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$M(r,s)$-IDEALS OF COMPACT OPERATORS

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Abstract. We study the position of compact operators in the space of all continuous linear operators and its subspaces in terms of ideals. One of our main results states that for Banach spaces X and Y the subspace of all compact operators $\mathscr{K}(X,Y)$ is an $M(r_1r_2,s_1s_2)$-ideal in the space of all continuous linear operators $\mathscr{L}(X,Y)$ whenever $\mathscr{K}(X,X)$ and $\mathscr{K}(Y,Y)$ are $M(r_1,s_1)$- and $M(r_2,s_2)$-ideals in $\mathscr{L}(X,X)$ and $\mathscr{L}(Y,Y)$, respectively, with $r_1+s_1/2>1$ and $r_2+s_2/2>1$. We also prove that the $M(r,s)$-ideal $\mathscr{K}(X,Y)$ in $\mathscr{L}(X,Y)$ is separably determined. Among others, our results complete and improve some well-known results on M-ideals.

Keywords: $M(r,s)$-ideal and M-ideal of compact operators, property $M^*(r,s)$, compact approximation property

MSC 2010: 46B20, 46B04, 46B28, 47L05

1. Introduction

Let X and Y be Banach spaces (over $\mathbb{K}=\mathbb{R}$ or $\mathbb{C}$). We denote by $\mathscr{L}(X,Y)$ the Banach space of all continuous linear operators from X to Y and by $\mathscr{K}(X,Y)$ its subspace of compact operators. Instead of $\mathscr{K}(X,X)$ and $\mathscr{L}(X,X)$ we write $\mathscr{K}(X)$ and $\mathscr{L}(X)$, respectively.

In this paper we study the position of $\mathscr{K}(X,Y)$ inside $\mathscr{L}(X,Y)$ and its subspaces in terms of ideals. Recall that a closed subspace $\mathscr{K}$ of a Banach space $\mathscr{L}$ is said to be an *ideal* in $\mathscr{L}$ if there exists a norm one projection P on $\mathscr{L}^*$ with $\ker P=\mathscr{K}^\perp$, the annihilator of $\mathscr{K}$. We shall say that P is an *ideal projection.* If moreover, there are $r,s\in(0,1]$ so that $\|f\|\geqslant r\|Pf\|+s\|f-Pf\|$ for all $f\in\mathscr{L}^*$, then $\mathscr{K}$ is an $M(r,s)$-*ideal* in $\mathscr{L}$. (In [5] and subsequent works such a $\mathscr{K}$ was called an ideal satisfying

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the $M(r,s)$-inequality in $\mathscr{L}$.) Occasionally, following [10], we shall consider $M(r,s)$-ideals also for positive numbers $r \leqslant 1$ and s. It is clear that $\mathscr{K} = \mathscr{L}$ if and only if $\mathscr{K}$ is an $M(r,s)$-ideal for some $s > 1$ (or for all $(r,s) \in (0,1] \times (0,\infty)$).

Well-studied M-ideals (see the monograph [11] and, e.g., [14], [20], [26], [33], [35], for results and references) are precisely $M(1,1)$-ideals. Examples of $M(r,s)$-ideals which are not M-ideals can be found, e.g., in [3] and [5]. For instance, it is a well-known result of Hennefeld [13] (see, e.g., [11, p. 305]) that for the Lorentz sequence space $d(v,p)$, $p > 1$, the space of compact operators $\mathscr{K}(d(v,p))$ is not an M-ideal in $\mathscr{L}(d(v,p))$. But $\mathscr{K}(d(v,p))$ is an $M(r,s)$-ideal in $\mathscr{L}(d(v,p))$ if $r^p + s^p \leqslant 1$ (see [5, Example 4.2]).

The starting point of our investigations was the following result which allowed to produce, departing from Banach spaces X such that $\mathscr{K}(X)$ is an M-ideal in $\mathscr{L}(X)$, new classes of M-ideals of compact operators.

Theorem 1.1 (see [31]). *Let X and Y be Banach spaces. If $\mathscr{K}(X)$ and $\mathscr{K}(Y)$ are M-ideals in $\mathscr{L}(X)$ and $\mathscr{L}(Y)$, then $\mathscr{K}(X,Y)$ is an M-ideal in $\mathscr{L}(X,Y)$.*

The extension of Theorem 1.1 from M-ideals to $M(r,s)$-ideals presents difficulties since the main techniques from the theory of M-ideals involving the 3-ball property do not work in this more general case (in [11, p. 301], e.g., Theorem 1.1 is proven using the 3-ball property). In [9], extending and developing methods from [18] and [31], and relying on results of [35], we extended Theorem 1.1 as follows.

Theorem 1.2 (see [9]). *Let X and Y be Banach spaces. Let $r_1, s_1, r_2, s_2 \in (0,1]$ satisfy $r_1 + s_1/2 > 1$ and $r_2 + s_2/2 > 1$. If $\mathscr{K}(X)$ is an $M(r_1,s_1)$-ideal in $\mathscr{L}(X)$ and $\mathscr{K}(Y)$ is an $M(r_2,s_2)$-ideal in $\mathscr{L}(Y)$, then $\mathscr{K}(X,Y)$ is an $M(r_1^2 r_2, s_1^2 s_2)$- and an $M(r_1 r_2^2, s_1 s_2^2)$-ideal in $\mathscr{L}(X,Y)$.*

The parameters $r_1^2 r_2$ and $s_1^2 s_2$, or $r_1 r_2^2$ and $s_1 s_2^2$ seem to be not optimal. In this paper, we propose a different approach which improves the parameters to $r_1 r_2$ and $s_1 s_2$ (see Theorem 3.11 for the case when X or Y is separable and Theorem 4.14 for the general non-separable case).

The key concepts of our approach are "the ideal projection preserving elementary functionals" (introduced in Section 2) and "property $M^*(r,s)$ for operators" (see Section 3). Sections 2 and 3 contain necessary auxiliary results on these concepts which lead, relying on a vector-valued version of Simons's inequality, to the main results in the case when one of the spaces X or Y is separable (see Theorems 3.6 and 3.11). Section 3 also provides corollaries of Theorem 3.6 which complete and improve some well-known results on M-ideals.

In Section 4, we prove that $M(r,s)$-ideals of compact operators $\mathscr{K}(X,Y)$ are separably determined for distinct spaces X and Y (see Theorem 4.1; the result seems to be new even for M-ideals). This fact together with Theorem 3.6 leads to the main result of the present paper (Theorem 4.7) asserting that $M^*(r_1, s_1)$-property of X and $M^*(r_2, s_2)$-property of Y imply that $\mathscr{K}(X, Y)$ is an $M(r_1r_2, s_1s_2)$-ideal in $\mathscr{L}(X, Y)$, and to Theorem 4.14 improving Theorem 1.2.

One important tool, we are basing on, is the Feder-Saphar description [7] of the dual space of $\mathscr{K}(X, Y)$ which holds whenever X^{**} or Y^* has the Radon-Nikodým property. The reader may notice that this hypothesis is often present also implicitly (as can be seen from Proposition 3.1).

Let us fix some more notation. The identity operator, the closed unit ball, and the unit sphere of a Banach space X are denoted by I_X, B_X, and S_X, respectively. For a set $A \subset X$, its norm closure is denoted by $\overline{A}$, its linear span by $\operatorname{span} A$, and its convex hull by $\operatorname{conv} A$. Let $\mathscr{L}$ be a subspace of $\mathscr{L}(X, Y)$, where X and Y are Banach spaces, and let $x^{**} \in X^{**}$ and $y^* \in Y^*$. Then the functional $x^{**} \otimes y^* \in \mathscr{L}^*$ is defined by $(x^{**} \otimes y^*)(T) = x^{**}(T^*y^*)$, where $T \in \mathscr{L}$. Note that $\|x^{**} \otimes y^*\| = \|x^{**}\|\|y^*\|$ whenever $\mathscr{L}$ contains the finite-rank operators. By $A \otimes B$, where $A \subset X^{**}$ and $B \subset Y^*$, we mean the set of all $x^{**} \otimes y^*$ such that $x^{**} \in A$ and $y^* \in B$. Thus $A \otimes B \subset \mathscr{L}(X, Y)^*$.

Recall that a net $(K_\alpha) \subset \mathscr{K}(X)$ is a *compact approximation of the identity* (*CAI*) provided $K_\alpha \longrightarrow I_X$ strongly (that is, $K_\alpha x \longrightarrow x$ for all $x \in X$). If, moreover, $K_\alpha^* \longrightarrow I_{X^*}$ strongly, then (K_α) is called a *shrinking CAI*. If X has a CAI such that the convergence is uniform on compact sets, then X is said to have the *compact approximation property* (*CAP*), and in the case of shrinking CAI, X^* is said to have the *CAP with conjugate operators*. If, moreover, $\|K_\alpha\| \leqslant \lambda$ for some $\lambda \geqslant 1$ and for all α, then (K_α) is called a *bounded CAI* (*BCAI*) and a *shrinking BCAI*, respectively, and X, and X^* are said to have the *BCAP* and the *BCAP with conjugate operators*. In the special case, when $\lambda = 1$, (K_α) is called a *metric CAI* (*MCAI*), and X is said to have the *MCAP*.

2. Ideal projections preserving elementary functionals

Let $\mathscr{K}$ be an ideal in a Banach space $\mathscr{L}$ with respect to an ideal projection P. It is well known and straightforward to verify that for every $f \in \mathscr{L}^*$, $Pf \in \mathscr{L}^*$ is a norm-preserving extension of the restriction $f|_{\mathscr{K}} \in \mathscr{K}^*$. Therefore, $\operatorname{ran} P$ is canonically isometric to $\mathscr{K}^*$ and we shall identify them, whenever convenient, identifying Pf and $f|_{\mathscr{K}}$ for all $f \in \mathscr{L}^*$. More precisely, if one defines $\Phi\colon \mathscr{K}^* \longrightarrow \operatorname{ran} P$ by $\Phi g = Pf$, $g \in \mathscr{K}^*$, where $f \in \mathscr{L}^*$ is any extension of g, then Φ is an isometric isomorphism such that $\Phi(f|_{\mathscr{K}}) = Pf$, $f \in \mathscr{L}^*$.

Let X and Y be Banach spaces. Let $\mathscr{L}$ be a closed subspace of $\mathscr{L}(X, Y)$ containing $\mathscr{K} := \mathscr{K}(X, Y)$. Assume that $\mathscr{K}$ is an ideal in $\mathscr{L}$ with respect to an ideal projection P. If $P(x^{**} \otimes y^*) = x^{**} \otimes y^*$ for all $x^{**} \in X^{**}$ and $y^* \in Y^*$, then we say that P *preserves elementary functionals.*

Suppose that (K_α) is a shrinking MCAI of X (respectively, of Y). Then, using a well-known result of Johnson (see [15, proof of Lemma 1]), by passing to a subnet of (K_α), we may assume that $\mathscr{K}$ is an ideal in $\mathscr{L}$ with respect to the projection P on $\mathscr{L}^*$ defined by

$$Pf(T) = \lim_\alpha f(TK_\alpha), \quad f \in \mathscr{L}^*, \quad T \in \mathscr{L}$$

(respectively,

$$Pf(T) = \lim_\alpha f(K_\alpha T), \quad f \in \mathscr{L}^*, \quad T \in \mathscr{L}).$$

Let us call P the *Johnson projection.* (This is essentially the same concept as in [26] and [39].)

Example 2.1. The Johnson projection is an ideal projection preserving elementary functionals.

By the above, the Johnson projection P is an ideal projection. It preserves elementary functionals. Indeed, consider any $x^{**} \otimes y^* \in \mathscr{L}^*$, and let $T \in \mathscr{L}$. If $K_\alpha^* x^* \longrightarrow x^*$ for all $x^* \in X^*$, then

$$\begin{aligned}(P(x^{**} \otimes y^*))(T) &= \lim_\alpha (x^{**} \otimes y^*)(TK_\alpha) \\ &= \lim_\alpha x^{**}(K_\alpha^* T^* y^*) = x^{**}(T^* y^*) \\ &= (x^{**} \otimes y^*)(T).\end{aligned}$$

If, respectively, $K_\alpha^* y^* \longrightarrow y^*$ for all $y^* \in Y^*$, then

$$\begin{aligned}(P(x^{**} \otimes y^*))(T) &= \lim_\alpha (x^{**} \otimes y^*)(K_\alpha T) \\ &= \lim_\alpha x^{**}(T^* K_\alpha^* y^*) = x^{**}(T^* y^*) \\ &= (x^{**} \otimes y^*)(T).\end{aligned}$$

In contrast with the Johnson projection, an ideal projection preserving elementary functionals may also be defined departing from a (generally) unbounded net of compact operators, as the following example shows.

Example 2.2 (see [23, Theorem 5.1] and [24, proof of Theorem 4.6]). Let X and Y be Banach spaces such that X^{**} or Y^* has the Radon-Nikodým property. Let $\mathscr{L}$ be a closed subspace of $\mathscr{L}(X, Y)$ containing $\mathscr{K} := \mathscr{K}(X, Y)$. If X^* or Y^* has the CAP with conjugate operators, then $\mathscr{K}$ is an ideal in $\mathscr{L}$ with respect to an ideal projection preserving elementary functionals.

Following [24], we say that a closed subspace $\mathscr{K}$ of a Banach space $\mathscr{L}$ has the *unique ideal property* if there is at most one ideal projection (that is, at most one norm one projection P on $\mathscr{L}^*$ with $\ker P = \mathscr{K}^{\perp}$). An obvious example of subspaces having the unique ideal property is presented by subspaces having Phelps's property U: $\mathscr{K}$ is said to have *property* U in $\mathscr{L}$, if every $g \in \mathscr{K}^*$ has a unique norm-preserving extension $f \in \mathscr{L}^*$. Ideals with property U have been studied, e.g., in [13], [27], [28], [29], [32], [37], [39].

It is well known that M-ideals, and more generally, $M(1,s)$-ideals (see [2]), have property U and therefore they also have the unique ideal property. However, e.g., for $r \neq 1$, $M(r,1)$-ideals of compact operators $\mathscr{K}(X)$ need not have property U in $\mathscr{L}(X)$ even if X^* has the Radon-Nikodým property [5, Example 4.5].

In the sequel, we shall need the fact that in many important cases, the subspace of compact operators enjoys the unique ideal property with respect to ideal projections preserving elementary functionals.

Proposition 2.3. *Let X and Y be Banach spaces. Let $\mathscr{L}$ be a closed subspace of $\mathscr{L}(X,Y)$ containing $\mathscr{K} := \mathscr{K}(X,Y)$. If X^{**} or Y^* has the Radon-Nikodým property, then for $\mathscr{K}$ in $\mathscr{L}$ there is at most one ideal projection preserving elementary functionals.*

P r o o f. Let Q and P be ideal projections on $\mathscr{L}^*$ preserving elementary functionals with $\ker Q = \ker P = \mathscr{K}^{\perp}$. Let $\Phi\colon \mathscr{K}^* \longrightarrow \operatorname{ran} Q$ and $\Psi\colon \mathscr{K}^* \longrightarrow \operatorname{ran} P$ be the corresponding isometric isomorphisms such that $Pf = \Phi(f|_{\mathscr{K}})$ and $Qf = \Psi(f|_{\mathscr{K}})$, where $f \in \mathscr{L}^*$. Therefore we need to prove that

$$\Phi g = \Psi g \quad \forall\, g \in \mathscr{K}^*.$$

The desired equality is immediate from the fact that

$$\mathscr{K}^* = \overline{\operatorname{span}}\{(x^{**} \otimes y^*)|_{\mathscr{K}}\colon\ x^{**} \in X^{**},\ y^* \in Y^*,\ x^{**} \otimes y^* \in \mathscr{L}^*\}$$

(implied by [7, Theorem 1] since X^{**} or Y^* has the Radon-Nikodým property) and the equality

$$\Phi((x^{**} \otimes y^*)|_{\mathscr{K}}) = P(x^{**} \otimes y^*) = x^{**} \otimes y^* = Q(x^{**} \otimes y^*) = \Psi((x^{**} \otimes y^*)|_{\mathscr{K}}),$$

which holds for all $x^{**} \in X^{**}$ and $y^* \in Y^*$. □

Recall that the *dual weak operator topology* on $\mathscr{L}(X,Y)$ is defined by the functionals $A \longmapsto x^{**}(A^*y^*)$, $y^* \in Y^*$, $x^{**} \in X^{**}$. Clearly, the dual weak operator topology is not weaker than the *weak operator topology* which is defined by the functionals $A \longmapsto y^*(Ax)$, $y^* \in Y^*$, $x \in X$.

Proposition 2.4. *Let X and Y be Banach spaces, and suppose that X^{**} or Y^* has the Radon-Nikodým property. Let $\mathscr{L}$ be a closed subspace of $\mathscr{L}(X,Y)$ containing $\mathscr{K} := \mathscr{K}(X,Y)$, and suppose that $\mathscr{K}$ is an ideal in $\mathscr{L}$ with respect to an ideal projection P preserving elementary functionals. If for an operator $T \in \mathscr{L}$ there exists a bounded net $(T_\alpha) \subset \mathscr{L}$ such that $T_\alpha \longrightarrow T$ in the dual weak operator topology, then*

$$(Pf)(T_\alpha) \underset{\alpha}{\longrightarrow} (Pf)(T) \quad \forall\, f \in \mathscr{L}^*.$$

P r o o f. As in the previous proof, we shall apply the fact that $\overline{\operatorname{span}}\{(x^{**}\otimes y^*)|_{\mathscr{K}} \colon x^{**} \in X^{**},\ y^* \in Y^*,\ x^{**} \otimes y^* \in \mathscr{L}^*\}$ is dense in $\mathscr{K}^*$. Using the associated isomorphism $\Phi\colon\ \mathscr{K}^* \longrightarrow \operatorname{ran} P$ satisfying $\Phi(f|_{\mathscr{K}}) = Pf$, $f \in \mathscr{L}^*$, and that P preserves the elementary functionals, we get that $\operatorname{span}\{x^{**} \otimes y^* \colon\ x^{**} \in X^{**},\ y^* \in Y^*\} \subset \operatorname{ran} P$ is dense in $\operatorname{ran} P \subset \mathscr{L}^*$.

Every $A \in \mathscr{L}$ can be viewed as an element of $(\operatorname{ran} P)^*$ with the same norm, defining

$$\langle A, h\rangle = h(A), \quad h \in \operatorname{ran} P.$$

Since the net (T_α) is bounded and for all $x^{**} \in X^{**}$, $y^* \in Y^*$,

$$\begin{aligned}\langle T_\alpha, x^{**} \otimes y^*\rangle &= (x^{**} \otimes y^*)(T_\alpha)\\ &= x^{**}(T_\alpha^* y^*) \underset{\alpha}{\longrightarrow} x^{**}(T^* y^*)\\ &= \langle T, x^{**} \otimes y^*\rangle,\end{aligned}$$

we have $\langle T_\alpha, h\rangle \underset{\alpha}{\longrightarrow} \langle T, h\rangle$ for all $h \in \operatorname{ran} P$. This means that $(Pf)(T_\alpha) \underset{\alpha}{\longrightarrow} (Pf)(T)$ for all $f \in \mathscr{L}^*$. □

Remark 2.5. Proposition 2.4 extends Lemma 1.2 of [39] from the case of the Johnson projection (involving the shrinking MCAI assumptions) to an arbitrary ideal projection preserving elementary functionals.

We shall apply Proposition 2.4 to deduce the following criteria for $M(r,s)$-ideals of compact operators which will be needed in Sections 3 and 4.

Theorem 2.6. *Let X and Y be Banach spaces, and suppose that X^{**} or Y^* has the Radon-Nikodým property. Let $\mathscr{L}$ be a closed subspace of $\mathscr{L}(X,Y)$ containing $\mathscr{K} := \mathscr{K}(X,Y)$, and suppose that $\mathscr{K}$ is an ideal in $\mathscr{L}$ with respect to an ideal projection P preserving elementary functionals. Let $r \leqslant 1$ and s be positive numbers. If for every operator $T \in S_{\mathscr{L}}$ there exists a bounded net $(T_\alpha) \subset \mathscr{K}$ such that*

$T_\alpha \longrightarrow T$ in the dual weak operator topology, then the following assertions are equivalent.

(a) *$\mathscr{K}$ is an $M(r,s)$-ideal in $\mathscr{L}$ with respect to P.*

(b) *For every $\varepsilon > 0$, $S \in B_{\mathscr{K}}$, $T \in B_{\mathscr{L}}$, and every index α (in the corresponding net (T_α)) there exists*
$$K \in \operatorname{conv}\{T_\beta \colon \beta \geqslant \alpha\}$$
such that
$$\|rS + s(T-K)\| \leqslant 1+\varepsilon.$$

(c) *For every $S \in S_{\mathscr{K}}$ and $T \in S_{\mathscr{L}}$ there exists a net $(K_\nu) \subset \mathscr{K}$ such that $K_\nu \longrightarrow T$ in the dual weak operator topology and*
$$\limsup_\nu \|rS + s(T-K_\nu)\| \leqslant 1.$$

Proof. (a) $\Rightarrow$ (b). This implication follows from a general $M(r,s)$-inequality criterion (see [10, Proposition, (a) $\Rightarrow$ (b)]. We include a proof for the sake of completeness. If the conclusion is false, then there are $\varepsilon > 0$, $S \in B_{\mathscr{K}}$, $T \in B_{\mathscr{L}}$, and α such that for $C := \operatorname{conv}\{T_\beta \colon \beta \geqslant \alpha\}$, we have
$$sC \cap B(rS + sT, 1+\varepsilon) = \emptyset,$$
where $B(rS+sT, 1+\varepsilon)$ is the open ball with center $rS+sT$ and radius $1+\varepsilon$. By the Hahn-Banach theorem, there exists $f \in S_{\mathscr{L}^*}$ such that
$$\begin{aligned}\operatorname{Re} f(rS+sT) - (1+\varepsilon) = {} & \inf\{\operatorname{Re} f(U) \colon U \in B(rS+sT, 1+\varepsilon)\} \\ & \geqslant s \operatorname{Re} f(K) = s \operatorname{Re} Pf(K) \quad \forall\, K \in C,\end{aligned}$$
because $C \subset \mathscr{K}$ and $f - Pf \in \ker P = \mathscr{K}^\perp$. Hence,
$$\begin{aligned}1+\varepsilon \leqslant {} & \operatorname{Re} f(rS+sT) - s\operatorname{Re} Pf(K) \\ & = r\operatorname{Re} Pf(S) + s\operatorname{Re}(f-Pf)(T) + s\operatorname{Re} Pf(T-K) \\ & \leqslant 1 + s\operatorname{Re} Pf(T-K) \quad \forall\, K \in C.\end{aligned}$$
Since $Pf(T) = \lim\limits_\alpha Pf(T_\alpha)$ (see Proposition 2.4), this implies that $\varepsilon \leqslant 0$, a contradiction.

(b) $\Rightarrow$ (c). Consider the set of all pairs $\nu = (\varepsilon, \alpha)$, where $\varepsilon > 0$ and where (T_α) corresponds to T, directed in the natural way, and choose $K_\nu \in \operatorname{conv}\{T_\beta \colon \beta \geqslant \alpha\}$ from condition (b).

(c) ⇒ (a). Let us fix $f \in \mathscr{L}^*$ and $\varepsilon > 0$. Recalling that $\|Pf\| = \|f|_{\mathscr{K}}\|$, we choose $S \in S_{\mathscr{K}}$ and $T \in S_{\mathscr{L}}$ so that

$$r\|Pf\| + s\|f - Pf\| - \varepsilon \leqslant rf(S) + s(f - Pf)(T).$$

Let (K_ν) be given by (c). By passing to a subnet, we may assume that (K_ν) is bounded. By Proposition 2.4, $(Pf)(T) = \lim_\nu (Pf)(K_\nu) = \lim_\nu f(K_\nu)$, because $K_\nu \in \mathscr{K}$ and $Pf - f \in \ker P = \mathscr{K}^\perp$. It follows that

$$\begin{aligned} r\|Pf\| + s\|f - Pf\| - \varepsilon &\leqslant rf(S) + sf(T) - s\lim_\nu f(K_\nu) \\ &= \lim_\nu f(rS + s(T - K_\nu)) \\ &\leqslant \|f\| \limsup_\nu \|rS + s(T - K_\nu)\| \leqslant \|f\|. \end{aligned}$$

□

Remark 2.7. Historically, for M-ideals, conditions similar to (b) and (c) of Theorem 2.6 were first considered in [25, Proposition 2.8], [40, Theorem 3.1 and Remark], and [33, proof of Theorem 2].

3. Properties $M(r,s)$ and $M^*(r,s)$ for spaces and operators; main results involving separability assumptions

Let $r, s \in (0,1]$. According to [4], we say that a Banach space X has *property $M(r,s)$* if

$$\limsup_\nu \|ru + sx_\nu\| \leqslant \limsup_\nu \|v + x_\nu\|$$

whenever $u, v \in X$ satisfy $\|u\| \leqslant \|v\|$ and $(x_\nu) \subset X$ is a bounded net converging weakly to null in X. We say that X has *property $M^*(r,s)$* if

$$\limsup_\nu \|ru^* + sx_\nu^*\| \leqslant \limsup_\nu \|v^* + x_\nu^*\|$$

whenever $u^*, v^* \in X^*$ satisfy $\|u^*\| \leqslant \|v^*\|$ and $(x_\nu^*) \subset X^*$ is a bounded net converging weak* to null in X^*.

Properties $M(1,1)$ and $M^*(1,1)$ clearly coincide with their prototypical properties (M) and (M^*), introduced by Kalton in [19] (see also [18]) (where the sequential version was used; see [31] for the general version). A much more general version of property (M^*), namely property $M^*(a,B,c)$, was introduced and studied in [35] (see also [34]). It can be easily seen that property $M^*(s,\{-s\},r)$ is precisely property $M^*(r,s)$.

Analogously to [19, Proposition 2.3] (see also [31, Proposition 2] or [11, Proposition VI.4.15] and [35, Proposition 1.3]), one can prove that property $M^*(r,s)$ implies property $M(r,s)$ and, moreover, it implies that X is an $M(r,s)$-ideal in X^{**} with respect to the canonical ideal projection on X^{***}. In the latter case, one says (following [2] or [10]) that X *satisfies the* $M(r,s)$*-inequality.*

Proposition 3.1 (see [5, Proposition 2.1] and [35, proof of Corollary 1.7])**.** *Let* $r,s \in (0,1]$*. If a Banach space* X *satisfies the* $M(r,s)$*-inequality (in particular, if* X *has property* $M^*(r,s)$*) for* $r+s>1$*, then* X^* *has the Radon-Nikodým property and every MCAI of* X *is shrinking.*

In [20, Section 6], an operator version of property (M) was introduced and studied (see also [14] and [16] for applications of this property). We need to extend its (M^*) prototype as follows.

Let X and Y be Banach spaces and let $r,s \in (0,1]$. We say that an operator $T \in B_{\mathscr{L}(X,Y)}$ has property $M^*(r,s)$ if

$$\limsup_{\nu} \|rx^* + sT^*y^*_\nu\| \leqslant \limsup_{\nu} \|y^* + y^*_\nu\|$$

whenever $x^* \in X^*$, $y^* \in Y^*$ satisfy $\|x^*\| \leqslant \|y^*\|$ and $(y^*_\nu) \subset Y^*$ is a bounded net converging weak* to null in Y^*.

If Y is separable, then $T \in B_{\mathscr{L}(X,Y)}$ has property $M^*(r,s)$ if and only if T has the *sequential version of property* $M^*(r,s)$ (i.e., the nets (y^*_ν) being replaced with the weak* null sequences (y^*_n)). This can be easily checked using the fact that the bounded subsets of Y^* are weak* metrizable.

Clearly, an operator T has property (M^*) if and only if T has $M^*(1,1)$, and a Banach space X has property $M^*(r,s)$ if and only if its identity operator I_X has $M^*(r,s)$. A much more general notion, namely an operator having property $M^*(a,B,c)$, was introduced and studied in [35] (see also [34]). As in the case of spaces, property $M^*(r,s)$ for operators is precisely property $M^*(s,\{-s\},r)$.

As in the (M^*) case (see [31, Lemma 4]), properties $M^*(r,s)$ for spaces and operators are related as follows.

Proposition 3.2. *Let* X *and* Y *be Banach spaces and let* $r_1,s_1,r_2,s_2 \in (0,1]$*. If* X *has property* $M^*(r_1,s_1)$ *and* Y *has property* $M^*(r_2,s_2)$*, then every* $T \in B_{\mathscr{L}(X,Y)}$ *has property* $M^*(r_1r_2, s_1s_2)$*.*

P r o o f. It is similar to the (M) case (see, e.g., [11, Lemma 4.14]). □

Proposition 3.3. *Let* X *and* Y *be Banach spaces and let* $r,s \in (0,1]$*. Let* $\mathscr{L}$ *be a closed subspace of* $\mathscr{L}(X,Y)$ *containing* $\mathscr{K} := \mathscr{K}(X,Y)$*. If an operator* $T \in B_{\mathscr{L}}$

has property $M^*(r,s)$ *and there is a net* $(T_\alpha)\subset\mathscr{K}$ *such that* $T_\alpha^*\longrightarrow T^*$ *strongly, then*

$$\limsup_\alpha |f(rS+s(T-T_\alpha))|\leqslant 1$$

for all $S\in B_{\mathscr{K}}$ *and* $f\in\overline{B_{X^{**}}\otimes B_{Y^*}}^{\text{weak}^*}\subset\mathscr{L}^*$.

Proof. Let $f=w^*\text{-}\lim x_\nu^{**}\otimes y_\nu^*$, i.e., $x_\nu^{**}(A^*y_\nu^*)\longrightarrow f(A)$, $A\in\mathscr{L}$, with $x_\nu^{**}\in B_{X^{**}}$, $y_\nu^*\in B_{Y^*}$. By passing to a subnet, we may assume that (y_ν^*) converges weak* to some $y^*\in B_{Y^*}$. From property $M^*(r,s)$, we get that

$$\limsup_\nu \|rS^*y^*+sT^*y_\nu^*-sT^*y^*\|\leqslant\limsup_\nu\|y_\nu^*\|\leqslant 1.$$

Hence, for any fixed α,

$$\begin{aligned}|f(rS+s(T-T_\alpha))| &= \lim_\nu |x_\nu^{**}((rS+s(T-T_\alpha))^*y_\nu^*)|\\ &\leqslant \limsup_\nu \|(rS+s(T-T_\alpha))^*y_\nu^*\|\\ &\leqslant \limsup_\nu(\|rS^*y_\nu^*-rS^*y^*\|\\ &\quad+\|rS^*y^*+sT^*y_\nu^*-sT^*y^*\|\\ &\quad+\|sT^*y^*-sT_\alpha^*y^*\|+\|sT_\alpha^*y_\nu^*-sT_\alpha^*y^*\|)\\ &\leqslant 1+\|sT^*y^*-sT_\alpha^*y^*\|,\end{aligned}$$

which implies

$$\limsup_\alpha |f(rS+s(T-T_\alpha))|\leqslant 1.$$

□

In the sequential case in Proposition 3.3, one may go further, applying the following vector-valued version of Simons's inequality due to [21], to obtain a similar norm condition: see Lemma 3.5 below.

Lemma 3.4 (see [21, Corollary 4] and its proof)**.** *Let* X *and* Y *be Banach spaces. Let* $\mathscr{L}$ *be a closed subspace of* $\mathscr{L}(X,Y)$ *and let* (A_n) *be a bounded sequence in* $\mathscr{L}$. *If*

$$\limsup_n \operatorname{Re} f(A_n)\leqslant\lambda$$

for some $\lambda\geqslant 0$ *and for all* $f\in\overline{S_X\otimes S_{Y^*}}^{\text{weak}^*}\subset\mathscr{L}^*$, *then there exists* $B_n\in\operatorname{conv}\{A_n,A_{n+1},\dots\}$ *such that*

$$\limsup_n\|B_n\|\leqslant\lambda.$$

Lemma 3.5. *Let X and Y be Banach spaces and let $r, s \in (0,1]$. Let $\mathscr{L}$ be a closed subspace of $\mathscr{L}(X,Y)$ containing $\mathscr{K} := \mathscr{K}(X,Y)$. If $T \in B_{\mathscr{L}}$ has property $M^*(r,s)$ and there is a sequence $(T_n) \subset \mathscr{K}$ such that $T_n^* \longrightarrow T^*$ strongly, then for all $S \in B_{\mathscr{K}}$ there exists $S_n \in \operatorname{conv}\{T_n, T_{n+1}, \ldots\}$ such that*

$$\limsup_n \|rS + s(T - S_n)\| \leqslant 1.$$

The next theorem is one of our main results. As we shall see below, in the M-ideal case, its Corollary 3.8 complements [20, Theorem 6.3], and its Corollaries 3.9 and 3.10 improve the dual version of [20, Theorem 6.3; see p. 171] and [14, Theorem 2.4].

Theorem 3.6. *Let X and Y be Banach spaces. Suppose that X^{**} or Y^* has the Radon-Nikodým property and that X or Y has a shrinking compact approximating sequence. Let $\mathscr{L}$ be a closed subspace of $\mathscr{L}(X,Y)$ containing $\mathscr{K} := \mathscr{K}(X,Y)$ and let $r, s \in (0,1]$. If every $T \in S_{\mathscr{L}}$ has property $M^*(r,s)$, then $\mathscr{K}$ is an $M(r,s)$-ideal in $\mathscr{L}$ with respect to an ideal projection preserving elementary functionals.*

Remark 3.7. The assumptions enforce X^* (and X) or Y^* (and Y) to be separable. In the latter case, Y^* automatically has the Radon-Nikodým property and, as was mentioned before, property $M^*(r,s)$ for operators is equivalent to its sequential version.

P r o o f of Theorem 3.6. By Example 2.2, $\mathscr{K}$ is an ideal in $\mathscr{L}$ with respect to an ideal projection P preserving elementary functionals.

For every operator $T \in S_{\mathscr{L}}$, let us define $T_n = TK_n$ (respectively, $T_n = K_nT$) if (K_n) is the shrinking compact approximating sequence of X (respectively, of Y). Then clearly $T_n^* \longrightarrow T^*$ strongly. Let $S \in S_{\mathscr{K}}$. By Lemma 3.5, there exists $S_n \in \operatorname{conv}\{T_n, T_{n+1}, \ldots\}$ such that

$$\limsup_n \|rS + s(T - S_n)\| \leqslant 1.$$

Since also $S_n^* \longrightarrow T^*$ strongly, by Theorem 2.6, (c) $\Rightarrow$ (a), $\mathscr{K}$ is an $M(r,s)$-ideal in $\mathscr{L}$ with respect to P. □

According to a theorem due to Kalton and Werner [20, Theorem 6.3], *if X is Banach space having an unconditional shrinking compact approximating sequence and Y is a Banach space such that every $T \in S_{\mathscr{L}(X,Y)}$ has property (M), then $\mathscr{K}(X,Y)$ is an M-ideal in $\mathscr{L}(X,Y)$.* The following immediate special case of our Theorem 3.6 completes the Kalton-Werner theorem showing that the unconditionality assumption is superfluous if one assumes that Y^* has the Radon-Nikodým property and strengthens property (M) up to (M^*).

Corollary 3.8. *Let X and Y be Banach spaces. Suppose that X^{**} or Y^* has the Radon-Nikodým property and that X has a shrinking compact approximating sequence. If every $T \in S_{\mathscr{L}(X,Y)}$ has property (M^*) then $\mathscr{K}(X,Y)$ is an M-ideal in $\mathscr{L}(X,Y)$.*

The dual version of the Kalton-Werner theorem states (see [20, p. 171] and [14, pp. 54–55]): *if Y is a Banach space having an unconditional shrinking compact approximating sequence and X is a Banach space such that every $T \in S_{\mathscr{L}(X,Y)}$ has property (M^*), then $\mathscr{K}(X,Y)$ is an M-ideal in $\mathscr{L}(X,Y)$.* The following immediate special case of Theorem 3.6 improves this theorem showing that the unconditionality assumption is superfluous.

Corollary 3.9. *Let X and Y be Banach spaces. Suppose that Y has a shrinking compact approximating sequence. If every $T \in S_{\mathscr{L}(X,Y)}$ has property (M^*), then $\mathscr{K}(X,Y)$ is an M-ideal in $\mathscr{L}(X,Y)$.*

Recall that a Banach space Y has *property* (wM^*) (introduced by Lima [22]) if

$$\limsup_{\nu} \|y^*_\nu\| = \limsup_{\nu} \|2y^* - y^*_\nu\|$$

whenever $y^* \in Y^*$ and $(y^*_\nu) \subset Y^*$ is a bounded net converging weak* to y^* in Y^*.

Corollary 3.10 below is an improvement of a theorem due to John and Werner [14, Theorem 2.4]: its assumption that Y has an unconditional shrinking compact approximating sequence (which easily implies property (wM^*) of Y) will be weakened up to the assumption that Y has property (wM^*), showing, e.g., that there is no need for a separability requirement of Y^*.

If Y is separable, then again (due to the weak* metrizability of bounded subsets of Y^*) the sequential version of (wM^*) is equivalent to property (wM^*), and the same concerns the property of Y (introduced by John and Werner [14]) described in the following.

Corollary 3.10. *Let $1 < p < \infty$ and $1/p + 1/q = 1$. Let Y be a Banach space having property (wM^*) and let*

$$\limsup_{\nu} (\|y^*\|^q + \|y^*_\nu\|^q)^{1/q} \leqslant \limsup_{\nu} \Big(\frac{\|y^* + y^*_\nu\|^q + \|y^* - y^*_\nu\|^q}{2} \Big)^{1/q}$$

whenever $y^* \in Y^*$ and $(y^*_\nu) \subset Y^*$ is a bounded net converging weak to null in Y^*. Then $\mathscr{K}(\ell_p, Y)$ is an M-ideal in $\mathscr{L}(\ell_p, Y)$.*

P r o o f. Based on Corollary 3.8, it is sufficient to show that every $T \in S_{\mathscr{L}(\ell_p,Y)}$ has property (M^*).

Let $x^* \in \ell_q$ and $y^* \in Y^*$ be such that $\|x^*\| \leqslant \|y^*\|$, and let $(y^*_\nu) \subset Y^*$ be a bounded net such that $y^*_\nu \longrightarrow 0$ weak*. Then for every $T \in S_{\mathscr{L}(\ell_p,Y)}$,

$$\begin{aligned}\limsup_\nu \|x^* + T^* y^*_\nu\| &= \limsup_\nu (\|x^*\|^q + \|T^* y^*_\nu\|^q)^{1/q}\\ &\leqslant \limsup_\nu (\|y^*\|^q + \|y^*_\nu\|^q)^{1/q}\\ &\leqslant \limsup_\nu \Big(\frac{\|y^* + y^*_\nu\|^q + \|y^* - y^*_\nu\|^q}{2}\Big)^{1/q}.\end{aligned}$$

Since Y has property (wM^*),

$$\limsup_\nu \|y^* + y^*_\nu\| = \limsup_\nu \|y^* - y^*_\nu\|.$$

Hence,

$$\limsup_\nu \|x^* + T^* y^*_\nu\| \leqslant \limsup_\nu \|y^* + y^*_\nu\|.$$

□

If one of the Banach spaces X or Y is *separable*, then, using Proposition 3.2 and Theorem 3.6, we can already now prove the desired quantitative extension of Theorem 1.1 from M-ideals to $M(r,s)$-ideals (see the Introduction).

Theorem 3.11. *Let X and Y be Banach spaces such that X or Y is separable. Let $r_1, s_1, r_2, s_2 \in (0,1]$ satisfy $r_1 + s_1/2 > 1$ and $r_2 + s_2/2 > 1$. If $\mathscr{K}(X)$ is an $M(r_1,s_1)$-ideal in $\mathscr{L}(X)$ and $\mathscr{K}(Y)$ is an $M(r_2,s_2)$-ideal in $\mathscr{L}(Y)$, then $\mathscr{K}(X,Y)$ is an $M(r_1r_2, s_1s_2)$-ideal in $\mathscr{L}(X,Y)$.*

P r o o f. If $r + s/2 > 1$ and $\mathscr{K}(X)$ is an $M(r,s)$-ideal in $\mathscr{L}(X)$, then, by [5, Lemma 2.3 and Proposition 2.1], $X^* = \overline{\operatorname{span}}(w^*\text{-}\operatorname{sexp} B_{X^*})$ (i.e., the weak* strongly exposed points of B_{X^*} span a norm dense subspace of X^*) and X^* has the Radon-Nikodým property. Therefore, by [5, Proposition 3.2] and [35, Theorem 4.1, $1^\circ \Rightarrow 2^\circ$], X has the MCAP and property $M^*(r,s)$. Hence, in our case, both X and Y have the MCAP, X has property $M^*(r_1,s_1)$, and Y has property $M^*(r_2,s_2)$. From Proposition 3.2 we get that every $T \in B_{\mathscr{L}(X,Y)}$ has property $M^*(r_1r_2, s_1s_2)$.

We can now apply Theorem 3.6 to show that $\mathscr{K}(X,Y)$ is an $M(r_1r_2,s_1s_2)$-ideal in $\mathscr{L}(X,Y)$. Indeed, as we saw above, Y^* has the Radon-Nikodým property. If, e.g., X is separable, since X has the MCAP, X clearly has a metric compact approximating sequence $(K_n)_{n=1}^\infty$. Then $(K_n)_{n=1}^\infty$ is shrinking because $X^* = \overline{\operatorname{span}}(w^*\text{-}\operatorname{sexp} B_{X^*})$ (this fact is well known and can be easily checked). □

The proof of Theorem 3.11 clearly shows that *if* Theorem 3.6 held true also in the non-separable case (i.e., with the assumption "X or Y has a shrinking compact approximating sequence" being replaced by "X^* or Y^* has the BCAP with conjugate operators"), *then* in Theorem 3.11 the separability assumption ("X or Y is separable") could be dropped. However, we do not know whether the non-separable case of Theorem 3.6 is true. Nevertheless, in Section 4, we shall establish the general non-separable case of Theorem 3.11 (see Theorem 4.14) using different methods.

4. Main results: the non-separable case

It is known that M-ideals of compact operators are separably determined [33]: *if a Banach space X has the MCAP and $\mathscr{K}(E)$ is an M-ideal in $\mathscr{L}(E)$ for all separable closed subspaces E of X having the MCAP, then $\mathscr{K}(X)$ is an M-ideal in $\mathscr{L}(X)$*. This theorem and its proof have served as a prototype to obtain similar results on certain general approximations of the identity [35] (see also [34]) and ideals of compact operators having Phelps's uniqueness property U [39]. The next result shows that $M(r,s)$-ideals of compact operators are also separably determined. For its proof, we shall develop ideas from [33] and [35, proofs of Lemmas 3.2 and 4.2] but (following an idea in [39, proofs of Theorems 2.2 and 2.3]) we do not make precise ε-nets of certain compact subsets. One inconvenience to be overcome is that in the $M(r,s)$-ideal case, unlike the M-ideal and property U cases, the ideal projection need not be unique.

Theorem 4.1. *Let X and Y be Banach spaces. Let positive numbers $r \leqslant 1$ and s satisfy $r+s>1$, and let $\varrho,\sigma \in (0,1]$ satisfy $\varrho+\sigma>1$. Suppose that Y satisfies the $M(\varrho,\sigma)$-inequality and has the MCAP. If $\mathscr{K}(E,F)$ is an $M(r,s)$-ideal in $\mathscr{L}(E,F)$ with respect to an ideal projection preserving elementary functionals for all separable closed subspaces E of X and F of Y such that F has the MCAP, then $\mathscr{K}(X,Y)$ is an $M(r,s)$-ideal in $\mathscr{L}(X,Y)$.*

P r o o f. We are going to apply Theorem 2.6. Let (K_α) be an MCAI of Y. By Proposition 3.1, (K_α) is shrinking and Y^* has the Radon-Nikodým property. Further, $\mathscr{K}(X,Y)$ is an ideal in $\mathscr{L}(X,Y)$ with respect to an ideal projection preserving elementary functionals (see Example 2.1) and $K_\alpha T \longrightarrow T$ in the dual weak operator topology.

Assume for contradiction that $\mathscr{K}(X,Y)$ is not an $M(r,s)$-ideal in $\mathscr{L}(X,Y)$. Then condition (b) of Theorem 2.6 is not satisfied: there are $\varepsilon>0$, $S \in B_{\mathscr{K}(X,Y)}$, $T \in B_{\mathscr{L}(X,Y)}$, and α_0 such that

$$\|rS + s(T-KT)\| > 1+3\varepsilon \quad \forall\, K \in \operatorname{conv}\{K_\alpha\colon\ \alpha \geqslant \alpha_0\}.$$

We shall define separable closed subspaces E of X and F of Y such that F has the MCAP, but $\mathscr{K}(E,F)$ cannot be an $M(r,s)$-ideal in $\mathscr{L}(E,F)$ with respect to any ideal projection preserving elementary functionals. This will contradict the assumption and complete the proof.

To begin, let $E_0 = \{0\} \subset X$ and $F_0 = \{0\} \subset Y$. Pick $x_0 \in B_X$ such that

$$\|(rS + s(T - K_{\alpha_0}T))x_0\| > \|rS + s(T - K_{\alpha_0}T)\| - \varepsilon > 1 + 2\varepsilon.$$

Denote $E_1 = E_0 \cup \{x_0\}$ and $F_1 = F_0 \cup K_{\alpha_0}(F_0) \cup S(E_1) \cup T(E_1)$. Then choose $\alpha_1 \geqslant \alpha_0$ such that

$$\|K_{\alpha_1}y - y\| < 1 \quad \forall\, y \in F_1.$$

Also choose a finite ε/s-net Λ_1 in $\operatorname{conv}\{K_{\alpha_0}, K_{\alpha_1}\}$, and for every $L \in \Lambda_1$ pick $x_L \in B_X$ such that

$$\|(rS + s(T - LT))x_L\| > \|rS + s(T - LT)\| - \varepsilon > 1 + 2\varepsilon.$$

Denote

$$E_2 = E_1 \cup \{x_L\colon\ L \in \Lambda_1\}$$

and

$$F_2 = F_1 \cup K_{\alpha_0}(F_1) \cup K_{\alpha_1}(F_1) \cup S(E_2) \cup T(E_2).$$

Continuing similarly, we obtain, for all $n \in \mathbb{N}$, an index α_n, a finite ε/s-net Λ_n in $\operatorname{conv}\{K_{\alpha_0}, \dots, K_{\alpha_n}\}$, a finite subset $\{x_L\colon\ L \in \Lambda_n\} \subset B_X$ such that

$$\|(rS + s(T - LT))x_L\| > 1 + 2\varepsilon, \quad L \in \Lambda_n,$$

and finite subsets $E_n \subset X$ and $F_n \subset Y$ such that

$$E_{n+1} = E_n \cup \{x_L\colon\ L \in \Lambda_n\},$$
$$F_{n+1} = F_n \cup K_{\alpha_0}(F_n) \cup \ldots \cup K_{\alpha_n}(F_n) \cup S(E_{n+1}) \cup T(E_{n+1}),$$

and

$$\|K_{\alpha_n}y - y\| < \frac{1}{n} \quad \forall\, y \in F_n.$$

Denote $E = \overline{\operatorname{span}} \bigcup_{n=1}^{\infty} E_n$ and $F = \overline{\operatorname{span}} \bigcup_{n=1}^{\infty} F_n$. It can be easily seen that $S(E) \subset F$, $T(E) \subset F$, $K_{\alpha_n}(F) \subset F$ for all $n \in \mathbb{N}$, and $K_{\alpha_n}y \longrightarrow y$ for all $y \in F$. Consider $S|_E \in B_{\mathscr{K}(E,F)}$, $T|_E \in B_{\mathscr{L}(E,F)}$, and $K_{\alpha_n}|_F \in B_{\mathscr{K}(F)}$.

Since Y satisfies the $M(\varrho,\sigma)$-inequality, also F does (this fact which is similar to that of the M-embedded spaces (see, e.g., [11, p. 111]) was observed in [2, Proposition 2.1]). Consequently, as in the beginning of the proof, we are in position to apply

Theorem 2.6 to $\mathscr{K}(E,F)$ in $\mathscr{L}(E,F)$. According to Theorem 2.6, if $\mathscr{K}(E,F)$ were an $M(r,s)$-ideal in $\mathscr{L}(E,F)$ with respect to an ideal projection preserving elementary functionals, then there would exist $K \in \operatorname{conv}\{K_{\alpha_1},\ldots,K_{\alpha_n}\}$, for some $n \in \mathbb{N}$, such that

$$\|(rS + s(T - KT))|_E\| \leqslant 1 + \varepsilon.$$

Let $L \in \Lambda_n$ satisfy $\|K - L\| < \varepsilon/s$. Then

$$1 + 2\varepsilon < \|(rS + s(T - LT))|_E\| \leqslant \|(rS + s(T - KT))|_E\| + \varepsilon \leqslant 1 + 2\varepsilon,$$

a contradiction. □

Remark 4.2. From the proof of Theorem 4.1 it is clear that the assumption "Y satisfies the $M(\varrho,\sigma)$-inequality with $\varrho + \sigma > 1$" can be replaced by any assumption guaranteeing that Y^* has the Radon-Nikodým property and every MCAI of any closed subspace F of $,Y$ is shrinking. This is well known to be true if Y has property U in its bidual Y^{**} (see [36, Corollary 5] and, e.g., [39, Lemma 2.1]).

The following result, which shows that M-ideals of compact operators $\mathscr{K}(X,Y)$ are separably determined not only for $X = Y$ but also for *distinct* spaces X and Y, seems to be new.

Corollary 4.3. *Let X and Y be Banach spaces. Suppose that Y has property U in its bidual and has the MCAP. If $\mathscr{K}(E,F)$ is an M-ideal in $\mathscr{L}(E,F)$ for all separable closed subspaces E of X and F of Y such that F has the MCAP, then $\mathscr{K}(X,Y)$ is an M-ideal in $\mathscr{L}(X,Y)$.*

Proof. This is immediate from Theorem 4.1 and Remark 4.2 because M-ideals enjoy the unique ideal property, and under the assumptions on E and F, $\mathscr{K}(E,F)$ is an ideal in $\mathscr{L}(E,F)$ with respect to an ideal projection preserving elementary functionals (see Example 2.1 or 2.2). □

Remark 4.4. The prototype of Corollary 4.3 is [39, Theorem 2.3] asserting that property U of $\mathscr{K}(X,Y)$ in $\mathscr{L}(X,Y)$ is separably determined.

There exist infinite-dimensional Banach spaces X and Y for which $\mathscr{K}(X,Y) = \mathscr{L}(X,Y)$. This is the case, for instance, when $X = \ell_p$, $Y = \ell_q$ with $p > q$ (Pitt's theorem); $X = \ell_p$, $Y = d(w,q)$ with $p > q$ and $w \notin \ell_{p/(p-q)}$ [30] (other Pitt's type theorems for Lorentz and Orlicz sequence spaces can be found in [1]). A consequence of Theorem 4.1 is that the property $\mathscr{K}(X,Y) = \mathscr{L}(X,Y)$ is also separably determined.

Corollary 4.5. *Let X and Y be Banach spaces. Suppose that Y has property U in its bidual and has the MCAP. If $\mathscr{K}(E,F) = \mathscr{L}(E,F)$ for all separable closed subspaces E of X and F of Y such that F has the MCAP, then $\mathscr{K}(X,Y) = \mathscr{L}(X,Y)$.*

P r o o f. Apply Remark 4.2 and Theorem 4.1 to any $s > 1$. $\square$

It is a well-known consequence of the Eberlein-Šmulian theorem that a Banach space is reflexive whenever all its separable closed subspaces are (for an alternative easy proof see [10, Corollary 2]). The next corollary shows that for $\mathscr{L}(X,Y)$ to be reflexive, it suffices that the separable subspaces of the form $\mathscr{K}(E,F)$ are reflexive.

Corollary 4.6. *Let X and Y be reflexive Banach spaces. Suppose that Y has the CAP. If $\mathscr{K}(E,F)$ is reflexive for all separable closed subspaces E of X and F of Y such that F has the CAP, then $\mathscr{L}(X,Y)$ is reflexive.*

P r o o f. It is known (see [6] or [8]) that a reflexive Banach space with the CAP actually has the MCAP. Since F has the CAP, by [8, Corollary 1.3], $\mathscr{K}(E,F)^{**} = \mathscr{L}(E,F)$, and by this identification, $j_{\mathscr{K}(E,F)}(T) = T$, for all $T \in \mathscr{K}(E,F)$. Since $\mathscr{K}(E,F)$ is reflexive, we have $\mathscr{K}(E,F) = \mathscr{L}(E,F)$. By Corollary 4.5, $\mathscr{K}(X,Y) = \mathscr{L}(X,Y)$. Hence, according to a classical theorem proved independently by Heinrich [12] and Kalton [17], $\mathscr{L}(X,Y)$ is reflexive. Alternatively, we have as above, $\mathscr{K}(X,Y)^{**} = \mathscr{L}(X,Y) = \mathscr{K}(X,Y)$, meaning that $\mathscr{K}(X,Y)$ is reflexive, and also so is $\mathscr{L}(X,Y)$. $\square$

Let us now turn to the promised main results of the present paper.

Theorem 4.7. *Let X and Y be Banach spaces. Assume that Y has the MCAP. Let $r_1, s_1, r_2, s_2 \in (0,1]$ satisfy $r_1 + s_1 > 1$ and $r_2 + s_2 > 1$. If X has property $M^*(r_1,s_1)$ and Y has property $M^*(r_2,s_2)$, then $\mathscr{K}(X,Y)$ is an $M(r_1r_2, s_1s_2)$-ideal in $\mathscr{L}(X,Y)$.*

P r o o f. Property $M^*(r_2,s_2)$ of Y implies that Y satisfies the $M(r_2,s_2)$-inequality (see the beginning of Section 3). Let $E \subset X$ and $F \subset Y$ be separable closed subspaces, and assume that F has the MCAP. Property $M^*(r,s)$ is inherited by closed subspaces (see [35, p. 2804]). Hence, E has property $M^*(r_1,s_1)$ and F has property $M^*(r_2,s_2)$. From Proposition 3.2 we know that then every $T \in B_{\mathscr{L}(E,F)}$ has property $M^*(r_1r_2, s_1s_2)$. Since F is separable and has the MCAP, it has a metric compact approximating sequence which is shrinking, because F satisfies the $M(r_2,s_2)$-inequality (see Proposition 3.1). It follows that F^* is separable. Applying Theorem 3.6 we get that $\mathscr{K}(E,F)$ is an $M(r_1r_2, s_1s_2)$-ideal in $\mathscr{L}(E,F)$ with respect to an ideal projection preserving elementary functionals. Hence, according to Theorem 4.1, $\mathscr{K}(X,Y)$ is an $M(r_1r_2, s_1s_2)$-ideal in $\mathscr{L}(X,Y)$. $\square$

A basic theorem of the theory of M-ideals of compact operators asserts that *$\mathscr{K}(X)$ is an M-ideal in $\mathscr{L}(X)$ if and only if X has property (M^*) and the MCAP.* It was established in [20] for separable X, in [22] for reflexive X, and extended to arbitrary (non-separable) X in [33]. A self-contained and "the shortest known proof" (we quote [26] here) is given in [35], another self-contained proof based on a new structure theorem for Borel probability measures can be found in a very recent paper [26]. The above theorem together with [31, Theorem 8] immediately yields a more general result: *$\mathscr{K}(X,Y)$ is an M-ideal in $\mathscr{L}(X,Y)$ whenever X and Y have property (M^*), and Y has the MCAP.* A self-contained measure-theoretic proof of this result is given in [26]. Keeping in mind that property (M^*) is precisely property $M^*(1,1)$, Theorem 4.7 contains the latter result as a special case, yielding another self-contained proof of it. It would be interesting to study whether the measure-theoretic approach by Nygaard and Põldvere [26] could be used to give an alternative proof of Theorem 4.7.

Recall that a Banach space X is said to have the *λ-commuting BCAP* (with $\lambda \geqslant 1$) if X has a CAI (K_α) such that $K_\alpha K_\beta = K_\beta K_\alpha$ for all indices α and β, and $\limsup \|K_\alpha\| \leqslant \lambda$. It follows from [38, Theorem 4.4] that X has the MCAP whenever X satisfies the $M(r,s)$-inequality and has the λ-commuting BCAP with $\lambda < r+s$. Therefore we can make the following essential remark.

Remark 4.8. The assumption of the MCAP of Y in Theorems 4.1 and 4.7 can be replaced by the assumption that Y has the λ-commuting BCAP with $\lambda < \varrho + \sigma$ and $\lambda < r_2 + s_2$, respectively.

Both results described in Remark 4.8 are new even for M-ideals. Since a corollary of Theorem 4.7 represents a version of the basic theorem of the theory of M-ideals of compact operators discussed above, let us spell it out as follows.

Corollary 4.9. *Let X and Y be Banach spaces having property (M^*). If Y has the λ-commuting BCAP with $\lambda < 2$, then $\mathscr{K}(X,Y)$ is an M-ideal $\mathscr{L}(X,Y)$.*

We remark that the special case of Corollary 4.9 when $X = Y$ is proven in [38, Corollary 4.10].

Let us denote $\mathscr{I}(X) = \operatorname{span}(\mathscr{K}(X) \cup \{I_X\})$ where X is a Banach space.

Lemma 4.10 (see [35, Corollary 4.4])**.** *Let X be a Banach space. If $r,s \in (0,1]$ satisfy $r + s/2 > 1$, then the following assertions are equivalent.*

1° *$\mathscr{K}(X)$ is an $M(r,s)$-ideal in $\mathscr{I}(X)$.*

2° *X has an MCAI and property $M^*(r,s)$.*

For the M-ideal prototype of the next result, see [31, Theorem 8].

Corollary 4.11. *Let X and Y be Banach spaces. Let $r_1, s_1, r_2, s_2 \in (0,1]$ satisfy $r_1 + s_1 > 1$ and $r_2 + s_2/2 > 1$. If X has property $M^*(r_1,s_1)$ and $\mathscr{K}(Y)$ is an $M(r_2,s_2)$-ideal in $\mathscr{I}(Y)$, then $\mathscr{K}(X,Y)$ is an $M(r_1r_2, s_1s_2)$-ideal in $\mathscr{L}(X,Y)$.*

P r o o f. This is immediate from Theorem 4.7 and Lemma 4.10. □

The next theorem, which is one of the main results of the present paper, is also immediate from Theorem 4.7 and Lemma 4.10.

Theorem 4.12. *Let X and Y be Banach spaces. Let $r_1, s_1, r_2, s_2 \in (0,1]$ satisfy $r_1+s_1/2 > 1$ and $r_2+s_2/2 > 1$. If $\mathscr{K}(X)$ is an $M(r_1,s_1)$-ideal in $\mathscr{I}(X)$ and $\mathscr{K}(Y)$ is an $M(r_2,s_2)$-ideal in $\mathscr{I}(Y)$, then $\mathscr{K}(X,Y)$ is an $M(r_1r_2, s_1s_2)$-ideal in $\mathscr{L}(X,Y)$.*

Using Lemma 4.10 and [5, Theorem 3.1], the following was observed in [9, Corollary 7].

Proposition 4.13. *Let $r, s \in (0,1]$ satisfy $r + s/2 > 1$. If $\mathscr{K}(X)$ is an $M(r,s)$-ideal in $\mathscr{L}(X)$, then $\mathscr{K}(X)$ is an $M(r,s)$-ideal in $\mathscr{I}(X)$.*

From Theorem 4.12 and Proposition 4.13 we immediately get the desired extension of Theorem 3.11 to arbitrary (non-separable) spaces. Let us spell it out.

Theorem 4.14. *Let X and Y be Banach spaces. Let $r_1, s_1, r_2, s_2 \in (0,1]$ satisfy $r_1+s_1/2 > 1$ and $r_2+s_2/2 > 1$. If $\mathscr{K}(X)$ is an $M(r_1,s_1)$-ideal in $\mathscr{L}(X)$ and $\mathscr{K}(Y)$ is an $M(r_2,s_2)$-ideal in $\mathscr{L}(Y)$, then $\mathscr{K}(X,Y)$ is an $M(r_1r_2, s_1s_2)$-ideal in $\mathscr{L}(X,Y)$.*

We remark that Theorem 4.14 extends [31, Corollary 9] (which is [11, Corollary 4.18]) from M-ideals to $M(r,s)$-ideals.

Concerning Proposition 4.13, let us recall Kalton's theorem [19, Theorem 2.6] (see [31, Theorem 5] or [11, Theorem 4.17] for its non-separable case): *$\mathscr{K}(X)$ is an M-ideal in $\mathscr{L}(X)$ if and only if $\mathscr{K}(X)$ is an M-ideal in $\mathscr{I}(X)$.*

This means that the converse of Proposition 4.13 holds for $r = s = 1$. We do not know whether it holds for other parameters than $r = s = 1$. The best we can do is the following result which is immediate from Theorem 4.12 (and which, in the special case when $r = s = 1$, reduces to Kalton's theorem).

Corollary 4.15. *Let X be Banach space and let $r, s \in (0,1]$ satisfy $r + s/2 > 1$. If $\mathscr{K}(X)$ is an $M(r,s)$-ideal in $\mathscr{I}(X)$, then $\mathscr{K}(X)$ is an $M(r^2,s^2)$-ideal in $\mathscr{L}(X)$.*

Corollary 4.15 improves [9, Corollary 13] where the claim is that $\mathscr{K}(X)$ is an $M(r^3,s^3)$-ideal in $\mathscr{L}(X)$.

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