresponding to N .

Thus $e(z) = 0$, which in turn implies $z \in N' \cap Z$.

Conversely, assume $z \in N' \cap Z$. Decompose

$$z = \lambda p_1 + (1-\lambda)p_2$$

where $p_1 \in F$, $p_2 \in F'$ and $0 \leq \lambda \leq 1$.

Hence

$$z - (1-\lambda)p_2 = \lambda p_1 \in N \cap N' = \{0\} ,$$

and so $z = p_2 \in F'$. Thus we have proved that $F' = N' \cap Z$. In particular, F' is convex.

From the above results we may establish the splitting property by proving

$$\mu \in A(Z)^{\perp} \cap M(\partial_e Z) \Longrightarrow \mu|_F \in A(Z)^{\perp} .$$

To this end we consider $\mu \in A(Z)^{\perp} \cap M(\partial_e Z)$. As before $\mu \in M(\partial_e K)$, and also

$$\int_K \theta a \, d\mu = \int_Z \theta a \, d\mu = 0 \qquad \forall a \in A ,$$

i.e. $\mu \in A_o(K)^{\perp} \cap M(\partial_e K)$, where $A_o(K)$ is the space of all real-valued w*-continuous linear functions on K . By virtue of [AE, Th.4.5] $\mu|_F \in A_o(K)^{\perp}$, or equivalently $\mu|_F \in A(Z)^{\perp}$. q.e.d.

<u>Remark</u>: Under the hypothesis of Theorem 2.3 we have:

$$F = \mathrm{conv}((F \cap S_A) \cup -i(F \cap S_A))$$

Following Ellis [E] we shall say that a subset of Z of the form

$$\mathrm{conv}(C \cup -i\,C) \ , \ C \subseteq S_A$$

is <u>symmetric</u>.

Let F be a closed face of S_A , and put

(2.5) $$E = \Phi^{-1}(F \cap \Phi(X)) ,$$

Then $F = \overline{\text{conv}}(\Phi(E))$ and $F \cap \Phi(X) = \Phi(E)$.

Proposition 2.4. Let F be a closed face of S_A for which $S_F = \text{conv}(F \cup -iF)$ is a split face of Z . Then E satisfies the condition:

$$\mu \in A^{\perp} \cap M(\partial_A X) \Longrightarrow \mu|_E \in A^{\perp} .$$

Proof: Let $\mu \in A^{\perp} \cap M(\partial_A X)$ and put $\sigma = \Phi\mu$. Then σ is a complex maximal measure on S_A . Decompose σ as

$$\sigma = \lambda_1\sigma_1 - \lambda_2\sigma_2 + i\lambda_3\sigma_3 - i\lambda_4\sigma_4$$

where $\sigma_i \in M_1^+(\partial_e S_A)$ and $\lambda_i \geq 0$ for $i = 1,2,3,4$. Since $\mu \in A^{\perp}$, $\lambda_1 = \lambda_2$ and $\lambda_3 = \lambda_4$.

Define p_i = barycenter of σ_i for $i = 1,2,3,4$. Since $\mu \in A^{\perp}$ it follows that

(2.6) $$0 = \lambda_1 p_1 - \lambda_2 p_2 + i\lambda_3 p_3 - i\lambda_4 p_4$$

Rewrite (2.6) as

(2.7) $$\lambda_1 p_1 + \lambda_4(-ip_4) = \lambda_2 p_2 + \lambda_3(-ip_3) \in Z$$

if we assume $\lambda_1 + \lambda_4 = \lambda_2 + \lambda_3 = 1$.

Define $\psi: S_A \to -iS_A$ by

$$\psi(p) = -ip \qquad \forall p \in S_A$$

Let $a \in A$ and put $\theta a|_{S_F} = b \in A_s(S_F)$.

Since S_F is assumed to be a split face of Z we can find a function $\tilde{b} \in A_s(Z)$ which extends b and such that $\tilde{b} \equiv 0$ on S_F' . Moreover,

$$\lambda_1 \tilde{b}(p_1) + \lambda_4 \tilde{b}(-ip_4) = \lambda_2 \tilde{b}(p_2) + \lambda_3 \tilde{b}(-ip_3) ,$$

and since $\tilde{b}$ satisfies the barycentric calculus we may rewrite this as

$$(2.8) \qquad \lambda_1 \int_Z \tilde{b}\, d\sigma_1 + \lambda_4 \int_Z \tilde{b}\, d(\psi\sigma_4) - \lambda_2 \int_Z \tilde{b}\, d\sigma_2 - \lambda_3 \int_Z \tilde{b}\, d(\psi\sigma_3) = 0 .$$

Since every maximal measure on Z is carried by S_F and S_F' and since $S_F \cap S_A = F$, we may rewrite (2.8) as

$$(2.9) \qquad \lambda_1 \int_F \theta a\, d\sigma_1 + \lambda_4 \int_F \theta a \circ \psi\, d\sigma_4 - \lambda_2 \int_F \theta a\, d\sigma_2 - \lambda_3 \int_F \theta a \circ \psi\, d\sigma_3 = 0$$

The measure μ can be decomposed as

$$(2.10) \qquad \mu = \lambda_1\mu_1 - \lambda_2\mu_2 + i\lambda_3\mu_3 - i\lambda_4\mu_4$$

where $\mu_i = \Phi^{-1}\sigma_i$ for $i = 1,2,3,4$. Now,

$$\int_E a\, d\mu = (\lambda_1 \int_E \mathrm{Re}\, a\, d\mu_1 - \lambda_2 \int_E \mathrm{Re}\, a\, d\mu_2 - \lambda_3 \int_E \mathrm{Im}\, a\, d\mu_3 + \lambda_4 \int_E \mathrm{Im}\, a\, d\mu_4)$$

$$+ i(\lambda_1 \int_E \mathrm{Im}\, a\, d\mu_1 - \lambda_2 \int_E \mathrm{Im}\, a\, d\mu_2 + \lambda_3 \int_E \mathrm{Re}\, a\, d\mu_3 - \lambda_4 \int_E \mathrm{Re}\, a\, d\mu_4)$$

Transforming the above integrals by the embedding map Φ and using the identity $\theta a(-ip) = \mathrm{Im}\, a(p)$, we rewrite this as follows:

$$(2.11) \qquad \int_E a\, d\mu = (\lambda_1 \int_F \theta a\, d\sigma_1 - \lambda_2 \int_F \theta a\, d\sigma_2 - \lambda_3 \int_F \theta a \circ \psi\, d\sigma_3 + \lambda_4 \int_F \theta a \circ \psi\, d\sigma_4)$$

$$+ i(\lambda_1 \int_F \theta(-ia)\, d\sigma_1 - \lambda_2 \int_F \theta(-ia)\, d\sigma_2 - \lambda_3 \int_F \theta(-ia) \circ \psi\, d\sigma_3 +$$

$$+ \lambda_4 \int_F \theta(-ia) \circ \psi\, d\sigma_1)$$

Combining (2.11) with (2.9) we get

$$\int_E a\, d\mu = 0 \qquad \forall a \in A .$$

Theorem 2.5. Let F be a closed face of S_A for which $S_F = \mathrm{conv}(F \cup -iF)$ is a split face of Z. Then

$$N = \mathrm{lin}_{\mathbb{C}} F$$

is a w*-closed L-ideal in A^*.

Proof: Since S_F is a split face, N may be considered as a w*-closed real linear subspace of $A(Z)^*$ and from the connection between A and $A(Z)$ cf. §1 it follows that N is w*-closed in A^*.

According to proposition 2.4 the following definition is legitimate,

$$e(p)(a) = \int_E a\, d\mu \qquad \forall a \in A ,$$

where E is as in (2.5) and μ is a maximal complex measure representing the point $p \in A^*$.

Clearly $e(A^*) \subseteq N$. Let $p \in N$ i.e.

$$p = \lambda_1 p_1 + \lambda_2(-p_2) + \lambda_3(ip_3) + \lambda_4(-ip_4)$$

where $p_i \in F$ and $\lambda_i \geq 0$ for $i = 1,2,3,4$.

Choose measures $\sigma_i \in M^+_{p_i}(\partial_e S_A)$ for $i = 1,2,3,4$. Then $\mathrm{Supp}(\sigma_i) \subseteq \Phi(E)$ since F is a face of S_A. Define $\mu_i = \Phi^{-1}\sigma_i$ for $i = 1,2,3,4$ and

$$\mu = \lambda_1\mu_1 - \lambda_2\mu_2 + i\lambda_3\mu_3 - i\lambda_4\mu_4$$

Now μ is a complex representing measure for p and $\mathrm{Supp}(\mu) \subseteq E$, i.e.

$$e(p) = p .$$

To prove that e is an L-projection, we shall need the fact that we may represent $p \in A^*$ by a measure $\mu \in M(\partial_A X)$ such that $\|p\| = \|\mu\|$. This follows by a slight modification of a theorem of Hustad [Hu].

Having chosen $\mu \in M(\partial_A X)$ representing $p \in A^*$ with $\|p\| = \|\mu\|$, we shall have

$$\|p\| \leq \|e(p)\| + \|p - e(p)\| \leq \|\mu\|_E + \|\mu\|_{X \setminus E} = \|\mu\| = \|p\| ,$$

which implies

$$\|p\| = \|e(p)\| + \|p - e(p)\| \qquad \forall p \in A^*$$

i.e. e in an L-projection with range N .

A compact subset $E \subseteq X$ is said to be <u>A-convex</u> if it satisfies:

$$E = \{x \in X \mid |a(x)| \leq \|a\|_E \ \ \forall a \in A\}$$

If F is a closed face of S_A such that $S_F = \mathrm{conv}(F \cup -iF)$ is a split face of Z then the set $E = \Phi^{-1}(F \cap \Phi(X))$ is A-convex and has the following properties:

(i) $\mu \in M_1^+(\partial_A X)$, $\nu \in M_1^+(E)$, $\mu - \nu \in A^\perp \Longrightarrow \mathrm{Supp}(\mu) \subseteq E$.

(ii) $\mu \in A^\perp \cap M(\partial_A X) \Longrightarrow \mu|_E \in A^\perp$.

If an A-convex subset E of X satisfies (i) and (ii) then we say that E is an <u>M-set</u> .

If $E \subseteq X$ is a compact subset then we denote by S_E the following subset of S_A ,

(2.12) $$S_E = \overline{\mathrm{conv}}(\Phi(E)) .$$

Clearly, if E is an M-set S_E is a closed face of S_A and $S_E \cap \Phi(X) = \Phi(E)$.

Moreover,

<u>Corollary 2.6</u>. Let E be an M-set of X . Then

$$N = \overline{\mathrm{lin}_C\, \Phi(E)}^{w^*}$$

is a w*-closed L-ideal of A^* .

Proof: Observe that $\mathrm{conv}(S_E \cup -i\, S_E)$ is a split face of Z and define

$$e(p)(a) = \int_E a\, d\mu \qquad \forall a \in A ,$$

where μ is a maximal representing measure for $p \in A^*$. Proceed as in the proof of Th. 2.5.

Corollary 2.7. Let E be an A-convex subset of X . Then the following statements are equivalent:

(i) E is an M-set.

(ii) $\mathrm{conv}(S_E \cup -i\, S_E)$ is a split face of Z .

(iii) $N = \mathrm{lin}_{\mathbb{C}}\, S_E$ is a w*-closed L-ideal.

Proof: Combine Th. 2.3 and Cor. 2.6.

Remark. Cf. [Az, Th.2,3] and [E] for similar results.

Remark. A closed face F of S_A is a split face of Z if and only if the following condition is satisfied:

$$\mu \in A^{\perp} \cap M(\partial_A X) \Longrightarrow \begin{cases} (\mu_1 - \mu_2)|_E \in A^{\perp} \\ (\mu_3 - \mu_4)|_E \in A^{\perp} \end{cases}$$

where $\mu = \mu_1 - \mu_2 + i(\mu_3 - \mu_4)$ and E as in (2.5).

Thus we see that not all split faces of Z are symmetric. Cf. [E].

Turning to the M-ideals in A we now have the following

<u>Theorem 2.8</u>. Let J be a closed subspace of A. Then the following statements are equivalent:

(i) J is an M-ideal.

(ii) $J = \{a \in A \mid a \equiv 0 \text{ on } E\}$,

where E is an M-set of X.

<u>Proof</u>: Assume J is an M-ideal of A, then $J^{\circ} \cap Z$ is a split face of Z since J° is an L-ideal. Moerover, we claim that

$$J^{\circ} = \text{lin}_{\mathbb{C}}(J^{\circ} \cap S_A)$$

Trivially, $\text{lin}_{\mathbb{C}}(J^{\circ} \cap S_A) \subseteq J^{\circ}$. If $p \in \partial_e(J^{\circ} \cap K)$ then

$$p \in \partial_e(J^{\circ} \cap K) = J^{\circ} \cap \partial_e K$$

Hence

$$p = \lambda q \ , \quad |\lambda| = 1 \ , \quad q \in \partial_e S_A$$

Thus

$$q = \lambda^{-1} p \in J^{\circ} \cap S_A$$

such that

$$p \in \text{lin}_{\mathbb{C}}(J^{\circ} \cap S_A)$$

It follows from theorem 2.5 that $\text{lin}_{\mathbb{C}}(J^{\circ} \cap S_A)$ is w*-closed and hence

$$\overline{\text{conv}}(\partial_e(J^{\circ} \cap K)) \subseteq \text{lin}_{\mathbb{C}}(J^{\circ} \cap S_A) \ .$$

This in turn implies

$$J^{\circ} = \text{lin}_{\mathbb{C}}(J^{\circ} \cap S_A)$$

Equivalently

$$J^{\circ} = \overline{\text{lin}_{\mathbb{C}}(\Phi(E))}^{w^*} \ ,$$

where $E = \Phi^{-1}(J^{\circ} \cap \Phi(X))$.

Thus we see that

$$J = \{a \in A \mid a \equiv 0 \text{ on } E\} ,$$

and clearly E is an M-set.

Conversely, if J is of the form

$$J = \{a \in A \mid a \equiv 0 \text{ on } E\} ,$$

where E is an M-set, then $J^{o} = \overline{\mathrm{lin}_{\mathbb{C}}\, \Phi(E)}^{w^*}$ is an L-ideal according to Corollary 2.6.

3. The uniform algebra case.

In this section we make the further assumption that A is a uniform algebra [G].

A peak set E for A is a subset of X for which there exists a function $a \in A$ such that

$$a(x) = 1 \quad \forall x \in E , \quad |a(x)| < 1 \quad \forall x \in X \setminus E$$

A p-set (generalized peak set) for A is an intersection of peak-sets for A. If X is metrizable then every p-set is a peak set [G, §12].

It follows from [G, Th.12.7] that the following is equivalent for a compact subset E of X :

(i) E is a p-set.

(ii) $\mu \in A^{\perp} \Rightarrow \mu|_E \in A^{\perp}$.

Clearly, p-sets are M-sets.
Moreover, since M-sets are A-convex it follows by a slight modification of [AH, Th.7.4] that M-sets are p-sets i.e. we may state

Theorem 3.1. Let A be a uniform algebra and J a closed subspace of A . Then the following statements are equivalent:

(i) J is an M-ideal.

(ii) $J = \{a \in A \mid a \equiv 0 \text{ on } E\}$,

where E is a p-set for A .

Turning to the M-summands of A we shall have,

Theorem 3.2. Let J be a closed subspace of A . Then the following statements are equivalent:

(i) J is an M-summand

(ii) $J = \{a \in A \mid a \equiv 0 \text{ on } E\}$ where E is an open-closed p-set for A .

Proof: Trivially ii) ⟹ i) by virtue of theorem 3.1.

Conversely, assume J is an M-summand. Then

$$J = \{a \in A \mid a \equiv 0 \text{ on } E\} ,$$

where E is a p-set for A . To prove that E is open it suffices to prove that

$$\{x \in X \mid e(\mathbb{1})(x) = 1\} = X \setminus E$$

where e is the M-projection corresponding to J . Clearly

$$\{x \in X \mid e(\mathbb{1})(x) = 1\} \subseteq X \setminus E .$$

Let $x \notin E$, and μ a maximal measure on X representing x . Then $(\mu - \varepsilon_x) \in A^\perp$ and hence $\mu(E) = 0$.
Moreover, if e^* denotes the adjoint of e then $(eA)^o = (I - e^*)A^*$ and hence

$$\mathbb{1} \circ (I - e^*)(\Phi(x)) = \int_E \mathbb{1}\, d\mu = 0$$

Thus

$$0 = (I - e^*)(\Phi(x))(\mathbb{1}) = 1 - e(\mathbb{1})(x)$$

and we are done, cf [AE, Cor.5.16].

Finally we point out that since every point $x \in \partial_A X$ is a p-set for A and

$$J_x = \{a \in A \mid a(x) = 0\}$$

is the largest M-ideal contained in the kernel of $\Phi(x)$ then the <u>Structure-topology</u> [AE, §6] on $\partial_e K$ restricted to $\partial_e S_A$ coincides with the <u>symmetric facial topology</u> studied by Ellis in [E]. This follows from theorems 2.3 - 2.5.

Moreover, this topology coincides with the well known <u>p-set topology</u>.

Specifically, if $p \in \partial_e K$ then there exists a unique point $x_p \in \partial_A X$ and $\lambda_p \in \{z \in C \mid |z| = 1\}$ such that $p = \lambda_p \Phi(x_p)$ and hence the largest M-ideal contained in the kernel of p is J_{x_p} i.e. the above can be summed up in the following diagram:

$$\begin{array}{ccccc}
\partial_A X & \overset{\Phi}{\longleftrightarrow} & \partial_e S_A & \overset{i}{\longrightarrow} & \partial_e K \\
\text{(p-set topology)} & & \text{(symmetric facial topology)} & & \text{(structure topology)} \\
 & & & \overset{s}{\nwarrow\!\!\searrow} & \downarrow q \\
 & & & & \text{Prim } A \\
 & & & & \text{(structure topology)}
\end{array}$$

where all the maps are continuous, q open, Φ and s homeomorphisms.

References.

[A] E.M. Alfsen, Compact convex sets and boundary integrals, Ergebnisse der Math. 57 Springer Verlag, 1971.

[AE] E.M. Alfsen and E. Effros, Structure in real Banach Spaces University of Oslo, Math.Inst.Preprint Series 1971 no.5.

[AH] E.M. Alfsen and B. Hirsberg, On dominated extensions in linear subspaces of $\mathcal{C}_{\mathbb{C}}(X)$, Pacific Journal of Mathematics 36 (3) 1971, 567-584.

[An] T.B. Andersen, On Banach space value extensions from split faces, University of Oslo, Math.Inst.Preprint Series 1971 no.6.

[Az] Azimo , Decomposable compact convex sets and peak sets for function spaces, Proc.Amer.Math.Soc 25 (1) 1970, 75-79.

[DS] N. Dumford and . Schwarz, Linear operators, Part I, Interscience Publishers, New York.

[E] A.J. Ellis, On split faces and function algebras, submitted to Math.Annalen.

[G] T.W. Gamelin, Uniform algebras, Prentice Hall Series.

[Hi] B. Hirsberg, A measure theoretic characterization of parallel- and split faces and their connections with function spaces and algebras, Univ, of Århus, Math.Inst. Various Publication Series no.16 (1970).

[Hu] O. Hustad, A norm preserving complex Choquet Theorem, to appear in Math.Scand.

[P] R.R. Phelps, Lectures on Choquet's theorem, Van Nostrand, New York 1966.

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