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M/M/1 RETRIAL QUEUE WITH COLLISIONS AND WORKING VACATION INTERRUPTION UNDER N-POLICY

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Abstract. Consider an M/M/1 retrial queue with collisions and working vacation interruption under N-policy. We use a quasi birth and death process to describe the considered system and derive a condition for the stability of the model. Using the matrix-analytic method, we obtain the stationary probability distribution and some performance measures. Furthermore, we prove the conditional stochastic decomposition for the queue length in the orbit. Finally, some numerical examples are presented.

Keywords. Retrial, collision, working vacation interruption, N-policy.

Mathematics Subject Classification. 60k25, 90B22.

1. Introduction

Queueing systems with server's vacations have been studied extensively, general models can be found in Tian and Zhang [15]. In 2002, Servi and Finn [12] first introduced a new vacation policy and studied an M/M/1 queue, the server commits a lower service rate rather than completely stopping the service during a vacation, which we call working vacation. Liu *et al.* [18] obtained the stochastic decompositions in the M/M/1 queue with working vacations by the matrix-analytic method. Subsequently, Wu and Takagi [4] generalized results in [12] to an M/G/1 queue. Baba [19] considered a GI/M/1 queue with working vacations by the matrix-analytic method. Recently, Li and Tian [8] investigated the GI/M/1 queue

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with single working vacation. Furthermore, during the working vacation period, the server can stop the vacation if there are customers at a service completion instant. For the vacation interruption models, Li and Tian [9] first introduced and studied an M/M/1 queue with working vacations and vacation interruption. Subsequently, Li *et al.* [10] analyzed a GI/M/1 queue with working vacations and vacation interruption by the matrix-analytic method. Using the method of a supplementary variable, Zhang and Hou [14] investigated the M/G/1 queue. Baba [20] even studied the M/PH/1 queue with working vacations and vacation interruption where the vacation time follows a phase type distribution.

In recent years there have been significant contributions to the retrial queues. Retrial queueing systems are described by the feature that the arriving customers who find the server busy join the retrial orbit to try again for their requests. Choi *et al.* [1] studied an M/M/1 retrial queue. Martin and Corral [13] investigated an M/G/1 retrial queue with liner control policy. Next, Lillo [16] considered a GI/M/1 retrial queue. More models can be found in Artalejo and Corral [6]. Many authors also analyzed the retrial system with collisions, since some networks can be treated as these models. Choi *et al.* [2] studied a retrial queue with collision arising from unslotted CSMA/CD protocol. Kim [7] analyzed an M/M/1 retrial queue with collision and impatience. Kumar *et al.* [3] discussed an M/M/1 retrial queue with feedback and collisions. Wu *et al.* [11] investigated a Geo/G/1 retrial queue with preemptive resume and collisions.

Do [17] first studied an M/M/1 queue with both retrials and working vacations. In this paper, we generalize the model in [17, 21], and also consider vacation interruption and collisions. In our model, upon the arrival of requests, if the server is busy, requests are forced to wait in the orbit of infinite size. Assume requests in the orbit try to get service from the server with a constant retrial rate. And, it is more reasonable to assume that the server can stop the vacation at a service completion instant, if the number of customers in the orbit achieves N. When a vacation ends, a regular busy period starts only if there are at least N customers in the orbit.

This paper is organized as follows. In Section 1, we introduce the model and obtain the infinitesimal generator. In Section 2, we derive the stability condition and the minimal non-negative solution R. Using the matrix-analytic method, the stationary probability distribution is obtained in Section 3. In Section 4, we introduce two random variables and give the conditional stochastic decomposition for the queue length. In Section 5, we obtain some important and interesting performance measures. Some numerical results are presented in Section 6. Finally, Section 7 concludes this paper.

2. Quasi birth and death (QBD) process model

In this paper, we consider an M/M/1 retrial queue with collisions, working vacations, vacation interruption and N-policy at the same time. Request retrials from the orbit of infinite size follow a Poisson process with rate α. And, if the server is busy, the retrial customer collides with the customer in service resulting in both being shifted to the orbit. During the normal service period, the server begins a working vacation each time when the system becomes empty, *i.e.*, when the server is free and there is no customer in the orbit. At a service completion instant, if there are at least N customers in the orbit, the server will stop the vacation and come back to the normal working level, which means vacation interruption happens. If the number of customers in the orbit is less than N, on the other hand, the server will stay in the working vacation period. Meanwhile, if there are at least N customers in the orbit when a vacation ends, the server switches to the normal busy period, otherwise, the server continues the vacation.

We assume that inter-arrival times, inter-retrial times, service times and vacation times are mutually independent. And, the inter-arrival times, the inter-retrial times, the service times during a normal period, the service times during a vacation period and the vacation times are exponentially distributed with parameters λ, α, μ, η and θ, respectively.

Let $Q(t)$ be the number of customers in the orbit at time t, and $J(t)$ be the state of server at time t. There are four possible states of the single server as follows:

(a) the server is on a working vacation at time t and the server is free. When the server is in this state $J(t) = 0$.
(b) the server is on a working vacation at time t and the server is busy. If the server is in this state $J(t) = 1$.
(c) the server is in a regular busy period at time t and the server is free. When the server is in this state $J(t) = 2$.
(d) the server is in a regular busy period at time t and the server is busy. If the server is in this state $J(t) = 3$.

Then $\{Q(t), J(t)\}$ is a Markov process with state space

$$\Omega = \{(k, j), k \geq 0, j = 0, 1, 2, 3\}.$$

The system states and one step transitions are shown in Figure 1. Using the lexicographical sequence for the states, the infinitesimal generator can be written as

$$\widetilde{Q} = \begin{matrix} 0 \\ 1 \\ \vdots \\ N-1 \\ N \\ N+1 \\ \vdots \end{matrix} \begin{pmatrix} A_0 & C_0 & & & & & \\ B & A_1 & C & & & & \\ & \ddots & \ddots & \ddots & & & \\ & & B & A_1 & C & & \\ & & & B & A & C & \\ & & & & B & A & C \\ & & & & & \ddots & \ddots & \ddots \end{pmatrix},$$

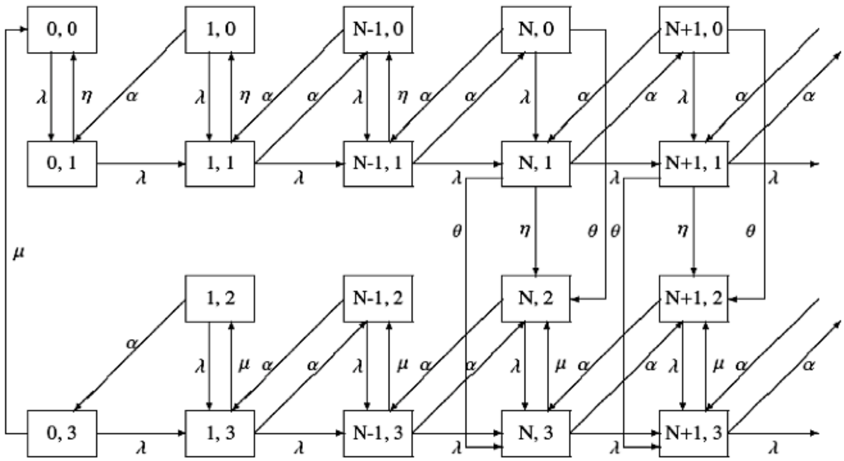

FIGURE 1. Transition diagram of the states when N=3.

where

$$A_0 = \begin{pmatrix} -\lambda & \lambda & 0 & 0 \\ \eta & -\lambda-\eta & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \mu & 0 & 0 & -\lambda-\mu \end{pmatrix}; \quad C_0 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda \end{pmatrix};$$

$$B = \begin{pmatrix} 0 & \alpha & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha \\ 0 & 0 & 0 & 0 \end{pmatrix}; \quad A_1 = \begin{pmatrix} -\lambda-\alpha & \lambda & 0 & 0 \\ \eta & -\lambda-\alpha-\eta & 0 & 0 \\ 0 & 0 & -\lambda-\alpha & \lambda \\ 0 & 0 & \mu & -\lambda-\alpha-\mu \end{pmatrix};$$

$$A = \begin{pmatrix} -\lambda-\alpha-\theta & \lambda & \theta & 0 \\ 0 & -\lambda-\alpha-\eta-\theta & \eta & \theta \\ 0 & 0 & -\lambda-\alpha & \lambda \\ 0 & 0 & \mu & -\lambda-\alpha-\mu \end{pmatrix}; \quad C = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \alpha & \lambda & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha & \lambda \end{pmatrix}.$$

Due to the block structure of matrix $\widetilde{Q}$, $\{Q(t), J(t)\}$ is called a QBD process.

Note that when there is no customer in the orbit, the probability that the server is in a busy period and does not serve a customer is zero.

3. Stability condition and rate matrix R

In this section, we derive the stability condition and the rate matrix R.

Theorem 3.1. *The QBD process $\{Q(t), J(t)\}$ is positive recurrent if and only if $(\mu - 2\lambda)\alpha > \lambda^2$.*

Proof. First, we assume

$$D = B + A + C = \begin{pmatrix} -\lambda-\alpha-\theta & \lambda+\alpha & \theta & 0 \\ \alpha & -\alpha-\eta-\theta & \eta & \theta \\ 0 & 0 & -\lambda-\alpha & \lambda+\alpha \\ 0 & 0 & \alpha+\mu & -\alpha-\mu \end{pmatrix}.$$

Since matrix D is reducible, the Theorem 7.3.1 in [5] gives the condition for positive recurrence of the QBD. After permutation of rows and columns, the Theorem 7.3.1 states that the QBD is positive recurrent if and only if

$$\upsilon \begin{pmatrix} 0 & \alpha \\ 0 & 0 \end{pmatrix} e > \upsilon \begin{pmatrix} 0 & 0 \\ \alpha & \lambda \end{pmatrix} e,$$

where e is a column vector with all elements equal to one, and υ is the unique solution of the system $\upsilon \begin{pmatrix} -\lambda-\alpha & \lambda+\alpha \\ \alpha+\mu & -\alpha-\mu \end{pmatrix} = \mathbf{0}$, $\upsilon e = 1$. After some algebraic manipulation, the QBD process is positive recurrent if and only if $\alpha(\alpha+\mu) > (\lambda+\alpha)^2$, *i.e.*, $(\mu-2\lambda)\alpha > \lambda^2$. □

Remark 3.2. In order to obtain the stability condition, we can also use the method in [17]. It needs to guarantee that the number of eigenvalues of $Q(x) = Bx^2 + Ax + C$ inside the unit disk is 4. We can easily get that $Q(x)$ have five eigenvalues: $x_1 = 0, x_2 = \lambda(\lambda+2\alpha+\theta)/[(\lambda+\alpha+\theta)(\lambda+\alpha+\eta+\theta)-\alpha^2], x_3 = 0, x_4 = \lambda(\lambda+2\alpha)/\alpha\mu, x_5 = 1$. Thus, $x_4 < 1$ leads to the result in Theorem 2.1.

Next, we solve the minimal non-negative solution R of the matrix quadratic equation

$$R^2B + RA + C = \mathbf{0} \tag{3.1}$$

Theorem 3.3. *If* $(\mu-2\lambda)\alpha > \lambda^2$, *the matrix equation* (3.1) *has the minimal non-negative solution*

$$R = \begin{pmatrix} 0 & 0 & 0 & 0 \\ r_1 & r_2 & r_3 & r_4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & r_5 & r_6 \end{pmatrix},$$

where

$$r_1 = \frac{\alpha}{\lambda+\alpha+\theta},\ r_2 = \frac{\lambda(\lambda+2\alpha+\theta)}{(\lambda+\alpha+\theta)(\lambda+\alpha+\eta+\theta)-\alpha^2},\ r_3 = \frac{\theta r_1 + (\eta+\theta) r_2}{(1-r_2)\alpha},$$
$$r_4 = \frac{\lambda+\alpha}{\mu} r_3 - \frac{\theta r_1 + \eta r_2}{\mu},\ r_5 = \frac{\lambda+\alpha}{\alpha},\ r_6 = \frac{\lambda(\lambda+2\alpha)}{\alpha\mu}.$$

Proof. From the structure of B, A, C, we can assume $R = \begin{pmatrix} R_{11} & R_{12} \\ 0 & R_{22} \end{pmatrix}$, where R_{11}, R_{12} and R_{22} are all 2×2 matrices. Taking R into (3.1), we have

$$\begin{cases} \mathbf{0} = R_{11}^2 \begin{pmatrix} 0 & \alpha \\ 0 & 0 \end{pmatrix} + R_{11} \begin{pmatrix} -\lambda - \alpha - \theta & \lambda \\ 0 & -\lambda - \alpha - \eta - \theta \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ \alpha & \lambda \end{pmatrix}, \\ \mathbf{0} = (R_{11}R_{12} + R_{12}R_{22}) \begin{pmatrix} 0 & \alpha \\ 0 & 0 \end{pmatrix} + R_{11} \begin{pmatrix} \theta & 0 \\ \eta & \theta \end{pmatrix} \\ \qquad + R_{12} \begin{pmatrix} -\lambda - \alpha & \lambda \\ \mu & -\lambda - \alpha - \mu \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \\ \mathbf{0} = R_{22}^2 \begin{pmatrix} 0 & \alpha \\ 0 & 0 \end{pmatrix} + R_{22} \begin{pmatrix} -\lambda - \alpha & \lambda \\ \mu & -\lambda - \alpha - \mu \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ \alpha & \lambda \end{pmatrix}. \end{cases}$$

From the first equation, we get $R_{11} = \begin{pmatrix} 0 & 0 \\ r_1 & r_2 \end{pmatrix}$. Similarly, $R_{22} = \begin{pmatrix} 0 & 0 \\ r_5 & r_6 \end{pmatrix}$ can be derived from the third equation. Taking R_{11} and R_{22} into the second equation, we finally obtain $R_{12} = \begin{pmatrix} 0 & 0 \\ r_3 & r_4 \end{pmatrix}$ by some computation. □

Remark 3.4. In our model, B, A, C and R are all block upper triangular matrices. When we solve the equation (3.1), we don't solve it directly, but break the equation into three matrix equations. This method is more effective when the dimension of the matrix is higher, but does not always hold in general case.

4. Stationary probability distribution

If $(\mu - 2\lambda)\alpha > \lambda^2$, let (Q, J) be the stationary limit of the process $\{Q(t), J(t)\}$, and denote

$$\begin{aligned} \pi_k &= (\pi_{k0}, \pi_{k1}, \pi_{k2}, \pi_{k3}), \quad k \geq 0; \\ \pi_{kj} &= P\{Q = k, J = j\} = \lim_{t \to \infty} P\{Q(t) = k, J(t) = j\}, \quad (k, j) \in \Omega. \end{aligned}$$

Note that from the states we described before, $\pi_{02} = 0$.

Theorem 4.1. *If $(\mu - 2\lambda)\alpha > \lambda^2$, the stationary probability distribution of (Q,J) is given by*

$$\begin{cases} \pi_{k0} = \pi_{N-1,1} r_1 r_2^{k-N}, & k \geq N, \\ \pi_{k1} = \pi_{N-1,1} r_2^{k+1-N}, & k \geq N, \\ \pi_{k2} = \pi_{N-1,1} \left(r_3 r_2^{k-N} + \dfrac{r_4 r_5}{r_6 - r_2} \left(r_6^{k-N} - r_2^{k-N} \right) \right) + \pi_{N-1,3} r_5 r_6^{k-N}, & k \geq N, \\ \pi_{k3} = \pi_{N-1,1} \dfrac{r_4}{r_6 - r_2} \left(r_6^{k+1-N} - r_2^{k+1-N} \right) + \pi_{N-1,3} r_6^{k+1-N}, & k \geq N, \end{cases} \tag{4.1}$$

and

$$\begin{cases}
\pi_{k0} = \dfrac{\alpha+\eta}{\lambda+\alpha}\pi_{01} + \dfrac{\alpha}{\lambda+\alpha}(\pi_{11}-\pi_{01})\dfrac{1-q_1^{k-1}}{1-q_1} + \dfrac{\eta}{\lambda+\alpha}(\pi_{11}-\pi_{01})\dfrac{1-q_1^{k}}{1-q_1}, & (4.2)\\
\qquad 2 \le k \le N-2, & (4.3)\\
\pi_{k1} = \pi_{01} + (\pi_{11}-\pi_{01})\dfrac{1-q_1^k}{1-q_1}, \qquad 2 \le k \le N-2, & (4.4)\\
\pi_{k2} = \dfrac{\alpha+\mu}{\lambda+\alpha}\pi_{03} + \dfrac{\alpha}{\lambda+\alpha}(\pi_{13}-\pi_{03})\dfrac{1-q_2^{k-1}}{1-q_2} + \dfrac{\mu}{\lambda+\alpha}(\pi_{13}-\pi_{03})\dfrac{1-q_2^{k}}{1-q_2}, & (4.5)\\
\qquad 2 \le k \le N-2, & (4.6)\\
\pi_{k3} = \pi_{03} + (\pi_{13}-\pi_{03})\dfrac{1-q_2^k}{1-q_2}, \qquad 2 \le k \le N-2, & (4.7)\\
\pi_{N-1,0} = \dfrac{\alpha(r_1\alpha-\lambda-\alpha-\eta)-\lambda\eta}{\lambda\eta+(\lambda+\alpha)(r_1\alpha-\lambda-\alpha-\eta)}\pi_{N-2,1}, & (4.8)\\
\pi_{N-1,1} = \dfrac{\lambda+\alpha}{\eta}\pi_{N-1,0} - \dfrac{\alpha}{\eta}\pi_{N-2,1}, & (4.9)\\
\pi_{N-1,2} = r_3\pi_{N-1,1} + \dfrac{\lambda+\alpha}{\alpha}\pi_{N-2,3}, & (4.10)\\
\pi_{N-1,3} = \dfrac{\lambda+\alpha}{\mu}\pi_{N-1,2} - \dfrac{\alpha}{\mu}\pi_{N-2,3}, & (4.11)
\end{cases}$$

where

$$q_1 = \frac{\lambda(\lambda+2\alpha)}{\alpha\eta}, q_2 = \frac{\lambda(\lambda+2\alpha)}{\alpha\mu},$$

and

$$\begin{cases}
\pi_{11} = -K^{-1}\left[\dfrac{\lambda(\lambda+2\alpha+\eta)}{\lambda+\alpha} + \Delta - K\right]\pi_{01}, & (4.12)\\
\pi_{10} = \dfrac{\eta}{\lambda+\alpha}\pi_{11}, & (4.13)\\
\pi_{00} = \dfrac{\lambda+\eta}{\lambda}\pi_{01} - \dfrac{\alpha}{\lambda}\pi_{10}, & (4.14)\\
\pi_{03} = \dfrac{\lambda}{\mu}\pi_{00} - \dfrac{\eta}{\mu}\pi_{01}, & (4.15)\\
\pi_{12} = \dfrac{\lambda+\mu}{\alpha}\pi_{03}, & (4.16)\\
\pi_{13} = \dfrac{\lambda+\alpha}{\mu}\pi_{12}, & (4.17)
\end{cases}$$

where $\Delta = \dfrac{\alpha^2(r_1\alpha-\lambda-\alpha-\eta)-\lambda\alpha\eta}{\lambda\eta+(\lambda+\alpha)(r_1\alpha-\lambda-\alpha-\eta)} - \lambda - \alpha - \eta$, *and* $K = \dfrac{\lambda(\lambda+2\alpha)}{\lambda+\alpha}\dfrac{1-q_1^{N-3}}{1-q_1} + (\Delta + \dfrac{\lambda\eta}{\lambda+\alpha})\dfrac{1-q_1^{N-2}}{1-q_1}$.

Finally, π_{01} *can be determined by the normalization condition.*

Proof. Using the matrix-geometric solution method (see [5]), we have

$$\begin{aligned}\pi_k &= (\pi_{k0}, \pi_{k1}, \pi_{k2}, \pi_{k3}) = \pi_{N-1}R^{k+1-N} \\ &= (\pi_{N-1,0}, \pi_{N-1,1}, \pi_{N-1,2}, \pi_{N-1,3})R^{k+1-N}, \qquad k \geq N.\end{aligned}$$

And for $k \geq N$,

$$R^{k+1-N} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ r_1 r_2^{k-N} & r_2^{k+1-N} & r_3 r_2^{k-N} + \frac{r_4 r_5}{r_6 - r_2}\left(r_6^{k-N} - r_2^{k-N}\right) & \frac{r_4}{r_6 - r_2}\left(r_6^{k+1-N} - r_2^{k+1-N}\right) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & r_5 r_6^{k-N} & r_6^{k+1-N} \end{pmatrix},$$

taking R^{k+1-N} into the above equation, we get (4.1). On the other hand, $\pi_0, \pi_1, \cdots, \pi_{N-1}$ satisfies the next equation

$$(\pi_0, \pi_1, \cdots, \pi_{N-1})B[R] = \mathbf{0},$$

where

$$B[R] = \begin{matrix} 0 \\ 1 \\ \vdots \\ N-2 \\ N-1 \end{matrix} \begin{pmatrix} A_0 & C_0 & & & \\ B & A_1 & C & & \\ & \ddots & \ddots & \ddots & \\ & & B & A_1 & C \\ & & & B & RB + A_1 \end{pmatrix}.$$

From the infinitesimal generator $\widetilde{Q}$, we can obtain $\pi_{N-2}C + \pi_{N-1}A_1 + \pi_N B = \mathbf{0}$. Since $\pi_N = \pi_{N-1}R$, we can get $\pi_{N-2}C + \pi_{N-1}(A_1 + RB) = \mathbf{0}$. Thus, we have $RB + A_1$ in the last line of $B[R]$.

And,

$$RB + A_1 = \begin{pmatrix} -\lambda - \alpha & \lambda & 0 & 0 \\ \eta & r_1\alpha - \lambda - \alpha - \eta & 0 & r_3\alpha \\ 0 & 0 & -\lambda - \alpha & \lambda \\ 0 & 0 & \mu & r_5\alpha - \lambda - \alpha - \mu \end{pmatrix}.$$

Thus, we get

$$\begin{cases} -\lambda\pi_{00} + \eta\pi_{01} + \mu\pi_{03} = 0, & (4.18)\\ \lambda\pi_{00} - (\lambda+\eta)\pi_{01} + \alpha\pi_{10} = 0, & (4.19)\\ -(\lambda+\mu)\pi_{03} + \alpha\pi_{12} = 0, & (4.20)\\ -(\lambda+\alpha)\pi_{10} + \eta\pi_{11} = 0, & (4.21)\\ -(\lambda+\alpha)\pi_{12} + \mu\pi_{13} = 0, & (4.22)\\ \alpha\pi_{k-1,1} - (\lambda+\alpha)\pi_{k0} + \eta\pi_{k1} = 0, \quad 2 \le k \le N-2, & (4.23)\\ \lambda\pi_{k-1,1} + \lambda\pi_{k0} - (\lambda+\alpha+\eta)\pi_{k1} + \alpha\pi_{k+1,0} = 0, \quad 1 \le k \le N-2, & (4.24)\\ \alpha\pi_{k-1,3} - (\lambda+\alpha)\pi_{k2} + \mu\pi_{k3} = 0, \quad 2 \le k \le N-2, & (4.25)\\ \lambda\pi_{k-1,3} + \lambda\pi_{k2} - (\lambda+\alpha+\mu)\pi_{k3} + \alpha\pi_{k+1,2} = 0, \quad 1 \le k \le N-2, & (4.26)\\ \alpha\pi_{N-2,1} - (\lambda+\alpha)\pi_{N-1,0} + \eta\pi_{N-1,1} = 0, & (4.27)\\ \lambda\pi_{N-2,1} + \lambda\pi_{N-1,0} + (r_1\alpha - \lambda - \alpha - \eta)\pi_{N-1,1} = 0, & (4.28)\\ \alpha\pi_{N-2,3} - (\lambda+\alpha)\pi_{N-1,2} + \mu\pi_{N-1,3} = 0, & (4.29)\\ \lambda\pi_{N-2,3} + r_3\alpha\pi_{N-1,1} + \lambda\pi_{N-1,2} + (r_5\alpha - \lambda - \alpha - \mu)\pi_{N-1,3} = 0. & (4.30) \end{cases}$$

From (4.23) and (4.24), we get (4.4) by some computation. Taking (4.4) into (4.23), we get (4.2). In a similar way, we will obtain (4.5) and (4.7) from (4.25) and (4.26). From (4.27) and (4.28), we get $\pi_{N-1,0}$ and $\pi_{N-1,1}$ after some computation. Similarly, taking r_5 into (4.30), together with (4.29), we can derive $\pi_{N-1,2}$ and $\pi_{N-1,3}$. Then, π_{10}, π_{00}, π_{03}, π_{12} and π_{13} can be obtained from the equations (4.18)–(4.22). Next, we explain the equation (4.12). Let k take $N-2$ in equation (4.24), using the expressions of $\pi_{N-3,1}$, $\pi_{N-2,0}$, $\pi_{N-2,1}$ and $\pi_{N-1,0}$, we get (4.12) after some computation. Since $\sum_{j=0}^{3}\sum_{k=0}^{\infty}\pi_{kj} = 1$, we can finally get π_{01}. □

5. Conditional stochastic decomposition

Consider a retrial M/M/1 queue with collisions. This system at any time t can be described by two inter-valued random variables. Let $Q^*(t)$ represent the number of customers in the orbit at time t, and

$$J^*(t) = \begin{cases} 0, & \text{the server is free at time } t,\\ 1, & \text{the server is busy at time } t, \end{cases}$$

then $\{Q^*(t), J^*(t)\}$ is a Markov process with state space $\{(k,j), k \ge 0, j = 0,1\}$. And the infinitesimal generator can be written as

$$\widetilde{Q}^* = \begin{pmatrix} A_0 & C_0 & & & \\ B & A & C & & \\ & B & A & C & \\ & & \ddots & \ddots & \ddots \end{pmatrix},$$

where

$$A_0 = \begin{pmatrix} -\lambda & \lambda \\ \mu & -\lambda-\mu \end{pmatrix}; \quad C_0 = \begin{pmatrix} 0 & 0 \\ 0 & \lambda \end{pmatrix};$$

$$B = \begin{pmatrix} 0 & \alpha \\ 0 & 0 \end{pmatrix}; \quad A = \begin{pmatrix} -\lambda-\alpha & \lambda \\ \mu & -\lambda-\alpha-\mu \end{pmatrix}; \quad C = \begin{pmatrix} 0 & 0 \\ \alpha & \lambda \end{pmatrix}.$$

Following the steps we used before, the QBD process $\{Q^*(t), J^*(t)\}$ is positive recurrent if and only if $(\mu - 2\lambda)\alpha > \lambda^2$. Denote $\widetilde{\pi}_{kj} = P\{Q^* = k, J^* = j\} = \lim_{t\to\infty} P\{Q^*(t) = k, J^*(t) = j\}$. Then, the stationary probability distribution is given by

$$\begin{cases} \widetilde{\pi}_{10} = \dfrac{\lambda^2}{\alpha\mu}\widetilde{\pi}_{00}, \\ \widetilde{\pi}_{k0} = \widetilde{\pi}_{11} r_5 r_6^{k-2}, & k \geq 2, \\ \widetilde{\pi}_{01} = \dfrac{\lambda}{\mu}\widetilde{\pi}_{00}, \\ \widetilde{\pi}_{11} = \dfrac{\lambda^2(\lambda+\alpha)}{\alpha\mu^2}\widetilde{\pi}_{00}, \\ \widetilde{\pi}_{k1} = \widetilde{\pi}_{11} r_6^{k-1}, & k \geq 1, \end{cases}$$

where $\widetilde{\pi}_{00}$ can be determined by the normalization condition.

Let $Q_o = \{Q^* - 1 | Q^* \geq 1, J^* = 1\}$, and Q_o is a conditional queue length given that the server is busy and there are at least one customer in the orbit.

Lemma 5.1. *If $(\mu - 2\lambda)\alpha > \lambda^2$, then Q_o has a probability generating function*

$$G_{Q_o}(z) = \frac{1-r_6}{1-r_6 z}.$$

Proof.

$$G_{Q_o}(z) = \sum_{k=0}^{\infty} z^k P\{Q_o = k\} = \sum_{k=0}^{\infty} z^k \frac{P\{Q^* = k+1, J^* = 1\}}{P\{Q^* \geq 1, J^* = 1\}}$$

$$= \sum_{k=0}^{\infty} \frac{\widetilde{\pi}_{11} r_6^k z^k}{\sum_{k=1}^{\infty} \widetilde{\pi}_{11} r_6^{k-1}} = \frac{1-r_6}{1-r_6 z}. \qquad \square$$

We introduce a random variable $Q^N = \{Q - N | Q \geq N, J = 1or3\}$, and Q^N is a conditional queue length given that the server is busy and there are at least N customers in the orbit. Let P_b^* be the probability that the server is busy and there are at least N customers in the orbit. Clearly,

$$\begin{aligned}
P_b^* &= P\{Q \geq N, J = 1or3\} = \sum_{k=N}^{\infty} \pi_{k1} + \sum_{k=N}^{\infty} \pi_{k3} \\
&= \sum_{k=N}^{\infty} \pi_{N-1,1} r_2^{k+1-N} + \sum_{k=N}^{\infty} \pi_{N-1,1} \frac{r_4}{r_6 - r_2} (r_6^{k+1-N} - r_2^{k+1-N}) \\
&\quad + \sum_{k=N}^{\infty} \pi_{N-1,3} r_6^{k+1-N} \\
&= \frac{r_4 + r_2(1 - r_6)}{(1 - r_2)(1 - r_6)} \pi_{N-1,1} + \frac{r_6}{1 - r_6} \pi_{N-1,3}.
\end{aligned}$$

Theorem 5.2. *If $(\mu - 2\lambda)\alpha > \lambda^2$, the conditional queue length Q^N can be decomposed into the sum of two independent random variables: $Q^N = Q_o + Q_c$, where Q_o follows a geometric distribution with parameter $1 - r_6$. Additional queue length Q_c has a distribution*

$$\begin{aligned}
P\{Q_c = 0\} &= \frac{1}{P_b^*} \frac{(r_2 + r_4)\pi_{N-1,1} + r_6 \pi_{N-1,3}}{1 - r_6}, \\
P\{Q_c = k\} &= \frac{\pi_{N-1,1}}{P_b^*} \frac{r_2(r_2 + r_4 - r_6)}{1 - r_6} r_2^{k-1}, \quad k \geq 1.
\end{aligned}$$

Proof. The probability generating function of Q^N is as follows:

$$\begin{aligned}
G_{Q^N}(z) &= \sum_{k=0}^{\infty} z^k P\{Q^N = k\} = \frac{1}{P_b^*} \left(\sum_{k=0}^{\infty} z^k \pi_{N+k,1} + \sum_{k=0}^{\infty} z^k \pi_{N+k,3} \right) \\
&= \frac{1}{P_b^*} \left[\pi_{N-1,1} \frac{r_2}{1 - r_2 z} + \pi_{N-1,1} \frac{r_4}{(1 - r_2 z)(1 - r_6 z)} + \pi_{N-1,3} \frac{r_6}{1 - r_6 z} \right] \\
&= \frac{1}{P_b^*} \frac{1 - r_6}{1 - r_6 z} \left[\pi_{N-1,1} \frac{r_2(1 - r_6 z)}{(1 - r_6)(1 - r_2 z)} + \pi_{N-1,1} \frac{r_4}{(1 - r_6)(1 - r_2 z)} \right. \\
&\quad \left. + \pi_{N-1,3} \frac{r_6}{1 - r_6} \right] \\
&= \frac{1}{P_b^*} \frac{1 - r_6}{1 - r_6 z} \left[\frac{(r_2 + r_4)\pi_{N-1,1} + r_6 \pi_{N-1,3}}{1 - r_6} + \pi_{N-1,1} \frac{r_2(r_2 + r_4 - r_6) z}{(1 - r_6)(1 - r_2 z)} \right] \\
&= \frac{1 - r_6}{1 - r_6 z} \left[\frac{1}{P_b^*} \frac{(r_2 + r_4)\pi_{N-1,1} + r_6 \pi_{N-1,3}}{1 - r_6} + \pi_{N-1,1} \frac{1}{P_b^*} \frac{r_2(r_2 + r_4 - r_6) z}{(1 - r_6)(1 - r_2 z)} \right] \\
&= G_{Q_o}(z) G_{Q_c}(z).
\end{aligned}$$

□

6. Performance measures

From Theorem 4.1, the probability that the server is busy is

$$\begin{aligned} P_b = \sum_{k=0}^{\infty}\pi_{k1} + \sum_{k=0}^{\infty}\pi_{k3} &= (N-1)\left(\frac{\pi_{11}}{1-q_1} - \frac{q_1\pi_{01}}{1-q_1}\right) - \frac{\pi_{11}-\pi_{01}}{(1-q_1)^2}(1-q_1^{N-1}) \\ &+ (N-1)\left(\frac{\pi_{13}}{1-q_2} - \frac{q_2\pi_{03}}{1-q_2}\right) - \frac{\pi_{13}-\pi_{03}}{(1-q_2)^2}(1-q_2^{N-1}) \\ &+ \frac{1-r_6+r_4}{(1-r_2)(1-r_6)}\pi_{N-1,1} + \frac{1}{1-r_6}\pi_{N-1,3}, \end{aligned}$$

and the probability that the server is free is

$$P_f = \sum_{k=0}^{\infty}\pi_{k0} + \sum_{k=1}^{\infty}\pi_{k2} = 1 - P_b.$$

Let L be the number of customers in the orbit, then from Theorem 4.1,

$$\begin{aligned} E[L] &= \sum_{k=1}^{\infty} k(\pi_{k0}+\pi_{k1}+\pi_{k2}+\pi_{k3}) \\ &= \sum_{k=1}^{N-1} k(\pi_{k0}+\pi_{k2}) + \sum_{k=1}^{N-2} k(\pi_{k1}+\pi_{k3}) \\ &+ (N-1)\pi_{N-1,1}\frac{(1+r_1+r_3)(1-r_6)+r_4(1+r_5)}{(1-r_2)(1-r_6)} \\ &+ (N-1)\pi_{N-1,3}\frac{1+r_5}{1-r_6} + \pi_{N-1,3}\frac{r_5+r_6}{(1-r_6)^2} \\ &+ \pi_{N-1,1}\frac{(r_1+r_2+r_3)(1-r_6)^2 + r_4r_5(2-r_2-r_6) + r_4(1-r_2r_6)}{(1-r_2)^2(1-r_6)^2}. \end{aligned}$$

Let L_s be the number of customers in the system, we have

$$E[L_s] = \sum_{k=1}^{\infty} k(\pi_{k0}+\pi_{k2}) + \sum_{k=0}^{\infty}(k+1)(\pi_{k1}+\pi_{k3}) = E[L] + P_b.$$

Let W be the waiting time of a customer in the orbit, using Little's formula, $E[W] = E[L]/\lambda$. And, the expected sojourn time of a customer in the system $E[W_s] = E[L_s]/\lambda$.

The steady-state interrupted frequency IF of the service due to collisions is given by

$$IF = \sum_{k=1}^{\infty}\alpha(\pi_{k1}+\pi_{k3}) = \alpha(P_b - \pi_{01} - \pi_{03}).$$

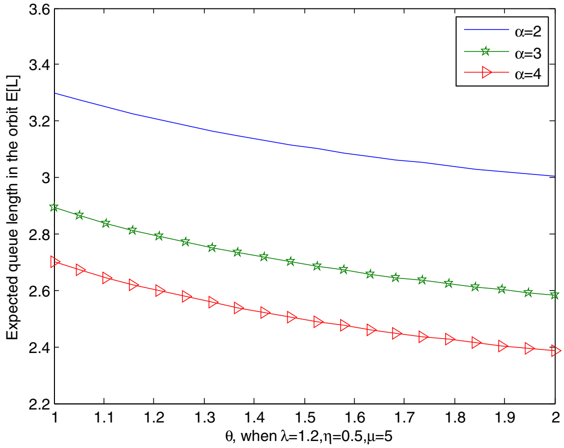

FIGURE 2. The expected queue length in the orbit with the change of θ.

Let T be the system busy period that starts at an epoch when an arriving customer finds an empty system and ends at the next departure epoch at which the system is empty. Using the theory of regenerative processes,

$$\pi_{00} = \frac{E[T_{00}]}{1/\lambda + E[T]},$$

where $E[T_{00}]$ is the amount of time in a regenerative cycle during which the system is in the state (0,0). Clearly, $E[T_{00}] = 1/\lambda$. Thus, $E[T] = \lambda^{-1}(\pi_{00}^{-1} - 1)$.

7. NUMERICAL RESULTS

In this section, taking N=3, λ=1.2 and μ=5, we present some numerical examples to illustrate the effect of the vacation rate θ, service rate η and retrial rate α.

Figure 2 illustrates the expected queue length in the orbit $E[L]$ with the change of θ at different retrial rate α. Let η=0.5<μ=5, we can find that $E[L]$ decreases with the rate θ increasing. And, it's easy to see that, if the other conditions are same, the larger retrial rate α is, the smaller $E[L]$ becomes. In Figure 3, with the change of vacation rate θ, the curves of $E[L_s]$ (the expected queue length in the system) and $E[W_s]$ (the expected sojourn time) are provided. $E[L_s]$ and $E[W_s]$ both decrease with an increasing value of θ.

Figure 4 shows the influence of service rate η on the expected queue length in the orbit $E[L]$. Taking θ=1, it's obvious that $E[L]$ decreases evidently with the rate η increasing. When η=0, there is no service during the vacation period, so the vacation interruption cannot happen. When service rate η approaches to μ=5, the model we considered will become a retrial M/M/1 queue with collisions. Under

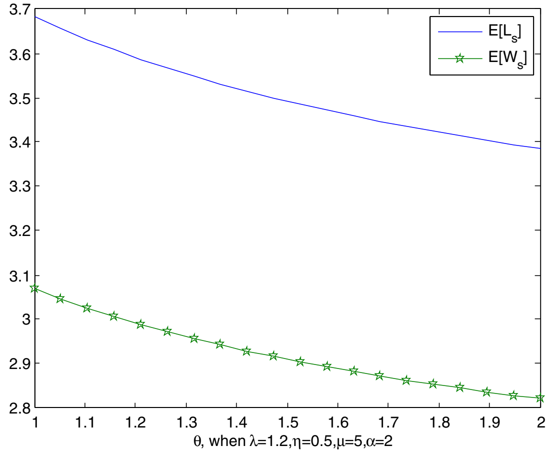

FIGURE 3. $E[L_s]$ and $E[W_s]$ with the change of θ.

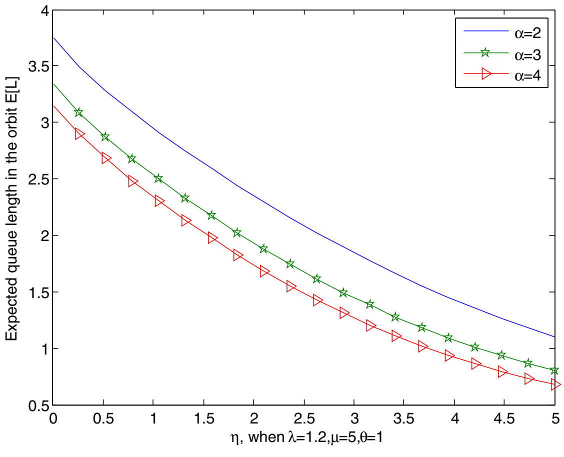

FIGURE 4. The expected queue length in the orbit with the change of η.

the vacation policy, if we want to develop a better service, we can consider working vacation policy that utilizes the server and decreases the waiting jobs effectively. In Figure 5, P_f (the probability that the server is free) and IF (the steady-state interrupted frequency) are plotted versus service rate η. We find that P_f increases as η increases while IF decreases as η increases.

From Theorem 3.1, we vary the retrial rate α from 2 to 4. Figure 6 shows the effect of retrial rate α on the expected queue length in the orbit $E[L]$. It's easy to see $E[L]$ decreases with the rate α increasing, this is due to the fact that the inter-retrial time becomes shorter. And, the larger service rate η is, the smaller $E[L]$

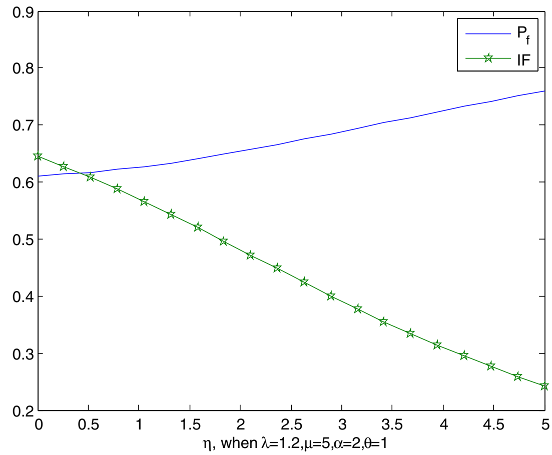

FIGURE 5. P_f and IF with the change of η.

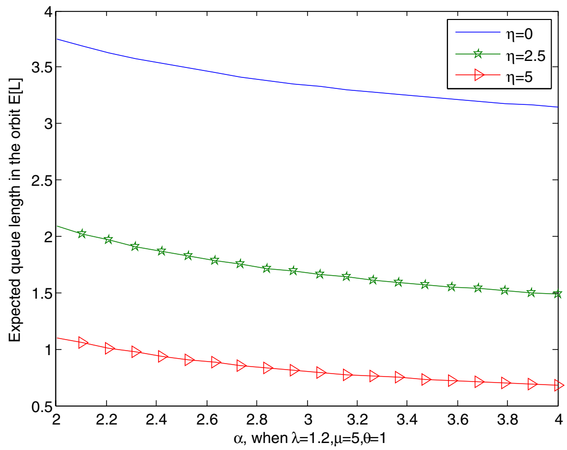

FIGURE 6. The expected queue length in the orbit with the change of α.

becomes. In Figure 7, the effect of retrial rate on $E[T]$ (the system busy period) and IF are presented. We can see that $E[T]$ decreases evidently with an increasing value of α. And as expected, IF increases with the retrial rate α increasing.

8. Conclusion

In this paper, an M/M/1 retrial queue with collisions and working vacation interruption under N-policy is analyzed. Using the matrix-analytic method, we obtain the stationary probability distribution. We also derive the conditional

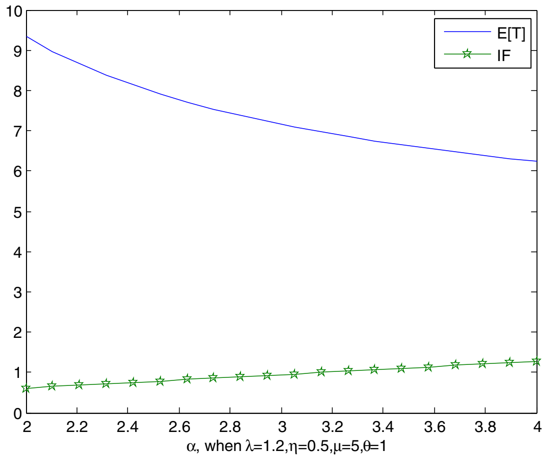

FIGURE 7. $E[T]$ and IF with the change of α.

stochastic decomposition and some performance measures. Under the stability condition, we perform some numerical examples to study the effect of various parameters on the system's characteristics. And, the model we considered without vacation interruption can be discussed in a similar way.

APPENDIX. A

If the matrix $A = A_0 + A_1 + A_2$ is reducible, and A is written, possibly after a permutation of its rows and columns, as

$$A = \begin{pmatrix} C^{(1)} & 0 & \cdots & 0 & 0 \\ 0 & C^{(2)} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & C^{(K)} & 0 \\ D^{(1)} & D^{(2)} & \cdots & D^{(K)} & D^{(0)} \end{pmatrix}, \tag{A.1}$$

where the blocks $C^{(k)}$, $1 \leq k \leq K$ are irreducible and satisfy $C^{(k)}e = \mathbf{0}$. The matrices A_0, A_1 and A_2 are similarly structured, and we have that

$$A_i = \begin{pmatrix} C_i^{(1)} & 0 & \cdots & 0 & 0 \\ 0 & C_i^{(2)} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & C_i^{(K)} & 0 \\ D_i^{(1)} & D_i^{(2)} & \cdots & D_i^{(K)} & D_i^{(0)} \end{pmatrix}.$$

The continuous time case of Theorem 7.3.1 in [5]: Assume that the matrix $A = A_0 + A_1 + A_2$ is partitioned as in (A.1), where $K \geq 1$ and the matrices $C^{(k)}$,

$1 \le k \le K$ are irreducible. The continuous time QBD is positive recurrent if and only if $\gamma^{(k)} C_2^{(k)} e > \gamma^{(k)} C_0^{(k)} e$ for all k: $1 \le k \le K$, where $\gamma^{(k)}$ is the unique solution of the system $\gamma^{(k)} C^{(k)} = \mathbf{0}$, $\gamma^{(k)} e = 1$.

Remark A.1. The matrices A_0, A_1, A_2 and A represent matrices C, A, B and D in our model, respectively.

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