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M5 brane and four dimensional $\mathcal{N} = 1$ theories I

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Abstract: Four dimensional $\mathcal{N} = 1$ theories are engineered by compactifying six dimensional $(2, 0)$ theory on a Riemann surface with regular punctures. A generalized Hitchin's equation involving two Higgs fields is proposed as the BPS equation for $\mathcal{N} = 1$ compactification. The puncture is interpreted as the singular boundary condition of this equation, and regular puncture is shown to be labeled by a nilpotent commuting pair. In this paper, we focus on a subset of regular puncture which is described by rotating branes representing $\mathcal{N} = 2$ puncture. As an application, we show that Seiberg duality of $SU(N)$ SQCD with $N_f = 2N$ and certain superpotential term is realized as different degeneration limits of the same punctured Riemann surface, and find four more dual theories.

Keywords: Supersymmetry and Duality, Brane Dynamics in Gauge Theories

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1 Introduction

Gauge theory can be engineered using brane systems of type II string theory, and various UV parameters can be understood nicely from brane configurations [1], moreover, the lift of the above brane configuration to M theory can usually lead to remarkable results about the IR behavior of the gauge theory. The above strategy has been successfully implemented for 4d $\mathcal{N} = 2$ [2] and $\mathcal{N} = 1$ theory [3, 4], and in both cases all the relevant type II branes become a single M5 brane and the IR behavior is controlled by a Riemann surface on which M5 brane wraps.

The type II brane construction usually involves D brane ending on NS brane, and two dimensional conformal field theory description is singular at the intersection, therefore, many questions about the UV theory, say S duality of 4d $\mathcal{N} = 2$ theory, is not easily understood since typically one need to move the branes to pass each other, and there are various difficult phase transition questions in this process.

Instead, one can engineer UV theory by directly starting with N multiple M5 branes, and compactify it on a Riemann surface with co-dimensional two defects (punctures), which basically represent various intersected branes. Gauge coupling is understood as the complex structure of the Riemann surface (as the gauge coupling is interpreted as the relative positions of the intersected branes [2]), and now nothing is singular in moving around the punctures to change the gauge coupling, so S duality of $\mathcal{N}=2$ theory [5] is manifest in this representation [6].

More generally, one could engineer new four dimensional $\mathcal{N}=2$ theory as follows: simply compactify 6d $(2,0)$ theory on a Riemann surface with various type of defects.[1] Usually, the most important ingredients are the local property of the defects, which basically provide all the richness of M5 brane engineering. The important tool is the Hitchin's equation defined on the Riemann surface: various defects are singular boundary conditions to Hitchin's equation. Using regular defects,[2] one can find lots of generalized superconformal quiver gauge theory [14–17]. Using the irregular singularity which is classified in [18] for A_{N-1} case, one can engineer many new Argyes-Douglas and asymptotical free theories.

Such M5 brane engineering is also extended to four dimensional $\mathcal{N}=1$ theories [19, 19–25], in particular, the global aspects of the compactification is found in [22]: Two line bundles L_1 and L_2 such that $L_1 \otimes L_2 = K$ (K is the canonical bundle) are defined on Riemann surface. However, two important ingredients are still missing: first, the analog of Hitchin's equation is not found in the literature, and secondly various defects which provide all the richness are not discussed in a systematical way. Because of the above two missing pieces, the matter system for $\mathcal{N}=1$ theory is basically $\mathcal{N}=2$ matter system.

Our main purpose in this paper is to fill in the above two gaps. We propose a generalized Hitchin's equation for $\mathcal{N}=1$ compactification:

$$\begin{aligned} D_{\bar{z}}\Phi_1 = D_{\bar{z}}\Phi_2 &= 0, \\ [\Phi_1, \Phi_2] &= 0, \\ F_{z\bar{z}} + [\Phi_1, \Phi_1^*]g_1 + [\Phi_2, \Phi_2^*]g_2 &= 0, \end{aligned} \tag{1.1}$$

here Φ_1 and Φ_2 are the sections of the line bundles L_1 and L_2 respectively, and they also transform in adjoint representation of the gauge group. Notice that because Φ_1, Φ_2 are not the sections of cotangent bundle, we use fixed Hermitian metric g_1 (g_2) of line bundle $K \otimes L_1^{-1}$ ($K \otimes L_2^{-1}$) to make the last equation coordinate invariant. This equation is conformal invariant and therefore one could find solution on compact Riemann surface.

The local regular singular solutions can then be easily found (for the local solution, we

[1] The history of constructing four dimensional field theories using M5 branes in the case of no defects goes back to [7, 8], see also [9] for constructing lower dimensional field theories using M5 branes.

[2] A_{N-1} regular defects are classified in [6], D_N regular defects are classified in [10], other type of regular defects including twisted lines are discussed in [11–13].

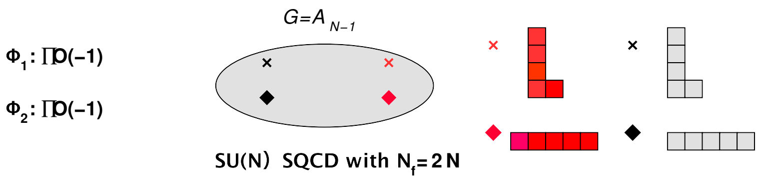

Figure 1. The $M5$ brane compactification data for $SU(N)$ SQCD with $N_f = 2N$. Black Young tableaux means that Φ_1 is singular at the puncture while Φ_2 is zero, and the similar interpretation applies to the red Young tableaux.

can ignore the Hermitian metric):

$$\begin{aligned}
&\Phi_1 = \frac{e_1}{z}, \quad \Phi_2 = \frac{e_2}{z}, \quad A_{\bar{z}} = \frac{h_1}{\bar{z}} + \frac{h_2}{\bar{z}}, \\
&[e_1, e_2] = 0, \quad [h_1, h_2] = 0, \\
&[h_1, e_1] = e_1, \quad [h_2, e_2] = e_2,
\end{aligned} \tag{1.2}$$

here e_1, e_2 is taken to be nilpotent. Therefore the regular singularity is specified by the orbit of a nilpotent commuting pair, which is studied by Ginzburg in [26]. When one of the nilpotent element is zero, we get the usual $\mathcal{N} = 2$ regular puncture.

In summary, Our theories are derived by starting with six dimensional $(2, 0)$ theory and compactify it on a Riemann surface with the following data:

- A punctured Riemann surface $M_{g,n}$.
- A rank two line bundle $L_1 \bigoplus L_2$,[3] such that $L_1 \otimes L_2 = K$ with K the canonical bundle, and two complex scalars Φ_1, Φ_2 are holomorphic sections of them.
- The local puncture types: a commuting nilpotent pair.

We conjecture that in the IR the theory flows to a 4d $\mathcal{N} = 1$ fixed point. See figure 1 for the description of $SU(N)$ SQCD with $N_f = 2N$.

In this paper, we focus on regular punctures which could be derived by rotating $\mathcal{N} = 2$ puncture: $\mathcal{N} = 2$ puncture can be realized as the half-BPS boundary condition of $\mathcal{N} = 4$ Super Yang-Mills (SYM) theory on a segment, and can be represented nicely by D3-D5 system.[4] Similarly, $\mathcal{N} = 1$ puncture can be realized as the quarter-BPS boundary condition of $\mathcal{N} = 4$ SYM,[5] and most simple ones are formed by rotating some of D5 branes to D5$'$ branes [29, 30], which could be labeled by a colored Young tableaux.

As an application, we study $\mathcal{N} = 1$ dualities using the Riemann surface picture, and we find that Seiberg duality [31] is also realized as different degeneration limit of the same Riemann surface much as the story of $\mathcal{N} = 2$ S duality. We identify the complex structure moduli of the Riemann surface as the quartic superpotential couplings:

$$W = ctr(\mu_1 \mu_2), \tag{1.3}$$

[3]More generally, one can consider an arbitrary holomorphic rank two bundle with fixed determinant bundle K. For simplicity, we only consider the split bundle in this paper.

[4]Such descriptions are available for punctures of A_N and D_N theories [27].

[5]Those quarter-BPS boundary conditions are studied in [28].

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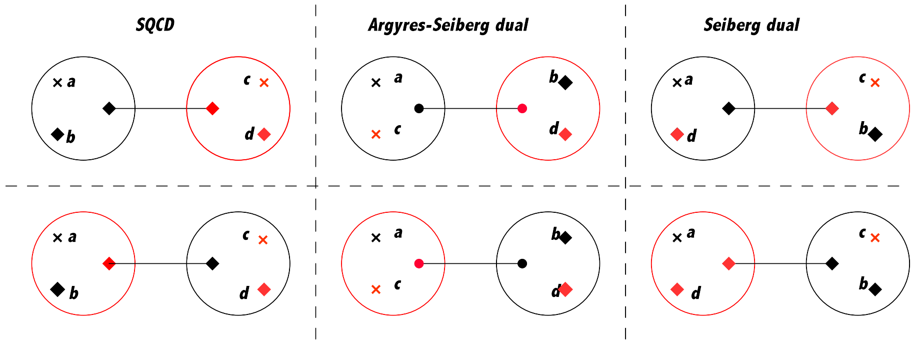

Figure 2. Different duality frames of $SU(N)$ SQCD with $N_f = 2N$, and the quartic superpotential couplings are the same for the duality frames in the same column.

here μ_1, μ_2 are the momentum map of two gluing punctures in the degeneration limit. After doing Seiberg duality, we have $c \to -\frac{1}{c}$, which matches the Riemann surface picture. Such quartic coupling is exactly marginal in the case of $SU(N)$ with $2N$ flavor [32] and we conjecture that it is exactly marginal for the theory considered above. This type of quartic superpotential is important for duality to work nicely in our picture: the matter in various duality frames can be represented by three punctured sphere. Notice that for $\mathcal{N} = 2$ gluing, there is a cubic superpotential term:

$$W = \tau tr\Phi(\mu_1 - \mu_2), \tag{1.4}$$

and again this coupling is identified as the complex structure moduli, and under S duality $\tau \to -\frac{1}{\tau}$. Therefore, $\mathcal{N} = 1$ duality works in the same way as $\mathcal{N} = 2$ S duality. Various duality frames for $SU(N)$ SQCD are shown in figure 2. Similar dualities are discussed in [25], but in our description, matter system is represented by a three punctured sphere with black and red punctures and therefore admits a M5 brane construction, it would be interesting to identify our matter system with theirs.

This paper is organized as follows: in section II, we discuss the basic ingredients of engineering $\mathcal{N} = 1$ theory from M5 brane: a generalized Hitchin's equation and a classification of regular punctures. In section III, we discuss $\mathcal{N} = 1$ duality for theories defined using 6d A_{N-1} type theory and fully rotated punctures; in section IV, we study theories engineered using partial rotated puncture and D_N theory. Finally, a conclusion is given in section V.

2 $\mathcal{N} = 1$ compactification

2.1 Topological partial twist and global breaking to $\mathcal{N} = 1$

One can get lower dimensional supersymmetric field theory by doing partial topological twist of a higher dimensional field theory [33]. In particular, one can do topological partial twist on 6d $(2, 0)$ theory to get four dimensional $\mathcal{N} = 1$ theory as discussed in [7, 34]. Here let's give a brief review. The global symmetry groups of 6d $(2, 0)$ theory are: the $Sp(4)$

R symmetry and $SO(6) \simeq SU(4)$ space-time symmetry. The supercharge Q transforms under $SU(4) \times Sp(4)$ as

$$Q: \ 4 \otimes 4, \tag{2.1}$$

and the five adjoint scalars ϕ_i transforms as

$$\phi: \ 1 \otimes 5. \tag{2.2}$$

Let's consider a six manifold with product structure $R^4 \times \Sigma$ so that the space-time symmetry group is decomposed as $SO(4) \times SO(2)$, and we use a R symmetry subgroup $U(1)_{45} \times U(1)_{89} \subset SO(5)$ to do the partial twist. Before the twist, the supercharges transform under the above subgroup $SO(4) \times SO(2) \times U(1)_{45} \times U(1)_{89}$ as

$$Q: \ 4 \otimes 4 \rightarrow ((2,1)_{\frac{1}{2}} + (1,2)_{-\frac{1}{2}}) \otimes \left(\left(\frac{1}{2}, \frac{1}{2}\right) + \left(\frac{1}{2}, -\frac{1}{2}\right) + \left(-\frac{1}{2}, \frac{1}{2}\right) + \left(-\frac{1}{2}, -\frac{1}{2}\right) \right) \tag{2.3}$$

and the scalars decompose as

$$\phi: \ 1 \otimes 5 \rightarrow 1_0 \otimes \left(\left(\frac{1}{2}\right)_0 + \left(-\frac{1}{2}\right)_0 + 0_{\frac{1}{2}} + 0_{-\frac{1}{2}} + 1_0 \right). \tag{2.4}$$

Let's twist the theory by embedding the R symmetry into the $SO(2)$ symmetry on Riemann surface: the new rotational symmetry group $SO(2)^{'} = SO(2) + aU(1)_{45} + bU(1)_{89}$. The supercharges have charges under $SO(2)^{'}$:

$$\pm \frac{1}{2} \pm \frac{a}{2} \pm \frac{b}{2}, \tag{2.5}$$

so if $a + b = 1$, we have invariant supercharges. If $b = 0, a = 1$ or $b = 1, a = 0$, one has eight unbroken supercharges, and we get four dimensional $\mathcal{N} = 2$ theory whose R symmetry group is $U(1)_{45} \times SU(2)$ and $U(1)_{89} \times SU(2)$, we call them NS type theory and NS$^{'}$ type theory whose meaning will be clear once we consider the type IIA brane configurations in next subsection. The scalars which have nonzero charges under $SO(2)^{'}$ are

$$\begin{aligned} &\Phi_1: \ \frac{1}{2}, \quad \Phi_2: \ 0, \qquad \text{NS theory}, \\ &\Phi_1: \ 0, \quad \Phi_2: \ \frac{1}{2}, \qquad \text{NS}^{'} \text{ theory}, \end{aligned} \tag{2.6}$$

here Φ_1 and Φ_2 is related to the original five scalars as $\Phi_1 = \phi_4 + i\phi_5$ and $\Phi_2 = \phi_8 + i\phi_9$. In our normalizations, ϕ_4 and ϕ_5 are vectors under $SO(2)^{'}$ (let's take NS theory as an example), so Φ_1 is a section of canonical bundle of Riemann surface.

If both a and b are nonzero and satisfy equation $a + b = 1$, there are four unbroken supercharges and we get 4d $\mathcal{N} = 1$ theory. The two complex scalars Φ_1 and Φ_2 have charges

$$\Phi_1: \ \frac{a}{2}, \qquad \Phi_2: \ \frac{b}{2}. \tag{2.7}$$

The above topological twisting only tells us the local behavior of the deformations. To get the global information of the normal deformations, we have to use full M theory picture [9]: the M5 branes are wrapping a holomorphic curve Σ inside a Calabi-Yau three-fold. Because of Calabi-Yau condition, the rank two normal bundle of Σ satisfies the following condition

$$\det(N_\Sigma) = K, \tag{2.8}$$

where K is the canonical bundle of Σ. So unlike $\mathcal{N} = 2$ theory where the normal deformation is associated with the Riemann surface itself, we have to choose a rank two bundle with fixed determinant K for $\mathcal{N} = 1$ theory.

In this paper, we only consider the case where the rank two bundle is split, and we have two fields describing the normal deformations:

$$\Phi_1 \in \Omega^0(\Sigma, E \otimes L_1), \quad \Phi_2 \in \Omega^0(\Sigma, E \otimes L_2), \tag{2.9}$$

and $L_1 \otimes L_2 = K$, where K is cotangent bundle of Riemann surface and E is the endmorphism bundle of a rank N vector bundle.

2.2 Punctures and local breaking to $\mathcal{N} = 1$

As we saw in last subsection, the $\mathcal{N} = 1$ compactification of $(2, 0)$ theory introduces two line bundles whose tensor product is the canonical bundle of Riemann surface. In this subsection, We would like to discuss the classification of local regular punctures. In the case of $\mathcal{N} = 2$ compactification, Hitchin's equation plays a crucial role in classifying the punctures; Following similar lines, we propose a generalized Hitchin's equation, and then use it to classify the regular puncture of $\mathcal{N} = 1$ compactification.

2.2.1 Hitchin's equation and $\mathcal{N} = 2$ punctures

Let's first review the regular punctures for $\mathcal{N} = 2$ compactification. The BPS equation for $\mathcal{N} = 2$ compactification is the so-called Hitchin equation, which can be derived in the following way: one further compactify four dimensional theory on T^2, and then consider the compactification in different order: first on T^2 and then on Riemann surface Σ. In the first T^2 compactification, one get four dimensional $\mathcal{N} = 4$ SYM, and in the second step, we do topological twist on Σ(it turns out the twist is the GL twist studied in [35]) and find the Hitchin's equation [35].

The local form of Hitchin's equation can be derived from dimensional reduction of four dimensional self-dual Yang-Mills equation:

$$F_{12} = F_{34}, \quad F_{13} = F_{42}, \quad F_{14} = F_{23}. \tag{2.10}$$

After dimensional reduction to two dimensions [36], i.e. all the fields are independent of coordinates x^3, x^4, we have

$$F_{12} = [\phi_3, \phi_4], \quad D_1\phi_3 = -D_2\phi_4, \quad D_1\phi_4 = D_2\phi_3. \tag{2.11}$$

Define $\Phi = \frac{1}{2}(\phi_3 - i\phi_4)$ and use the complex coordinate $z = x^1 + ix^2$, we have the familiar Hitchin's equation:

$$F_{z\bar{z}} + [\Phi, \Phi^*] = 0, \quad D_{\bar{z}}\Phi = D_z\Phi^* = 0. \tag{2.12}$$

To make this equation coordinate invariant on a curved Riemann surface, it is suggested by Hitchin that the scalar field Φ should be a one form. As we see from the topological twisting, there is indeed one scalar which lives in the cotangent bundle of the Riemann surface.

The regular puncture means the singular boundary condition of Hitchin's equation, and various fields have the following form [37]:

$$\begin{aligned}\Phi_z &= \frac{e}{z}dz, \quad A_{\bar{z}} = \frac{h}{\bar{z}}, \\ [h,e] &= e,\end{aligned} \tag{2.13}$$

here e is a nilpotent element whose orbit is labeled by a Young tableaux $\mathrm{Y} = [n_1, n_2, \ldots, n_s]$,[6] with $\sum n_i = N$: e can be put into the standard form using Jordan blocks with size $n_i \times n_i$. The moduli space of Hitchin's moduli space $\mathcal{M}_H$ is identified with the Coulomb branch of four dimensional theory compactified on a circle. $\mathcal{M}_H$ has a distinguished complex structure which is independent of the radius of compactification circle, and one can define a spectral curve [38] which is then identified with the Seiberg-Witten curve. The local Coulomb branch (the local contribution to $\mathcal{M}_H$) could be identified as the nilpotent orbit specified by e.

The local Higgs branch is described by the moduli space of Nahm's equation on a semi-infinite segment $[0, \infty]$ whose boundary condition at 0 is specified by a dual Young tableaux Y_D (Y_D is derived by transposing Y, i.e. the rows of Y_D are the columns of Y.):

$$\begin{aligned} &Y = [n_1, n_2, \ldots, n_s], \quad n_1 \geq n_2 \ldots \geq n_s \quad \text{Local Coulomb} \\ &Y^D = [r_1, r_2, \ldots, r_t]. \quad r_1 \geq r_2 \ldots \geq r_t \quad \text{Local Higgs} \end{aligned} \tag{2.14}$$

The Nahm's equation can be derived by further assuming all the fields in Hitchin's equation to be independent of coordinate x^2, and the equation reads

$$D_1\phi_2 = [\phi_3, \phi_4], \quad D_1\phi_3 = [\phi_4, \phi_2], \quad D_1\phi_4 = [\phi_2, \phi_3]. \tag{2.15}$$

Define $\alpha = (A_1 - i\phi_2)$ and $\beta = \frac{1}{2}(\phi_3 + i\phi_4)$, and Nahm's equation becomes a complex and a real equation:

$$\begin{aligned} &\frac{d\beta}{ds} + [\alpha, \beta] = 0, \qquad\qquad\qquad \text{Complex} \\ &\frac{d(\alpha^* - \alpha)}{ds} + [\alpha, \alpha^*] + [\beta, \beta^*] = 0, \quad \text{Real,} \end{aligned} \tag{2.16}$$

here $s = x^1$. The complex equation is invariant under the complex gauge group G_c. By standard argument, for one special complex structure of the moduli space, imposing real equation is equivalent to divide the complex equation by G_c.

Let's study the Nahm's equation on semi-infinite line $[0, \infty]$, and it has the following singular solution

$$\alpha = \frac{h_D}{s}, \quad \beta = \frac{e_D}{s}, \tag{2.17}$$

[6] Let's take $G = SU(N)$ for example; The regular punctures of D and E type group are similar, for more details, see [10, 12].

	x^0	x^1	x^2	x^3	x^4	x^5	x^6	x^7	x^8	x^9
D3	$\circ$	$\circ$	$\circ$				$\circ$			
NS5	$\circ$	$\circ$	$\circ$	$\circ$	$\circ$	$\circ$				
$\mathrm{NS5}'$	$\circ$	$\circ$	$\circ$	$\circ$					$\circ$	$\circ$
D5	$\circ$	$\circ$	$\circ$					$\circ$	$\circ$	$\circ$
$\mathrm{D5}'$	$\circ$	$\circ$	$\circ$		$\circ$	$\circ$		$\circ$		

Table 1. The brane configuration used in describing local $\mathcal{N} = 2$ punctures.

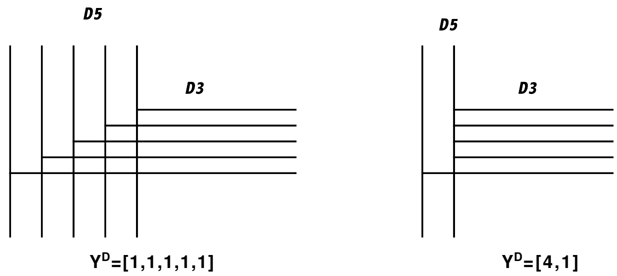

Figure 3. The puncture is represented by half-BPS boundary condition of $\mathcal{N} = 4$ SYM theory: left figure represents full puncture and right represents simple puncture.

with e^D a nilpotent element labeled by Young tableaux Y^D, i.e. the standard form has Jordan block $r_i \times r_i$; and h is a semi-simple element which satisfies $[h^D, e^D] = e^D$. Each such nilpotent orbit defines a $SU(2)$ homomorphism $\rho_{Y^D} : SU(2) \to SU(N)$; and the commutant subgroup H of ρ is then identified as the flavor symmetry group

$$H = S[\prod_l U(p_h)), \tag{2.18}$$

where p_h means the number of columns with height h in Young tableaux Y^D.

This Nahm pole boundary condition specified by Y^D can be nicely represented by a brane configuration of type IIB theory [39], here let's review it for the later use. The brane configurations are summarized in table 1. The above boundary condition is represented by a total of t D5 branes, and the number of D3 branes suspended between ith and $(i+1)$th D5 branes (counting from right to left) are

$$N_i = N - (r_1 + \ldots + r_i), \tag{2.19}$$

see figure 3 for illustration. The boundary condition on the other side of the segment is trivial, and the local Higgs branch can be identified with the moduli space of Nahm's equation with trivial boundary condition at the infinity and Nahm's pole boundary condition specified by Y^D at $s = 0$.

By doing S-dual on above brane configuration, one find a three dimensional quiver A whose Higgs branch gives the local Coulomb branch. The Higgs branch of three dimensional mirror B of quiver A describes the local Higgs branch, see figure 4. Using the 3d quiver descriptions, we can see that the local Coulomb branch has a symmetry $SU(N)$ and local

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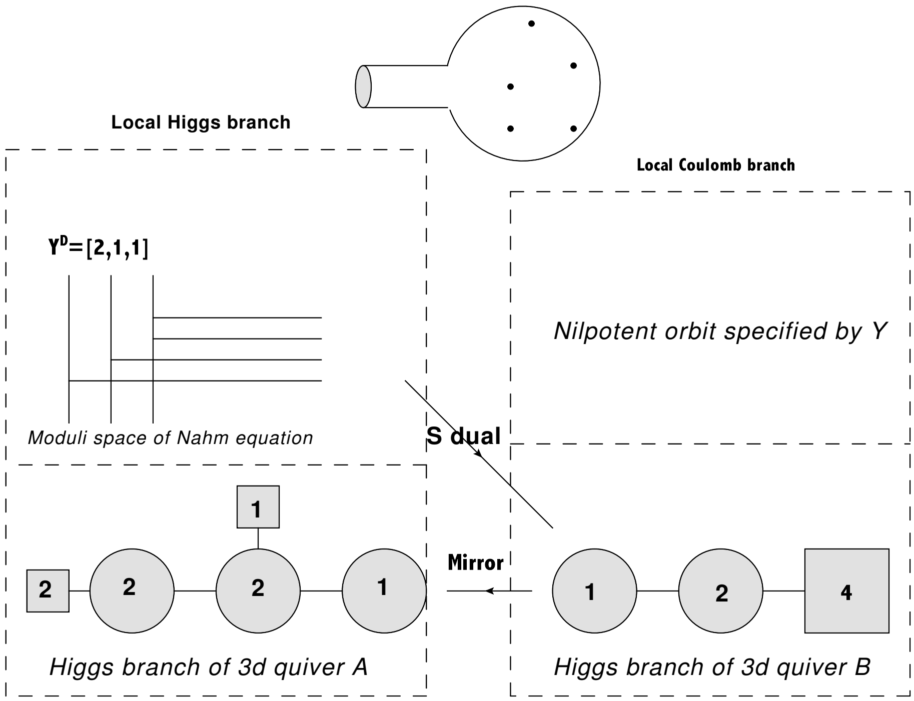

Figure 4. Local Higgs and Coulomb branch of four dimensional $\mathcal{N} = 2$ compactification can be identified with the Higgs branch of 3d quivers.

Higgs branch has a symmetry H. The most basic ingredients for constructing 4d $\mathcal{N} = 2$ theory using M5 brane are the three punctured sphere which can be represented by a three junction [40]. The Nahm's pole boundary condition is used to represent the puncture, see figure 5, which is called a NS type three sphere as we use D5 – NS branes. This type of three sphere preserves $U(1)_{45}$ symmetry. Notice that we have the same story by replacing NS5 brane and D5 brane with NS5$'$ and D5$'$ branes, which preserves $U(1)_{89}$ symmetry, and it is called NS$'$ type sphere. For the consideration of $\mathcal{N} = 2$ theory, these two types of three spheres are completely equivalent as we only use one type of three spheres to preserve $\mathcal{N} = 2$ supersymmetry. In the next subsection, when we consider $\mathcal{N} = 1$ theory, the distinction between these two type of spheres becomes important.

2.2.2 Generalized Hitchin's equation and $\mathcal{N} = 1$ punctures

We would like to consider the BPS equations for $\mathcal{N} = 1$ compactification. The derivation of the equation is similar to $\mathcal{N} = 2$ case: one further compactify the theory on T^2 and then reverse the compactification order to first get four dimensional $\mathcal{N} = 4$ SYM theory; Using the Langrangian and topological twist of 4d SYM theory, one can get the desired equation. The local form of the equation is actually quite simple to get by using the $D4$ branes suspended between $D6$ and $D6'$ branes as shown in appendix A. Here we derive the local equation as the dimensional reduction of higher dimensional equation: The $\mathcal{N} = 1$

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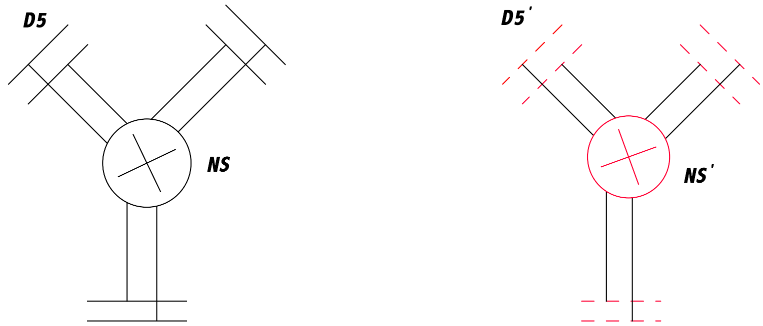

Figure 5. NS and NS$'$ type three sphere are represented by three junctions with specific boundary condition on three legs, and both lead to same $\mathcal{N}=2$ theory.

BPS equation can be derived from dimensional reduction of 6d self-dual equations [41, 42]:

$$\begin{aligned}
&F_{13}=F_{24}, \quad F_{14}=F_{32},\\
&F_{15}=F_{26}, \quad F_{16}=F_{52},\\
&F_{35}=F_{46}, \quad F_{36}=F_{54},\\
&F_{12}=F_{34}+F_{56}.
\end{aligned} \tag{2.20}$$

Let's do dimensional reduction to two dimensions which has coordinates x^1, x^2, and we get:

$$\begin{aligned}
D_1\phi_3=D_2\phi_4, \quad D_1\phi_4=-D_2\phi_3,\\
D_1\phi_5=D_2\phi_6, \quad D_1\phi_6=-D_2\phi_5,\\
[\phi_3,\phi_5]=[\phi_4,\phi_6] \quad [\phi_3,\phi_6]=[\phi_5,\phi_4],\\
F_{12}=[\phi_3,\phi_4]+[\phi_5,\phi_6].
\end{aligned} \tag{2.21}$$

Define $\Phi_1=\frac{1}{2}(\phi_3-i\phi_4)$ and $\Phi_2=\frac{1}{2}(\phi_5-i\phi_6)$, then the above equation becomes:

$$D_{\bar z}\Phi_1=0, \qquad D_{\bar z}\Phi_2=0, \tag{2.22}$$

$$[\Phi_1,\Phi_2]=0, \tag{2.23}$$

$$F_{z\bar z}+[\Phi_1,\Phi_1^*]+[\Phi_2,\Phi_2^*]=0. \tag{2.24}$$

As we learn from 4d $\mathcal{N}=1$ twist, there are two scalar fields which are sections of different bundles L_1 and L_2. These two scalar fields are actually Φ_1 and Φ_2 appearing in above equations, and the last equation is not coordinate invariant. To make the last equation coordinate invariant on a curved Riemann surface, we need to use the fixed Hermitian metric g_1 (g_2) of line bundle $L_1^{-1}\otimes K$ ($L_2^{-1}\otimes K$), and the last equation becomes

$$F_{z\bar z}+[\Phi_1,\Phi_1^*]g_1+[\Phi_2,\Phi_2^*]g_2=0. \tag{2.25}$$

There are some obvious solutions, i.e. if $\Phi_1=\Phi_2=0$, then the moduli space of above equation is the moduli space of stable holomorphic bundle [43]. If Φ_1 or Φ_2 equals to zero, one get the moduli space of stable twisted Higgs bundle as studied in [44].

Using the above equations in the local form, one can find the following regular singular boundary condition:

$$
\begin{aligned}
\Phi_1 = \frac{e_1}{z}, \quad \Phi_2 = \frac{e_2}{z}, \quad A_{\bar{z}} = \frac{h_1}{\bar{z}} + \frac{h_2}{\bar{z}}, \\
[e_1, e_2] = 0, \quad [h_1, h_2] = 0, \\
[h_1, e_1] = e_1, \quad [h_2, e_2] = e_2,
\end{aligned}
\tag{2.26}
$$

here e_1, e_2 are nilpotent and h_1, h_2 can be taken as semi-simple. Interestingly such nilpotent pair with the corresponding semi-simple pair has been studied in detail by Ginzburg [26], and we conjecture the above ones are the most general regular singular boundary condition. The local Coulomb branch is then described by orbit of nilpotent commuting pair.

Again, the local Higgs branch might be described by moduli space of generalized Nahm's equation. Similarly, such BPS equation can be derived by further reducing 6d self-dual equation down to 1d, then we get a generalized Nahm's equation:

$$
\begin{aligned}
D_1\phi_3 &= [\phi_2, \phi_4], \quad D_1\phi_4 = -[\phi_2, \phi_3] \\
D_1\phi_5 &= [\phi_2, \phi_6], \quad D_1\phi_6 = -[\phi_2, \phi_5] \\
[\phi_3, \phi_5] &= [\phi_4, \phi_6] \quad [\phi_3, \phi_6] = [\phi_5, \phi_4] \\
D_1\phi_2 &= [\phi_3, \phi_4] + [\phi_5, \phi_6].
\end{aligned}
\tag{2.27}
$$

We want to consider singular solutions to above generalized Nahm equation. Let's rewrite the above equations into the real part and complex part by defining

$$
\alpha = A_1 - i\phi_2, \quad \beta = \frac{1}{2}(\phi_3 + i\phi_4), \quad \gamma = \frac{1}{2}(\phi_5 + i\phi_6), \tag{2.28}
$$

and we have

$$
\begin{aligned}
&\frac{d\beta}{ds} + [\alpha, \beta] = 0, \quad \frac{d\gamma}{ds} + [\alpha, \gamma] = 0, \quad [\beta, \gamma] = 0, \qquad \text{Complex} \\
&\frac{d(\alpha^* - \alpha)}{ds} + [\alpha, \alpha^*] + [\beta, \beta^*] + [\gamma, \gamma^*] = 0, \qquad \text{Real}
\end{aligned}
\tag{2.29}
$$

Again the imposition of the real equation is equivalent to dividing the complex equation by complex gauge transformation. Rgular singular solution to above complex equation can be easily found:

$$
\alpha = \frac{h_1^D + h_2^D}{s}, \quad \beta = \frac{e_1^D}{s}, \quad \gamma = \frac{e_2^D}{s}, \tag{2.30}
$$

here e_1^D, e_2^D are nilpotent elements and h_1^D, h_2^D are semi-simple elements:

- $[e_1^D, e_2^D]$=0 $[h_1^D, h_2^D] = 0$,
- $[h_1^D, e_1^D] = e_1^D$, $[h_2^D, e_2^D] = e_2^D$.

Again, the moduli space of generalized Nahm's equation with the above singular boundary condition on $s = 0$ and trivial boundary condition at $s = \infty$ describes the local Higgs

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branch, and the label of commuting nilpotent orbits at $s = 0$ should be the transpose (in proper sense) of the ones used in the boundary condition of generalized Hitchin's equation.

In this paper, we will focus on the puncture which can be derived by rotating $\mathcal{N} = 2$ punctures: change some of D5 branes to D5$^{'}$ branes by rotating nighty degrees in the brane description of $\mathcal{N} = 2$ puncture, and we will have the quarter BPS boundary condition instead of half-BPS boundary condition for $\mathcal{N} = 4$ SYM theory. The study of this set of puncture is already quite rich, and we call them rotated puncture. The puncture is then still labeled by a Young tableaux with colors as the D3 branes can suspend between different types of D5 branes:

- A column is colored as black if it is represented by D3 branes suspended between two D5 branes.
- A column is colored as red if it is represented by D3 branes suspended between two $D5^{'}$ branes
- A column is colored as blue if it is represented by D3 branes suspended between $D5$ and $D5^{'}$ branes.

Some examples of rotated punctures are shown in figure 6. In particular, we call a puncture D (D$^{'}$) type if the Young tableaux is completely black (red). The flavor symmetry could be read from the 3d quiver derived from doing S duality of $D3 - D5 - D5^{'}$ system, i.e. the flavor symmetry is the symmetry on the Coulomb branch of 3d $\mathcal{N} = 2$ quiver. Generically, each quiver node contributes a $U(1)$ factor, and we have an enhanced $SU(r)$ symmetry if there is a chain of $(r - 1)$ 3d $\mathcal{N} = 4$ quiver nodes satisfying the balanced condition: $n_f = 2n_c$. See figure 7 for an example.

2.3 $\mathcal{N} = 1$ theory from M5 brane

In summary, 4d $\mathcal{N} = 1$ theory derived from compactifying six dimensional $(2, 0)$ theory of G type on a punctured Riemann surface is defined by the following data:

- A punctured Riemann surface $M_{g,n}$ with various type of punctures labeled by nilpotent commuting pair.
- A rank two bundle $L_1 \bigoplus L_2$ such that their direct product equal to the canonical bundle.

We conjecture that in the IR the theory flows to an interacting SCFT. It is nice to have some junction picture in mind about the local breaking and global breaking to $\mathcal{N} = 2$ theory. For local breaking, we only change the boundary condition and the NS core is not changed. For the global breaking, we use both NS and NS$^{'}$ three sphere. Brane picture suggests the following breaking pattern: since there are extra matter coming from fluctuation of D3 brane suspended between NS and D5$^{'}$ brane, the local breaking introduces extra matter and the cubic superpotential term, which breaks $\mathcal{N} = 2$ to $\mathcal{N} = 1$, and the rotation also completely changes the theory, i.e. the Coulomb branch deformations are not

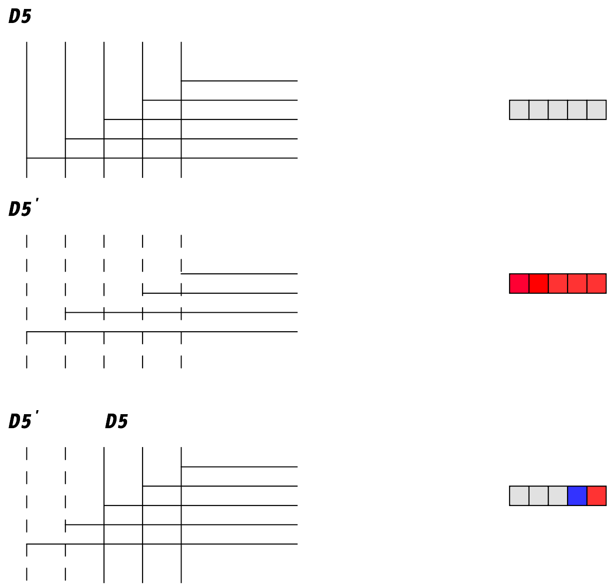

Figure 6. $\mathcal{N}=1$ rotated puncture is represented by $D3-D5-D5^{'}$ brane system and labeled by a colored Young tableaux.

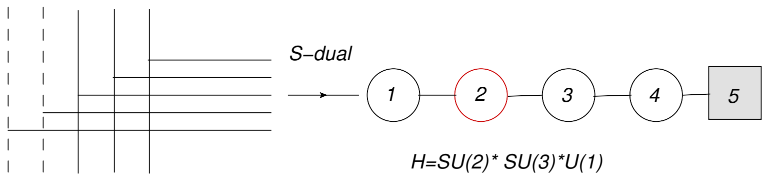

Figure 7. The flavor symmetry of a rotated puncture could be read from the Coulomb branch of S dual 3d quiver. Red circle means that we have 3d $\mathcal{N}=2$ gauge group.

there. The global breaking, on the other hand, break $\mathcal{N}=2$ vector multiplet to $\mathcal{N}=1$ vector multiplet.

Let's summarize some simple properties of the 4d theory which can be read from the above defining geometric objects:

a: The flavor symmetry is read from the local punctures, and the mass deformations are encoded locally at the punctures. The complex structure moduli of the gauge theory could be identified as the exactly marginal deformation whose detailed form will be discussed in next section.

b: There are $U(1)_{45}\times U(1)_{89}$ R symmetries rotating the fibres of two bundles. In the IR, one linear combination becomes the R symmetry, and the other one becomes the flavor symmetry.

c. The moduli space of generalized Hitchin's equation should describe the Coulomb branch of 4d theory compactified on a circle, and the details will appear elsewhere.

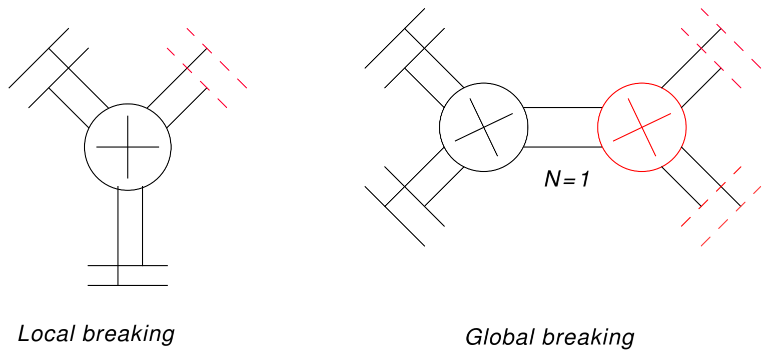

Figure 8. Left: Local breaking is achieved by rotating the branes corresponding to punctures. Right: Global breaking is achieved by gluing two different kinds of three sphere.

3 $\mathcal{N} = 1$ duality

Let's consider four dimensional $\mathcal{N} = 1$ SCFT derived from compactifying 6d A_{N-1} theory on a Riemann surface with D and D' type punctures. We are going to interpret the UV gauge theory description as the degeneration limit of the Riemann surface, and interpret the duality as taking different degeneration limit of the same Riemann surface. The natural question would be which coupling constant of the field theory would be identified as the complex structure moduli of the Riemann surface.

In $\mathcal{N} = 2$ case, S-duality exchanges the gauge coupling as $\tau \to \frac{-1}{\tau}$, and so if we identify gauge coupling as the complex structure moduli, then S-duality is naturally understood as taking different degeneration limits. Notice that the gauge coupling in the conformal case is exactly marginal, which nicely matches the fact that the complex structure moduli is dimensionless.

By comparing with $\mathcal{N} = 2$ S-duality, it is then natural to find the exactly marginal deformations of $\mathcal{N} = 1$ theory. For $\mathcal{N} = 1$ $SU(N)$ SQCD with $2N$ flavors, there is indeed an exactly marginal deformation [32]:

$$W = cQ\tilde{Q}Q\tilde{Q}. \tag{3.1}$$

The gauge group in the Seiberg dual theory is still $SU(N)$ and the superpotential term is

$$W^{'} = Mq\tilde{q} + cMM; \tag{3.2}$$

after integrating out the massive meson, the superpotentail becomes:

$$W^{'} = -\frac{1}{c}q\tilde{q}q\tilde{q}; \tag{3.3}$$

so under Seiberg duality, $c \to -\frac{1}{c}$. Therefore it is suggestive to identify the coupling constant of quartic superpotential term as the complex structure moduli of the Riemann surface, and Seiberg duality could also be interpreted as different degeneration limits of the same Riemann surface.

In general, one can consider quartic superpotential terms preserving certain sub-group of full global symmetries, in the following, we will find out what specific superpotential term should be allowed for theories engineered from M5 branes.

	x^0	x^1	x^2	x^3	x^4	x^5	x^6	x^7	x^8	x^9
D4	$\circ$	$\circ$	$\circ$	$\circ$			$\circ$			
NS5	$\circ$	$\circ$	$\circ$	$\circ$	$\circ$	$\circ$				
D6	$\circ$	$\circ$	$\circ$	$\circ$				$\circ$	$\circ$	$\circ$
$\mathrm{NS5}^{'}$	$\circ$	$\circ$	$\circ$	$\circ$					$\circ$	$\circ$
$\mathrm{D6}^{'}$	$\circ$	$\circ$	$\circ$	$\circ$	$\circ$	$\circ$		$\circ$		

Table 2. Brane configuration for engineering four dimensional $\mathcal{N} = 2$ and $\mathcal{N} = 1$ theories.

3.1 Insights from type IIA brane construction

To get some insights about the superpotential term and matter content, it is useful to review how Seiberg duality is discovered using type IIA brane configurations, then we will find out how Seiberg duality works M5 brane framework by lifting the IIA picture to M5 brane description.

Let's first start with $\mathcal{N} = 2$ theory engineered using D4 $-$ D6 $-$ NS5 brane system of Type IIA string theory. The summary of various brane configuration is listed in table 2. The brane set-up for $\mathcal{N} = 2$ SU(N) gauge theory with $N_f = 2N$ is shown in figure 9, in which the $2N$ flavors are separated into two equal parts. Notice that we use a configuration in which all D6 branes are put on $x_6 = \pm\infty$.

The above brane system is described by N M5 brane wrapping on a sphere with four punctures: two simple punctures describing the intersections of two NS5 branes, and two full punctures describing the boundary condition specified by the D6 brane system, see figure 9. The gauge coupling constant is identified with the complex structure moduli of fourth punctured sphere.

Next let's rotate one of NS brane to $\mathrm{NS}^{'}$ brane, and we get $\mathcal{N} = 1$ $SU(N)$ gauge theory with $N_f = 2N$ (plus singlets and cubic superpotential, so it is a mixed electric-magnetic theory). Now the world volume of NS brane is in x_4, x_5 direction and $\mathrm{NS}^{'}$ brane is in x_8, x_9 direction, and the isometries $U(1)_{45}$ and $U(1)_{89}$ correspond to two R symmetries of $\mathcal{N} = 1$ compactifications. In M5 brane description, the rotation corresponds to rotating one of simple puncture from D type to $\mathrm{D}^{'}$ type, see figure 9, and we also represent two matter systems as the NS and NS' three spheres to match the field theory expectation. This naive lift of type IIA configuration to M5 brane configuration matches our general story: the rotated puncture, and $\mathcal{N} = 1$ vector multiplet from gluing two different types of three punctured spheres.

Seiberg duality corresponds to exchanging the positions of NS and $\mathrm{NS}^{'}$ brane [29], which in M5 brane picture corresponds to exchanging the positions of two simple punctures, and we also need to exchange the NS and $\mathrm{NS}^{'}$ three sphere at the same time. So in this sense the Seiberg duality has the same interpretation of $\mathcal{N} = 2$ S-duality [6]: it corresponds to exchanging the position of punctures or corresponds to taking different degeneration limits.

We have argued that the complex structure moduli should be identified as the quartic superpotential couplings, but the specific form is not known. Let's look at the duality in figure 9 more closely to determine the exact superpotential term. The electric theory has

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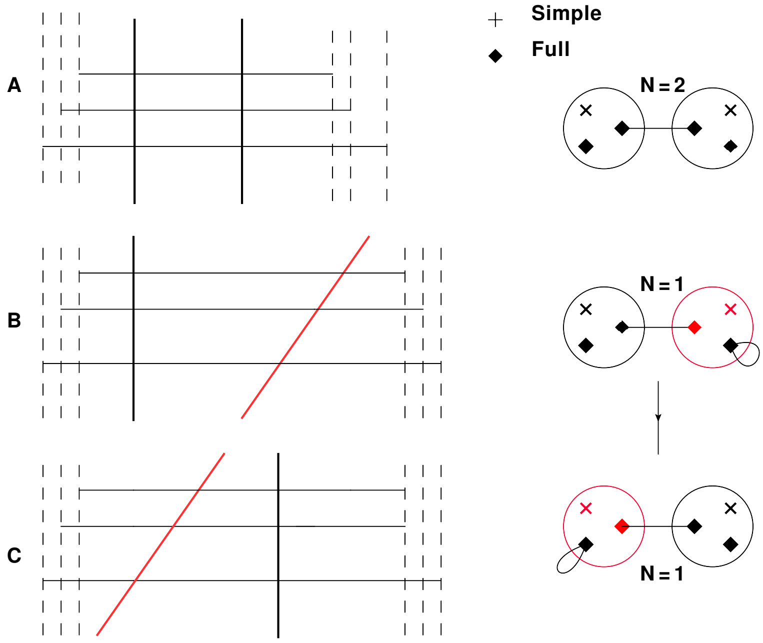

Figure 9. Top: Type IIA brane configuration for $\mathcal{N}=2$ SU(N) with $N_f=2N$ and M theory lift. Middle: Type IIA brane configuration for $\mathcal{N}=1$ SU(N) with $N_f=2N$ which is achieved by rotating one of NS brane to NS$^{'}$ brane, and the M5 brane description corresponds to rotating one of simple puncture. Bottom: Seiberg duality is realized as exchanging the position of NS and NS$^{'}$ brane, which corresponds to exchange the position of simple punctures in M5 brane description, we also need to exchange two three spheres.

two sets of quarks $(q^i, \tilde{q}_{\tilde{j}})$ and $(p^k, \tilde{p}_{\tilde{l}})$ with $i, \tilde{j}, k, \tilde{l} = 1, \ldots, N$. There are gauge singlets $M_k^{\tilde{l}}$ coming from D4 brane suspended between D6 and NS5$^{'}$ brane, and the cubic superpotential between quark and the meson is

$$W_1 = M_k^{\tilde{l}}\left(p^k \tilde{p}_{\tilde{l}} - \frac{1}{N} tr(p\tilde{p})\delta_{\tilde{l}}^k\right), \tag{3.4}$$

here we use the fact that the $U(1)$ on the D4 branes is frozen. Notice that the term inside the bracket is the momentum map for the flavor symmetry $SU(N)_L$, which we denote it as μ_L, and the superpotential can be written as $W = M_L \mu_L$ where M_L is the adjoint on $SU(N)_L$.

In the brane configuration corresponding to Seiberg-dual description, one has the interaction $W^{'} = M_R \mu_R$ from figure 9. However, the two theories with the above interactions are not Seiberg-dual to each other, therefore we need to add some more interaction terms to it. We propose the following extra superpotential term (see [45] for related issue)

$$W_2 = \lambda tr(\mu_1 \mu_2) = \lambda \tilde{q}^a{}_{\tilde{j}} q^j{}_b \tilde{p}^b{}_{\tilde{l}} p^l{}_a - \frac{\lambda}{N} tr(p\tilde{p}) tr(q\tilde{q}), \tag{3.5}$$

where μ_1 is the moment map for the $SU(N)$ gauge group action on $(q^i{}_a, \tilde{q}^a{}_j)$, and μ_2 is the moment map for $SU(N)$ gauge group action on $(p^k{}_a, \tilde{p}^a{}_l)$. The full potential for electric theory is then

$$W = W_1 + W_2 = M_k^{\tilde{l}}\left(p^k\tilde{p}_{\tilde{l}} - \frac{1}{N}tr(p\tilde{p})\delta_{\tilde{l}}^k\right) + \lambda\tilde{q}^a{}_{\tilde{j}}q^j{}_b\tilde{p}^b{}_{\tilde{l}}p^l{}_a - \frac{\lambda}{N}tr(p\tilde{p})tr(q\tilde{q}). \tag{3.6}$$

After performing Seiberg duality, the fundamentals (anti-fundamentals) become as anti-fundamentals (fundamentals), and new mesons appear, the superpotential changes as

$$W = M_k^{\tilde{l}}A_{\tilde{l}}^k + \lambda B_{\tilde{l}}^jC_{\tilde{j}}^l + A_{\tilde{l}}^k p_k^*\tilde{p}^{*\tilde{l}} + D_{\tilde{j}}^i q_i^*\tilde{q}^{*\tilde{j}} + B_{\tilde{l}}^j q_j^*\tilde{p}^{*\tilde{l}} + C_{\tilde{j}}^l p_l^*\tilde{q}^{*\tilde{j}} - \frac{\lambda}{N}trAtrD. \tag{3.7}$$

Off diagonal meson and M is massive, and integrate out them out we get the potential

$$W^d = D_{\tilde{j}}^{'i}\left(q_i^*\tilde{q}^{*\tilde{j}} - \frac{1}{N}tr(q^*\tilde{q^*})\delta_l^k\right) - \frac{1}{\lambda}q_j^*\tilde{p}^{*\tilde{l}}p_l^*\tilde{q}^{*\tilde{j}} + \frac{1}{N\lambda}tr(p^*\tilde{p*})tr(q^*\tilde{q^*}) \tag{3.8}$$

here $D^{'}$ is the traceless part of D, this potential has precisely the same form as the electric theory with the coupling constant reversed: $\lambda \to -\frac{1}{\lambda}$! Therefore, the superpotential of our theory can be written in a simple way:

$$W = ctr\mu_1\mu_2 + tr\mu_A M_A \tag{3.9}$$

where μ_1 and μ_2 are the moment map for two flavor symmetries which are gauged, and M_A is the adjoint on flavor symmetry $SU(N)_A$. The exact marginal coupling constant c might be interpreted as the length of the long tube in the degeneration limit.

One can rotate other branes to get different $\mathcal{N}=1$ theories: for instance, we can rotate D6 and NS branes on far left, which seems to be a good possibility. The brane configuration and its M theory representation is shown in figure 10, and the theory is SQCD without any gauge singlets. Seiberg duality is once again understood as different degeneration limits of the same Riemann surface, i.e. by exchanging the positions of two simple punctures and two types of three sphere. Again, there is a quartic superpotential term for each of the duality frame to match the brane picture.

3.2 $\mathcal{N}=1$ duality from M5 brane

3.2.1 A short review of $\mathcal{N}=2$ duality

S duality of 4d $\mathcal{N}=2$ theory corresponds to different degeneration limits of the same punctured Riemann surface (let's assume all the punctures are full for simplicity):

- In the complete degeneration limit, there are two new full punctures appearing in degenerating a long tube, which represents a $N=2$ vector multiplet.
- The Riemann surface is degenerated into several three punctured spheres representing T_N theory. The gauge theory is derived by gauging the $SU(N)$ flavor symmetries of T_N theory.

When the punctures are general, one have the similar story except that newly appearing puncture is not the full puncture and three punctured sphere is generically an isolated SCFT. The understanding of the S duality is reduced to identify the weakly coupled gauge group and matter system, which is answered in [16, 17] for A_{N-1} case, and [46] for D_N theories.

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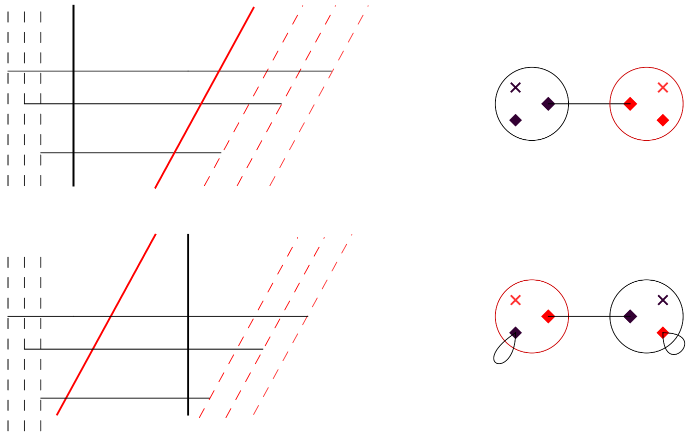

Figure 10. A different $\mathcal{N}=1$ theory is described by rotating both NS and D6 branes, and the M theory lift is also shown. Seiberg duality again corresponds to exchange the position of simple punctures and two types of three spheres.

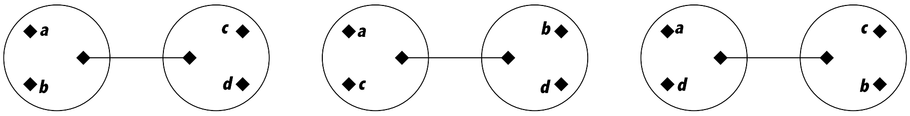

Figure 11. S duality for $\mathcal{N}=2$ theory corresponds to different degeneration limit of the same Riemann surface.

3.2.2 $\mathcal{N}=1$ duality: three punctured sphere and gluing

Motivated by the type IIA brane construction and it's lift to M theory, we would like to conjecture that $\mathcal{N}=1$ duality is realized the same way as $\mathcal{N}=2$ theory: different duality frames correspond to different degeneration limits of the same punctured Riemann surface whose defining data is explained in last section.

The first question is that how the global breaking, i.e. different bundle structure is reflected in the degeneration limit? Motivated by the theory without punctures which is considered in [23], we propose that the global breaking is encoded by having both NS and NS$^{'}$ type three spheres whose number is determined by the degree of bundles and puncture types. Let's assume there are n_{NS} NS sphere and $n_{NS'}$ NS$^{'}$ sphere, which might be determined by the following formula:

$$n_{NS}=p+n_D, \quad n_{NS'}=q+n_{D'}, \tag{3.10}$$

where $p=deg(L_1)$ and $q=deg(L_2)$ with $p+q=2g-2$.

The heuristic interpretation for the above formula is that: the Euler number $2g-2+n$ is separated two parts: NS (NS$^{'}$) part is due to the number of D (D$^{'}$) type punctures and the line bundle L_1 (L_2). If we assume each NS (NS$^{'}$) three sphere contributes one unit to NS (NS$^{'}$) part of Euler number, then we got the above formula. In this paper, we restrict to the case where both n_{NS} and $n_{NS'}$ are non-negative.

Consider a three sphere with only maximal punctures, then we have the following possibilities

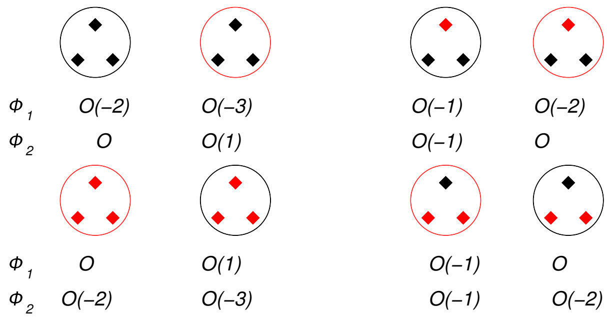

Figure 12. The bundle structures of different three punctured sphere with full punctures.

1. $n_D = 3, n_{D'} = 0, n_{NS} = 1$, then the line bundle structure is $\mathcal{O}(-2) \bigoplus \mathcal{O}(0)$.
2. $n_D = 3, n_{D'} = 0, n_{NS'} = 1$, then the line bundle structure is $\mathcal{O}(-3) \bigoplus \mathcal{O}(1)$.
3. $n_D = 2, n_{D'} = 1, n_{NS} = 1$, then the line bundle structure is $\mathcal{O}(-1) \bigoplus \mathcal{O}(-1)$.
4. $n_D = 2, n_{D'} = 1, n_{NS'} = 1$, then the line bundle structure is $\mathcal{O}(-2) \bigoplus \mathcal{O}(0)$.

There are other four type of three spheres derived by exchanging D type puncture with D$'$ type puncture, and NS type sphere with NS$'$ type three sphere, see figure 12.

We depict the NS three sphere as black circle and NS$'$ three sphere as red circle. Based on the brane picture studied in last subsection, we can easily find the gauging rules:

- If two three spheres with same color are glued together, there is a $\mathcal{N} = 2$ vector multiplet and usual $\mathcal{N} = 2$ coupling:

$$W = \tau tr\Phi(\mu_1 - \mu_2). \tag{3.11}$$

- If two three spheres with opposite color are glued together, there is a $\mathcal{N} = 1$ vector multiplet and a superpotential term:

$$W = ctr(\mu_1\mu_2), \tag{3.12}$$

  where μ_1, μ_2 are the moment maps for two gluing punctures.

we also assume that a new D (D$'$) type puncture appears on the black (red) three sphere in the complete degeneration limit.

The remaining task is to determine the matter system corresponding to three sphere with rotated puncture. The local effects can be easily seen using the brane construction. Let's assume the three sphere is NS type, and look at a rotated full puncture: a D$'$ type full puncture. Because of the rotated puncture, one need to use generalized Nahm's equation. The scalar $Y = x_8 + ix_9$ now has trivial Nahm pole, and therefore $X = x_4 + ix_5$ which represents the fluctuation for D3 brane suspended between D5$'$ and NS branes is unconstrained and give a gauge singlet M which is the adjoint of $SU(N)$ flavor symmetry. We also have the cubic superpotential

$$W = Tr(\mu_G M), \tag{3.13}$$

here μ_G is the moment map for flavor symmetry $SU(N)$, see figure 13.

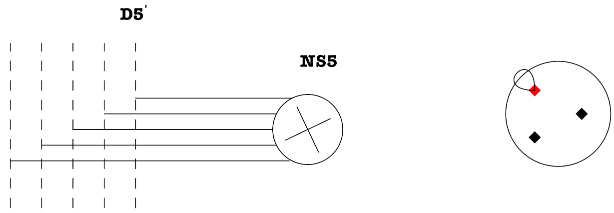

Figure 13. There are new gauge singlets coming from the fluctuation of D3 branes suspended between $D5^{'}$ branes and NS brane.

For general rotated punctures specified by a Young tableaux and a $SU(2)$ homomorphism ρ, the $D5^{'}$ brane system gives a Nahm pole boundary condition on scalar Y, which break the flavor symmetry to the commutant H, and the moment map for the flavor symmetry is μ_H. From the generalized Nahm's equation, only the X component commuting with ρ is preserved, namely

$$X \in g^{\rho}, \tag{3.14}$$

and the superpotential becomes

$$W = Tr(\mu_H X_H). \tag{3.15}$$

For each nilpotent element Y, one can associate a standard triple (e, f, h) where $e = Y$, and h is a semi-simple element. h provides a grading of the lie algebra g which decomposes as

$$g = \bigoplus g_i \tag{3.16}$$

where $g_i = \{g|[h, g] = ig\}$. Therefore the centralizer of Y is also factorized as

$$g^Y = \bigoplus g_i^Y \tag{3.17}$$

and the centralizer of the homomorphism ρ is actually given by $g^{\rho} = g_0^Y$, which gives the reductive group identified with flavor group.

In summary, the local effect of rotating a general puncture specified by a homomorphism ρ is the following: the flavor symmetry is the commutant of ρ and there are g^{ρ} number of gauge singlets, which is equal to the dimension of the flavor group. For example, in the case of full puncture, there are $N^2 - 1$ gauge singlets, and in the case of simple puncture, there is only one gauge singlet. Notice that in our treatment, the number of gauge singlets we see is different from what was found in [25].[7] Here we only look at the puncture locally, and it might be possible that we can recover the result in [25] by completing a global analysis of the three punctured sphere theory. We leave this question to the future.

Globally, after rotating the puncture, the three puncture theory is very different from the original there punctured sphere, as they are represented by completely different data: different bundles, different puncture types. In some cases, one can express the new three

[7]They count the number of gauge singlets as g^Y, namely the centralizer of Nilpotent element, here according to our generalized Nahm's equation, the number of gauge singlets is equal to g^{ρ}.

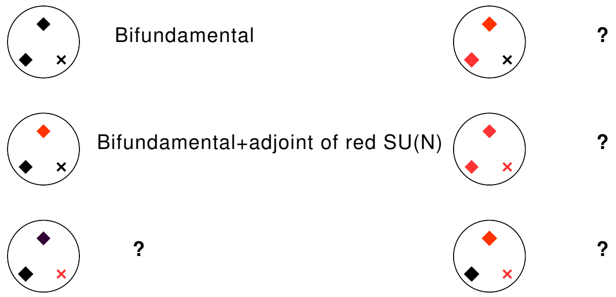

Figure 14. Various three punctured sphere theory defined using two full punctures and one simple puncture. In the first and second case, one can express the matter content in terms of free fields using type IIA brane picture.

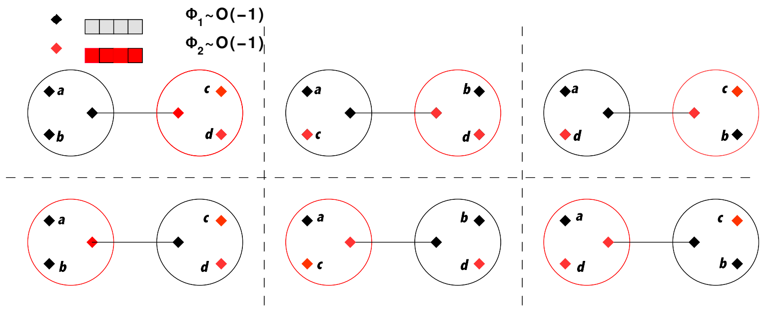

Figure 15. Different duality frames of a theory defined by full punctures.

punctured sphere theory in terms of original $\mathcal{N}=2$ matter content plus the gauge singlet introduced above from local analysis, but in general, the new theory is not related to the original theory in a simple way, see figure 14, some of them is even strongly coupled. In this paper, we simply assume the fact that each $\mathcal{N}=1$ matter system is uniquely specified by a three punctured sphere whose defining data are bundles and puncture types. It would be really interesting to further study these matter systems.

3.2.3 Examples

Using the above rules, we are going to study the dualities of several interesting examples involving punctures.

Example 1: Let's consider a sphere with two full D type punctures and two full D' type punctures, and the line bundle is $L_1 = L_2 = \mathcal{O}(-1)$, then according to formula [3.10], there are one NS sphere and one NS' sphere. There are six duality frames as shown in figure 15, and one of them is described by $\mathcal{N}=1$ vector multiplet coupled with two T_N theories, and there is a quartic superpotential. Notice that the coupling constant is proportional for duality frames in the same column, as we only exchange the NS and NS' type spheres without moving the punctures (without changing the complex structure moduli and therefore the quartic coupling).

Example 2: SU(N) SQCD with $N_f = 2N$ could be realized by a sphere with four punctures: D type full puncture and simple puncture, plus D' type full and simple puncture.

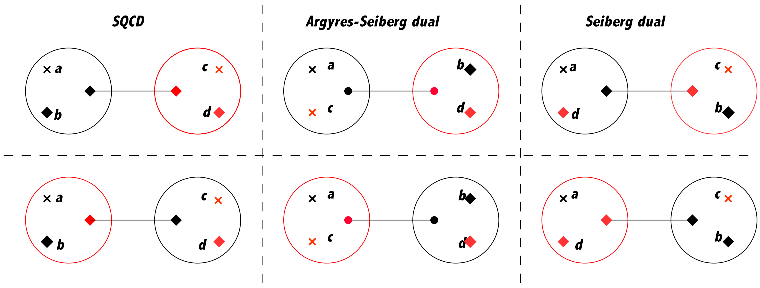

Figure 16. Different duality frames of SU(N) SQCD with $N_f = 2N$.

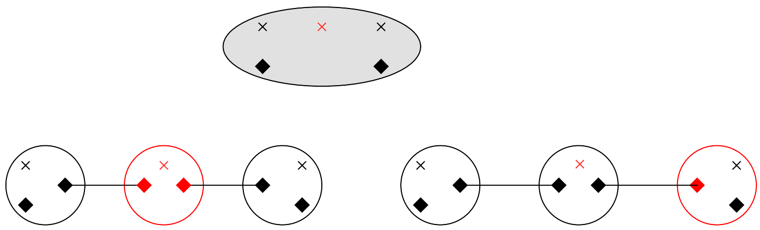

Figure 17. A theory is defined by a sphere with five punctures, and we show two duality frames.

The line bundle structure is also $L_1 = L_2 = \mathcal{O}(-1)$. In the degeneration limit, there are one NS and one NS$'$ sphere. There are also six duality frames as discussed in [25], but the matter system in our treatment is quite different, i.e. we do not know if there is a free field representation for various three punctured sphere appearing in other duality frames except the standard Seiberg-dual in which the three punctured sphere could be represented by a bifundamental fields and gauge singlets transforming as adjoint of one of $SU(N)$ flavor group.

Example 3: Let's consider a sphere with two D type full punctures, and two D type simple punctures, and one D$'$ type puncture, and the line bundle structure is $\mathcal{O}(-2)\bigoplus\mathcal{O}$, so there are two NS spheres and one NS$'$ sphere. In one duality frame, one get a linear quiver with two $\mathcal{N} = 1$ $SU(N)$ gauge groups and quartic superpotential terms. In another duality frame, one has $\mathcal{N} = 2$ gauge group and various mesons and cubic superpotential term.

Example 4: Let's consider a torus with one D type simple puncture and one D$'$ type simple puncture, and the line bundle is $L_1 = L_2 = \mathcal{O}(-1)$. There are one NS sphere and one NS$'$ sphere in weakly coupled gauge theory description. In one duality frame, we have a Klebanov-Witten [47] like theory with superpotential:

$$W = c_1 Tr(\mu_1\mu_2) + c_2 Tr(\mu_{1'}\mu_{2'}). \tag{3.18}$$

The conventional Seiberg duality corresponds to exchange both the simple puncture and three spheres. If we only exchange the simple punctures, we find a different duality frame where we have gauge singlets and cubic superpotential term as shown in figure 18B, which is a new dual of the Klebanov-Witten like theory.

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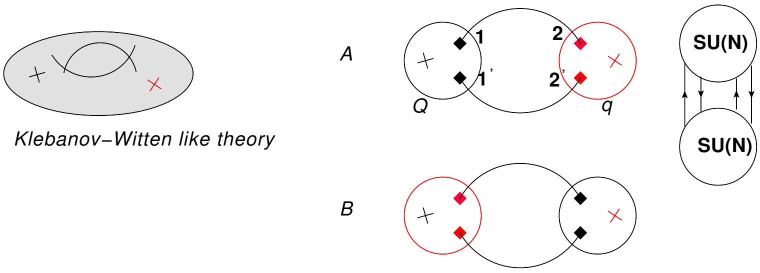

Figure 18. The new dual of Klebanov-Witten like theories.

4 $\mathcal{N} = 1$ duality for other theories

4.1 Partially rotated puncture

In last section, we studied $\mathcal{N} = 1$ duality of theories defined using D and D$^{'}$ type punctures. In this part, we briefly discuss the theory defined using partial rotated puncture. The story is quite similar: in the degeneration limit, the number of NS or NS$^{'}$ three spheres are given by the formula [3.10], and now n_D ($n_{D'}$) means the number of punctures which have $D5$ ($D5^{'}$) brane as the starting one in their brane construction; and the gluing is also the same as discussed in last section. The remaining question is to determine the matter content, namely, how many gauge singlets are there. This question can be similarly answered by looking at the flavor symmetry generated by this puncture.

For simplicity, let's consider a rotated full puncture which can be represented by a sequence of numbers $(n_1, n_2, \ldots, n_r)$, which means that there are n_1 D5 branes followed by n_2 $D5^{'}$ branes. Then the flavor symmetry read from S dual brane configuration is

$$H = S[\sum_i U(n_i)]. \tag{4.1}$$

Now suppose the above puncture is put on a NS three sphere, then there are new fields coming from D3 branes suspended between NS and D5$^{'}$ branes. So there are adjoints on $\sum_{i\in even} SU(n_i)$ and a corresponding cubic superpotential. If the above puncture is put on a NS$^{'}$ three sphere, then there are adjoints on $\sum_{i\in odd} SU(n_i)$.

4.2 D_N theories

It is also straightforward to generalize the study of dualities to $\mathcal{N} = 1$ theory engineered using six dimensional D_N theory. We already discussed the generalized Hitchin's equation and regular punctures which are valid for any group. One also has a brane construction for various $\mathcal{N} = 2$ regular punctures [27], and therefore we can also define the rotated puncture in similar way.

Here let's consider two theories for illustration: $SO(2n)$ theory with $N_f = 4n - 4$ and $USp(2n-2)$ theory with $N_f = 4n$, notice that the above two theories are self-dual (the dual gauge group has the same rank) under Seiberg duality, and the quartic superpotential is exactly marginal since the quark has R charge $\frac{1}{2}$.

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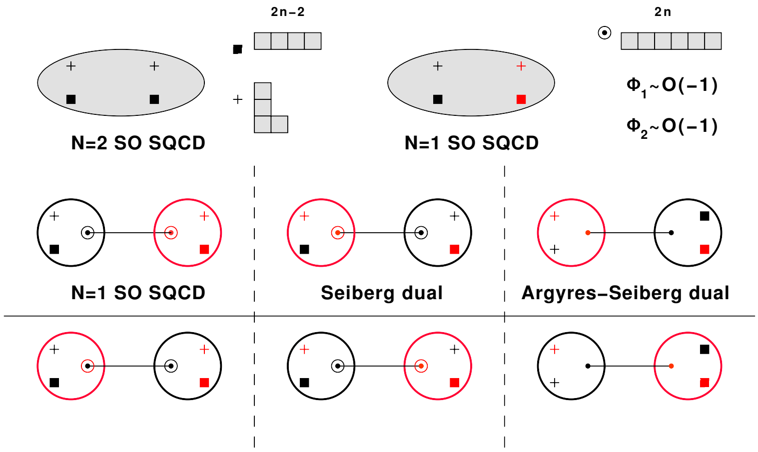

Figure 19. The $M5$ brane compactification data for $SO(2n)$ SQCD with $N_f = 4n - 4$. One find five other duality frames and one of them is Seiberg dual.

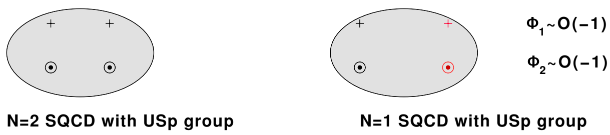

Figure 20. The $M5$ brane compactification data for $Usp(2n-2)$ SQCD with $N_f = 4n$. One can study dualities by looking at different degeneration limit.

The M5 brane description for $\mathcal{N} = 2$ [10] and $\mathcal{N} = 1$ SO SQCD is shown in figure 19, here again, we introduce the quartic superpotential term in $\mathcal{N} = 1$ gluing. Again, the Seiberg duality is interpreted as taking different degeneration limits of the same Riemann surface, see figure 19. We also find five other duality frames, and one of them is the standard Seiberg dual. The M5 brane description for USp SQCD is shown in figure 20, and one can study dualities in exactly same way.

5 Conclusion

We found two important missing ingredients in studying $\mathcal{N} = 1$ theory from M5 brane: a. Generalized Hitchin's equation involving two Higgs fields; b. Regular punctures which are classified by orbits of commuting nilpotent pair. We then study $\mathcal{N} = 1$ dualities by looking at various degeneration limits of Riemann surface. There are many open questions:

- In this paper, we focus on regular punctures which have brane representations. It is interesting to study more general local puncture i.e. using the algebraic tools developed in [26]. The regular puncture of $\mathcal{N} = 1$ theory is much more fruitful than $\mathcal{N} = 2$ theory, for example, the number of regular puncture is infinite, and the details will appear in a separate publication [48].

- It is interesting to further study the theories defined in this paper, such as the basic three puncture sphere theory, a maximization, chiral ring, central charges and super-

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conformal index, etc. The M5 brane construction presented in this paper is expected to be very helpful.

- It is interesting to study the irregular puncture of $\mathcal{N}=1$ theory, which locally should be classified by commuting first order differential operator with higher order singularity. Using these irregular singularities, one can find a lot of new $\mathcal{N}=1$ Argyres-Douglas type theories [49], and the details will appear in [48]. Irregular singularities are also needed for describing confining theories.

- One can compactify 4d theory on a circle to get 3d $\mathcal{N}=2$ theory, and try to understand 3d Seiberg duality from 4d duality follow [50], our M5 brane construction should be very helpful. It is also interesting to learn mirror symmetry of 3d $\mathcal{N}=2$ theory, probably along [40].

- There are some other interesting dynamical question one might be interested to study: Seiberg-Witten curve on Coulomb branch , phase structure, extended objects such as line operators, surface operators and domain walls, dynamically generated superpotential [51].

- The generalized Hitchin's equation plays a key role in our construction, and this equation is not studied before. Understanding the property of this equation like it's moduli space is perhaps the most important question in trying to learn $\mathcal{N}=1$ gauge dynamics from M5 brane [52].

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A A derivation of local generalized Hitchin's equation

The local form of the generalized Hitchin's equation can be derived by studying the world volume action of $D4$ branes suspended between $D6$ and $D6'$ branes (so one only consider the fluctuation of the scalar fields in $(x_4, x_5), (x_8, x_9)$ directions), moreover, we assume that all the fields have only coordinate dependence on x_1, x_2, and the bosonic action is

$$L = \int d^3x \left(\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} D_\mu \phi_i D^\mu \phi_i + \frac{1}{4} \sum_{i \neq j} [\phi_i, \phi_j]^2\right). \tag{A.1}$$

For our interest, $i = 4, 5, 8, 9$. The energy of the static configuration is

$$
\begin{aligned}
E &= \frac{1}{2}\int d^2x(F_{12}F_{12} + D_a\phi_i D_a\phi_i + \sum_{i<j}[\phi_i,\phi_j]^2) \\
&= \frac{1}{2}\int d^2x[(F_{12} - [\phi_4,\phi_5] - [\phi_8,\phi_9])^2 + (D_1\phi_4 - D_2\phi_5))^2 + (D_2\phi_4 + D_1\phi_5)^2 \\
&(D_1\phi_8 - D_2\phi_9))^2 + (D_2\phi_8 + D_1\phi_9)^2 + ([\phi_4,\phi_8] - [\phi_5,\phi_9])^2 + ([\phi_4,\phi_9] + [\phi_5,\phi_8])^2 + T]
\end{aligned} \tag{A.2}
$$

where T is a topological term, so the minimization of energy gives us the following generalized Hitchin's equations

$$
\begin{aligned}
D_1\phi_4 &= D_2\phi_5, \quad D_1\phi_5 = -D_2\phi_4, \\
D_1\phi_8 &= D_2\phi_9, \quad D_1\phi_9 = -D_2\phi_8, \\
[\phi_4,\phi_8] &= [\phi_5,\phi_9] \quad [\phi_4,\phi_9] = [\phi_8,\phi_5], \\
F_{12} &= [\phi_4,\phi_5] + [\phi_8,\phi_9].
\end{aligned} \tag{A.3}
$$

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