μ Synthesis and LFT Gain Scheduling with Mixed Uncertainties

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Abstract

This paper presents the solution to the mixed μ synthesis problem, and how to design gain scheduling controllers with linear fractional transformations (LFTs). The system is assumed to have a parameter dependences described by LFTs The goal of the controller is, with real-time knowledge of the parameters, to provide disturbance and error attenuation. The paper treats both complex and real parametric dependencies.

Keywords: Gain sceduling, linear fractional transforms, structured singular values, parametric dependent systems, synthesis.

1 Introduction

During the last couples of year synthesis methods for gain scheduling using linear fractional transformations (LFTs) and structured singular values have been developed, see e.g [10, 9, 7]. The idea behind this approach is to let the controller have access to some of the uncertainties or perturbations of the system to be controlled. The design leads to two linear matrix inequalities (LMIs), which are coupled by a third LMI and non-convex rank conditions. This is analogous with the two-Riccati algorithm for solving H_∞ synthesis.

In the referred papers [10, 9, 7], the shared uncertainties have been assumed to be complex, possibly repeated scalar blocks. In this paper the mixed μ design problem (that is with both real and complex uncertainties) is elaborated. The synthesis algorithm has the same structure as in the pure complex case with two LMIs coupled by a third LMI and rank conditions. The difference to the pure

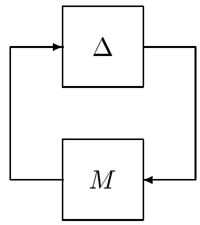

Figure 1: System with uncertainties.

complex case is that blocks corresponding to real uncertainties in the connecting LMI disappear; only the rank conditions remain.

The structure of the problem is treated in a unified and general way covering both discrete and continuous systems, which are equivalent using bilinear transformation between the z-domain and the s-domain [2].

1.1 Notations

X^* denotes the complex conjugate transpose of X; $X > (\geq)\ 0$ a hermitian ($X = X^*$) positive definite (semidefinite) matrix; $X^{-*} = (X^*)^{-1}$; $X^\dagger$ is the Moore-Penrose pseudo inverse of X; $\operatorname{diag}[X_1, X_2]$ a block-diagonal matrix composed of X_1 and X_2; $\operatorname{rank} X$ denotes the rank of the matrix X; $\operatorname{herm} X = \frac{1}{2}(X + X^*)$; $\mathcal{S}(.,.)$ denotes the Redheffer star product; $\bar{\sigma}(X)$ the maximal singular value of X.

2 μ-analysis and LFTs

This section gives a short review on structured singular values and linear fractional transformations (LFTs), see also e.g. [2].

2.1 Definitions

We will here use a similar notations and definitions as is used in [12, 13]. The definition of μ depends upon the underlying block structure $\mathcal{N}$ of the uncertainties Δ, which could be either real or complex, see Figure 1. For notational convenience we assume that all uncertainty blocks are square. This can be done without loss of generality by adding dummy inputs or outputs.

Given a matrix $M \in \mathbb{C}^{n\times n}$ and three non-negative integers f_r, f_c and f_C, with $f = f_r + f_c + f_C \leq n$ the block structure is an f-tuple of positive integers

$$\mathcal{N} = [n_1, \ldots, n_{f_r}, n_{f_r+1}, \ldots, n_{f_r+f_c}, n_{f_r+f_c+1}, \ldots, n_{f_r+f_c+f_C}] \tag{1}$$

where $\sum_{i=1}^{f} n_i = n$ for dimensional compatibility. The set of allowable perturbations is defined by a set of block diagonal matrices $\mathcal{X} \in \mathbb{C}^{n\times n}$ defined by

$$\mathcal{X} = \{\Delta = \operatorname{diag}[\delta_1^r I_{n_1}, \dots, \delta_{n_{f_r}}^r I_{n_{f_r}}, \\ \delta_1^c I_{n_{f_r+1}}, \dots, \delta_{f_c}^c I_{n_{f_r+f_c}}, \Delta_1^C, \dots, \Delta_{nC}^C] : \\ \delta_i^r \in \mathbb{R}, \delta_i^c \in \mathbb{C}, \Delta_i^C \in \mathbb{C}^{n_{f_r+f_c+i} \times n_{f_r+f_c+i}}\}. \tag{2}$$

Assuming the uncertainty structure $\mathcal{N}$, the structured singular value μ of a matrix $M \in \mathbb{C}^{n\times n}$ is defined by

$$\mu = \left(\min_{\Delta\in\mathcal{X}} \{\bar{\sigma}(\Delta) : \det(I - \Delta M) = 0\} \right)^{-1} \tag{3}$$

and if no $\Delta \in \mathcal{X}$ satisfies $\det(I - \Delta M) = 0$ then $\mu(M) = 0$.

2.2 Upper and Lower Bounds

Generally the structured singular value cannot be exactly computed, and instead we have to resort to upper and lower bounds, which are usually sufficient for most practical applications. A tutorial review of the complex structured singular value is given in [8].

An upper bound can be determined using convex methods, either involving minimization of singular values with respect to a scaling matrix or by solving a linear matrix inequality (LMI) problem. The upper bound is conservative in the general case, but can be improved by branch and bound schemes.

A lower bound can be found by maximizing the real eigenvalue of a scaled matrix. This bound is nonconservative in the sense that if the true global maximum is found it is equal to μ. However, since the problem is not convex, we cannot guarantee that we find the global maximum.

We will here focus on the computation of the upper bound, which we here denote $\nu \geq \mu$, in order to distinguish it from the true μ function. The upper bound ν can be computed as a convex optimization problem. For complex uncertainties it is determined by

$$\nu(M) = \inf_{D\in\mathcal{D}} \bar{\sigma}(DMD^{-1}) \tag{4}$$

where $\mathcal{D}$ is the set of block diagonal Hermitian matrices that commute with $\mathcal{X}$, that is

$$\mathcal{D} = \{0 < D = D^* \in \mathbb{C}^{n\times n} : D\Delta = \Delta D, \forall \Delta \in \mathcal{X}\}. \tag{5}$$

This problem is equivalent to an LMI problem

$$\nu(M) = \inf_{\substack{\nu>0 \\ P\in\mathcal{D}}} \{\nu : M^*PM < \nu^2 P\}. \tag{6}$$

Real uncertainties can be included in the LMI problem for computing the upper bound (see e.g. [3, 12, 13]):

$$\nu(M) = \inf_{\substack{\nu>0 \\ P\in\mathcal{D} \\ G\in\mathcal{G}}} \{\nu : M^*PM + j(GM - M^*G) < \nu^2 P\} \tag{7}$$

where

$$\mathcal{G} = \{G : G = G^* \in \mathbb{C}^{n\times n}, G\Delta = \Delta^* G, \forall \Delta \in \mathcal{X}\}. \tag{8}$$

Every $G \in \mathcal{G}$ is block diagonal with zero blocks for complex uncertainties. If we let $\mathcal{G} = \{0\}$ in (7) we recover the complex upper bound (6).

We can reformulated (7) as a positive real property

$$\nu(M) = \inf_{\substack{\nu>0 \\ W\in\mathcal{W}}} \left\{\nu : \operatorname{herm}\left((I + \tfrac{1}{\nu}M)\, W\, (I - \tfrac{1}{\nu}M)\right) > 0\right\} \tag{9}$$

where $\operatorname{herm} X = \frac{1}{2}(X + X^*)$ and $\mathcal{W} = \{W = P + jG : P \in \mathcal{D}, G \in \mathcal{G}\}$. Note that $\operatorname{herm} W > 0$ always and that $W = W^*$ for complex uncertainties. Another equivalent reformulation of (7) is

$$\nu(M) = \inf_{\substack{\nu>0 \\ P\in\mathcal{D} \\ G\in\mathcal{G}}} \{\nu : \bar{\sigma}\left(\left(\tfrac{1}{\nu}DMD^{-1} - jG\right)(I + G^2)^{-\frac{1}{2}}\right) < 1\}. \tag{10}$$

2.3 Linear Fractional Transformations (LFTs)

Suppose M is a complex matrix partitioned as

$$M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \in \mathbb{C}^{(p_1+p_2)\times(m_1+m_2)}, \tag{11}$$

and let $\Delta_u \in \mathbb{C}^{m_1\times p_1}$ and $\Delta_l \in \mathbb{C}^{m_2\times p_2}$. The upper and lower linear fractional transformations (LFTs) are defined by

$$\mathcal{F}_u(M, \Delta_u) = M_{22} + M_{21}\Delta_u(I - M_{11}\Delta_u)^{-1}M_{12} \tag{12}$$

and

$$\mathcal{F}_l(M, \Delta_l) = M_{11} + M_{12}\Delta_l(I - M_{22}\Delta_l)^{-1}M_{21} \tag{13}$$

respectively. Clearly, the existence of the LFTs depends on the invertibility of $I - M_{11}\Delta_u$ and $I - M_{22}\Delta_l$ respectively.

The Redheffer star product [11] is a generalization of the LFTs. Assume that Q is partitioned similarly to M. Then the star product is defined by

$$\mathcal{S}(Q, M) = \begin{bmatrix} \mathcal{F}_l(Q, M_{11}) & Q_{12}(I - M_{11}Q_{22})^{-1}M_{12} \\ M_{21}(I - Q_{22}M_{11})^{-1}Q_{21} & \mathcal{F}_u(M, Q_{22}) \end{bmatrix}. \tag{14}$$

A block diagram illustrating the star product is given in Figure 2.

Note that the definition above is dependent on the partitioning of the matrices Q and M. The LFTs can be defined by the $\mathcal{S}$ notation, as

$$\mathcal{F}_u(M, \Delta_u) = \mathcal{S}(\Delta_u, M)$$

and

$$\mathcal{F}_l(M, \Delta_l) = \mathcal{S}(M, \Delta_l).$$

The star product is associative, that is

$$\mathcal{S}(A, \mathcal{S}(B, C)) = \mathcal{S}(\mathcal{S}(A, B), C).$$

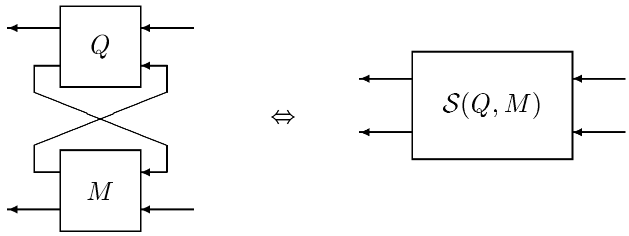

Figure 2: The Redheffer star product.

3 Synthesis

In this section we will treat the μ synthesis problem when uncertainties in the controller may be shared by the controller.

3.1 Structure of shared uncertainties

Formally we treat this by letting the controller K have access to a copy of the system's uncertainties. Not all blocks in the uncertainty matrix Δ are accessible and we denote the accessible uncertainties by

$$\mathcal{L}_{\mathcal{R}}(\Delta) = \operatorname{diag}[\delta_1 I_{r_1}, \delta_2 I_{r_2}, \ldots, \delta_{f_1} I_{r_{f_1}}] \tag{15}$$

where $f_1 = f_r + f_c$. We only allow blocks that are scalar repeated blocks of the type $\delta_i I_{r_i}$ where δ could be either real or complex. The structure of the controller accessible uncertainties are given by

$$\mathcal{R} = [r_1, \ldots, r_{f_r}, r_{f_r+1}, \ldots, r_{f_r+f_c}, 0, \ldots, 0] \tag{16}$$

Note that all blocks corresponding to full complex blocks are not allowed and, consequently, $r_i = 0$ for those.

3.2 The General Synthesis Problem

The general synthesis problem can be depicted as in Figure 3. Find a controller K, possibly with a given order, to the system M (or the augmented system $\tilde{M}$), such that $\mu(\mathcal{S}(M, K)) = \mu(\mathcal{S}(\tilde{M}, K))$ is minimized or below a specified value. The diagrams below show three equivalent representations of the μ-synthesis problem. In the left diagram M and K have separated uncertainty blocks, Δ and $\mathcal{L}_{\mathcal{R}}(\Delta)$ respectively. In the middle diagram the uncertainties are explicitly shared. Here $\tilde{\Delta} = \operatorname{diag}[\Delta, \mathcal{L}_{\mathcal{R}}(\Delta)]$. In the right diagram through connections between K and Δ have been included in $\tilde{M}$.

Here we use a somewhat more general structure of $\tilde{\Delta}$ than used so far. With this formulation it is possible to catch most of the relevant synthesis problems in the μ setting. The generalized structure of $\tilde{\Delta}$ is nothing more than a reordering of rows and columns, so that both M and K may share the same uncertainties. For instance, if Δ includes frequency, either s or z, we can handle continuous-time and discrete-time problems respectively.

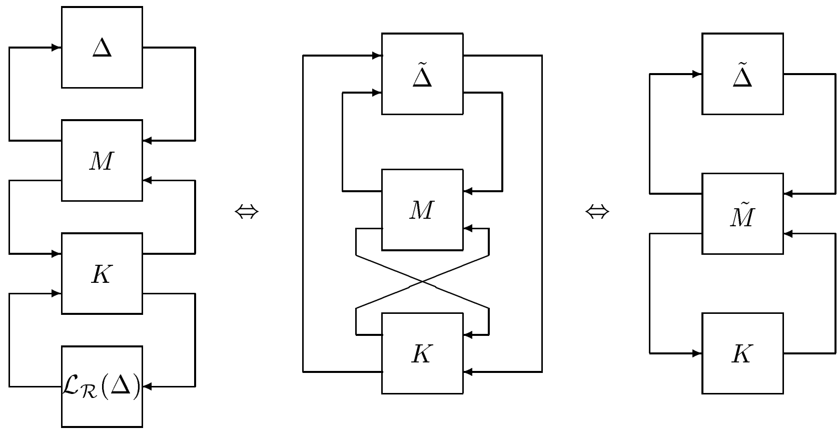

Figure 3: Equivalent representations of a system M controlled by K that have access to a reduced copy $\mathcal{L}_{\mathcal{R}}(\Delta)$ of the system's uncertainties Δ. The augmented uncertainty block $\tilde{\Delta} = \operatorname{diag}[\Delta, \mathcal{L}_{\mathcal{R}}(\Delta)]$.

3.3 An Affine Problem

We assume that M is partitioned as:

$$M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \in \mathbb{C}^{(n+m)\times(n+p)}. \tag{17}$$

Without restriction we may assume that $M_{22} = 0$ (M is strictly proper). If $M_{22} \neq 0$, we replace M with

$$\bar{M} = \mathcal{S}(M, N_M) = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & 0 \end{bmatrix} \tag{18}$$

with

$$N_M = \begin{bmatrix} 0 & I \\ I & -M_{22} \end{bmatrix} \tag{19}$$

Thus, if we can find a controller $\bar{K}$ to the modified system $\bar{M}$, then $K = \mathcal{S}(N_M, \bar{K})$ will solve the original problem. Since the star product inverse of N_M exists for all M, the modified problem is equivalent to the original one as long as $I - M_{22}\bar{K}$ is not singular.

Due to the simpler structure of M, the star product can be rewritten as the following matrix expression, which is affine in $\bar{K}$:

$$\mathcal{S}(\bar{M}, \bar{K}) = M_{11} + M_{12}\bar{K}M_{21} = Q + U\bar{K}V. \tag{20}$$

3.4 Mathematical Preliminaries

In this section we give two important and related lemmas to be used for solving μ synthesis problems. The lemmas, theorems and proofs are made on the assumption that the matrices are complex. The real case can be obtained by replacing $\mathbb{C}$ with $\mathbb{R}$ without affecting the validity of the theory.

We first consider an affine optimization problem [1].

Lemma 3.1 *Assume that* $Q \in \mathbb{C}^{n\times n}$ *and that both* $U \in \mathbb{C}^{n\times p}$ *and* $V \in \mathbb{C}^{m\times n}$ *have full column rank and row rank respectively. Suppose* $U_\perp \in \mathbb{C}^{n\times(n-p)}$ *and* $V_\perp \in \mathbb{C}^{(n-m)\times n}$ *are chosen such that* $\begin{bmatrix} U & U_\perp \end{bmatrix}$, $\begin{bmatrix} V \\ V_\perp \end{bmatrix}$ *are both invertible, and that* $U^* U_\perp = 0$ *and* $V V_\perp^* = 0$. *Then*

$$\inf_{K\in\mathbb{C}^{p\times m}} \bar{\sigma}\left(Q + UKV\right) < \nu \tag{21}$$

if and only if

$$V_\perp \left(Q^*Q - \nu^2 I\right) V_\perp^* < 0 \tag{22}$$

and

$$U_\perp^* \left(QQ^* - \nu^2 I\right) U_\perp < 0. \tag{23}$$

Proof: *See [1].* □

A related lemma [4, 5], goes as follows.

Lemma 3.2 *Given a hermitian matrix* $Q \in \mathbb{C}^{n\times n}$ *and* U, V, $U_\perp$ *and* $V_\perp$ *as above. Then*

$$Q + UKV + (UKV)^* < 0 \tag{24}$$

is solvable for $K \in \mathbb{C}^{p\times m}$ *if and only if*

$$\begin{cases} U_\perp^* Q U_\perp & < & 0 \\ V_\perp Q V_\perp^* & < & 0. \end{cases} \tag{25}$$

Proof: *Necessity of (25) is clear: for instance,* $U_\perp^* U = 0$ *implies* $U_\perp^* Q U_\perp < 0$ *when pre- and post-multiplying (24) by* $U_\perp^*$ *and* $U_\perp$ *respectively. For details on the sufficiency part, see [4, 5].* □

In [5] the set of K solving (24) is parametrized.

3.5 Complex μ-synthesis

If we restrict the problem to only have complex uncertainties (real uncertainties can be conservatively treated us complex ones), the following lemma from [10] applies.

Lemma 3.3 *Let* Q, U, V, $U_\perp$ *and* $V_\perp$ *be given as above. Then*

$$\inf_{\substack{K\in\mathbb{C}^{p\times m}\\ D\in\mathcal{D}}} \bar{\sigma}\left(D(Q + UKV)D^{-1}\right) < \nu \tag{26}$$

if and only if there exists a $P \in \mathcal{D}$ *such that*

$$V_\perp \left(Q^*PQ - \nu^2 P\right) V_\perp^* < 0 \tag{27}$$

and

$$U_\perp^* \left(QP^{-1}Q^* - \nu^2 P^{-1}\right) U_\perp < 0. \tag{28}$$

Proof: *The necessity of (27) follows by rewriting the inequality in (26) as*

$$(Q + UKV)^* P (Q + UKV) - \nu^2 P < 0$$

and pre and post-multiplying by $V_\perp$ *and* $V_\perp^*$ *respectively. The necessity of (28) is obtained analogously. For the sufficiency part refer to [10].* □

3.6 Mixed μ-synthesis

If both real and complex uncertainties are involved, we have a mixed μ problem. The following lemma then applies.

Lemma 3.4 *Let Q, U, V, $U_\perp$ and $V_\perp$ be given as above. Then*

$$\inf_{\substack{K\in\mathbb{C}^{p\times m}\\ D\in\mathcal{D}\\ G\in\mathcal{G}}} \bar{\sigma}\left(\left(\tfrac{1}{\nu}D(Q+UKV)D^{-1}-jG\right)(I+G^2)^{-\frac{1}{2}}\right)<1, \tag{29}$$

if and only if there exists an $W\in\mathcal{W}$ where $\mathcal{W}=\{W=P+jG : P\in\mathcal{D}, G\in\mathcal{G}\}$, such that

$$\mathrm{herm}\ \left(V_\perp(I+\tfrac{1}{\nu}Q^*)\,W\,(I-\tfrac{1}{\nu}Q)V_\perp^*\right)>0 \tag{30}$$

and

$$\mathrm{herm}\ \left(U_\perp^*(I+\tfrac{1}{\nu}Q)\,W^{-1}\,(I-\tfrac{1}{\nu}Q^*)U_\perp\right)>0 \tag{31}$$

where $\mathrm{herm}\,(X)=\frac{1}{2}(X+X^*)$.

Proof: *Let* $\hat{Q}=\frac{1}{\nu}Q$, $\tilde{Q}=(D\hat{Q}D^{-1}-jG)(I+G^2)^{-\frac{1}{2}}$, $\tilde{U}=DU$, $\tilde{V}=VD^{-1}(I+G^2)^{-\frac{1}{2}}$ *and* $\hat{K}=\frac{1}{\nu}K$. *Then*

$$\bar{\sigma}\left(\tilde{Q}+\tilde{U}\hat{K}\tilde{V}\right)<1 \tag{32}$$

if and only if

$$\begin{aligned}
&\tilde{U}_\perp^*(\tilde{Q}\tilde{Q}^*-I)\tilde{U}_\perp\\
&=U_\perp^*D^{-1}\left((D\hat{Q}D^{-1}-jG)(I+G^2)^{-1}(D^{-*}\hat{Q}^*D^*+jG)-I\right)D^{-*}U_\perp\\
&=U_\perp^*\left(\hat{Q}P_U\hat{Q}^*+j(\hat{Q}G_U-G_U\hat{Q}^*)-P_U\right)U_\perp\\
&=\ \mathrm{herm}\ \left(U_\perp^*(\hat{Q}+I)(P_U-jG_U)(\hat{Q}^*-I)U_\perp\right)<0
\end{aligned} \tag{33}$$

with $P_U=D^{-1}(I+G^2)^{-1}D^{-*}$ *and* $G_U=D^{-1}(I+G^2)^{-1}GD^{-*}$. *Analogously we obtain*

$$\begin{aligned}
&\tilde{V}_\perp(\tilde{Q}^*\tilde{Q}-I)\tilde{V}_\perp^*\\
&=V_\perp D^{-1}\left((D^{-*}\hat{Q}^*D^*+jG)(D\hat{Q}D^{-1}-jG)-(I+G^2)\right)D^{-*}V_\perp\\
&=V_\perp\left(\hat{Q}^*P_V\hat{Q}+j(G_V\hat{Q}-\hat{Q}^*G_V)-P_V\right)V_\perp^*\\
&=\ \mathrm{herm}\ \left(V_\perp(\hat{Q}^*+I)(P_V+jG_V)(\hat{Q}-I)V_\perp^*\right)<0
\end{aligned} \tag{34}$$

with $P_V=D^*D$ *and* $G_U=D^*GD$. *Next, by introducing* $X=P_V+jG_V\in\mathcal{W}$ *and observing that* $W^{-1}=P_U-jG_U$, *we can conclude the proof.* □

Remark 3.1 *Lemma 3.4 follows by letting* $W=W^*=P$.

3.7 Shared uncertainties

We will now consider the problem with shared uncertainties. We augment the system $\tilde{M}$ with through connections between the uncertainty block $\tilde{\Delta}$ and K. We assume that $M_{22} = 0$.

$$\tilde{M} = \left[\begin{array}{c|c} \tilde{M}_{11} & \tilde{M}_{12} \\ \hline \tilde{M}_{21} & 0 \end{array}\right] = \left[\begin{array}{cc|cc} M_{11} & 0 & M_{12} & 0 \\ 0 & 0 & 0 & I_r \\ \hline M_{21} & 0 & 0 & 0 \\ 0 & I_r & 0 & 0 \end{array}\right] = \left[\begin{array}{cc|cc} Q & 0 & U & 0 \\ 0 & 0 & 0 & I_r \\ \hline V & 0 & 0 & 0 \\ 0 & I_r & 0 & 0 \end{array}\right]. \tag{35}$$

Thus, $\tilde{U} = \operatorname{diag}[M_{12}, I_r]$ and $\tilde{V} = \operatorname{diag}[M_{21}, I_r]$. Then, $\tilde{U}_\perp = \operatorname{diag}[U_\perp, 0]$ and $\tilde{V}_\perp = \operatorname{diag}[V_\perp, 0]$. Applying Lemma 3.4 we obtain

$$\operatorname{herm}\left(V_\perp (I + \tfrac{1}{\nu} Q^*)\, S\, (I - \tfrac{1}{\nu} Q) V_\perp^*\right) > 0 \tag{36}$$

and

$$\operatorname{herm}\left(U_\perp^* (I + \tfrac{1}{\nu} Q)\, R\, (I - \tfrac{1}{\nu} Q^*) U_\perp\right) > 0 \tag{37}$$

where $S,\, R \in \mathbb{C}^{n \times n}$ are the upper left block of W and W^{-1} respectively.

$$W = \begin{bmatrix} S & * \\ * & * \end{bmatrix} \tag{38}$$

and

$$W^{-1} = \begin{bmatrix} R & * \\ * & * \end{bmatrix}. \tag{39}$$

The matrices R and S are mutually constrained by these two equations. Next, we will show that this constraint can be reformulated as a rank condition on $I - RS$ and, in the case of complex uncertainties, an additional LMI.

Before going much further we need to specify the structure of $\tilde{\Delta}$, W and W^{-1} more explicitly. The uncertainty matrix $\tilde{\Delta}$ is assumed to be a block diagonal matrix such that

$$\begin{aligned} \Delta &= \operatorname{diag}[\Delta, \mathcal{L}_{\mathcal{R}}(\Delta)] \\ &= \operatorname{diag}[\delta_1 I_{n_1}, \dots, \delta_{f_1} I_{n_{f_1}}, \Delta_{f_1+1}, \dots, \Delta_f, \delta_1 I_{r_1}, \dots, \delta_{f_1} I_{r_{f_1}}] \end{aligned} \tag{40}$$

where $f_1 = f_r + f_c$ is the number of scalar blocks and f is the total number of blocks. Shared blocks are restricted to the type $\delta_i I$ where δ_i is either a real or a complex scalar.

We assume that W have the following block structure.

$$W = \left[\begin{array}{cccc|cccc} W_{111} & 0 & \cdots & 0 & W_{121} & 0 & \cdots & 0 \\ 0 & W_{112} & \cdots & 0 & 0 & W_{122} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & W_{11f} & 0 & 0 & \cdots & W_{12f} \\ \hline W_{211} & 0 & \cdots & 0 & W_{221} & 0 & \cdots & 0 \\ 0 & W_{212} & \cdots & 0 & 0 & W_{222} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & W_{21f} & 0 & 0 & \cdots & W_{22f} \end{array}\right] \tag{41}$$

For notational convenience, all W_{12i}, W_{21i} and W_{22i} are included even if they are not shared and are in such case empty matrices. The structure of W^{-1} will be identical to the structure of W.

We now denote

$$W_i = \begin{bmatrix} W_{11i} & W_{12i} \\ W_{21i} & W_{22i} \end{bmatrix}.$$

Specifically, for uncertainties that are not shared we have $W_i = W_{11i}$.

The structure of W_i is given by the type of uncertainty it refers to. We will now look specifically at two kinds of uncertainties that may be shared: complex and real. They are both of the type $\Delta_i = \delta_i I$, where δ_i is either complex or real. For complex uncertainties we have $W_i = W_i^* > 0$; for real uncertainties W_i could be any square matrix such that $\operatorname{herm} W_i > 0$.

We will now assume that R and S are diagonal block matrices with a structure identical to W_{11}. Depending on the structure of Δ, and consequently of W, the matrices R and S are related to each other. The next two sections treats the two cases of shared uncertainties: complex and real. For notational convenience we drop the suffix i and consider each block at a time.

3.7.1 Complex uncertainties

If we are dealing with complex uncertainties we restrict W to be hermitian and positive definite $W = W^* = P > 0$. Then we have the following lemma from [7].

Lemma 3.5 *Suppose that $R = R^* \in \mathbb{C}^{n\times n}$, $S = S^* \in \mathbb{C}^{n\times n}$, with $R > 0$ and $S > 0$. Let r be a positive integer. Then there exists matrices $N \in \mathbb{C}^{n\times r}$ and $L \in \mathbb{C}^{r\times r}$ such that $L = L^*$ and*

$$P = \begin{bmatrix} S & N \\ N^* & L \end{bmatrix} > 0 \quad \textit{and} \quad P^{-1} = \begin{bmatrix} R & * \\ * & * \end{bmatrix},$$

if and only if

$$\begin{bmatrix} R & I_n \\ I_n & S \end{bmatrix} \geq 0 \quad \textit{and} \quad \operatorname{rank}\left(S - R^{-1}\right) \leq r.$$

Proof: *($\Rightarrow$) Using Schur complements, $R = S^{-1} + S^{-1}N(L - N^*S^{-1}N)^{-1}N^*S^{-1}$. Inverting, using the matrix inversion lemma, gives that $R^{-1} = S - NL^{-1}N^*$. Hence $S - R^{-1} = NL^{-1}N^* \geq 0$, and indeed, $\operatorname{rank}(S - R^{-1}) = \operatorname{rank}(NL^{-1}N^*) \leq r$. ($\Leftarrow$) By assumption, there is a matrix $N \in \mathbb{C}^{n\times r}$ such that $S - R^{-1} = NN^*$. Defining $L = I_r$ completes the construction.* □

3.7.2 Real uncertainties

For real uncertainties we have a more general set of matrices such that $\operatorname{herm} W > 0$. Note that $\operatorname{herm} W > 0$ implies that W^{-1} exists and that $\operatorname{herm}(W^{-1}) > 0$. The following lemma then applies.

Lemma 3.6 *Suppose that $R \in \mathbb{C}^{n\times n}$, $S \in \mathbb{C}^{n\times n}$, with $\operatorname{herm} R > 0$ and $\operatorname{herm} S > 0$. Let r be a positive integer. Then there exists matrices $N \in \mathbb{C}^{n\times r}$, $M \in \mathbb{C}^{r\times n}$ and $L \in \mathbb{C}^{r\times r}$ such that L is nonsingular and*

$$W = \begin{bmatrix} S & N \\ M & L \end{bmatrix}, \quad \operatorname{herm} W > 0 \quad \textit{and} \quad W^{-1} = \begin{bmatrix} R & * \\ * & * \end{bmatrix},$$

if and only if

$$\operatorname{rank}\left(S - R^{-1}\right) \le r.$$

Proof: *(⇒) Using Schur complements,* $R = S^{-1} + S^{-1}N(L - MS^{-1}N)^{-1}MS^{-1}$. *Inverting, using the matrix inversion lemma, gives that* $R^{-1} = S - NL^{-1}M$. *Hence,* $S - R^{-1} = NL^{-1}M$, *and indeed,* $\operatorname{rank}(S - R^{-1}) = \operatorname{rank}(NLM) \le r$. *(⇐) Define* $t = \operatorname{rank}(S - R^{-1})$. *Find any* $M \in \mathbb{C}^{t\times n}$ *of full row rank, such that* $S - R^{-1} = KM_t$, *with* $K \in \mathbb{C}^{n\times t}$ *having full column rank. Choose* N_t *as a linear function of* L_t *by letting* $N_t = (S - R^{-1})M_t^{\dagger}L_t$ *where* $M_t^{\dagger} = M_t^*(M_tM_t^*)^{-1} \in \mathbb{C}^{n\times t}$ *is the Moore-Penrose inverse of* M_t. *By defining* $M = \begin{bmatrix} M_t \\ 0 \end{bmatrix}$, $N = \begin{bmatrix} N_t & 0 \end{bmatrix}$ *and* $L = \operatorname{diag}[L_t, I_{r-t}]$, *we obtain*

$$W = \begin{bmatrix} S & N \\ M & L \end{bmatrix} = \begin{bmatrix} S & 0 & 0 \\ M_t & 0 & 0 \\ 0 & 0 & I_{r-t} \end{bmatrix} + \begin{bmatrix} (S - R^{-1})M_t^{\dagger} \\ I_t \\ 0 \end{bmatrix} L_t \begin{bmatrix} 0 & I_t & 0 \end{bmatrix}.$$

Then, $\operatorname{herm} W_i > 0$ *is equivalent to*

$$\begin{bmatrix} S + S^* & M_t^* \\ M_t & 0 \end{bmatrix} + \begin{bmatrix} (S - R^{-1})M_t^{\dagger} \\ I_t \end{bmatrix} L_t \begin{bmatrix} 0 & I_t \end{bmatrix} + [\,]^* > 0, \tag{42}$$

where $[\,]^*$ *denotes the complex conjugate transpose of the previous term. Using lemma 3.2 with* $U_\perp^* = \begin{bmatrix} I_n & (R^{-1} - S)M_t^{\dagger} \end{bmatrix}$ *and* $V_\perp = \begin{bmatrix} I_n & 0 \end{bmatrix}$, *this is in turn equivalent to*

$$\begin{aligned} & U_\perp^* \begin{bmatrix} S + S^* & M_t^* \\ M_t & 0 \end{bmatrix} U_\perp \\ &= S + S^* + (M_t^{\dagger}M_t)^*(R^{-1} - S)^* + (R^{-1} - S)M_t^{\dagger}M_t \\ &= R^{-1} + KM_t + R^{-*} + (KM_t)^* - (KM_tM_t^{\dagger}M_t)^* - KM_tM_t^{\dagger}M_t \\ &= R^{-1} + R^{-*} = 2\operatorname{herm}(R^{-1}) > 0, \end{aligned}$$

and

$$V_\perp \begin{bmatrix} S + S^* & M_t^* \\ M_t & 0 \end{bmatrix} V_\perp^* = S + S^* = 2\operatorname{herm} S > 0.$$

Thus (42) is satisfied implying that L *is nonsingular, which concludes the proof.*
□

3.8 Summary

It has been shown that the matrix inequality $S \ge R^{-1}$ or equivalently

$$\begin{bmatrix} S & I \\ I & R \end{bmatrix} \ge 0 \tag{43}$$

disappears in the case of real uncertainties. When used for gain scheduling the parameters are most naturally described by real uncertainites. Real uncertainties are parameters while complex uncertainties can be interpreted as single-input-single-output dynamics linear systems (rational transfer functions) with time-varying state-space matrices. Assuming real uncertainties LMI condition (43) used in for instance [9, 7] can be dispensed with.

Thus we have arrived at the following mixed-μ synthesis theorem

Theorem 3.1 *Assume that*

$$M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} = \begin{bmatrix} Q & U \\ V & M_{22} \end{bmatrix}$$

and $\mathcal{W} = \operatorname{diag}[\mathcal{W}_1, \mathcal{W}_2, \ldots, \mathcal{W}_f]$ *and* $\mathcal{R} = [r_1, r_2, \ldots, r_f]$ *be given, where* $r_i > 0$ *denotes shared uncertainties. Let* $U_\perp$ *and* $V_\perp$ *be given as in lemma 3.4. Then*

$$\inf_{\substack{K \in \mathbb{C}^{p\times m} \\ D\in\mathcal{D} \\ G\in\mathcal{G}}} \bar{\sigma}\left(\left(\tfrac{1}{\nu} D\, \mathcal{S}(M,K)\, D^{-1} - jG\right)(I+G^2)^{-\frac{1}{2}}\right) < 1, \tag{44}$$

if and only if there exists $R, S \in \mathcal{W}$ *such that*

$$\operatorname{herm}\left(V_\perp (I + \tfrac{1}{\nu}Q^*)\, S\, (I - \tfrac{1}{\nu}Q) V_\perp^*\right) > 0 \tag{45}$$

and

$$\operatorname{herm}\left(U_\perp^* (I + \tfrac{1}{\nu}Q)\, R\, (I - \tfrac{1}{\nu}Q^*) U_\perp\right) > 0, \tag{46}$$

such that $R = \operatorname{diag}[R_1, R_2, \ldots, R_f]$ *and* $S = \operatorname{diag}[S_1, S_2, \ldots, S_f]$ *with* $\operatorname{rank}(S_i - R_i^{-1}) \le r_i$ *and*

$$\begin{bmatrix} S_i & I \\ I & R_i \end{bmatrix} \ge 0$$

for complex blocks only.

The synthesis problem can also be approached and solved using the positive real Parrot theorem [6].

It is important to note that these equations are linear except for the rank conditions on $S - R^{-1}$, which makes the problem nonconvex in the general case. If we have uncertainties not shared $r_i = 0$ this is always the case. However, if we let $r_i = n_i$ for all uncertainty block, where n_i denotes the dimension of the Δ_i-block visible to M, the problem is linear and thus convex.

4 Conclusions

A novel design algorithm for handling the mixed μ LFT gain scheduling problem has been developed. The synthesis problem consists of two LMIs, which are of the positive real structure. These two LMIs are constrained by a connecting LMI and rank conditions. This is in line with previous results on the same problem with complex uncertainties. For real uncertainties the structure of the solution is very much that the same as for complex uncertainties with the exception that the connecting LMI for real uncertainty blocks can be dispensed with.

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