SIAM J. MATRIX ANAL. APPL.
Vol. 35, No. 2, pp. 437–452

M-TENSORS AND SOME APPLICATIONS*

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Abstract. We introduce M-tensors. This concept extends the concept of M-matrices. We denote Z-tensors as the tensors with nonpositive off-diagonal entries. We show that M-tensors must be Z-tensors and the maximal diagonal entry must be nonnegative. The diagonal elements of a symmetric M-tensor must be nonnegative. A symmetric M-tensor is copositive. Based on the spectral theory of nonnegative tensors, we show that the minimal value of the real parts of all eigenvalues of an M-tensor is its smallest H^+-eigenvalue and also is its smallest H-eigenvalue. We show that a Z-tensor is an M-tensor if and only if all its H^+-eigenvalues are nonnegative. Some further spectral properties of M-tensors are given. We also introduce strong M-tensors, and some corresponding conclusions are given. In particular, we show that all H-eigenvalues of strong M-tensors are positive. We apply this property to study the positive definiteness of a class of multivariate forms associated with Z-tensors. We also propose an algorithm for testing the positive definiteness of such a multivariate form.

Key words. M-tensors, H^+-eigenvalue, Z-tensors, positive definiteness, multivariate form

AMS subject classifications. 15A18, 15A69

DOI. 10.1137/130915339

1. Introduction.

Tensors are increasingly ubiquitous in various areas of applied, computational, and industrial mathematics and have wide applications in data analysis and mining, information science, signal/image processing, and computational biology as well [5, 9, 15, 17]. A tensor can be regarded as a higher-order generalization of a matrix, which takes the form

$$\mathcal{A} = (A_{i_1\cdots i_m}), \quad A_{i_1\cdots i_m} \in R, \quad 1 \le i_1, \dots, i_m \le n.$$

Such a multiarray $\mathcal{A}$ is said to be an *m-order n-dimensional square real tensor* with n^m entries $A_{i_1\cdots i_m}$. Eigenvalues of tensors were introduced in [17, 22] in 2005. Since then, much work has been done in spectral theory of tensors. In particular, theory of, and algorithms for calculating, eigenvalues of nonnegative tensors are well developed [6, 7, 8, 12, 16, 18, 19, 23, 26, 27, 28].

It is known that an mth degree homogeneous polynomial form of n variables $g(x)$, where $x \in R^n$, can be denoted as

$$g(x) := \sum_{i_1,i_2,\dots,i_m=1}^{n} A_{i_1 i_2 \dots i_m} x_{i_1} x_{i_2} \cdots x_{i_m}. \tag{1.1}$$

When m is even, $g(x)$ is called *positive definite* if

$$g(x) > 0 \quad \forall x \in R^n,\ x \ne 0.$$

*Received by the editors April 2, 2013; accepted for publication (in revised form) by L. De Lathauwer December 11, 2013; published electronically April 8, 2014.
http://www.siam.org/journals/simax/35-2/91533.html

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Testing positive definiteness of a multivariate form defined as (1.1) is an important problem in the stability study of nonlinear autonomous systems via Lyapunov's direct method in automatic control [20]. Researchers in automatic control have studied the conditions of such positive definiteness intensively [2, 3, 4, 13]. However, for $n \geq 3$ and $m \geq 4$, this is a hard problem in mathematics. There are only a few methods for solving the problem [3, 4, 20]. In practice, when $n > 3$ and $m \geq 4$, these methods are computationally expensive. Recently, some efficient methods based on eigenvalues of tensors were proposed to solve the problem [18, 20]. Moreover, the theory of M-matrices was used to certify avoidance conditions in stability autonomous systems [24]. Motivated by these observations, we extend the concept of M-matrices to tensors and then introduce M-tensors. Our purpose is to propose a new method for testing positive definiteness of a multivariate form using the spectral properties of M-tensors.

The concept of M-matrices, which have many applications in various fields such as computational mathematics, mathematical physics, mathematical economics, graph theory, and wireless communications [1, 10, 14, 24], was introduced by Ostrowski [21] in 1937 [1, 10, 14, 25]. M-matrices have the following form [1, 14].

Definition 1.1. *Any real matrix A of the form*

$$A = sI - B, \quad \text{where } s > 0 \text{ and } B \text{ is a nonnegative matrix,}$$

for which $s \geq \rho(B)$, the spectral radius of B, is called an M-matrix. If $s > \rho(B)$, then A is called a nonsingular M-matrix.

In this paper, we extend the concept in Definition 1.1 to tensors. We introduce M-tensors and strong M-tensors. By using spectral theory of nonnegative tensors [6, 12, 23, 26], we give some properties of M-tensors. We prove that the smallest H^+-eigenvalue of an M-tensor is nonnegative. We show that an M-tensor has at least one nonnegative H^+-eigenvalue, a weakly irreducible M-tensor has a unique H^{++}-eigenvalue, and an irreducible M-tensor has a unique H^+-eigenvalue. Similar to Z-matrices, we denote tensors with all nonpositive off-diagonal entries by *Z-tensors*. Note that M-tensors belong to the class of Z-tensors. We show that a Z-tensor is an M-tensor if and only if all its H^+-eigenvalues are nonnegative. Moreover, a Z-tensor is a strong M-tensor if and only if all its H^+-eigenvalues are positive. We show that the class of M-tensors can be viewed as the closure of the class of strong M-tensors. Some further spectral properties are also established. Finally, we apply some spectral properties of M-tensors to study the positive definiteness of a class of multivariate forms associated with Z-tensors. We propose an algorithm for testing the positive definiteness of such a multivariate form. It should be pointed out that the class of multivariate forms studied in [18] is a special case of our model. We do not need the assumption that the diagonal entries are positive.

This paper is organized as follows. In section 2, we recall some preliminary results. We introduce M-tensors and characterize some basic properties of M-tensors in section 3. In section 4, we discuss some applications of M-tensors. Finally, we conclude the paper with some remarks in section 5.

2. Preliminaries.

We start this section with some fundamental notions and properties on tensors. An m-order n-dimensional tensor $\mathcal{A} = (A_{i_1 \cdots i_m})$ is called *nonnegative* if each entry is nonnegative. The tensor $\mathcal{A}$ is called *symmetric* if its entries $A_{i_1 \cdots i_m}$ are invariant under any permutation of their indices $\{i_1 \cdots i_m\}$ [22]. The m-order n-dimensional identity tensor, denoted by $\mathcal{I} = (I_{i_1 \dots i_m})$, is the tensor with entries

$$I_{i_1\ldots i_m} = \begin{cases} 1 & \text{if } i_1 = \cdots = i_m, \\ 0 & \text{otherwise.} \end{cases}$$

A tensor $\mathcal{A}$ is called *reducible* [6] if there exists a nonempty proper index subset $I \subset \{1, 2, \ldots, n\}$ such that

$$A_{i_1\cdots i_m} = 0 \quad \forall i_1 \in I \quad \forall i_2, \ldots, i_m \notin I.$$

Otherwise, we say $\mathcal{A}$ is *irreducible*.

Let $\mathcal{A} = (A_{i_1\cdots i_m})$ be a nonnegative tensor. We call an $n \times n$ nonnegative matrix $R(\mathcal{A})$ the *representation* of $\mathcal{A}$ if the (i, j)th element of $R(\mathcal{A})$ is defined to be the summation of $A_{ii_2\cdots i_m}$ with indices $\{i_2 \cdots i_m\} \ni j$. We say that the tensor $\mathcal{A}$ is *weakly reducible* if its representation $R(\mathcal{A})$ is a reducible matrix. If $\mathcal{A}$ is not weakly reducible, then it is called *weakly irreducible* [12, 16].

The following definitions about eigenvalues of tensors were introduced by Qi [17, 22]. Let C (R) be the complex (real) field. The nonnegative orthant of R^n is denoted by R^n_+ and the interior of R^n_+ denoted by R^n_{++}. For a vector $x \in \mathrm{C}^n$, we use x_i to denote its components and $x^{[m-1]}$ to denote a vector in C^n such that

$$x_i^{[m-1]} = x_i^{m-1}$$

for all i. $\mathcal{A}x^{m-1}$ denotes a vector in C^n, whose ith component is

$$\sum_{i_2,\ldots,i_m=1}^{n} A_{ii_2\cdots i_m} x_{i_2} \cdots x_{i_m}.$$

And we write

$$\mathcal{A}x^m = \sum_{i_1,\ldots,i_m=1}^{n} A_{i_1\cdots i_m} x_{i_1} \cdots x_{i_m}.$$

If a pair $(\lambda, x) \in \mathrm{C} \times (\mathrm{C}^n \backslash \{0\})$ satisfies

$$\mathcal{A}x^{m-1} = \lambda x^{[m-1]}, \tag{2.1}$$

then we call λ an *eigenvalue* of $\mathcal{A}$ and x its corresponding *eigenvector*. In particular, if x is real, then λ is also real. In this case, we say that λ is an *H-eigenvalue* of $\mathcal{A}$ and x its corresponding *H-eigenvector*. If $x \in \mathrm{R}^n_+(\mathrm{R}^n_{++})$, then λ is called an H^+*-eigenvalue* (H^{++}*-eigenvalue*) of $\mathcal{A}$. The largest modulus of the eigenvalues of $\mathcal{A}$ is called the *spectral radius* of $\mathcal{A}$, denoted by $\rho(\mathcal{A})$.

When m is even and $\mathcal{A}$ is symmetric, we say that $\mathcal{A}$ is *positive definite (semidefinite)* if $\mathcal{A}x^m > 0$ ($\mathcal{A}x^m \geq 0$) for all $x \in \mathrm{R}^n$ and $x \neq 0$. It is proved in [22, Theorem 5] that $\mathcal{A}$ is positive definite (semidefinite) if and only if all its H-eigenvalues are positive (nonnegative).

We now recall some existing results on tensors which will be used in the next section. The following theorem summarizes the Perron–Frobenius theorem for nonnegative tensors; see, e.g., [6, 12, 23, 26].

THEOREM 2.1. *Let $\mathcal{A}$ be a nonnegative tensor. Then the spectral radius $\rho(\mathcal{A})$ is an H^+-eigenvalue of $\mathcal{A}$. If $\mathcal{A}$ is weakly irreducible, then $\rho(\mathcal{A})$ is the unique H^{++}-eigenvalue of $\mathcal{A}$. If $\mathcal{A}$ is irreducible, then $\rho(\mathcal{A})$ is the unique H^+-eigenvalue of $\mathcal{A}$.*

The following lemma was given by Qi [22, Corollary 3].

LEMMA 2.2. *Let $\mathcal{A}$ be an m-order n-dimensional tensor. Suppose that $\mathcal{B} = a(\mathcal{A}+b\mathcal{I})$, where a and b are two real numbers. Then μ is an eigenvalue (H-eigenvalue) of $\mathcal{B}$ if and only if $\mu = a(\lambda + b)$ and λ is an eigenvalue (H-eigenvalue) of $\mathcal{A}$. In this case, they have the same eigenvectors (H-eigenvectors).*

Let $\mathcal{A}$ be an m-order and n-dimensional tensor. Denote its smallest H-eigenvalues by $\lambda_{\min}(\mathcal{A})$ and define

$$R_{\min}(\mathcal{A}) = \min_{1\le i\le n} \sum_{i_2,\dots,i_m=1}^{n} A_{ii_2\dots i_m}.$$

If all the off-diagonal entries are nonpositive, then $\mathcal{A}$ is called a *Z-tensor*. Note that Z-tensors directly generalize Z-matrices. Some existing results on symmetric Z-tensors [23] are summarized in the following lemma.

LEMMA 2.3. *Let $\mathcal{A} = (A_{i_1\cdots i_m})$ be a symmetric Z-tensor. Then we have*

$$\lambda_{\min}(\mathcal{A}) = \min\left\{\mathcal{A}x^m : x \in R^n_+,\ \sum_{i=1}^{n} x_i^m = 1\right\}$$

and

$$R_{\min}(\mathcal{A}) \le \lambda_{\min}(\mathcal{A}) \le \min_{i=1,\dots,n} A_{i\cdots i}.$$

Qi [23] introduced copositive tensors and strictly copositive tensors, which extend the concept of copositive matrices. A real symmetric tensor $\mathcal{A} = (A_{i_1\cdots i_m})$ is called a *copositive tensor* if for any $x \in \mathrm{R}^n_+$, we have $\mathcal{A}x^m \ge 0$. We say that $\mathcal{A}$ is a *strictly copositive tensor* if for any $x \in \mathrm{R}^n_+$, $x \ne 0$, we have $\mathcal{A}x^m > 0$. The main characterization theorem for copositive tensors is summarized in the following lemma [23, Theorem 5].

LEMMA 2.4. *Let $\mathcal{A} = (A_{i_1\cdots i_m})$ be a symmetric tensor. Then, $\mathcal{A}$ is copositive if and only if*

$$\min\left\{\mathcal{A}x^m : x \in R^n_+,\ \sum_{i=1}^{n} x_i^m = 1\right\} \ge 0.$$

$\mathcal{A}$ is strictly copositive if and only if

$$\min\left\{\mathcal{A}x^m : x \in R^n_+,\ \sum_{i=1}^{n} x_i^m = 1\right\} > 0.$$

3. M-tensors and strong M-tensors. In this section, we introduce *M-tensors* and *strong M-tensors*, which extend Definition 1.1 from matrices to tensors. Based on the spectral theory of nonnegative tensors, we give characterization theorems for M-tensors and strong M-tensors.

DEFINITION 3.1. *Let $\mathcal{A}$ be an m-order and n-dimensional tensor. $\mathcal{A}$ is called an M-tensor if there exist a nonnegative tensor $\mathcal{B}$ and a positive real number $\eta \ge \rho(\mathcal{B})$ such that*

$$\mathcal{A} = \eta\mathcal{I} - \mathcal{B}.$$

If $\eta > \rho(\mathcal{B})$, then $\mathcal{A}$ is called a strong M-tensor.

This concept extends the concept of M-matrices given in Definition 1.1 and [1, Definition 6.1.2]. Clearly, when $m = 2$, if $\mathcal{A}$ is an M-tensor, then $\mathcal{A}$ is an M-matrix; if $\mathcal{A}$ is a strong M-tensor, then $\mathcal{A}$ is a nonsingular M-matrix.

Note that the off-diagonal entries of M-tensors are nonpositive, hence M-tensors belong to the class of Z-tensors. We begin with a theorem that shows that the class of M-tensors can be thought of as the closure of the class of strong M-tensors.

THEOREM 3.2. *$\mathcal{A}$ is an M-tensor if and only if $\mathcal{A}+\varepsilon\mathcal{I}$ is a strong M-tensor for all scalars $\varepsilon > 0$.*

Proof. Let $\mathcal{A}$ be an M-tensor of the form

$$\mathcal{A} = \eta\mathcal{I} - \mathcal{B}, \quad \eta > 0, \quad \mathcal{B} \geq 0.$$

Then, for any $\varepsilon > 0$

$$\mathcal{A} + \varepsilon\mathcal{I} = \eta\mathcal{I} - \mathcal{B} + \varepsilon\mathcal{I} = (\eta+\varepsilon)\mathcal{I} - \mathcal{B} = \eta'\mathcal{I} - \mathcal{B}, \tag{3.1}$$

where $\eta' = \eta + \varepsilon > \rho(\mathcal{B})$ since $\eta \geq \rho(\mathcal{B})$. Thus $\mathcal{A}+\varepsilon\mathcal{I}$ is a strong M-tensor.

Conversely, if $\mathcal{A}+\varepsilon\mathcal{I}$ is a strong M-tensor for all $\varepsilon > 0$, then it follows that $\mathcal{A}$ is an M-tensor by considering (3.1) and letting ε approach zero. ◻

We now analyze the spectral properties of M-tensors. By Theorem 2.1 and Lemma 2.2, the following theorem shows that an M-tensor has at least one nonnegative H$^+$-eigenvalue, and a strong M-tensor has at least one positive H$^+$-eigenvalue.

THEOREM 3.3. *Let $\mathcal{A}$ be an M-tensor and denote by $\sigma(\mathcal{A})$ the set of eigenvalues of $\mathcal{A}$. Let* $\mathrm{Re}\lambda$ *be the real part of eigenvalue $\lambda \in \sigma(\mathcal{A})$. Then* $\min_{\lambda\in\sigma(\mathcal{A})} \mathrm{Re}\lambda$ *is a nonnegative H$^+$-eigenvalue. If $\mathcal{A}$ is a strong M-tensor, then* $\min_{\lambda\in\sigma(\mathcal{A})} \mathrm{Re}\lambda > 0$.

Proof. Since $\mathcal{A}$ is an M-tensor, by Definition 3.1, there exist a nonnegative tensor $\mathcal{B}$ and a positive number $c \geq \rho(\mathcal{B})$ such that

$$\mathcal{A} = c\mathcal{I} - \mathcal{B}.$$

Denote $\iota(\mathcal{A}) = c - \rho(\mathcal{B})$; we have $\iota(\mathcal{A}) \geq 0$. By Theorem 2.1, $\rho(\mathcal{B})$ is an H$^+$-eigenvalue of $\mathcal{B}$. By Lemma 2.2, $\iota(\mathcal{A})$ is an eigenvalue of $\mathcal{A}$. Moreover, $\iota(\mathcal{A})$ and $\rho(\mathcal{B})$ have the same eigenvectors. Hence, $\iota(\mathcal{A})$ is an H$^+$-eigenvalue of $\mathcal{A}$.

Let $\lambda \in \sigma(\mathcal{A})$ and $\mathrm{Re}\lambda$ be the real part of λ. Then

$$\iota(\mathcal{A}) \geq \min_{\lambda\in\sigma(\mathcal{A})} \mathrm{Re}\lambda. \tag{3.2}$$

By Lemma 2.2, $c - \lambda$ is an eigenvalue of $\mathcal{B}$. Since $\rho(\mathcal{B})$ is the spectral radius of $\mathcal{B}$ and $c \geq \rho(\mathcal{B})$,

$$\rho(\mathcal{B}) \geq |c-\lambda| \geq c - \mathrm{Re}\lambda \geq \rho(\mathcal{B}) - \mathrm{Re}\lambda, \tag{3.3}$$

which implies that $\mathrm{Re}\lambda \geq 0$, and hence

$$\rho(\mathcal{B}) \geq \max_{\lambda\in\sigma(\mathcal{A})} \{c - \mathrm{Re}\lambda\} = c - \min_{\lambda\in\sigma(\mathcal{A})} \mathrm{Re}\lambda,$$

which, together with (3.2) and $\iota(\mathcal{A}) = c - \rho(\mathcal{B})$, yields

$$\iota(\mathcal{A}) = \min_{\lambda\in\sigma(\mathcal{A})} \mathrm{Re}\lambda.$$

That is, $\min_{\lambda\in\sigma(\mathcal{A})} \mathrm{Re}\lambda$ is a nonnegative H$^+$-eigenvalue of $\mathcal{A}$.

If $\mathcal{A}$ is a strong M-tensor, then $c > \rho(\mathcal{B})$. It follows from (3.3) that $\min_{\lambda\in\sigma(\mathcal{A})} \mathrm{Re}\lambda > 0$. ◻

By Theorems 3.3 and 2.1, we immediately have the following conclusions.

THEOREM 3.4. *Let $\mathcal{A} = (A_{i_1 \cdots i_m})$ be an (strong) M-tensor. Then*

(a) $\min_{\lambda \in \sigma(\mathcal{A})} \mathrm{Re}\lambda$ *is the smallest (positive) nonnegative H^+-eigenvalue of $\mathcal{A}$;*
(b) *any of its eigenvalues has a (positive) nonnegative real part;*
(c) *all its H-eigenvalues are (positive) nonnegative;*
(d) *if $\mathcal{A}$ is weakly irreducible, then $\mathcal{A}$ has a unique (positive) nonnegative H^{++}-eigenvalue;*
(e) *if $\mathcal{A}$ is irreducible, then $\mathcal{A}$ has a unique (positive) nonnegative H^+-eigenvalue.*

By Theorem 3.4, Lemma 2.3, and Lemma 2.4, we have the following further conclusions for symmetric (strong) M-tensors.

THEOREM 3.5. *Let $\mathcal{A} = (A_{i_1 \cdots i_m})$ be a symmetric (strong) M-tensor. Then*

(a) $\min_{\lambda \in \sigma(\mathcal{A})} \mathrm{Re}\lambda = \min\{\mathcal{A}x^m : \ x \in R^n_+, \ \sum_{i=1}^n x_i^m = 1\}$*;*
(b) $R_{\min}(\mathcal{A}) \leq \min_{\lambda \in \sigma(\mathcal{A})} \mathrm{Re}\lambda \leq \min_{i=1,\ldots,n} A_{i \cdots i}$*;*
(c) *all the diagonal entries are (positive) nonnegative;*
(d) *$\mathcal{A}$ is positive (definite) semidefinite when m is even;*
(e) *$\mathcal{A}$ is (strictly) copositive.*

Proof. Clearly, $\mathcal{A}$ is a symmetric Z-tensor. By Lemma 2.3 and Theorem 3.4, we immediately have (a), (b), (c), and (d). By Lemma 2.4, Theorem 3.5(a), and Theorem 3.4(b), (e) is obvious. ∎

For a symmetric M-tensor, it is known that its diagonal entries are nonnegative. But for an asymmetric M-tensor, we show that all largest diagonal entries are nonnegative.

THEOREM 3.6. *Let $\mathcal{A} = (A_{i_1 \cdots i_m})$ be an M-tensor. Then we have*

$$\max_{1 \leq i \leq n} A_{i \ldots i} \ \geq 0.$$

If $\mathcal{A} = (A_{i_1 \cdots i_m})$ is a strong M-tensor, then

$$\max_{1 \leq i \leq n} A_{i \ldots i} \ > 0.$$

Proof. Clearly, $\mathcal{A}$ is a Z-tensor. Define $a = \max_{1 \leq i \leq n} A_{i \ldots i}$ and $\mathcal{B} = a\mathcal{I} - \mathcal{A}$. Then $\mathcal{B} \geq 0$ and hence $\rho(\mathcal{B})$ is an H$^+$-eigenvalue of $\mathcal{B}$. So, by Lemma 2.2, $a - \rho(\mathcal{B})$ is an H$^+$-eigenvalue of $\mathcal{A}$.

Since $\mathcal{A}$ is an M-tensor, by Theorem 3.4(c), $a - \rho(\mathcal{B}) \geq 0$, which implies $a \geq \rho(\mathcal{B}) \geq 0$.

If $\mathcal{A}$ is a strong M-tensor, by Theorem 3.4(c), $a - \rho(\mathcal{B}) > 0$, i.e., $a > \rho(\mathcal{B}) \geq 0$. This completes the proof. ∎

Define $\iota(\mathcal{A}) := \min_{\lambda \in \sigma(\mathcal{A})} \mathrm{Re}\lambda$. The following theorem gives a way to obtain an M-tensor from a strong M-tensor.

THEOREM 3.7. *Let $\mathcal{A} = (A_{i_1 \cdots i_m})$ be a strong M-tensor. Then $\mathcal{A} - \iota(\mathcal{A})\mathcal{I}$ is an M-tensor. In particular, zero is an H^+-eigenvalue of $\mathcal{A} - \iota(\mathcal{A})\mathcal{I}$.*

Proof. Since $\mathcal{A}$ is a strong M-tensor, there exist a nonnegative tensor $\mathcal{B}$ and a real number $c > \rho(\mathcal{B})$ such that

$$\mathcal{A} = c\mathcal{I} - \mathcal{B}.$$

Hence,

$$\mathcal{A} - \iota(\mathcal{A})\mathcal{I} = c\mathcal{I} - \mathcal{B} - \iota(\mathcal{A})\mathcal{I} = (c - \iota(\mathcal{A}))\mathcal{I} - \mathcal{B}.$$

Let $\eta = c - \iota(\mathcal{A})$. By Theorem 3.4 and the proof of Theorem 3.3, $\iota(\mathcal{A}) = c - \rho(\mathcal{B})$. Hence, we have

$$\mathcal{A} - \iota(\mathcal{A})\mathcal{I} = \eta\mathcal{I} - \mathcal{B}, \quad \eta = \rho(\mathcal{B}),$$

which shows that $\mathcal{A} - \iota(\mathcal{A})\mathcal{I}$ is an M-tensor and that $\eta - \rho(\mathcal{B}) = 0$ is its H^+-eigenvalue. ∎

By Lemma 2.2, we have the following theorem that gives the region of all eigenvalues of an M-tensor.

THEOREM 3.8. *Let $\mathcal{A} = (A_{i_1\cdots i_m})$ be an M-tensor and denote $a = \max_{1\le i\le n} A_{i\ldots i}$. Then the circular region in the complex plane with center at a and radius $\rho(a\mathcal{I} - \mathcal{A})$ contains the entire spectrum of $\mathcal{A}$, i.e.,*

$$|\lambda - a| \le \rho(a\mathcal{I} - \mathcal{A}) \ \forall \lambda \in \sigma(\mathcal{A}).$$

Proof. Clearly, $a\mathcal{I} - \mathcal{A}$ is a nonnegative tensor. By Lemma 2.2, for any $\lambda \in \sigma(\mathcal{A})$, $a - \lambda$ is also an eigenvalue of $a\mathcal{I} - \mathcal{A}$. So, $|\lambda - a| \le \rho(a\mathcal{I} - \mathcal{A})$. ∎

We now give some necessary and sufficient conditions for a Z-tensor to be an M-tensor.

THEOREM 3.9. *Let $\mathcal{A} = (A_{i_1\cdots i_m})$ be a Z-tensor. Then,*

(a) *$\mathcal{A}$ is an M-tensor if and only if $\min_{\lambda\in\sigma(\mathcal{A})} \mathrm{Re}\lambda \ge 0$,*

(b) *$\mathcal{A}$ is a strong M-tensor if and only if $\min_{\lambda\in\sigma(\mathcal{A})} \mathrm{Re}\lambda > 0$.*

Proof. Necessity: Theorem 3.4(a) shows necessity.

Sufficiency: Let $a = \max_{1\le i\le n}\{A_{i\ldots i}\}$. Then $\mathcal{B} = a\mathcal{I} - \mathcal{A}$ is nonnegative. By Lemma 2.2 and Theorem 2.1, $a - \rho(\mathcal{B})$ is an H^+-eigenvalue of $\mathcal{A}$. Hence, $a - \rho(\mathcal{B}) \ge 0$ due to $\min_{\lambda\in\sigma(\mathcal{A})} \mathrm{Re}\lambda \ge 0$. So, $\mathcal{A} = a\mathcal{I} - \mathcal{B}$ is an M-tensor. The proof for strong M-tensors is similar. ∎

By Theorem 3.3, $\min_{\lambda\in\sigma(\mathcal{A})} \mathrm{Re}\lambda$ is the smallest H-eigenvalue (H^+-eigenvalue) of $\mathcal{A}$. Hence, by Theorem 3.9, we immediately have the following conclusions.

COROLLARY 3.10. *Let $\mathcal{A} = (A_{i_1\cdots i_m})$ be a Z-tensor. Then,*

(a) *$\mathcal{A}$ is a (strong) M-tensor if and only if all its H^+-eigenvalues are (positive) nonnegative,*

(b) *$\mathcal{A}$ is a (strong) M-tensor if and only if all its H-eigenvalues are (positive) nonnegative.*

Let $\mathcal{A} = (A_{i_1\cdots i_m})$ be a Z-tensor. Clearly, we can define a nonnegative tensor by

$$\mathcal{B} = a\mathcal{I} - \mathcal{A}, \quad a = \max_{1\le i\le n}\{A_{i\ldots i}\}.$$

Thus we have the following necessary and sufficient condition, which provides us an easy method for determining whether a Z-tensor $\mathcal{A}$ is an M-tensor. We only need to compute the spectral radius ρ of the tensor $a\mathcal{I} - \mathcal{A}$. If $a \ge \rho$, then $\mathcal{A}$ is an M-tensor. Otherwise, $\mathcal{A}$ is not an M-tensor.

THEOREM 3.11. *Let $\mathcal{A} = (A_{i_1\cdots i_m})$ be a Z-tensor. Then,*

(a) *$\mathcal{A}$ is an M-tensor if and only if $a \ge \rho(a\mathcal{I} - \mathcal{A})$,*

(b) *$\mathcal{A}$ is a strong M-tensor if and only if $a > \rho(a\mathcal{I} - \mathcal{A})$.*

Proof. Clearly, $\mathcal{B} = a\mathcal{I} - \mathcal{A} \ge 0$. If $a \ge \rho(\mathcal{B})$, then $\mathcal{A}$ is an M-tensor. Conversely, by Lemma 2.2 and Theorem 2.1, $a - \rho(\mathcal{B})$ is an H^+-eigenvalue of $\mathcal{A}$. Since $\mathcal{A}$ is an M-tensor, by Corollary 3.10, $a - \rho(\mathcal{B}) \ge 0$, i.e., $a \ge \rho(\mathcal{B})$. This completes the proof of (a). Similarly, we can prove (b). ∎

Theorem 3.5(e) shows that symmetric (strong) M-tensors are (strictly) copositive. We next show that for a symmetric Z-tensor, the converse propositions are also true. By Lemmas 2.3 and 2.4, Theorem 3.5, and Corollary 3.10, we immediately obtain the following conclusion.

THEOREM 3.12. *Let $\mathcal{A} = (A_{i_1\cdots i_m})$ be a symmetric Z-tensor. Then,*

(a) *$\mathcal{A}$ is an M-tensor if and only if it is copositive,*

(b) *$\mathcal{A}$ is a strong M-tensor if and only if it is strictly copositive.*

It is well known that [10] an M-matrix plus any nonnegative diagonal matrix is still an M-matrix. By Theorem 3.12, we easily extend the statement from matrices to symmetric Z-tensors in the following theorem. This theorem gives a way to obtain new M-tensors from a given M-tensor, namely, by increasing the diagonal entries.

THEOREM 3.13. *Let $\mathcal{D}$ be any nonnegative diagonal tensor. If $\mathcal{A}$ is a symmetric (strong) M-tensor, then $\mathcal{A}+\mathcal{D}$ is also a symmetric (strong) M-tensor.*

Proof. Clearly, $\mathcal{A}+\mathcal{D}$ is a symmetric Z-tensor. Since $\mathcal{A}$ is a symmetric M-tensor, by Theorem 3.12, $\mathcal{A}$ is copositive. Hence, $\mathcal{A}x^m \geq 0$ for all $x \in \mathrm{R}^n_+$. Since $\mathcal{D}$ is a nonnegative diagonal tensor, we have

$$\mathcal{D}x^m = \sum_{i=1}^{n} D_{i\ldots i}x_i^m \geq 0 \ \forall x \in \mathrm{R}^n_+.$$

Hence,

$$(\mathcal{A}+\mathcal{D})x^m = \mathcal{A}x^m + \mathcal{D}x^m \geq 0 \ \forall x \in \mathrm{R}^n_+.$$

That is, $\mathcal{A}+\mathcal{D}$ is copositive. By Theorem 3.12(a), $\mathcal{A}+\mathcal{D}$ is an M-tensor. The proof for strong M-tensors is similar. ∎

Finally, we give a sufficient condition for a tensor to be an M-tensor. First, we introduce the definition of *diagonally dominant*, which is an extension from matrices to tensors [10].

DEFINITION 3.14. *Let $\mathcal{A}$ be an m-order and n-dimensional tensor. $\mathcal{A}$ is diagonally dominant if for $i = 1,\ldots,n$,*

$$\sum_{(i,i_2,\ldots,i_m)\neq(i,i,\ldots,i)} |A_{ii_2\ldots i_m}| \leq |A_{ii\ldots i}|. \tag{3.4}$$

$\mathcal{A}$ is strictly diagonally dominant if the strict inequality holds in (3.4) *for all i. $\mathcal{A}$ is irreducibly diagonally dominant if $\mathcal{A}$ is irreducible and diagonally dominant and the strict inequality in* (3.4) *holds for at least one i.*

THEOREM 3.15. *Let $\mathcal{A} = (A_{i_1\cdots i_m})$ be a Z-tensor with nonnegative diagonal entries. If $\mathcal{A}$ is diagonally dominant, then $\mathcal{A}$ is an M-tensor. If $\mathcal{A}$ is strictly or irreducibly diagonally dominant, then $\mathcal{A}$ is a strong M-tensor.*

Proof. Let λ be an eigenvalue of $\mathcal{A}$ with a nonzero eigenvector x. Let x_i be the entry with largest modulus. Then

$$\sum_{i_2,\ldots,i_m=1}^{n} A_{ii_2\ldots i_m}x_{i_2}\cdots x_{i_m} = \lambda x_i^{m-1}, \tag{3.5}$$

which implies

$$|\lambda - A_{ii\ldots i}| \leq \sum_{(i,i_2,\ldots,i_m)\neq(i,i,\ldots,i)} |A_{ii_2\ldots i_m}|.$$

Hence, the diagonal dominance of $\mathcal{A}$ implies

$$|\mathrm{Re}\lambda - A_{i\ldots i}| \leq |\lambda - A_{i\ldots i}| \leq |A_{i\ldots i}|. \tag{3.6}$$

Since $A_{j\dots j} \geq 0$ for $j = 1, 2, \dots, n$, (3.6) yields

$$\mathrm{Re}\lambda - A_{i\dots i} \geq -A_{i\dots i}, \tag{3.7}$$

which implies $\mathrm{Re}\lambda \geq 0$. By Theorem 3.9, $\mathcal{A}$ is an M-tensor.

Suppose that $\mathcal{A}$ is strictly diagonally dominant. Then the strict inequality holds in (3.4) for all j, so the strict inequality holds in (3.7). This yields $\mathrm{Re}\lambda > 0$. By Theorem 3.9, $\mathcal{A}$ is a strong M-tensor.

Suppose now that $\mathcal{A}$ is irreducibly diagonally dominant. Define

$$J = \left\{ l : |x_l| = \max_{1 \leq i \leq n} |x_i|, \, |x_l| > |x_i| \text{ for some } i \right\}.$$

If $J = \emptyset$, then (3.5) and the diagonal dominance of $\mathcal{A}$ imply that for $i = 1, \dots, n$,

$$|\lambda - A_{ii\dots i}| \leq \sum_{(i,i_2,\dots,i_m) \neq (i,i,\dots,i)} |A_{ii_2\dots i_m}| \leq |A_{ii\dots i}|.$$

Let

$$|A_{kk\dots k}| > \sum_{(k,i_2,\dots,i_m) \neq (k,k,\dots,k)} |A_{ki_2\dots i_m}|$$

for some k. We have

$$|\mathrm{Re}\lambda - A_{k\dots k}| \leq |\lambda - A_{k\dots k}| < |A_{k\dots k}| = A_{k\dots k},$$

which implies $\mathrm{Re}\lambda > 0$.

If $J \neq \emptyset$, then the irreducibility of $\mathcal{A}$ implies that there exist $l \in J$ and $i_2, \dots, i_m \notin J$ such that$A_{li_2\dots i_m} \neq 0$. Hence (3.5) yields

$$\begin{aligned} |\lambda - A_{ll\dots l}| &\leq \sum_{(l,i_2,\dots,i_m) \neq (l,l,\dots,l)} |A_{li_2\dots i_m}| \frac{|x_{i_2}|}{|x_l|} \cdots \frac{|x_{i_m}|}{|x_l|} \\ &< \sum_{(l,i_2,\dots,i_m) \neq (l,l,\dots,l)} |A_{li_2\dots i_m}| \leq |A_{ll\dots l}|, \end{aligned}$$

which implies $\mathrm{Re}\lambda > 0$. By Theorem 3.9, $\mathcal{A}$ is a strong M-tensor. $\square$

By the above theorem, we may give another necessary and sufficient condition. Before presenting our theorem, we give the following lemma. Let $\mathcal{A}$ be an m-order and n-dimensional tensor and $D = \mathrm{diag}(d_1, \dots, d_n)$ be a positive diagonal matrix. Define a new tensor $\mathcal{B} = (B_{i_1 i_2 \dots i_m})$:

$$\mathcal{B} = \mathcal{A} \cdot D^{-(m-1)} \cdot \overbrace{D \cdots D}^{m-1} \tag{3.8}$$

with

$$B_{i_1 i_2 \dots i_m} = A_{i_1 i_2 \dots i_m} d_{i_1}^{-(m-1)} d_{i_2} \cdots d_{i_m}.$$

Then, we have this lemma given in [26].

LEMMA 3.16. *If λ is an eigenvalue of $\mathcal{A}$ with corresponding eigenvector x, then λ is also an eigenvalue of $\mathcal{B}$ with corresponding eigenvector $D^{-1}x$; if τ is an eigenvalue of $\mathcal{B}$ with corresponding eigenvector y, then τ is also an eigenvalue of $\mathcal{A}$ with corresponding eigenvector Dy;*

Based on the above lemma, we establish the following necessary and sufficient condition.

THEOREM 3.17. *Let $\mathcal{A} = (A_{i_1\cdots i_m})$ be a Z-tensor. Suppose that $\mathcal{A}$ is weakly irreducible and has all nonnegative diagonal elements. Then,*

(a) *$\mathcal{A}$ is an M-tensor if and only if there exists a positive diagonal matrix D such that $\mathcal{B}$ defined as* (3.8) *is diagonally dominant,*

(b) *$\mathcal{A}$ is a strong M-tensor if and only if there exists a positive diagonal matrix D such that $\mathcal{B}$ defined as* (3.8) *is strictly diagonally dominant.*

Proof. (a) *Sufficiency*: Since $A_{i\ldots i} \geq 0$ for each i, so is $B_{i\ldots i}$. Since $\mathcal{B}$ is diagonally dominant and it is an essentially nonpositive tensor, by Theorem 3.15, $\mathcal{B}$ is an M-tensor. Hence, by Theorem 3.9(a), we have $\min_{\lambda\in\sigma(\mathcal{B})} \mathrm{Re}\lambda \geq 0$. By Lemma 3.16, $\min_{\lambda\in\sigma(\mathcal{A})} \mathrm{Re}\lambda \geq 0$. Thus, by Theorem 3.9(a), $\mathcal{A}$ is an M-tensor.

Necessity: By Theorem 3.4(e), $\mathcal{A}$ has the unique nonnegative H^{++}-eigenvalue λ with corresponding eigenvector $x \in \mathrm{R}^n_{++}$. That is, for $i = 1, \ldots, n$,

$$\sum_{i_2,\ldots,i_m=1}^{n} A_{ii_2\ldots i_m} x_{i_2} \cdots x_{i_m} = \lambda x_i^{m-1},$$

which yields

$$A_{i\ldots i} - \lambda = - \sum_{(i,i_2,\ldots,i_m)\neq(i,i,\ldots,i)} A_{ii_2\ldots i_m} x_i^{-(m-1)} x_{i_2} \cdots x_{i_m}.$$

Since $\lambda \geq 0$, we have

$$\sum_{(i,i_2,\ldots,i_m)\neq(i,i,\ldots,i)} |A_{ii_2\ldots i_m}| x_i^{-(m-1)} x_{i_2} \cdots x_{i_m} \leq A_{i\ldots i}. \tag{3.9}$$

Define $D = \mathrm{diag}(x_1, \ldots, x_n)$. Then, (3.9) yields

$$\sum_{(i,i_2,\ldots,i_m)\neq(i,i,\ldots,i)} |B_{ii_2\ldots i_m}| \leq B_{i\ldots i}, \quad i = 1, \ldots, n.$$

This shows that $\mathcal{B}$ is diagonally dominant.

Similarly, we can prove (b). ∎

4. Applications of *M*-tensors. In this section, we give some applications of M-tensors based on the spectral properties given in the above section. Testing positive definiteness of a multivariate form is an important problem in the stability study of nonlinear autonomous systems [3, 4, 20]. We use the theory of strong M-tensors to test the positive definiteness of a multivariate form.

We now consider the following class of multivariate forms:

$$f(x) = \sum_{i_1,i_2,\ldots,i_m=1}^{n} A_{i_1i_2\ldots i_m} x_{i_1} x_{i_2} \cdots x_{i_m},$$

where $\mathcal{A} = (A_{i_1i_2\ldots i_m})$ is a symmetric Z-tensor. Qi [22, Theorem 5] proved that $f(x)$ is positive definite if and only if all its H-eigenvalues are positive. Theorem 3.9 shows that a Z-tensor $\mathcal{A}$ is a strong M-tensor if and only if the smallest H-eigenvalue of $\mathcal{A}$ is positive. Hence, we have the following criterion to test the positive definiteness of $f(x)$.

THEOREM 4.1. *Let $\mathcal{A} = (A_{i_1 \dots i_m})$ be a symmetric Z-tensor and m be even. Then $f(x) = \mathcal{A}x^m$ is positive definite if and only if $\mathcal{A}$ is a strong M-tensor.*

Based on Theorem 4.1, we next propose an algorithm for testing the positive definiteness of $f(x)$. The following lemma will be used.

LEMMA 4.2. *Let $\mathcal{A}$ be an m-order and n-dimensional tensor. Define*

$$L_{\mathcal{A}} = \min_{1 \le i \le n} \{A_{ii\dots i} - C_i\}, \quad U_{\mathcal{A}} = \max_{1 \le i \le n} \{A_{ii\dots i} + C_i\}, \tag{4.1}$$

where

$$C_i = \sum_{(i,i_2,\dots,i_m) \neq (i,i,\dots,i)} |A_{ii_2\dots i_m}|, \quad i = 1, 2, \dots, n.$$

Then $L_{\mathcal{A}}$ and $U_{\mathcal{A}}$ are the lower and upper bounds of H-eigenvalues of $\mathcal{A}$, respectively.

Proof. Let λ be an H-eigenvalue of $\mathcal{A}$ with an H-eigenvector $x \neq 0$. That is, for $i = 1, 2, \dots, n$,

$$\sum_{i_2,\dots,i_m=1}^{n} A_{ii_2\dots i_m} x_{i_2} \cdots x_{i_m} = \lambda x_i^{m-1}. \tag{4.2}$$

Let x_k be the entry of x with largest modulus. Then (4.2) implies that

$$\begin{aligned} &|\lambda - A_{kk\dots k}| \\ &\le \sum_{(k,i_2,\dots,i_m) \neq (k,k,\dots,k)} |A_{ki_2\dots i_m}| \frac{|x_{i_2}|}{|x_k|} \cdots \frac{|x_{i_m}|}{|x_k|} \\ &\le C_k, \end{aligned}$$

which yields $A_{kk\dots k} - C_k \le \lambda \le A_{kk\dots k} + C_k$. This shows $L_{\mathcal{A}} \le \lambda \le U_{\mathcal{A}}$. ☐

For a Z-tensor $\mathcal{A}$, we define a tensor $\mathcal{C}$ as

$$\mathcal{C} = U_{\mathcal{A}}\mathcal{I} - \mathcal{A}, \tag{4.3}$$

where $U_{\mathcal{A}}$ is defined in (4.1). Clearly, $\mathcal{C} \ge 0$ and $U_{\mathcal{A}} - \rho(\mathcal{C})$ is the smallest H-eigenvalue of $\mathcal{A}$. By Theorem 4.1, $U_{\mathcal{A}} - \rho(\mathcal{C}) > 0$ if and only if the corresponding multivariate form $f(x)$ is positive definite. Based on this observation, we next propose an iterative method for testing the the positive definiteness of $f(x)$ with a Z-tensor. For this purpose, we first design an algorithm for computing the spectral radius of the nonnegative tensor $\mathcal{C}$. This algorithm is a modified version of [18]. The substantial difference is that the modified version is always convergent for any nonnegative tensor, but the algorithm in [18] may be not convergent for some reducible nonnegative tensors. Our contribution is to add a perturbation term on $\mathcal{C}$. That is, we apply the algorithm in [18] to compute the large eigenvalue of the tensor

$$\mathcal{B} = \mathcal{C} + \gamma\mathcal{I} + \mathcal{E},$$

where $\gamma > 0$ is a parameter and $\mathcal{E}$ is a positive tensor with every entry being ε ($\varepsilon > 0$ is a sufficiently small number).

For convenience, we present the modified algorithm as follows.

ALGORITHM 4.1.

Step 0. *Choose* $x^{(1)} \in R^n_{++}$ *and* $\gamma > 0$. *Let* $\varepsilon > 0$ *be a sufficiently small number and* $\mathcal{E}$ *be a positive tensor whose every entry is* ε. *Let* $\mathcal{B} = \mathcal{C} + \gamma\mathcal{I} + \mathcal{E}$, *and set* $k := 1$.

Step 1. *Compute*

$$y^{(k)} = \mathcal{B}\left(x^{(k)}\right)^{m-1},$$

$$\underline{\lambda}_k = \min_{x_i^{(k)}>0} \frac{\left(y^{(k)}\right)_i}{\left(x_i^{(k)}\right)^{m-1}},$$

$$\bar{\lambda}_k = \max_{x_i^{(k)}>0} \frac{\left(y^{(k)}\right)_i}{\left(x_i^{(k)}\right)^{m-1}}.$$

Step 2. *If* $\bar{\lambda}_k = \underline{\lambda}_k$, *then let* $\lambda = \bar{\lambda}_k$ *and stop. Otherwise, compute*

$$x^{(k+1)} = \frac{\left(y^{(k)}\right)^{\left[\frac{1}{m-1}\right]}}{\left\|\left(y^{(k)}\right)^{\left[\frac{1}{m-1}\right]}\right\|},$$

replace k *by* $k+1$, *and go to Step* 1.

Step 3. *Output* $\lambda - \gamma$, *which is the largest eigenvalue of* $\mathcal{C}$.

To establish the convergence of Algorithm 4.1, we need the following lemma [26, Theorem 2.3].

LEMMA 4.3. *Let* $\mathcal{A}$ *be a nonnegative tensor of order* m *and dimension* n, *and let* $\varepsilon > 0$ *be a sufficiently small number. If* $\mathcal{A}_\varepsilon = \mathcal{A} + \mathcal{E}$ *where* $\mathcal{E}$ *denotes the tensor with every entry being* ε, *then*

$$\lim_{\varepsilon \to 0} \rho(\mathcal{A}_\varepsilon) = \rho(\mathcal{A}).$$

Note that for any nonnegative tensor $\mathcal{C}$, $\mathcal{B} = \mathcal{C} + \gamma\mathcal{I} + \mathcal{E}$ is an irreducible nonnegative tensor. Then by Lemma 4.3 and [18, Theorem 2.5], we immediately show that Algorithm 4.1 is convergent for any nonnegative tensors.

THEOREM 4.4. *Let* $\mathcal{C}$ *be a nonnegative tensor. Let* $\mathcal{B} = \mathcal{C} + \gamma\mathcal{I} + \mathcal{E}$ *with* $\gamma > 0$. *Then, Algorithm* 4.1 *produces a value of* $\rho(\mathcal{B})$ *in a finite number of steps or generates two sequences* $\{\underline{\lambda}_k\}$ *and* $\{\bar{\lambda}_k\}$ *which converge to* $\rho(\mathcal{B})$. *Furthermore,* $\rho(\mathcal{C}) = \lim_{\varepsilon\to 0} \rho(\mathcal{B}) - \gamma$.

For symmetric nonnegative tensors, we have the following error bound between the largest eigenvalues of $\mathcal{C} + \mathcal{E}$ and $\mathcal{C}$.

THEOREM 4.5. *Let* $\mathcal{C}$ *be a symmetric nonnegative* m*-order* n*-dimensional tensor and* $\mathcal{C}_\varepsilon = \mathcal{C} + \mathcal{E}$. *Then, we have*

$$0 \le \rho(\mathcal{C}_\varepsilon) - \rho(\mathcal{C}) \le \varepsilon n^{m-1}.$$

Proof. By Theorem 3.6 in [29], we have

$$\begin{aligned}\rho(\mathcal{C}_\varepsilon) &= \max\left\{(\mathcal{C}+\mathcal{E})x^m : \ x \in \mathrm{R}^n_+, \ \sum_{i=1}^n x_i^m = 1\right\}\\ &= \max\left\{\mathcal{C}x^m + \mathcal{E}x^m : \ x \in \mathrm{R}^n_+, \ \sum_{i=1}^n x_i^m = 1\right\}\\ &\le \max\left\{\mathcal{C}x^m : \ x \in \mathrm{R}^n_+, \ \sum_{i=1}^n x_i^m = 1\right\}\\ &\quad + \max\left\{\mathcal{E}x^m : \ x \in \mathrm{R}^n_+, \ \sum_{i=1}^n x_i^m = 1\right\}\\ &= \rho(\mathcal{C}) + \varepsilon \max\left\{\left(\sum_{i=1}^n x_i\right)^m : \ x \in \mathrm{R}^n_+, \ \sum_{i=1}^n x_i^m = 1\right\}.\end{aligned}$$

By simple computation, we obtain

$$\max\left\{\left(\sum_{i=1}^n x_i\right)^m : \ x \in \mathrm{R}^n_+, \ \sum_{i=1}^n x_i^m = 1\right\} = n^{m-1}.$$

Hence,

$$\rho(\mathcal{C}_\varepsilon) \le \rho(\mathcal{C}) + \varepsilon n^{m-1}.$$

So, we complete the proof. ∎

The above theorem shows that Algorithm 4.1 is convergent for any nonnegative tensor. We consider a three-order three-dimensional tensor $\mathcal{M}$ given by $m_{111} = m_{333} = 1, m_{222} = 2$, and zero elsewhere.

Clearly, tensor $\mathcal{M}$ is *reducible* and its largest eigenvalue is 2. We choose $x^{(1)} = [10, 10, 10]^T$. By the algorithm in [18], we cannot obtain the largest eigenvalue for this tensor within 1000 iterations. Let every entry of $\mathcal{E}$ be 10^{-8}. By Algorithm 4.1, we can get the largest eigenvalue of $\mathcal{M}$ within 47 iterations. This clearly shows that we can use Algorithm 4.1 to compute the largest eigenvalue for reducible nonnegative tensors but the algorithm in [18] may not work for reducible nonnegative tensors.

Algorithm 4.1 can be used to compute the largest eigenvalue of the tensor $\mathcal{C}$ in (4.3). We propose the following algorithm for testing the positive definiteness of the multivariate form $f(x) = \mathcal{A}x^m$ with a Z-tensor $\mathcal{A}$ and even m.

ALGORITHM 4.2.

Step 0. *Compute the upper bound $U_\mathcal{A}$ by the formula* (4.1) *and let $\mathcal{C} = U_\mathcal{A}\mathcal{I} - \mathcal{A}$ be defined as in* (4.3).

Step 1. *By using Algorithm* 4.1*, compute the spectral radius $\rho(\mathcal{C})$ of $\mathcal{C}$.*

Step 2. *Let $\mu = U_\mathcal{A} - \rho(\mathcal{C})$. If $\mu > 0$, then $f(x) = \mathcal{A}x^m$ is positive definite. Otherwise, it is not positive definite.*

We now use Algorithm 4.2 to test the positive definiteness of $f(x) = \mathcal{A}x^m$ with a Z-tensor $\mathcal{A}$. The Z-tensors in numerical examples are randomly generated by the following procedure.

Procedure 1.

(i) Give (m, n, A_d), where n and m are the dimension and the order of the randomly generated tensor, respectively, and $A_d > 0$.

TABLE 1
Output of Algorithm 4.2 for testing positive definiteness of the multivariate form $\mathcal{A}x^m$.

m	n	A_d	Yes	No	CPU(s)
4	10	5	0	100	0.0592
4	10	10	0	100	0.0603
4	10	100	100	0	0.0615
4	10	1000	100	0	0.0628
4	20	5	0	100	0.2945
4	20	10	0	100	0.3097
4	20	100	100	0	0.3195
4	20	1000	100	0	0.3116
4	30	5	0	100	1.3233
4	30	10	0	100	1.3125
4	30	100	0	100	1.3170
4	30	1000	100	0	1.3475
4	40	5	0	100	6.5375
4	40	10	0	100	6.5358
4	40	100	0	100	6.4925
4	40	1000	0	100	6.5520
4	50	5	0	100	15.2086
4	50	10	0	100	15.1844
4	50	100	0	100	15.2102
4	50	1000	0	100	15.2039

(ii) Randomly generate an m-order n-dimensional tensor $\mathcal{D}$ such that all elements of $\mathcal{D}$ are in the interval $(0,1)$.

(iii) Let $\mathcal{A} = (A_{i_1\cdots i_m})$, where $A_{i\cdots i} = A_d + D_{i\cdots i}$, $i = 1,2,\ldots,n$; otherwise, $A_{i_1\cdots i_m} = -D_{i_1\cdots i_m}, 1 \le i_1,\ldots,i_m \le n$.

In Algorithm 4.1, all entries of $\mathcal{E}$ are taken to be 10^{-8}. Since all entries of $\mathcal{E}$ are very small, we may think the eigenvalue obtained by Algorithm 4.1 is the largest eigenvalue of the tensor $\mathcal{C}$ in (4.3). Our numerical results are reported in Table 1. In this table, **n** and **m** specify the dimension and the order of the randomly generated tensor, respectively. A_d is a parameter in Procedure 1. Given (m,n,A_d), we generate 100 tensors and determine whether they are strong M-tensors by Algorithm 4.2. In the Yes column we show the number of multivariate forms which are positive definite. In the No column, we give the number of multivariate forms which are not positive definite. CPU(s) denotes the average computer time in seconds used for Algorithm 4.2. The results reported in Table 1 show that Algorithm 4.2 can test whether the multivariate forms with the randomly generated Z-tensors are positive definite.

5. Conclusion. We have introduced M-tensors and strong M-tensors. The simple definition is a natural generalization of the definition of M-matrices and nonsingular M-matrices. We have established some basic properties for M-tensors and strong M-tensors. We have proposed some sufficient and necessary conditions for Z-tensors to be M-tensors or strong M-tensors. We also have presented a sufficient condition for M-tensors and strong M-tensors. In particular, we have shown that a Z-tensor is a strong M-tensor if and only if its smallest H-eigenvalue is positive. Based on the necessary and sufficient condition, we use strong M-tensors to test the positive definiteness of a class of multivariate forms. We have proposed an algorithm for testing the positive definiteness of the class of multivariate forms. Numerical results are reported.

There are some questions which are still under study. For example, can we show whether the conditions "there exists $x \in R^n_+$ such that $\mathcal{A}x^{m-1} > 0$" and "the determinants of its principal subtensors are all positive" are necessary and sufficient conditions for a Z-tensor $\mathcal{A}$ to be a strong M-tensor? We know that [11, Theorem 3] gives a positive answer for the first condition. The second question is still open.

Acknowledgments. The authors would like to thank the editor and the anonymous referees for their constructive comments and suggestions, which lead to a significantly improved version of the paper.

REFERENCES

[1] A. Berman and R. J. Plemmons, *Nonnegative Matrices in the Mathematical Sciences*, Academic Press, New York, 1994.

[2] N. K. Bose, *Applied Multidimensional System Theory*, Van Nostrand Rheinhold, New York, 1982.

[3] N. K. Bose and P. S. Kamat, *Algorithm for stability test of multidimensional filters*, IEEE Trans. Acoust. Speech Signal Process., 22 (1974), pp. 307–314.

[4] N. K. Bose and A. R. Modarressi, *General procedure for multivariable polynomial positivity with control applications*, IEEE Trans. Automat. Control, 21 (1976), pp. 696–701.

[5] S. Rota Bulò and M. Pelillo, *New bounds on the clique number of graphs based on spectral hypergraph theory*, in Learning and Intelligent Optimization, T. Stützle ed., Springer-Verlag, Berlin, 2009, pp. 45–48.

[6] K. C. Chang, K. Pearson, and T. Zhang, *Perron Frobenius theorem for nonnegative tensors*, Commun. Math. Sci., 6 (2008), pp. 507–520.

[7] K.-C. Chang, K. J. Pearson, and T. Zhang, *Primitivity, the convergence of the NZQ method, and the largest eigenvalue for nonnegative tensors*, SIAM J. Matrix Anal. Appl., 32 (2011), pp. 806–819.

[8] K. C. Chang, K. Pearson, and T. Zhang, *On eigenvalue problems of real symmetric tensors*, J. Math. Anal. Appl., 350 (2009), pp. 416–422.

[9] J. Cooper and A. Dutle, *Spectra of uniform hypergraphs*, Linear Algebra Appl., 436 (2012), pp. 3268-3292.

[10] J. Ding and A. Zhou, *Nonnegative Matrices, Positive Operators and Applications*, World Scientific, River Edge, NJ, 2009.

[11] W. Ding, L. Qi, and Y. Wei, *M-tensors and nonsingular M-tensors*, Linear Algebra Appl., 439 (2013), pp. 3264–3278.

[12] S. Friedland, S. Gaubert, and L. Han, *Perron-Frobenius theorem for nonnegative multilinear forms and extensions*, Linear Algebra Appl., 438 (2013), pp. 738–749.

[13] M. A. Hasan and A. A. Hasan, *A procedure for the positive definiteness of forms of even-order*, IEEE Trans. Automat. Control, 41 (1996), pp. 615–617.

[14] R. Horn and C. H. Johnson, *Matrix Analysis*, Cambridge University Press, Cambridge, UK, 1996.

[15] S. Hu and L. Qi, *Algebraic connectivity of an even uniform hypergraph*, J. Combin. Optim., 24 (2012), pp. 564–579.

[16] S. Hu, Z. Huang, and L. Qi, *Strictly nonnegative tensors and nonnegative tensor partition*, Sci. China Math., 57 (2014), pp. 181–195.

[17] L.-H. Lim, *Singular values and eigenvalues of tensors: A variational approach*, in Proceedings of the IEEE International Workshop on Computational Advances in Multi-Sensor Adaptive Processing (CAMSAP'05), IEEE Computer Society Press, Piscataway, NJ, 2005, pp. 129–132.

[18] Y. Liu, G. Zhou, and N. F. Ibrahim, *An always convergent algorithm for the largest eigenvalue of an irreducible nonnegative tensor*, J. Comput. Appl. Math., 235 (2010), pp. 286–292.

[19] M. Ng, L. Qi, and G. Zhou, *Finding the largest eigenvalue of a nonnegative tensor*, SIAM J. Matrix Anal. Appl., 31 (2009), pp. 1090–1099.

[20] Q. Ni, L. Qi, and F. Wang, *An eigenvalue method for testing the positive definiteness of a multivariant form*, IEEE Trans. Automat. Control, 53 (2008), pp. 1096–1107.

[21] A. M. Ostrowski, *Über die determinanten mit überwiegender hauptdiagonale*, Comment. Math. Helv., 10 (1937), pp. 69–96.

[22] L. Qi, *Eigenvalues of a real supersymmetric tensor*, J. Symbolic Comput., 40 (2005), pp. 1302–1324.

[23] L. Qi, *Symmetric nonnegative tensors and copositive tensors*, Linear Algebra Appl., 439 (2013), pp. 228–238.

[24] D. M. Stipanović, S. Shankaran, and C. J. Tomlin, *Multi-agent avoidance control using an M-matrix property*, Electron. J. Linear Algebra, 22 (2005), pp. 64–72.

[25] R. Varga, *Matrix Iterative Analysis*, Prentice–Hall, Englewood Cliffs, NJ, 1962.
[26] Y. Yang and Q. Yang, *Further results for Perron–Frobenius theorem for nonnegative tensors*, SIAM J. Matrix Anal. Appl., 31 (2010), pp. 2517–2530.
[27] L. Zhang, L. Qi, and Y. Xu, *Weakly positive tensors and linear convergence of the LZI algorithm*, J. Comput. Math., 30 (2012), pp. 24–33.
[28] L. Zhang and L. Qi, *Linear convergence of an algorithm for computing the largest eigenvalue of a nonnegative tensor*, Numer. Linear Algebra Appl., 19 (2012), pp. 830–841.
[29] G. Zhou, L. Qi, and S.-Y. Wu, *On the largest eigenvalue of a symmetric nonnegative tensor*, Numer. Linear Algebra Appl., 20 (2013), pp. 913–928.