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MACLAURIN'S SERIES EXPANSIONS FOR POSITIVE INTEGER POWERS OF INVERSE (HYPERBOLIC) SINE AND TANGENT FUNCTIONS, CLOSED-FORM FORMULA OF SPECIFIC PARTIAL BELL POLYNOMIALS, AND SERIES REPRESENTATION OF GENERALIZED LOGSINE FUNCTION

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To Professor Shi-Ying Yuan, retired President of Henan Polytechnic University

In the paper, the authors find series expansions and identities for positive integer powers of inverse (hyperbolic) sine and tangent, for composite of incomplete gamma function with inverse hyperbolic sine, in terms of the first kind Stirling numbers, apply a newly established series expansion to derive a closed-form formula for specific partial Bell polynomials and to derive a series representation of generalized logsine function, and deduce combinatorial identities involving the first kind Stirling numbers.

1. OUTLINES

Basing on conventions in community of mathematics, we use the notations

$$\mathbb{N} = \{1, 2, \dots\}, \quad \mathbb{N}_- = \{-1, -2, \dots\}, \quad \mathbb{N}_0 = \{0, 1, 2, \dots\},$$

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$$\mathbb{Z}=\{0,\pm1,\pm2,\dots\},\quad \mathbb{R}=(-\infty,\infty),\quad \mathbb{C}=\bigl\{x+\mathrm{i}\,y:x,y\in\mathbb{R},\mathrm{i}=\sqrt{-1}\,\bigr\}.$$

In general, Maclaurin's series expansions of powers of elementary functions and hypergeometric functions are not widely available. This has been demonstrated in the papers [**22**, **32**, **47**], for example. Many special cases of Maclaurin's series expansions of the positive integer power $(\arcsin t)^m$ for $m\in\mathbb{N}$ have been reviewed and surveyed in the paper [**20**].

In Section 2 of this paper, we will discover Maclaurin's series expansions of the power functions

$$\biggl(\frac{\arcsin t}{t}\biggr)^m,\quad \frac{(\arcsin t)^m}{\sqrt{1-t^2}},\quad \biggl(\frac{\operatorname{arcsinh} t}{t}\biggr)^m,\quad \frac{(\operatorname{arcsinh} t)^m}{\sqrt{1-t^2}}$$

for $m\in\mathbb{N}_0$. Some of these series expansions simplify and unify previous results in [**20**, Section 2 and 5]. In this section, we will also derive two combinatorial identities for finite sums involving the first kind Stirling numbers $s(n,k)$ which can be generated [**10**, **14**] by

$$\frac{[\ln(1+x)]^k}{k!}=\sum_{n=k}^{\infty}s(n,k)\frac{x^n}{n!},\quad |x|<1 \tag{1}$$

and satisfy diagonal recursive relations

$$\frac{s(n+k,k)}{\binom{n+k}{k}}=\sum_{\ell=0}^{n}(-1)^\ell\frac{\langle k\rangle_\ell}{\ell!}\sum_{m=0}^{\ell}(-1)^m\binom{\ell}{m}\frac{s(n+m,m)}{\binom{n+m}{m}}$$

and

$$s(n,k)=(-1)^{n-k}\sum_{\ell=0}^{k-1}(-1)^\ell\binom{n}{\ell}\binom{\ell-1}{k-n-1}s(n-\ell,k-\ell)$$

in [**34**, p. 23, Theorem 1.1] and [**40**, p. 156, Theorem 4].

In Section 3, applying Maclaurin's series expansion of $\bigl(\frac{\arcsin t}{t}\bigr)^m$ established in Section 2, we will present a closed-form formula of specific values

$$\mathrm{B}_{2n,k}\biggl(0,\frac{1}{3},0,\frac{9}{5},0,\frac{225}{7},\dots,\frac{1+(-1)^{k+1}}{2}\frac{[(2n-k)!!]^2}{2n-k+2}\biggr) \tag{2}$$

for $2n\ge k\in\mathbb{N}$, where partial Bell polynomials $\mathrm{B}_{n,k}$ for $n\ge k\in\mathbb{N}_0$ are defined in [**10**, Definition 11.2] and [**14**, p. 134, Theorem A] by

$$\mathrm{B}_{n,k}(x_1,x_2,\dots,x_{n-k+1})=\sum_{\substack{1\le i\le n-k+1,\ell_i\in\mathbb{N}_0,\\ \sum_{i=1}^{n-k+1}i\ell_i=n,\\ \sum_{i=1}^{n-k+1}\ell_i=k}}\frac{n!}{\prod_{i=1}^{n-k+1}\ell_i!}\prod_{i=1}^{n-k+1}\biggl(\frac{x_i}{i!}\biggr)^{\ell_i}. \tag{3}$$

This kind of polynomials are important in combinatorics, number theory, analysis, and other areas in mathematical sciences. In recent years, some new conclusions and applications of specific values for partial Bell polynomials $\mathrm{B}_{n,k}$ have been reviewed and surveyed in [45, 46]. Our main result in Section 3 simplifies and unifies those results in [20, Sections 1 and 3].

In Section 4, applying Maclaurin's series expansion of the power $\left(\frac{\arcsin t}{t}\right)^m$ established in Section 2, we will derive a series representation of the generalized logsine function

$$\operatorname{Ls}_j^{(k)}(\theta) = -\int_0^\theta x^k \left(\ln\left|2\sin\frac{x}{2}\right|\right)^{j-k-1} \mathrm{d}\,x, \tag{4}$$

where j,k are integers with $j \ge k+1 \in \mathbb{N}$ and θ is an arbitrary real number. The generalized logsine function $\operatorname{Ls}_j^{(k)}(\theta)$ was originally introduced in [30, pp. 191–192]. This series representation of the generalized logsine function $\operatorname{Ls}_j^{(k)}(\theta)$ simplifies and unifies corresponding ones in [20, Section 4] and [15, 26].

In Section 5, by similar methods in Section 2, we will find Maclaurin's series expansion of the function $\mathrm{e}^{\operatorname{arcsinh} t}$ and three series identities involving $(\operatorname{arcsinh} t)^\ell$ and the first kind Stirling numbers $s(n,k)$. Moreover, we also discover Maclaurin's series expansions of the function $\Gamma(m, \operatorname{arcsinh} t)$ for $m \ge 2$, where the incomplete gamma function $\Gamma(z,x)$ is defined [23, 36, 43, 44] by

$$\Gamma(z,x) = \int_x^\infty \mathrm{e}^{-t}\, t^{z-1}\,\mathrm{d}\,t \tag{5}$$

for $\Re(z) > 0$ and $x \in \mathbb{N}_0$.

In Section 6, basing on Maclaurin's series expansions for positive integer powers of the inverse tangent function $\arctan t$ and the inverse hyperbolic tangent function $\operatorname{arctanh} t$, we will verify two explicit and general expressions of Maclaurin's series expansions of positive integer powers $(\arctan t)^n$ and $(\operatorname{arctanh} t)^n$ for $n \in \mathbb{N}$.

In Section 7, we state useful remarks on our main results and related stuffs, including infinite series representations of positive integer powers of the circular constant π.

2. MACLAURIN'S SERIES EXPANSION FOR POSITIVE INTEGER POWERS OF INVERSE (HYPERBOLIC) SINE FUNCTION

In [20, Remarks 5.2 to 5.5], a review and survey of special cases of Maclaurin's series expansions of $(\arcsin t)^\ell$ for $\ell \in \mathbb{N}$ was presented. In [20, Section 2], general expressions for Maclaurin's series expansions of $(\arcsin t)^{2\ell-1}$ and $(\arcsin t)^{2\ell}$ for $\ell \in \mathbb{N}$ were established respectively.

In this section, we find simpler and general expressions of Maclaurin's series expansions of the functions

$$(\arcsin t)^m, \quad \frac{(\arcsin t)^m}{\sqrt{1-t^2}}, \quad (\operatorname{arcsinh} t)^m, \quad \frac{(\operatorname{arcsinh} t)^m}{\sqrt{1-t^2}}$$

for $m \in \mathbb{N}_0$. We also derive two combinatorial identities for finite sums involving the first kind Stirling numbers $s(n,k)$.

Theorem 1. *For $m \in \mathbb{N}$ and $|t| < 1$, the function $\left(\frac{\arcsin t}{t}\right)^m$, whose value at $t = 0$ is defined to be 1, has Maclaurin's series expansion*

$$\left(\frac{\arcsin t}{t}\right)^m = 1 + \sum_{k=1}^{\infty} (-1)^k \frac{Q(m, 2k; 2)}{\binom{m+2k}{m}} \frac{(2t)^{2k}}{(2k)!}, \tag{6}$$

where

$$Q(m, k; \alpha) = \sum_{\ell=0}^{k} \binom{m+\ell-1}{m-1} s(m+k-1, m+\ell-1) \left(\frac{m+k-\alpha}{2}\right)^{\ell} \tag{7}$$

for $m, k \in \mathbb{N}$ and $\alpha \in \mathbb{R}$ such that $m + k \neq \alpha$ and $s(m+k-1, m+\ell-1)$ is generalized by (1).

First proof. In [6, p. 3, (2.7)] and [21, pp. 210–211, (10.49.33) and (10.49.34)], the formulas

$$\sum_{k=0}^{\infty} \frac{(\mathrm{i}\, a)_{k/2}}{(\mathrm{i}\, a + 1)_{-k/2}} \frac{(-\mathrm{i}\, x)^k}{k!} = \exp\left(2a \arcsin \frac{x}{2}\right) \tag{8}$$

and

$$\sum_{k=0}^{\infty} \frac{\left(\mathrm{i}\, a + \frac{1}{2}\right)_{k/2}}{\left(\mathrm{i}\, a + \frac{1}{2}\right)_{-k/2}} \frac{(-\mathrm{i}\, x)^k}{k!} = \frac{2}{\sqrt{4 - x^2}} \exp\left(2a \arcsin \frac{x}{2}\right) \tag{9}$$

were collected, where $\mathrm{i} = \sqrt{-1}$ is the imaginary unit, the extended Pochhammer symbol $(z)_\alpha$ for $z, \alpha \in \mathbb{C}$ such that $z + \alpha \neq 0, -1, -2, \dots$ is defined by

$$(z)_\alpha = \frac{\Gamma(z+\alpha)}{\Gamma(z)}, \tag{10}$$

and the Euler gamma function $\Gamma(z)$ is defined [57, Chapter 3] by

$$\Gamma(z) = \lim_{n\to\infty} \frac{n! n^z}{\prod_{k=0}^{n} (z+k)}, \quad z \in \mathbb{C} \setminus \{0, -1, -2, \dots\}.$$

In (8) and (9), replacing x by $2x$ and employing the extended Pochhammer symbol in (10) gives

$$\begin{aligned} \mathrm{e}^{2a \arcsin x} &= \sum_{k=0}^{\infty} (-2\,\mathrm{i})^k \frac{\Gamma\left(\mathrm{i}\, a + \frac{k}{2}\right)}{\Gamma(\mathrm{i}\, a)} \frac{\Gamma(\mathrm{i}\, a + 1)}{\Gamma\left(\mathrm{i}\, a - \frac{k}{2} + 1\right)} \frac{x^k}{k!} \\ &= 1 + \mathrm{i}\, a \sum_{k=1}^{\infty} (-2\,\mathrm{i})^k \binom{\mathrm{i}\, a + \frac{k}{2} - 1}{k-1} \frac{x^k}{k} \end{aligned} \tag{11}$$

and

$$\frac{\mathrm{e}^{2a \arcsin x}}{\sqrt{1-x^2}} = \sum_{k=0}^{\infty} (-2\,\mathrm{i})^k \frac{\Gamma\left(\mathrm{i}\, a + \frac{1+k}{2}\right)}{\Gamma\left(\mathrm{i}\, a + \frac{1-k}{2}\right)} \frac{x^k}{k!} = \sum_{k=0}^{\infty} (-2\,\mathrm{i})^k \binom{\mathrm{i}\, a + \frac{k-1}{2}}{k} x^k, \tag{12}$$

where the extended binomial coefficient $\binom{z}{w}$ is defined in [**58**] by

$$
\binom{z}{w}=\begin{cases}\dfrac{\Gamma(z+1)}{\Gamma(w+1)\Gamma(z-w+1)}, & z\notin\mathbb{N}_-,\quad w,z-w\notin\mathbb{N}_-\\ 0, & z\notin\mathbb{N}_-,\quad w\in\mathbb{N}_-\text{ or } z-w\in\mathbb{N}_-\\ \dfrac{\langle z\rangle_w}{w!}, & z\in\mathbb{N}_-,\quad w\in\mathbb{N}_0\\ \dfrac{\langle z\rangle_{z-w}}{(z-w)!}, & z,w\in\mathbb{N}_-,\quad z-w\in\mathbb{N}_0\\ 0, & z,w\in\mathbb{N}_-,\quad z-w\in\mathbb{N}_-\\ \infty, & z\in\mathbb{N}_-,\quad w\notin\mathbb{Z}\end{cases}\tag{13}
$$

in terms of the gamma function $\Gamma(z)$ and the falling factorial

$$
\langle z\rangle_k=\prod_{\ell=0}^{k-1}(z-\ell)=\begin{cases}z(z-1)\cdots(z-k+1), & k\in\mathbb{N};\\ 1, & k=0.\end{cases}\tag{14}
$$

Integrating on both sides of (12) with respect to $x\in(0,t)\subset(-1,1)$ leads to

$$
\frac{\mathrm{e}^{2a\arcsin t}-1}{2a}=t+at^2+\sum_{k=2}^{\infty}(-2\,\mathrm{i})^k\binom{\mathrm{i}\,a+\frac{k-1}{2}}{k}\frac{t^{k+1}}{k+1}.\tag{15}
$$

In [**51**, p. 165, (12.1)], the Stirling numbers of the first kind $s(k,\ell)$ produce Taylor's coefficients of the expansion of the binomial coefficient by

$$
k!\binom{z}{k}=\sum_{\ell=0}^{k}s(k,\ell)z^\ell,\quad z\in\mathbb{C}.\tag{16}
$$

Therefore, we acquire

$$
\begin{aligned}
\binom{\mathrm{i}\,a+\frac{k}{2}-1}{k-1}&=\frac{1}{(k-1)!}\sum_{\ell=0}^{k-1}s(k-1,\ell)\biggl(\mathrm{i}\,a+\frac{k}{2}-1\biggr)^\ell\\
&=\frac{1}{(k-1)!}\sum_{\ell=0}^{k-1}s(k-1,\ell)\sum_{m=0}^{\ell}\binom{\ell}{m}(\mathrm{i}\,a)^m\biggl(\frac{k-2}{2}\biggr)^{\ell-m}\\
&=\frac{1}{(k-1)!}\sum_{m=0}^{k-1}\Biggl[\sum_{\ell=m}^{k-1}\binom{\ell}{m}s(k-1,\ell)\biggl(\frac{k-2}{2}\biggr)^{\ell-m}\Biggr](\mathrm{i}\,a)^m
\end{aligned}
$$

for $k\ge3$. Substituting this into the right hand side of (11) yields

$$
1+\mathrm{i}\,a\sum_{k=1}^{\infty}(-2\,\mathrm{i})^k\binom{\mathrm{i}\,a+\frac{k}{2}-1}{k-1}\frac{x^k}{k}=1+2ax+2a^2x^2
$$

$$
\begin{aligned}
&+\sum_{k=3}^{\infty}(-2\,\mathrm{i})^k\sum_{m=0}^{k-1}\left[\sum_{\ell=m}^{k-1}\binom{\ell}{m}s(k-1,\ell)\left(\frac{k-2}{2}\right)^{\ell-m}\right](\mathrm{i}\,a)^{m+1}\frac{x^k}{k!}\\
&=1+\left[2x+\sum_{k=3}^{\infty}(-2)^k\,\mathrm{i}^{k+1}\,Q(1,k-1;2)\frac{x^k}{k!}\right]a\\
&+\left(2x^2-\sum_{k=3}^{\infty}(-2\,\mathrm{i})^k\left[\sum_{\ell=1}^{k-1}\ell s(k-1,\ell)\left(\frac{k-2}{2}\right)^{\ell-1}\right]\frac{x^k}{k!}\right)a^2\\
&+\sum_{m=3}^{\infty}\mathrm{i}^m\left(\sum_{k=m}^{\infty}(-2\,\mathrm{i})^k\left[\sum_{\ell=m}^{k}\binom{\ell-1}{m-1}s(k-1,\ell-1)\left(\frac{k-2}{2}\right)^{\ell-m}\right]\frac{x^k}{k!}\right)a^m.
\end{aligned}
$$

The left hand side of (11) can be expanded into

$$
\mathrm{e}^{2a\arcsin x}=\sum_{m=0}^{\infty}\frac{(2a\arcsin x)^m}{m!}=\sum_{m=0}^{\infty}\frac{(2\arcsin x)^m}{m!}a^m.
$$

Comparing coefficients of a^m for $m\in\mathbb{N}$ results in

$$
\begin{aligned}
2\arcsin x&=2x+\sum_{k=3}^{\infty}(-2)^k\,\mathrm{i}^{k+1}\,Q(1,k-1;2)\frac{x^k}{k!}\\
&=2x\left[1+\sum_{k=1}^{\infty}\frac{(-1)^k}{2k+1}Q(1,2k;2)\frac{(2x)^{2k}}{(2k)!}\right],
\end{aligned}
\tag{17}
$$

$$
\begin{aligned}
\frac{(2\arcsin x)^2}{2!}&=2x^2-\sum_{k=3}^{\infty}(-2\,\mathrm{i})^k\left[\sum_{\ell=1}^{k-1}\ell s(k-1,\ell)\left(\frac{k-2}{2}\right)^{\ell-1}\right]\frac{x^k}{k!}\\
&=2x^2\left[1+\sum_{k=1}^{\infty}\frac{2(-1)^k}{(2k+2)(2k+1)}Q(2,2k;2)\frac{(2x)^{2k}}{(2k)!}\right],
\end{aligned}
\tag{18}
$$

and

$$
\begin{aligned}
\frac{(2\arcsin x)^m}{m!}&=\mathrm{i}^m\sum_{k=m}^{\infty}\left[\sum_{\ell=m}^{k}\binom{\ell-1}{m-1}s(k-1,\ell-1)\left(\frac{k-2}{2}\right)^{\ell-m}\right]\frac{(-2\,\mathrm{i}\,x)^k}{k!}\\
&=\frac{(2x)^m}{m!}\sum_{k=0}^{\infty}\frac{(-1)^k}{\binom{m+2k}{m}}Q(m,2k;2)\frac{(2x)^{2k}}{(2k)!}
\end{aligned}
\tag{19}
$$

for $m\geq3$. Maclaurin's series expansion (6) in Theorem 1 is proved. The first proof of Theorem 1 is complete. □

Second proof. By virtue of (16) again, we obtain

$$
\binom{\mathrm{i}\,a+\frac{k-1}{2}}{k}=\frac{1}{k!}\sum_{\ell=0}^{k}s(k,\ell)\left(\mathrm{i}\,a+\frac{k-1}{2}\right)^{\ell}
$$

$$
\begin{aligned}
&= \frac{1}{k!} \sum_{\ell=0}^{k} s(k,\ell) \sum_{m=0}^{\ell} \binom{\ell}{m} (\mathrm{i}\, a)^m \left(\frac{k-1}{2}\right)^{\ell-m} \\
&= \frac{1}{k!} \sum_{m=0}^{k} \left[\sum_{\ell=m}^{k} \binom{\ell}{m} s(k,\ell) \left(\frac{k-1}{2}\right)^{\ell-m}\right] (\mathrm{i}\, a)^m
\end{aligned}
$$

for $k \ge 2$. Substituting this into the right hand side of (15) results in

$$
\begin{aligned}
& t + at^2 + \sum_{k=2}^{\infty} (-2\,\mathrm{i})^k \binom{\mathrm{i}\, a + \frac{k-1}{2}}{k} \frac{t^{k+1}}{k+1} \\
&= t + at^2 + \sum_{k=2}^{\infty} (-2)^k \left(\sum_{m=0}^{k} \mathrm{i}^{k+m} \left[\sum_{\ell=m}^{k} \binom{\ell}{m} s(k,\ell) \left(\frac{k-1}{2}\right)^{\ell-m}\right] a^m\right) \frac{t^{k+1}}{(k+1)!} \\
&= t + at^2 + \sum_{k=2}^{\infty} (-2)^k\, \mathrm{i}^k\, Q(1,k;2) \frac{t^{k+1}}{(k+1)!} \\
&\quad + \sum_{k=2}^{\infty} (-2)^k \left(\left[\mathrm{i}^{k+1} \sum_{\ell=1}^{k} \ell s(k,\ell) \left(\frac{k-1}{2}\right)^{\ell-1}\right] a\right) \frac{t^{k+1}}{(k+1)!} \\
&\quad + \sum_{k=2}^{\infty} (-2)^k \left(\sum_{m=2}^{k} \left[\mathrm{i}^{k+m} \sum_{\ell=m}^{k} \binom{\ell}{m} s(k,\ell) \left(\frac{k-1}{2}\right)^{\ell-m}\right] a^m\right) \frac{t^{k+1}}{(k+1)!} \\
&= t + \sum_{k=2}^{\infty} (-2)^k\, \mathrm{i}^k\, Q(1,k;2) \frac{t^{k+1}}{(k+1)!} \\
&\quad + \left(t^2 + \sum_{k=2}^{\infty} \mathrm{i}^{k+1} (-2)^k \left[\sum_{\ell=1}^{k} \ell s(k,\ell) \left(\frac{k-1}{2}\right)^{\ell-1}\right] \frac{t^{k+1}}{(k+1)!}\right) a \\
&\quad + \sum_{m=2}^{\infty} \left(\sum_{k=m}^{\infty} (-2)^k \left[\mathrm{i}^{k+m} \sum_{\ell=m}^{k} \binom{\ell}{m} s(k,\ell) \left(\frac{k-1}{2}\right)^{\ell-m}\right] \frac{t^{k+1}}{(k+1)!}\right) a^m.
\end{aligned}
$$

The series expansion of the left hand side in (15) is

$$
\frac{\mathrm{e}^{2a \arcsin t} - 1}{2a} = \sum_{m=1}^{\infty} \frac{(2a)^{m-1} (\arcsin t)^m}{m!} = \sum_{m=0}^{\infty} \frac{2^m (\arcsin t)^{m+1}}{(m+1)!} a^m. \tag{20}
$$

Equating coefficients of a^m for $m \in \mathbb{N}_0$ in the series (20) and its previous one produces

$$
\begin{aligned}
\arcsin t &= t + \sum_{k=2}^{\infty} (-2)^k\, \mathrm{i}^k\, Q(1,k;2) \frac{t^{k+1}}{(k+1)!} \\
&= t\left[1 + \sum_{k=1}^{\infty} (-2)^{2k}\, \mathrm{i}^{2k}\, Q(1,2k;2) \frac{t^{2k}}{(2k+1)!}\right] \qquad (21)
\end{aligned}
$$

$$= t\left[1+\sum_{k=1}^{\infty}\frac{(-1)^k}{2k+1}Q(1,2k;2)\frac{(2t)^{2k}}{(2k)!}\right],$$

$$(\arcsin t)^2 = t^2+\sum_{k=2}^{\infty}\mathrm{i}^{k+1}(-2)^k\left[\sum_{\ell=1}^{k}\ell s(k,\ell)\left(\frac{k-1}{2}\right)^{\ell-1}\right]\frac{t^{k+1}}{(k+1)!}$$

$$= t^2\left(1+\sum_{k=1}^{\infty}\mathrm{i}^{2k+2}(-2)^{2k+1}\left[\sum_{\ell=1}^{2k+1}\ell s(2k+1,\ell)\left(\frac{2k}{2}\right)^{\ell-1}\right]\frac{t^{2k}}{(2k+2)!}\right) \tag{22}$$

$$= t^2\left[1+\sum_{k=1}^{\infty}\frac{(-1)^k}{(k+1)(2k+1)}Q(2,2k;2)\frac{(2t)^{2k}}{(2k)!}\right],$$

$$\frac{2^m(\arcsin t)^{m+1}}{(m+1)!} = \frac{2^m t^{m+1}}{(m+1)!}\left((-1)^m\,\mathrm{i}^m\sum_{k=m}^{\infty}(m+1)!2^{k-m}\right.$$

$$\left.\times\left[\mathrm{i}^k\sum_{\ell=m}^{k}\binom{\ell}{m}s(k,\ell)\left(\frac{k-1}{2}\right)^{\ell-m}\right]\frac{t^{k-m}}{(k+1)!}\right)$$

$$= \frac{2^m t^{m+1}}{(m+1)!}\left(1+(-1)^m\,\mathrm{i}^m\sum_{k=m+1}^{\infty}(m+1)!2^{k-m}\right.$$

$$\left.\times\left[\mathrm{i}^k\sum_{\ell=m}^{k}\binom{\ell}{m}s(k,\ell)\left(\frac{k-1}{2}\right)^{\ell-m}\right]\frac{t^{k-m}}{(k+1)!}\right) \tag{23}$$

$$= \frac{2^m t^{m+1}}{(m+1)!}\left[1+\sum_{k=1}^{\infty}(m+1)!\,\mathrm{i}^k\,Q(m+1,k;2)\frac{(2t)^k}{(k+m+1)!}\right]$$

$$= \frac{2^m t^{m+1}}{(m+1)!}\left[1+\sum_{k=1}^{\infty}(-1)^k\frac{(m+1)!(2k)!}{(2k+m+1)!}Q(m+1,2k;2)\frac{(2t)^{2k}}{(2k)!}\right]$$

for $m \ge 2$. Maclaurin's series expansion (6) in Theorem 1 is proved once again. The second proof of Theorem 1 is complete. □

Corollary 1. *For $m \in \mathbb{N}_0$ and $|t| < 1$, we have Maclaurin's series expansion*

$$\frac{(\arcsin t)^m}{\sqrt{1-t^2}} = t^m\left[1+\sum_{k=1}^{\infty}(-1)^k\frac{Q(m+1,2k;2)}{\binom{m+2k}{m}}\frac{(2t)^{2k}}{(2k)!}\right], \tag{24}$$

where $Q(m,k;2)$ is defined by (7).

Proof. Multiplying by t^m and differentiating with respect to t on both sides of (6) in sequence yield

$$\frac{m(\arcsin t)^{m-1}}{\sqrt{1-t^2}} = mt^{m-1}+\sum_{k=1}^{\infty}\frac{(-1)^k}{\binom{2k+m}{m}}Q(m,2k;2)\frac{2^{2k}(2k+m)t^{2k+m-1}}{(2k)!}.$$

Replacing $m-1$ by m and simplifying lead to (24). Corollary 1 is thus proved. □

Corollary 2. *For $m \in \mathbb{N}$ and $|t| < 1$, the function $\left(\frac{\operatorname{arcsinh} t}{t}\right)^m$, whose value at $t = 0$ is defined to be 1, has Maclaurin's series expansion*

$$\left(\frac{\operatorname{arcsinh} t}{t}\right)^m = 1 + \sum_{k=1}^{\infty} \frac{Q(m, 2k; 2)}{\binom{m+2k}{m}} \frac{(2t)^{2k}}{(2k)!}, \tag{25}$$

where $Q(m, 2k; 2)$ is defined by (7).

For $m \in \mathbb{N}_0$ and $|t| < 1$, we have Maclaurin's series expansion

$$\frac{(\operatorname{arcsinh} t)^m}{\sqrt{1+t^2}} = t^m \left[1 + \sum_{k=1}^{\infty} \frac{Q(m+1, 2k; 2)}{\binom{m+2k}{m}} \frac{(2t)^{2k}}{(2k)!}\right], \tag{26}$$

where $Q(m+1, 2k; 2)$ is defined by (7).

Proof. Since the relation

$$\operatorname{arcsinh} t = -\mathrm{i} \arcsin(\mathrm{i}\, t), \tag{27}$$

the series expansion (25) follows readily from (6) in Theorem 1.

The series expansion (26) follows from an application of the relation (27) to Maclaurin's series expansion (24) in Corollary 1 and simplifying. The proof of Corollary 2 is complete. □

Corollary 3. *For $k, m \in \mathbb{N}$, we have the combinatorial identities*

$$Q(1, 2k+1; 2) = 0 \quad \text{and} \quad Q(m+1, 2k-1; 2) = 0, \tag{28}$$

where $Q(m, k; 2)$ is defined by (7).

Proof. This follows from the disappearance of imaginary parts in the equalities (17), (18), (19), (21), (22), and (23) and from reformulation. Corollary 3 is proved. □

3. CLOSED-FORM FORMULA OF SPECIFIC PARTIAL BELL POLYNOMIALS

The values of specific partial Bell polynomials $\mathrm{B}_{2n,k}$ in (2) were represented in [**20**, Sections 1 and 3] by two explicit formulas for two cases $\mathrm{B}_{2n,2k-1}$ and $\mathrm{B}_{2n,2k}$. In this section, applying Maclaurin's series expansion (6) in Theorem 1, we give a simpler, nicer, unified, and closed-form formula of $\mathrm{B}_{2n,k}$ in (2) in terms of the quantity $Q(m, k; 2)$ defined in (7).

Theorem 2. *For $k, n \in \mathbb{N}$ such that $2n \ge k \in \mathbb{N}$, we have*

$$\begin{aligned} \mathrm{B}_{2n,k}&\left(0, \frac{1}{3}, 0, \frac{9}{5}, 0, \frac{225}{7}, \dots, \frac{1+(-1)^{k+1}}{2} \frac{[(2n-k)!!]^2}{2n-k+2}\right) \\ &= (-1)^{n+k} \frac{(4n)!!}{(2n+k)!} \sum_{q=1}^{k} (-1)^q \binom{2n+k}{k-q} Q(q, 2n; 2), \end{aligned} \tag{29}$$

where $Q(q, 2n; 2)$ is given by (7).

Proof. It is well known that the power series expansion

$$\arcsin t=\sum_{\ell=0}^{\infty}[(2\ell-1)!!]^2\frac{t^{2\ell+1}}{(2\ell+1)!},\quad |t|<1$$

is valid, where $(-1)!!=1$. This implies that

$$\begin{aligned}
&\mathrm{B}_{2n,k}\left(0,\frac{1}{3},0,\frac{9}{5},0,\frac{225}{7},\dots,\frac{1+(-1)^{k+1}}{2}\frac{[(2n-k)!!]^2}{2n-k+2}\right)\\
&=\mathrm{B}_{2n,k}\left(\frac{(\arcsin t)''|_{t=0}}{2},\frac{(\arcsin t)'''|_{t=0}}{3},\dots,\frac{(\arcsin t)^{(2n-k+2)}|_{t=0}}{2n-k+2}\right).
\end{aligned}$$

Employing the formula

$$\mathrm{B}_{n,k}\left(\frac{x_2}{2},\frac{x_3}{3},\dots,\frac{x_{n-k+2}}{n-k+2}\right)=\frac{n!}{(n+k)!}\mathrm{B}_{n+k,k}(0,x_2,x_3,\dots,x_{n+1})$$

in [**14**, p. 136], we acquire

$$\begin{aligned}
&\mathrm{B}_{2n,k}\left(0,\frac{1}{3},0,\frac{9}{5},0,\frac{225}{7},\dots,\frac{1+(-1)^{k+1}}{2}\frac{[(2n-k)!!]^2}{2n-k+2}\right)\\
&=\frac{(2n)!}{(2n+k)!}\mathrm{B}_{2n+k,k}\big(0,(\arcsin t)''|_{t=0},(\arcsin t)'''|_{t=0},\dots,(\arcsin t)^{(2n+1)}|_{t=0}\big).
\end{aligned}$$

Making use of the formula

$$\frac{1}{k!}\left(\sum_{m=1}^{\infty}x_m\frac{t^m}{m!}\right)^k=\sum_{n=k}^{\infty}\mathrm{B}_{n,k}(x_1,x_2,\dots,x_{n-k+1})\frac{t^n}{n!}$$

for $k\in\mathbb{N}_0$ in [**14**, p. 133] yields

$$\begin{aligned}
\sum_{n=0}^{\infty}\mathrm{B}_{n+k,k}(x_1,x_2,\dots,x_{n+1})\frac{k!n!}{(n+k)!}\frac{t^{n+k}}{n!}&=\left(\sum_{m=1}^{\infty}x_m\frac{t^m}{m!}\right)^k,\\
\sum_{n=0}^{\infty}\frac{\mathrm{B}_{n+k,k}(x_1,x_2,\dots,x_{n+1})}{\binom{n+k}{k}}\frac{t^{n+k}}{n!}&=\left(\sum_{m=1}^{\infty}x_m\frac{t^m}{m!}\right)^k,\\
\mathrm{B}_{n+k,k}(x_1,x_2,\dots,x_{n+1})&=\binom{n+k}{k}\lim_{t\to0}\frac{\mathrm{d}^n}{\mathrm{d}\,t^n}\left[\sum_{m=0}^{\infty}x_{m+1}\frac{t^m}{(m+1)!}\right]^k,\\
\mathrm{B}_{2n+k,k}(x_1,x_2,\dots,x_{2n+1})&=\binom{2n+k}{k}\lim_{t\to0}\frac{\mathrm{d}^{2n}}{\mathrm{d}\,t^{2n}}\left[\sum_{m=0}^{\infty}x_{m+1}\frac{t^m}{(m+1)!}\right]^k.
\end{aligned}$$

Setting $x_1=0$ and $x_m=(\arcsin t)^{(m)}|_{t=0}$ for $m\ge2$ gives

$$\mathrm{B}_{2n+k,k}\big(0,(\arcsin t)''|_{t=0},(\arcsin t)'''|_{t=0},\dots,(\arcsin t)^{(2n+1)}|_{t=0}\big)$$

$$
\begin{aligned}
&= \binom{2n+k}{k}\frac{\mathrm{d}^{2n}}{\mathrm{d}\,t^{2n}}\left[\frac{1}{t}\sum_{m=2}^{\infty}(\arcsin t)^{(m)}|_{t=0}\frac{t^m}{m!}\right]^k\\
&= \binom{2n+k}{k}\frac{\mathrm{d}^{2n}}{\mathrm{d}\,t^{2n}}\left(\frac{\arcsin t - t}{t}\right)^k\\
&= \binom{2n+k}{k}\frac{\mathrm{d}^{2n}}{\mathrm{d}\,t^{2n}}\sum_{q=0}^{k}(-1)^{k-q}\binom{k}{q}\left(\frac{\arcsin t}{t}\right)^q\\
&= \binom{2n+k}{k}\sum_{q=1}^{k}(-1)^{k-q}\binom{k}{q}\frac{\mathrm{d}^{2n}}{\mathrm{d}\,t^{2n}}\left(\frac{\arcsin t}{t}\right)^q.
\end{aligned}
$$

By virtue of the series expansion (6) in Theorem 1, we obtain

$$
\lim_{t\to0}\frac{\mathrm{d}^{2n}}{\mathrm{d}\,t^{2n}}\left(\frac{\arcsin t}{t}\right)^q = (-1)^n\frac{2^{2n}}{\binom{2n+q}{q}}Q(q,2n;2)
$$

for $n \ge q \in \mathbb{N}$. In conclusion, we arrive at

$$
\begin{aligned}
&\mathrm{B}_{2n,k}\left(0,\frac{1}{3},0,\frac{9}{5},0,\frac{225}{7},\dots,\frac{1+(-1)^{k+1}}{2}\frac{[(2n-k)!!]^2}{2n-k+2}\right)\\
&= \frac{(-1)^n(2n)!}{(2n+k)!}\binom{2n+k}{k}\sum_{q=1}^{k}\binom{k}{q}\frac{(-1)^{k-q}2^{2n}}{\binom{2n+q}{q}}Q(q,2n;2)\\
&= (-1)^{n+k}\frac{(4n)!!}{(2n+k)!}\sum_{q=1}^{k}(-1)^q\binom{2n+k}{k-q}Q(q,2n;2).
\end{aligned}
$$

The proof of Theorem 2 is complete. □

4. SERIES REPRESENTATION OF GENERALIZED LOGSINE FUNCTION

In [20, Section 4 and Remark 5.7], two series representations for generalized logsine function $\mathrm{Ls}_j^{(k)}(\theta)$ defined by (4) were established by two cases $\mathrm{Ls}_j^{(2\ell-1)}(\theta)$ and $\mathrm{Ls}_j^{(2\ell)}(\theta)$ for $\ell \in \mathbb{N}$ respectively. In this section, applying Maclaurin's series expansion (6) in Theorem 1, we derive a simpler and unified series representation of $\mathrm{Ls}_j^{(k)}(\theta)$ for $k \in \mathbb{N}$ in terms of the quantity $Q(m,k;2)$ defined in (7).

Theorem 3. *In the region $0 < \theta \le \pi$ and for $j,k \in \mathbb{N}$, generalized logsine function $\mathrm{Ls}_j^{(k)}(\theta)$ has the series representation*

$$
\mathrm{Ls}_j^{(k)}(\theta) = (\ln 2)^j\left(\frac{2\sin\frac{\theta}{2}}{\ln 2}\right)^{k+1}\left[k!\sum_{q=1}^{\infty}\frac{(-1)^{q+1}}{(k+2q)!}\left(2\sin\frac{\theta}{2}\right)^{2q}Q(k+1,2q;2)\right. \tag{30}
$$

$$\times \sum_{\ell=0}^{j-k-1} \binom{j-k-1}{\ell} \left(\frac{\ln \sin \frac{\theta}{2}}{\ln 2}\right)^{\ell} \sum_{p=0}^{\ell} \frac{(-1)^p \langle \ell \rangle_p}{(k+2q+1)^{p+1} \left(\ln \sin \frac{\theta}{2}\right)^p}$$
$$- \sum_{\ell=0}^{j-k-1} \binom{j-k-1}{\ell} \left(\frac{\ln \sin \frac{\theta}{2}}{\ln 2}\right)^{\ell} \sum_{p=0}^{\ell} \frac{(-1)^p \langle \ell \rangle_p}{(k+1)^{p+1} \left(\ln \sin \frac{\theta}{2}\right)^p} \Bigg],$$

where the falling factorial $\langle z \rangle_k$ *is defined by* (14) *and* $Q(k+1, 2q; 2)$ *is defined by* (7).

Proof. In [**15**, p. 308], it was derived that

$$\operatorname{Ls}_j^{(k)}(\theta) = -2^{k+1} \int_0^{\sin(\theta/2)} \frac{(\arcsin x)^k}{\sqrt{1-x^2}} \ln^{j-k-1}(2x) \,\mathrm{d}\, x \tag{31}$$

for $0 < \theta \le \pi$ and $j \ge k+1 \in \mathbb{N}$. Applying the series expansion (24) in Corollary 1 to (31) gives

$$\operatorname{Ls}_j^{(k)}(\theta) = \sum_{q=1}^{\infty} \frac{(-1)^{q+1}}{(2q)!} \frac{2^{2q+k+1}}{\binom{2q+k}{k}} Q(k+1, 2q; 2) \int_0^{\sin(\theta/2)} x^{2q+k} \ln^{j-k-1}(2x) \,\mathrm{d}\, x$$
$$- 2^{k+1} \int_0^{\sin(\theta/2)} x^k \ln^{j-k-1}(2x) \,\mathrm{d}\, x.$$

Making use of the formula

$$\int x^n \ln^m x \,\mathrm{d}\, x = x^{n+1} \sum_{p=0}^{m} (-1)^p \langle m \rangle_p \frac{\ln^{m-p} x}{(n+1)^{p+1}}, \quad m, n \in \mathbb{N}_0$$

in [**18**, p. 238, 2.722] results in

$$\int_0^{\sin(\theta/2)} x^{2q+k} \ln^{j-k-1}(2x) \,\mathrm{d}\, x = \int_0^{\sin(\theta/2)} x^{2q+k} (\ln 2 + \ln x)^{j-k-1} \,\mathrm{d}\, x$$
$$= \int_0^{\sin(\theta/2)} x^{2q+k} \sum_{\ell=0}^{j-k-1} \binom{j-k-1}{\ell} (\ln 2)^{j-k-\ell-1} (\ln x)^{\ell} \,\mathrm{d}\, x$$
$$= \sum_{\ell=0}^{j-k-1} \binom{j-k-1}{\ell} (\ln 2)^{j-k-\ell-1} \int_0^{\sin(\theta/2)} x^{2q+k} (\ln x)^{\ell} \,\mathrm{d}\, x$$
$$= \sum_{\ell=0}^{j-k-1} \binom{j-k-1}{\ell} (\ln 2)^{j-k-\ell-1} \left(\sin \frac{\theta}{2}\right)^{2q+k+1} \sum_{p=0}^{\ell} (-1)^p \langle \ell \rangle_p \frac{\left(\ln \sin \frac{\theta}{2}\right)^{\ell-p}}{(2q+k+1)^{p+1}}$$

and

$$\int_0^{\sin(\theta/2)} x^k \ln^{j-k-1}(2x) \,\mathrm{d}\, x$$
$$= \sum_{\ell=0}^{j-k-1} \binom{j-k-1}{\ell} (\ln 2)^{j-\ell} \left(\frac{\sin \frac{\theta}{2}}{\ln 2}\right)^{k+1} \sum_{p=0}^{\ell} (-1)^p \langle \ell \rangle_p \frac{\left(\ln \sin \frac{\theta}{2}\right)^{\ell-p}}{(k+1)^{p+1}}.$$

Consequently, it follows that

$$
\begin{aligned}
\operatorname{Ls}_j^{(k)}(\theta) &= \sum_{q=1}^{\infty} \frac{(-1)^{q+1}}{(2q)!} \frac{2^{2q+k+1}}{\binom{2q+k}{k}} Q(k+1,2q;2) \\
&\times \sum_{\ell=0}^{j-k-1} \binom{j-k-1}{\ell} (\ln 2)^{j-\ell} \left(\frac{\sin\frac{\theta}{2}}{\ln 2}\right)^{2q+k+1} \sum_{p=0}^{\ell} (-1)^p \langle \ell\rangle_p \frac{\left(\ln\sin\frac{\theta}{2}\right)^{\ell-p}}{(2q+k+1)^{p+1}} \\
&-2^{k+1} \sum_{\ell=0}^{j-k-1} \binom{j-k-1}{\ell} (\ln 2)^{j-\ell} \left(\frac{\sin\frac{\theta}{2}}{\ln 2}\right)^{k+1} \sum_{p=0}^{\ell} (-1)^p \langle \ell\rangle_p \frac{\left(\ln\sin\frac{\theta}{2}\right)^{\ell-p}}{(k+1)^{p+1}} \\
&= (\ln 2)^j k! \left(\frac{2\sin\frac{\theta}{2}}{\ln 2}\right)^{k+1} \sum_{q=1}^{\infty} \frac{(-1)^{q+1}}{(2q+k)!} \left(2\sin\frac{\theta}{2}\right)^{2q} Q(k+1,2q;2) \\
&\times \sum_{\ell=0}^{j-k-1} \binom{j-k-1}{\ell} \left(\frac{\ln\sin\frac{\theta}{2}}{\ln 2}\right)^{\ell} \sum_{p=0}^{\ell} \frac{(-1)^p \langle \ell\rangle_p}{(2q+k+1)^{p+1}\left(\ln\sin\frac{\theta}{2}\right)^p} \\
&-(\ln 2)^j \left(\frac{2\sin\frac{\theta}{2}}{\ln 2}\right)^{k+1} \sum_{\ell=0}^{j-k-1} \binom{j-k-1}{\ell} \left(\frac{\ln\sin\frac{\theta}{2}}{\ln 2}\right)^{\ell} \sum_{p=0}^{\ell} \frac{(-1)^p \langle \ell\rangle_p}{(k+1)^{p+1}\left(\ln\sin\frac{\theta}{2}\right)^p}.
\end{aligned}
$$

The proof of Theorem 3 is complete. □

5. SERIES IDENTITIES INVOLVING INVERSE HYPERBOLIC SINE FUNCTION AND MACLAURIN'S EXPANSIONS FOR INCOMPLETE GAMMA FUNCTION

In this section, by similar methods and arguments used in Section 2, we present Maclaurin's series expansion of the exponential function $e^{\operatorname{arcsinh} t}$ and three series identities involving $(\operatorname{arcsinh} t)^\ell$ for $\ell \ge 2$ and the quantity $Q(m,k;\alpha)$ for $\alpha = 2,3$. In this section, we also discover Maclaurin's series expansions of the function $\Gamma(m, \operatorname{arcsinh} t)$ for $m \ge 2$.

Theorem 4. *The inverse hyperbolic sine function* $\operatorname{arcsinh} t$ *satisfies the series identities*

$$
\sum_{\ell=0}^{\infty} (-1)^\ell (\ell+1) \frac{(\operatorname{arcsinh} t)^{\ell+2}}{(\ell+2)!} = \frac{1}{2}t^2 - \frac{1}{3}t^3 + \frac{1}{4}\sum_{k=3}^{\infty} Q(2,k-1;3)\frac{(2t)^{k+1}}{(k+1)!}, \tag{32}
$$

$$
\sum_{\ell=0}^{\infty} (-1)^\ell (\ell+1)(\ell+2) \frac{(\operatorname{arcsinh} t)^{\ell+3}}{(\ell+3)!} = \frac{1}{3}t^3 + \frac{1}{8}\sum_{k=3}^{\infty} Q(3,k-2;3)\frac{(2t)^{k+1}}{(k+1)!}, \tag{33}
$$

and, for $m \ge 3$,

$$
\sum_{\ell=0}^{\infty} (-1)^\ell \binom{\ell+m}{m} \frac{(\operatorname{arcsinh} t)^{\ell+m+1}}{(\ell+m+1)!} = \frac{1}{2^{m+1}} \sum_{k=m}^{\infty} Q(m+1,k-m;3)\frac{(2t)^{k+1}}{(k+1)!}, \tag{34}
$$

where $Q(m,k;3)$ is defined by (7).

The exponential function $\mathrm{e}^{\operatorname{arcsinh} t}$ has Maclaurin's series expansion

$$\mathrm{e}^{\operatorname{arcsinh} t}=1+t-\frac{1}{4}\sum_{k=0}^{\infty}\binom{\frac{2k-1}{2}}{2k+1}\frac{(2t)^{2(k+1)}}{k+1}, \tag{35}$$

where extended binomial coefficient $\binom{z}{w}$ is defined by (13).

Proof. In [**21**, pp. 210–211, (10.49.32) and (10.49.35)], the formulas

$$\sum_{k=0}^{\infty}\frac{(a)_{k/2}}{(a+1)_{-k/2}}\frac{x^k}{k!}=\exp\biggl(2a\operatorname{arcsinh}\frac{x}{2}\biggr) \tag{36}$$

and

$$\sum_{k=0}^{\infty}\frac{(a)_{k/2}}{(a)_{-k/2}}\frac{x^k}{k!}=\frac{2}{\sqrt{4+x^2}}\exp\biggl[(2a-1)\operatorname{arcsinh}\frac{x}{2}\biggr] \tag{37}$$

were listed. In (36) and (37), replacing x by $2x$, making use of the extended Pochhammer symbol in (10), and employing extended binomial coefficient in (13) result in

$$\mathrm{e}^{2a\operatorname{arcsinh} x}=a\sum_{k=0}^{\infty}\frac{\Gamma\bigl(a+\frac{k}{2}\bigr)}{\Gamma\bigl(a+1-\frac{k}{2}\bigr)}\frac{(2x)^k}{k!}=1+a\sum_{k=1}^{\infty}\binom{a-1+\frac{k}{2}}{k-1}\frac{(2x)^k}{k} \tag{38}$$

and

$$\frac{\exp[(2a-1)\operatorname{arcsinh} x]}{\sqrt{1+x^2}}=\sum_{k=0}^{\infty}\frac{\Gamma\bigl(a+\frac{k}{2}\bigr)}{\Gamma\bigl(a-\frac{k}{2}\bigr)}\frac{(2x)^k}{k!}=\sum_{k=0}^{\infty}\binom{a-1+\frac{k}{2}}{k}(2x)^k. \tag{39}$$

Integrating on both sides of (39) with respect to $x\in(0,t)$ produces

$$\frac{\exp[(2a-1)\operatorname{arcsinh} t]-1}{2a-1}=\sum_{k=0}^{\infty}\binom{a-1+\frac{k}{2}}{k}\frac{2^k t^{k+1}}{k+1}. \tag{40}$$

By virtue of the formula (16), we obtain

$$\begin{aligned}
\binom{a-1+\frac{k}{2}}{k-1}&=\frac{1}{(k-1)!}\sum_{\ell=0}^{k-1}s(k-1,\ell)\biggl(a-1+\frac{k}{2}\biggr)^{\ell}\\
&=\frac{1}{(k-1)!}\sum_{\ell=0}^{k-1}s(k-1,\ell)\sum_{m=0}^{\ell}\binom{\ell}{m}\biggl(\frac{k-2}{2}\biggr)^{\ell-m}a^m\\
&=\frac{1}{(k-1)!}\sum_{m=0}^{k-1}\Biggl[\sum_{\ell=m}^{k-1}\binom{\ell}{m}s(k-1,\ell)\biggl(\frac{k-2}{2}\biggr)^{\ell-m}\Biggr]a^m
\end{aligned}$$

$$= \frac{1}{(k-1)!} \sum_{m=0}^{k-1} Q(m+1, k-m-1; 2) a^m$$

and

$$\begin{aligned}
\binom{a-1+\frac{k}{2}}{k} &= \frac{1}{k!} \sum_{\ell=0}^{k} s(k,\ell) \left(a-1+\frac{k}{2}\right)^{\ell} \\
&= \frac{1}{k!} \sum_{\ell=0}^{k} s(k,\ell) \sum_{m=0}^{\ell} \binom{\ell}{m} \left(\frac{k-2}{2}\right)^{\ell-m} a^m \\
&= \frac{1}{k!} \sum_{m=0}^{k} \left[\sum_{\ell=m}^{k} \binom{\ell}{m} s(k,\ell) \left(\frac{k-2}{2}\right)^{\ell-m} \right] a^m \\
&= \frac{1}{k!} \sum_{m=0}^{k} Q(m+1, k-m; 3) a^m
\end{aligned}$$

for $k \ge 3$. Substituting these two finite sums into the right hand sides of (38) and (40) gives

$$\begin{aligned}
1 + a \sum_{k=1}^{\infty} \binom{a-1+\frac{k}{2}}{k-1} \frac{(2x)^k}{k} &= 1 + 2xa + 2x^2a^2 + a \sum_{k=3}^{\infty} \binom{a-1+\frac{k}{2}}{k-1} \frac{(2x)^k}{k} \\
&= 1 + 2xa + 2x^2a^2 + \sum_{k=3}^{\infty} \left[\sum_{m=0}^{k-1} Q(m+1, k-m-1; 2) a^{m+1} \right] \frac{(2x)^k}{k!} \\
= 1 + \left[2x + \sum_{k=3}^{\infty} Q(1, k-1; 2) \frac{(2x)^k}{k!} \right] a &+ \left[2x^2 + \sum_{k=3}^{\infty} Q(2, k-2; 2) \frac{(2x)^k}{k!} \right] a^2 \\
&+ \sum_{m=3}^{\infty} \left[\sum_{k=m}^{\infty} Q(m, k-m; 2) \frac{(2x)^k}{k!} \right] a^m
\end{aligned}$$

and

$$\begin{aligned}
\sum_{k=0}^{\infty} \binom{a-1+\frac{k}{2}}{k} \frac{2^k t^{k+1}}{k+1} &= t - \frac{1}{2}t^2 + \sum_{k=3}^{\infty} Q(1, k; 3) \frac{2^k t^{k+1}}{(k+1)!} \\
&+ \left[t^2 - \frac{2}{3}t^3 + \sum_{k=3}^{\infty} Q(2, k-1; 3) \frac{2^k t^{k+1}}{(k+1)!} \right] a \\
&+ \left[\frac{2}{3}t^3 + \sum_{k=3}^{\infty} Q(3, k-2; 3) \frac{2^k t^{k+1}}{(k+1)!} \right] a^2 \\
&+ \sum_{m=3}^{\infty} \left[\sum_{k=3}^{\infty} Q(m+1, k-m; 3) \frac{2^k t^{k+1}}{(k+1)!} \right] a^m.
\end{aligned}$$

On the other hand, the left hand sides of (38) and (40) can be expanded into

$$\mathrm{e}^{2a\operatorname{arcsinh}x}=\sum_{m=0}^{\infty}\frac{(2\operatorname{arcsinh}x)^m}{m!}a^m$$

and

$$\begin{aligned}
\frac{\exp[(2a-1)\operatorname{arcsinh}t]-1}{2a-1}&=\sum_{m=1}^{\infty}\frac{(2a-1)^{m-1}(\operatorname{arcsinh}t)^m}{m!}\\
&=\sum_{m=0}^{\infty}\frac{(\operatorname{arcsinh}t)^{m+1}}{(m+1)!}\sum_{q=0}^{m}(-1)^{m-q}\binom{m}{q}(2a)^q\\
&=\sum_{q=0}^{\infty}\left[\sum_{m=q}^{\infty}(-1)^{m-q}\binom{m}{q}\frac{(\operatorname{arcsinh}t)^{m+1}}{(m+1)!}\right](2a)^q\\
&=\sum_{m=0}^{\infty}\left[\sum_{\ell=m}^{\infty}(-1)^{\ell-m}\binom{\ell}{m}\frac{(\operatorname{arcsinh}t)^{\ell+1}}{(\ell+1)!}\right](2a)^m.
\end{aligned}$$

Accordingly, equating coefficients of a^m for $m\in\mathbb{N}_0$, we obtain

$$\begin{aligned}
2\operatorname{arcsinh}x&=2x+\sum_{k=3}^{\infty}Q(1,k-1;2)\frac{(2x)^k}{k!},\\
\frac{(2\operatorname{arcsinh}x)^2}{2!}&=2x^2+\sum_{k=3}^{\infty}Q(2,k-2;2)\frac{(2x)^k}{k!},\\
\frac{(2\operatorname{arcsinh}x)^m}{m!}&=\sum_{k=m}^{\infty}Q(m,k-m;2)\frac{(2x)^k}{k!},\\
\sum_{\ell=0}^{\infty}(-1)^{\ell}\frac{(\operatorname{arcsinh}t)^{\ell+1}}{(\ell+1)!}&=t-\frac{t^2}{2}+\sum_{k=3}^{\infty}Q(1,k;3)\frac{2^kt^{k+1}}{(k+1)!}\\
\left[\sum_{\ell=1}^{\infty}(-1)^{\ell-1}\binom{\ell}{1}\frac{(\operatorname{arcsinh}t)^{\ell+1}}{(\ell+1)!}\right]2&=t^2-\frac{2}{3}t^3+\sum_{k=3}^{\infty}Q(2,k-1;3)\frac{2^kt^{k+1}}{(k+1)!},\\
\left[\sum_{\ell=2}^{\infty}(-1)^{\ell-2}\binom{\ell}{2}\frac{(\operatorname{arcsinh}t)^{\ell+1}}{(\ell+1)!}\right]2^2&=\frac{2}{3}t^3+\sum_{k=3}^{\infty}Q(3,k-2;3)\frac{2^kt^{k+1}}{(k+1)!},\\
\left[\sum_{\ell=m}^{\infty}(-1)^{\ell-m}\binom{\ell}{m}\frac{(\operatorname{arcsinh}t)^{\ell+1}}{(\ell+1)!}\right]2^m&=\sum_{k=3}^{\infty}Q(m+1,k-m;3)\frac{2^kt^{k+1}}{(k+1)!}
\end{aligned}$$

for $m \ge 3$. Reformulating these series expansions and series identities arrives at

$$\begin{aligned}
\frac{\operatorname{arcsinh} x}{x} &= 1 + \sum_{k=3}^{\infty} Q(1, k-1; 2) \frac{(2x)^{k-1}}{k!} \\
&= 1 + \sum_{k=1}^{\infty} \frac{Q(1, 2k; 2)}{\binom{2k+1}{1}} \frac{(2x)^{2k}}{(2k)!} + \sum_{k=1}^{\infty} \frac{Q(1, 2k+1; 2)}{\binom{2k+2}{1}} \frac{(2x)^{2k+1}}{(2k+1)!} \\
&= 1 + \sum_{k=1}^{\infty} \frac{Q(1, 2k; 2)}{\binom{2k+1}{1}} \frac{(2x)^{2k}}{(2k)!},
\end{aligned}$$

$$\begin{aligned}
\left(\frac{\operatorname{arcsinh} x}{x}\right)^2 &= 1 + \frac{1}{2x^2} \sum_{k=3}^{\infty} Q(2, k-2; 2) \frac{(2x)^k}{k!} \\
&= 1 + \sum_{k=1}^{\infty} \frac{2Q(2, k; 2)}{(k+2)(k+1)} \frac{(2x)^k}{k!} \\
&= 1 + \sum_{k=1}^{\infty} \frac{Q(2, 2k; 2)}{\binom{2k+2}{2}} \frac{(2x)^{2k}}{(2k)!} + \sum_{k=1}^{\infty} \frac{Q(2, 2k-1; 2)}{\binom{2k+1}{2}} \frac{(2x)^{2k-1}}{(2k-1)!} \\
&= 1 + \sum_{k=1}^{\infty} \frac{Q(2, 2k; 2)}{\binom{2k+2}{2}} \frac{(2x)^{2k}}{(2k)!},
\end{aligned}$$

$$\left(\frac{\operatorname{arcsinh} x}{x}\right)^m = \sum_{k=0}^{\infty} \frac{Q(m, k; 2)}{\binom{m+k}{m}} \frac{(2x)^k}{k!}$$

$$= 1 + \sum_{k=1}^{\infty} \frac{1}{\binom{m+2k}{m}} Q(m, 2k; 2) \frac{(2x)^{2k}}{(2k)!} + \sum_{k=1}^{\infty} \frac{Q(m, 2k-1; 2)}{\binom{m+2k-1}{m}} \frac{(2x)^{2k-1}}{(2k-1)!}$$

$$= 1 + \sum_{k=1}^{\infty} \frac{1}{\binom{m+2k}{m}} Q(m, 2k; 2) \frac{(2x)^{2k}}{(2k)!},$$

$$\sum_{\ell=0}^{\infty} (-1)^\ell \frac{(\operatorname{arcsinh} t)^{\ell+1}}{(\ell+1)!} = t - \frac{t^2}{2} + \sum_{k=3}^{\infty} Q(1, k; 3) \frac{2^k t^{k+1}}{(k+1)!}$$

$$= t - \frac{t^2}{2} + \sum_{k=1}^{\infty} Q(1, 2k+1; 3) \frac{2^{2k+1} t^{2k+2}}{(2k+2)!} + \sum_{k=1}^{\infty} Q(1, 2k+2; 3) \frac{2^{2k+2} t^{2k+3}}{(2k+3)!},$$

and

$$\begin{aligned}
\sum_{\ell=0}^{\infty} (-1)^\ell \binom{\ell+1}{1} \frac{(\operatorname{arcsinh} t)^{\ell+2}}{(\ell+2)!} &= \frac{1}{2} t^2 - \frac{1}{3} t^3 + \frac{1}{2^2} \sum_{k=3}^{\infty} Q(2, k-1; 3) \frac{(2t)^{k+1}}{(k+1)!}, \\
\sum_{\ell=0}^{\infty} (-1)^\ell \binom{\ell+2}{2} \frac{(\operatorname{arcsinh} t)^{\ell+3}}{(\ell+3)!} &= \frac{1}{6} t^3 + \frac{1}{2^3} \sum_{k=3}^{\infty} Q(3, k-2; 3) \frac{(2t)^{k+1}}{(k+1)!}, \\
\sum_{\ell=0}^{\infty} (-1)^\ell \binom{\ell+m}{m} \frac{(\operatorname{arcsinh} t)^{\ell+m+1}}{(\ell+m+1)!} &= \frac{1}{2^{m+1}} \sum_{k=3}^{\infty} Q(m+1, k-m; 3) \frac{(2t)^{k+1}}{(k+1)!}
\end{aligned}$$

for $m \ge 3$. Consequently, from the first three equations and the last three equations above, we conclude Maclaurin's series expansion (25) again and conclude the series identities (32), (33), and (34).

In the fourth formula above, by virtue of (16), we obtain

$$Q(1,2k+1;3)=(2k+1)!\binom{\frac{2k-1}{2}}{2k+1} \quad \text{and} \quad Q(1,2k+2;3)=0. \tag{41}$$

These two combinatorial identities imply

$$\sum_{\ell=0}^{\infty}(-1)^{\ell}\frac{(\operatorname{arcsinh} t)^{\ell+1}}{(\ell+1)!}=t-\frac{t^2}{2}+\sum_{k=1}^{\infty}2^{2k+1}\binom{\frac{2k-1}{2}}{2k+1}\frac{t^{2k+2}}{2k+2}.$$

Furthermore, since

$$\begin{aligned}\sum_{\ell=0}^{\infty}(-1)^{\ell}\frac{(\operatorname{arcsinh} t)^{\ell+1}}{(\ell+1)!}&=-\sum_{\ell=1}^{\infty}\frac{(-\operatorname{arcsinh} t)^{\ell}}{\ell!}\\&=1-\sum_{\ell=0}^{\infty}\frac{(-\operatorname{arcsinh} t)^{\ell}}{\ell!}=1-\mathrm{e}^{-\operatorname{arcsinh} t},\end{aligned}$$

we acquire

$$1-\mathrm{e}^{-\operatorname{arcsinh} t}=t-\frac{t^2}{2}+\sum_{k=1}^{\infty}2^{2k+1}\binom{\frac{2k-1}{2}}{2k+1}\frac{t^{2k+2}}{2k+2}.$$

Replacing t by $-t$ in the above equation leads to Maclaurin's series expansion (35). The proof of Theorem 4 is complete. □

Theorem 5. *For $m \ge 2$, the composite $\Gamma(m,\operatorname{arcsinh} t)$ has Maclaurin's series expansions*

$$\Gamma(2,\operatorname{arcsinh} t)=1-\frac{1}{2}t^2+\frac{1}{3}t^3-\frac{1}{4}\sum_{k=3}^{\infty}Q(2,k-1;3)\frac{(2t)^{k+1}}{(k+1)!}, \tag{42}$$

$$\Gamma(3,\operatorname{arcsinh} t)=2-\frac{1}{3}t^3-\frac{1}{8}\sum_{k=3}^{\infty}Q(3,k-2;3)\frac{(2t)^{k+1}}{(k+1)!}, \tag{43}$$

and, for $m \ge 3$,

$$\Gamma(1+m,\operatorname{arcsinh} t)=m!-\frac{m!}{2^{m+1}}\sum_{k=m}^{\infty}Q(m+1,k-m;3)\frac{(2t)^{k+1}}{(k+1)!}. \tag{44}$$

where $Q(m,k;3)$ is given by (7) and the incomplete gamma function $\Gamma(a,x)$ is defined by (5).

Proof. In [18, p. 908, 8.352.2] and [23, Theorem 3], the formula

$$\Gamma(1+m,x)=m!\,\mathrm{e}^{-x}\sum_{k=0}^{m}\frac{x^k}{k!}, \quad m=0,1,2,\dots$$

was given. Hence, it follows that

$$\begin{aligned}
(-1)^m\left[1-\frac{\Gamma(1+m,x)}{m!}\right]&=(-1)^m\left(1-\mathrm{e}^{-x}\sum_{k=0}^{m}\frac{x^k}{k!}\right)\\
&=(-1)^m\left[1-\mathrm{e}^{-x}\left(\mathrm{e}^{x}-\sum_{k=m+1}^{\infty}\frac{x^k}{k!}\right)\right]\\
&=(-1)^m\,\mathrm{e}^{-x}\sum_{k=m+1}^{\infty}\frac{x^k}{k!}\\
&=(-1)^m\left[\sum_{k=0}^{\infty}(-1)^k\frac{x^k}{k!}\right]\left[\sum_{k=0}^{\infty}\frac{x^{k+m+1}}{(k+m+1)!}\right]\\
&=(-1)^m x^{m+1}\sum_{k=0}^{\infty}\left[\sum_{\ell=0}^{k}\frac{(-1)^\ell}{\ell!}\frac{1}{(k-\ell+m+1)!}\right]x^k\\
&=(-1)^m\sum_{k=0}^{\infty}\left[\sum_{\ell=0}^{k}(-1)^\ell\binom{k+m+1}{\ell}\right]\frac{x^{k+m+1}}{(k+m+1)!}\\
&=(-1)^m\sum_{k=0}^{\infty}\frac{(-1)^k(k+1)}{k+m+1}\binom{k+m+1}{k+1}\frac{x^{k+m+1}}{(k+m+1)!}\\
&=\sum_{k=m}^{\infty}\frac{(-1)^k(k-m+1)}{k+1}\binom{k+1}{k-m+1}\frac{x^{k+1}}{(k+1)!}\\
&=\sum_{k=m}^{\infty}(-1)^k\binom{k}{m}\frac{x^{k+1}}{(k+1)!}\\
&=\sum_{k=0}^{\infty}(-1)^{k+m}\binom{k+m}{m}\frac{x^{k+m+1}}{(k+m+1)!}
\end{aligned}$$

for $m\in\mathbb{N}_0$, where we used the combinatorial identity

$$\sum_{\ell=0}^{k}(-1)^\ell\binom{k+m+1}{\ell}=\frac{(-1)^k(k+1)}{k+m+1}\binom{k+m+1}{k+1}=(-1)^k\binom{k+m}{k}$$

which can be derived from the identity

$$\sum_{k=0}^{n}(-1)^k\binom{x}{k}=(-1)^n\binom{x-1}{n}=\prod_{k=1}^{n}\left(1-\frac{x}{k}\right) \tag{45}$$

in [56, p. 18, (1.5)]. Substituting this result into the series identities (32), (33), and (34) in Theorem 4 and rearranging yield series expansions (42), (43), and (44). The proof of Theorem 5 is complete. □

6. MACLAURIN'S SERIES EXPANSIONS FOR POSITIVE INTEGER POWERS OF INVERSE (HYPERBOLIC) TANGENT

In this section, we discuss Maclaurin's series expansions of the inverse tangent function $\arctan t$ and the inverse hyperbolic tangent function $\operatorname{arctanh} t$.

6.1 Maclaurin's series expansion for positive integer powers of inverse tangent function

It is well known that

$$\arctan t=\sum_{k=0}^{\infty}(-1)^k\frac{t^{2k+1}}{2k+1},\quad |t|<1.$$

In [25, pp. 152–153, (820) and (821)], Maclaurin's series expansions

$$\begin{aligned}\frac{(\arctan t)^2}{2!}&=\sum_{k=0}^{\infty}(-1)^k\left(\sum_{\ell=0}^{k}\frac{1}{2\ell+1}\right)\frac{t^{2k+2}}{2k+2}\\&=\frac{t^2}{2}-\left(1+\frac{1}{3}\right)\frac{t^4}{4}+\left(1+\frac{1}{3}+\frac{1}{5}\right)\frac{t^6}{6}-\cdots\end{aligned}$$

and

$$\begin{aligned}\frac{(\arctan t)^3}{3!}&=\sum_{k=0}^{\infty}(-1)^k\left(\sum_{\ell_2=0}^{k}\frac{1}{2\ell_2+2}\sum_{\ell_1=0}^{\ell_2}\frac{1}{2\ell_1+1}\right)\frac{t^{2k+3}}{2k+3}\\&=\frac{1}{2}\frac{t^3}{3}-\left[\frac{1}{2}+\frac{1}{4}\left(1+\frac{1}{3}\right)\right]\frac{t^5}{5}+\left[\frac{1}{2}+\frac{1}{4}\left(1+\frac{1}{3}\right)+\frac{1}{6}\left(1+\frac{1}{3}+\frac{1}{5}\right)\right]\frac{t^7}{7}-\cdots\end{aligned}$$

for $|t|<1$ were collected. What is the general expression of Maclaurin's series expansion of $(\arctan t)^n$ for $n>3$ and $|t|<1$? We guess that it should be

$$\begin{aligned}\frac{(\arctan t)^n}{n!}&=\sum_{k=0}^{\infty}(-1)^k\left(\sum_{\ell_{n-1}=0}^{k}\frac{1}{2\ell_{n-1}+n-1}\sum_{\ell_{n-2}=0}^{\ell_{n-1}}\frac{1}{2\ell_{n-2}+n-2}\right.\\&\qquad\left.\cdots\sum_{\ell_2=0}^{\ell_3}\frac{1}{2\ell_2+2}\sum_{\ell_1=0}^{\ell_2}\frac{1}{2\ell_1+1}\right)\frac{t^{2k+n}}{2k+n}\\&=\sum_{k=0}^{\infty}(-1)^k\left(\prod_{m=1}^{n-1}\sum_{\ell_m=0}^{\ell_{m+1}}\frac{1}{2\ell_m+m}\right)\frac{t^{2k+n}}{2k+n}\end{aligned}\tag{46}$$

for $|t| < 1$ and all $n \in \mathbb{N}$ with $\ell_n = k$, where the product is understood to be 1 if the starting index exceeds the finishing index. For example, when $n = 4$, we have

$$\frac{(\arctan t)^4}{4!} = \sum_{k=0}^{\infty}(-1)^k \left(\sum_{\ell_3=0}^{k} \frac{1}{2\ell_3+3} \sum_{\ell_2=0}^{\ell_3} \frac{1}{2\ell_2+2} \sum_{\ell_1=0}^{\ell_2} \frac{1}{2\ell_1+1}\right) \frac{t^{2k+4}}{2k+4}$$

for $|t| < 1$.

In [2, p. 122, 6.42.3], Maclaurin's series expansion

$$(\arctan x)^p = p! \sum_{k_0=1}^{\infty} (-1)^{k_0-1} \frac{x^{2k_0+p-2}}{2k_0+p-2} \prod_{\alpha=1}^{p-1} \left(\sum_{k_\alpha=1}^{k_{\alpha-1}} \frac{1}{2k_\alpha+p-\alpha-2}\right) \tag{47}$$

for $p \in \mathbb{N}$ can be found. The Maclaurin's series expansion (47) was proved in [53] and is obviously equivalent to (46). Hence, Maclaurin's series expansion (46) is true.

Maclaurin's series expansion (46) was cited in [11, Proposition 4.2].

6.2 Maclaurin's series expansion for positive integer powers of inverse hyperbolic tangent function

It is also well known that

$$\operatorname{arctanh} t = \sum_{k=0}^{\infty} \frac{t^{2k+1}}{2k+1}, \quad |t| < 1.$$

Motivated by the difference between (6) and (25), basing on (46), and utilizing the relation

$$\arctan z = -\mathrm{i} \operatorname{arctanh}(\mathrm{i}\, z), \quad z^2 \neq -1$$

from [1, p. 80, 4.4.22], we derive that

$$\begin{aligned}\frac{(\operatorname{arctanh} t)^n}{n!} &= \sum_{k=0}^{\infty} \left(\sum_{\ell_{n-1}=0}^{k} \frac{1}{2\ell_{n-1}+n-1} \sum_{\ell_{n-2}=0}^{\ell_{n-1}} \frac{1}{2\ell_{n-2}+n-2}\right. \\ &\qquad \left. \cdots \sum_{\ell_2=0}^{\ell_3} \frac{1}{2\ell_2+2} \sum_{\ell_1=0}^{\ell_2} \frac{1}{2\ell_1+1}\right) \frac{t^{2k+n}}{2k+n} \\ &= \sum_{k=0}^{\infty} \left(\prod_{m=1}^{n-1} \sum_{\ell_m=0}^{\ell_{m+1}} \frac{1}{2\ell_m+m}\right) \frac{t^{2k+n}}{2k+n}\end{aligned} \tag{48}$$

for $|t| < 1$ and all $n \in \mathbb{N}$ with $\ell_n = k$, where the product is understood to be 1 if the starting index exceeds the finishing index. For example, when $n = 2, 3, 4$, we have

$$\frac{(\operatorname{arctanh} t)^2}{2!} = \sum_{k=0}^{\infty} \left(\sum_{\ell=0}^{k} \frac{1}{2\ell+1}\right) \frac{t^{2k+2}}{2k+2},$$

$$\frac{(\operatorname{arctanh} t)^3}{3!} = \sum_{k=0}^{\infty}\left(\sum_{\ell_2=0}^{k}\frac{1}{2\ell_2+2}\sum_{\ell_1=0}^{\ell_2}\frac{1}{2\ell_1+1}\right)\frac{t^{2k+3}}{2k+3},$$

and

$$\frac{(\operatorname{arctanh} t)^4}{4!} = \sum_{k=0}^{\infty}\left(\sum_{\ell_3=0}^{k}\frac{1}{2\ell_3+3}\sum_{\ell_2=0}^{\ell_3}\frac{1}{2\ell_2+2}\sum_{\ell_1=0}^{\ell_2}\frac{1}{2\ell_1+1}\right)\frac{t^{2k+4}}{2k+4}$$

for $|t| < 1$.

Alternative proof of the series expansion (48). By simulating the proof of (47) in the paper [**53**], we give an alternative proof of Maclaurin's series expansion (48).

It is clear that

$$\operatorname{arctanh} t = \int_0^t \frac{\mathrm{d}\,x}{1-x^2} = \sum_{k=0}^{\infty}\int_0^t x^{2k}\,\mathrm{d}\,x = \sum_{k=0}^{\infty}\frac{t^{2k+1}}{2k+1}$$

and

$$\begin{aligned}(\operatorname{arctanh} t)^2 &= 2\int_0^t \frac{\operatorname{arctanh} x}{1-x^2}\,\mathrm{d}\,x = 2\int_0^t\left(\sum_{k=0}^{\infty}\frac{x^{2k+1}}{2k+1}\right)\left(\sum_{k=0}^{\infty}x^{2k}\right)\mathrm{d}\,x\\
&= 2\int_0^t\sum_{k=0}^{\infty}\frac{1}{2k+1}\left(\sum_{\ell=0}^{\infty}x^{2\ell+2k+1}\right)\mathrm{d}\,x = 2\int_0^t\sum_{k=0}^{\infty}\frac{1}{2k+1}\left(\sum_{\ell=k}^{\infty}x^{2\ell+1}\right)\mathrm{d}\,x\\
&= 2!\sum_{k=0}^{\infty}\frac{1}{2k+1}\left(\sum_{\ell=k}^{\infty}\frac{t^{2\ell+2}}{2\ell+2}\right) = 2!\sum_{\ell=0}^{\infty}\left(\sum_{k=0}^{\ell}\frac{1}{2k+1}\right)\frac{t^{2\ell+2}}{2\ell+2}.\end{aligned}$$

If Maclaurin's series expansion (48) is true, then

$$\begin{aligned}(\operatorname{arctanh} t)^{n+1} &= (n+1)\int_0^t \frac{(\operatorname{arctanh} x)^n}{1-x^2}\,\mathrm{d}\,x\\
&= (n+1)!\int_0^t\sum_{k=0}^{\infty}\left(\prod_{m=1}^{n-1}\sum_{\ell_m=0}^{\ell_{m+1}}\frac{1}{2\ell_m+m}\right)\frac{x^{2k+n}}{2k+n}\sum_{\ell=0}^{\infty}x^{2\ell}\,\mathrm{d}\,x\\
&= (n+1)!\sum_{k=0}^{\infty}\left(\prod_{m=1}^{n-1}\sum_{\ell_m=0}^{\ell_{m+1}}\frac{1}{2\ell_m+m}\right)\frac{1}{2k+n}\sum_{\ell=0}^{\infty}\int_0^t x^{2\ell+2k+n}\,\mathrm{d}\,x\\
&= (n+1)!\sum_{k=0}^{\infty}\left(\prod_{m=1}^{n-1}\sum_{\ell_m=0}^{\ell_{m+1}}\frac{1}{2\ell_m+m}\right)\frac{1}{2k+n}\sum_{\ell=0}^{\infty}\frac{x^{2\ell+2k+n+1}}{2\ell+2k+n+1}\\
&= (n+1)!\sum_{k=0}^{\infty}\frac{1}{2k+n}\left(\sum_{\ell_{n-1}=0}^{k}\frac{1}{2\ell_{n-1}+n-1}\prod_{m=1}^{n-2}\sum_{\ell_m=0}^{\ell_{m+1}}\frac{1}{2\ell_m+m}\right)\sum_{\ell=k}^{\infty}\frac{x^{2\ell+n+1}}{2\ell+n+1}\end{aligned}$$

$$= (n+1)!\sum_{\ell=0}^{\infty}\sum_{k=0}^{\ell}\frac{1}{2k+n}\left(\sum_{\ell_{n-1}=0}^{k}\frac{1}{2\ell_{n-1}+n-1}\prod_{m=1}^{n-2}\sum_{\ell_m=0}^{\ell_{m+1}}\frac{1}{2\ell_m+m}\right)\frac{x^{2\ell+n+1}}{2\ell+n+1}$$
$$= (n+1)!\sum_{k=0}^{\infty}\left(\prod_{m=1}^{n}\sum_{\ell_m=0}^{\ell_{m+1}}\frac{1}{2\ell_m+m}\right)\frac{t^{2k+n+1}}{2k+n+1},$$

where $\ell_{n+1}=k$. By induction, Maclaurin's series expansion (48) is proved. □

7. USEFUL REMARKS

In this section, we state several useful remarks on our main results and related stuffs, including a possibly new combinatorial identity similar to those two in (28) in Corollary 3.

Remark 1. Maclaurin's series expansion (6) in Theorem 1 is recovered in [**38**, Section 6] and is generalized in [**35**, Theorem 4.1]. The closed-form formula (29) in Theorem 2 is reconsidered in [**35**, Theorem 2.2].

Remark 2. In order to avoid the indefinite case 0^0, we do not include the terms 1 behind equal signs in (6), (24), and (25), the terms $\frac{1}{2}t^2-\frac{1}{3}t^3$ in (32), the terms $\frac{1}{3}t^3$ in (33), the terms $1-\frac{1}{2}t^2+\frac{1}{3}t^3$ in (42), and the terms $2-\frac{1}{3}t^3$ in (43) into their corresponding sums, while we do not include the first identity into the second one in (28). This idea has been reflected in the proofs of Theorems 1 and 4.

Remark 3. Theorem 1, Theorem 2, and Theorem 3 give answers to three unification problems posed in [**20**, Remark 5.3].

Remark 4. When $m=1$, by virtue of (16), Maclaurin's series expansion (6) in Theorem 1 and Maclaurin's series expansion (25) in Theorem 4 become

$$\frac{\arcsin t}{t}=1+\sum_{k=1}^{\infty}(-1)^k\binom{\frac{2k-1}{2}}{2k}\frac{(2t)^{2k}}{2k+1}$$

and

$$\frac{\operatorname{arcsinh} x}{x}=1+\sum_{k=1}^{\infty}\binom{\frac{2k-1}{2}}{2k}\frac{(2x)^{2k}}{(2k+1)!}.$$

These forms are expressed in terms of the extended binomial coefficients.

Remark 5. If $k=0$, by virtue of (16), the series representation (30) in Theorem 3 becomes

$$\operatorname{Ls}_j(\theta)=2(\ln 2)^{j-1}\sin\left(\frac{\theta}{2}\right)\left[\sum_{q=1}^{\infty}(-1)^{q+1}\binom{\frac{2q-1}{2}}{2q}\left(2\sin\frac{\theta}{2}\right)^{2q}\right.$$
$$\times\sum_{\ell=0}^{j-1}\binom{j-1}{\ell}\left(\frac{\ln\sin\frac{\theta}{2}}{\ln 2}\right)^{\ell}\sum_{p=0}^{\ell}\frac{(-1)^p\langle\ell\rangle_p}{(2q+1)^{p+1}\left(\ln\sin\frac{\theta}{2}\right)^p}$$

$$-\sum_{\ell=0}^{j-1}\binom{j-1}{\ell}\left(\frac{\ln\sin\frac{\theta}{2}}{\ln 2}\right)^{\ell}\sum_{p=0}^{\ell}\frac{(-1)^p\langle\ell\rangle_p}{\left(\ln\sin\frac{\theta}{2}\right)^p}\Bigg],$$

where $\mathrm{Ls}_j(\theta)=\mathrm{Ls}_j^{(0)}(\theta)$ is the logsine function defined by (4) and $\langle\ell\rangle_p$ is defined by (14).

Remark 6. In [**2**, p. 122, 6.42], [**3**, pp. 262–263, Proposition 15], [**4**, pp. 50–51 and p. 287], [**5**, p. 384], [**6**, p. 2, (2.1)], [**8**, p. 188, Example 1], [**12**, Lemma 2], [**15**, p. 308], [**16**, pp. 88–90], [**18**, p. 61, 1.645], [**25**, pp. 124–125, (666); pp. 146–147, (778); pp. 148–149, (783) and (784); pp. 154–155, (832) and (834); pp. 176–177, (956)], [**28**, p. 1011], [**29**, p. 453], [**42**, Section 6.3], [**52**, p. 126], [**54**], [**59**, p. 59, (2.56)], or [**61**, p. 676, (2.2)], one can find Maclaurin's series expansions

$$\arcsin x=\sum_{\ell=0}^{\infty}\frac{1}{2^{2\ell}}\binom{2\ell}{\ell}\frac{x^{2\ell+1}}{2\ell+1},\quad |x|<1,$$

$$\left(\frac{\arcsin x}{x}\right)^2=2!\sum_{k=0}^{\infty}[(2k)!!]^2\frac{x^{2k}}{(2k+2)!},\quad |x|<1, \tag{49}$$

$$(\arcsin x)^3=3!\sum_{\ell=0}^{\infty}[(2\ell+1)!!]^2\left[\sum_{k=0}^{\ell}\frac{1}{(2k+1)^2}\right]\frac{x^{2\ell+3}}{(2\ell+3)!},\quad |x|<1,$$

or their variants.

In the paper [**39**], those three series expansions in (49) were applied to recover and establish several known and new combinatorial identities containing the ratio of two central binomial coefficients $\binom{2k}{k}$. The central binomial coefficient $\binom{2k}{k}$ is related to the Catalan numbers [**42**] in combinatorial number theory. In the paper [**27**], those three series expansions in (49) and Maclaurin's series expansion of $(\arcsin x)^4$ were applied several times.

In the monograph [**55**], Identities 151–153 and 163 read that

$$\frac{4}{\sqrt{4x-x^2}}\arcsin\frac{\sqrt{x}}{2}=\sum_{n=0}^{\infty}\frac{x^n}{\binom{2n}{n}(2n+1)},$$

$$\frac{\arcsin x}{x\sqrt{1-x^2}}=\sum_{n=0}^{\infty}\frac{4^n x^{2n}}{\binom{2n}{n}(2n+1)},$$

$$\frac{1}{1-x^2}+\frac{x\arcsin x}{(1-x^2)^{3/2}}=\sum_{n=0}^{\infty}\frac{4^n x^{2n}}{\binom{2n}{n}},$$

and

$$\frac{4}{4-x}+\frac{\sqrt{x}\arcsin\frac{\sqrt{x}}{2}}{2(1-x/4)^{3/2}}=\sum_{n=0}^{\infty}\frac{x^n}{\binom{2n}{n}}.$$

These four series expansions can be analytically derived from differentiating on both sides of the second series expansion for $\left(\frac{\arcsin x}{x}\right)^2$ in (49) and from straightforward manipulations.

Comparing the second series expansion in (49) with the series expansion (6) for $m = 2$ in Theorem 1, we obtain the identity

$$Q(2, 2k; 2) = (-1)^k (k!)^2, \quad k \in \mathbb{N}. \tag{50}$$

This combinatorial identity is recovered in [**38**, Lemma 3.1 and Remark 3.3].

The combinatorial identity (50) and those in (41) are possibly new. For further discussion, please refer to Remark 16 below.

Remark 7. By virtue of the formula (45), we can reformulated the equation (2.1) in [**45**, Theorem 2.1] and the equations (1.5) and (1.6) in [**46**, Section 1.3] as

$$\mathrm{B}_{n,k}(\langle\alpha\rangle_1, \langle\alpha\rangle_2, \dots, \langle\alpha\rangle_{n-k+1}) = (-1)^k \frac{n!}{k!} \sum_{\ell=0}^{k} (-1)^\ell \binom{k}{\ell} \binom{\alpha\ell}{n}$$

and

$$\mathrm{B}_{n,k}\left(1, 1-\lambda, (1-\lambda)(1-2\lambda), \dots, \prod_{\ell=0}^{n-k}(1-\ell\lambda)\right) = \begin{cases} (-1)^k \dfrac{\lambda^n n!}{k!} \displaystyle\sum_{\ell=0}^{k} (-1)^\ell \binom{k}{\ell} \binom{\ell/\lambda}{n}, & \lambda \ne 0 \\ S(n,k), & \lambda = 0 \end{cases}$$

for $n \ge k \in \mathbb{N}_0$ and $\alpha, \lambda \in \mathbb{C}$, where $\mathrm{B}_{n,k}$ is defined by (3), the falling factorial $\langle\alpha\rangle_p$ is defined by (14), the second kind Stirling numbers $S(n,k)$ for $n \ge k \in \mathbb{N}_0$ can be analytically generated [**14**, p. 51] by

$$\frac{(\mathrm{e}^x - 1)^k}{k!} = \sum_{n=k}^{\infty} S(n,k) \frac{x^n}{n!}$$

and can be explicitly computed [**14**, p. 204, Theorem A] by

$$S(n,k) = \begin{cases} \dfrac{(-1)^k}{k!} \displaystyle\sum_{\ell=0}^{k} (-1)^\ell \binom{k}{\ell} \ell^n, & n > k \in \mathbb{N}_0; \\ 1, & n = k \in \mathbb{N}_0, \end{cases}$$

and extended binomial coefficient $\binom{z}{w}$ is defined by (13). See also the paper [**24**]. These two identities and those collected in [**46**, Section 1.3 to Section 1.5] on closed-form formulas for specific partial Bell polynomials $\mathrm{B}_{n,k}$ supply approaches to establish explicit and general formulas of the mth derivatives and Maclaurin's series expansions for composite functions $f((a+bx)^\alpha)$, such as e^{x^α} and $\sin[(a+bx)^\alpha]$, with $\alpha \in \mathbb{R}$, if the mth derivatives of the function f can be explicitly or recursively computed for $m \in \mathbb{N}$.

In [33, Theorem 1.2], the formulas

$$\begin{aligned}(51)\quad &\mathrm{B}_{n,k}\left(-\sin x,-\cos x,\sin x,\cos x,\dots,\cos\left[x+\frac{(n-k+1)\pi}{2}\right]\right)\\ &=\frac{(-1)^k\cos^k x}{k!}\sum_{\ell=0}^{k}\binom{k}{\ell}\frac{(-1)^\ell}{(2\cos x)^\ell}\sum_{q=0}^{\ell}\binom{\ell}{q}(2q-\ell)^n\cos\left[(2q-\ell)x+\frac{n\pi}{2}\right]\end{aligned}$$

and

$$\begin{aligned}(52)\quad &\mathrm{B}_{n,k}\left(\cos x,-\sin x,-\cos x,\sin x,\dots,\sin\left[x+\frac{(n-k+1)\pi}{2}\right]\right)\\ &=\frac{(-1)^k\sin^k x}{k!}\sum_{\ell=0}^{k}\frac{\binom{k}{\ell}}{(2\sin x)^\ell}\sum_{q=0}^{\ell}(-1)^q\binom{\ell}{q}(2q-\ell)^n\cos\left[(2q-\ell)x+\frac{(n-\ell)\pi}{2}\right]\end{aligned}$$

for $n\ge k\in\mathbb{N}$ were obtained. See also [46, Section 1.6] and closely related references listed therein. These closed-form formulas (51) and (52) provide methods to establish explicit and general formulas of the mth derivatives and Maclaurin's series expansions for composite functions $f(\sin x)$ and $f(\cos x)$, such as $\sin^\alpha x$, $\cos^\alpha x$, $\sec^\alpha x$, $\csc^\alpha x$, $\mathrm{e}^{\pm\sin x}$, $\mathrm{e}^{\pm\cos x}$, $\ln\cos x$, $\ln\sin x$, $\ln\sec x$, $\ln\csc x$, $\sin\sin x$, $\cos\sin x$, $\sin\cos x$, $\cos\cos x$, $\tan x$, and $\cot x$ with $\alpha\in\mathbb{R}$, if the mth derivatives of the function f can be explicitly or recursively computed for $m\in\mathbb{N}$.

In the paper [9], earlier than [33], among other things, the mth derivatives and Maclaurin's series expansions of the positive integer powers $\sin^n z$, $\cos^n z$, $\tan^n z$, $\cot^n z$, $\sec^n z$, and $\csc^n z$ for $m,n\in\mathbb{N}$ were computed and investigated.

It is not difficult to see that, by virtue of the formulas (51) and (52), we can deal with explicit and general formulas of the mth derivatives and Maclaurin's series expansions of more general functions.

Remark 8. Replacing $\operatorname{arcsinh} t$ by t in Maclaurin's series expansions (35), (42), (43), and (44) leads to

$$\begin{aligned}\mathrm{e}^t&=1+\sinh t-(\sinh t)^2\sum_{k=0}^{\infty}\binom{\frac{2k-1}{2}}{2k+1}\frac{(2\sinh t)^{2k}}{k+1},\\ \Gamma(2,t)&=1-\frac{1}{2}(\sinh t)^2+\frac{1}{3}(\sinh t)^3-\frac{1}{4}\sum_{k=3}^{\infty}Q(2,k-1;3)\frac{(2\sinh t)^{k+1}}{(k+1)!},\\ \Gamma(3,t)&=2-\frac{1}{3}(\sinh t)^3-\frac{1}{8}\sum_{k=3}^{\infty}Q(3,k-2;3)\frac{(2\sinh t)^{k+1}}{(k+1)!},\end{aligned}$$

and, for $m\ge3$,

$$\Gamma(1+m,t)=m!-\frac{m!}{2^{m+1}}\sum_{k=m}^{\infty}Q(m+1,k-m;3)\frac{(2\sinh t)^{k+1}}{(k+1)!},$$

where $Q(m,k;3)$ is defined by (7) and the incomplete gamma function $\Gamma(a,x)$ is defined by (5).

Remark 9. In [**25**, pp. 168–169, (901); pp 176–177, (956)], there exist Maclaurin's series expansions

$$\frac{(\operatorname{arcsinh}\theta)^2}{2!}=\sum_{k=1}^{\infty}(-1)^{k+1}\frac{[2(k-1)]!!}{(2k-1)!!}\frac{\theta^{2k}}{2k}=\frac{\theta^2}{2}-\frac{2}{3}\frac{\theta^4}{4}+\frac{2}{3}\frac{4}{5}\frac{\theta^6}{6}-\cdots$$

and

$$\frac{\operatorname{arcsinh}\theta}{\sqrt{1+\theta^2}}=\sum_{k=1}^{\infty}(-1)^{k+1}\frac{[2(k-1)]!!}{(2k-1)!!}\theta^{2k-1}=\theta-\frac{2}{3}\theta^3+\frac{2}{3}\frac{4}{5}\theta^5-\cdots$$

Comparing the first one with (25) for $m=2$ deduces the identity (50) again.

Remark 10. Maclaurin's series expansion (25) in Theorem 4 can be applied to find a closed-form formula for the central factorial numbers of the first kind $t(n,k)$ which can be generated [**49**] by

$$\frac{1}{k!}\Bigl(2\operatorname{arcsinh}\frac{x}{2}\Bigr)^k=\sum_{n=k}^{\infty}t(n,k)\frac{x^n}{n!},\quad |x|\le 2.$$

Remark 11. The Faà di Bruno formula can be described in terms of partial Bell polynomials $\mathrm{B}_{n,k}(x_1,x_2,\dots,x_{n-k+1})$ by

$$\frac{\mathrm{d}^n}{\mathrm{d}\,x^n}f\circ h(x)=\sum_{k=0}^{n}f^{(k)}(h(x))\,\mathrm{B}_{n,k}\bigl(h'(x),h''(x),\dots,h^{(n-k+1)}(x)\bigr)\tag{53}$$

for $n\in\mathbb{N}_0$. See [**10**, Theorem 11.4] and [**14**, p. 139, Theorem C]. It is clear that

$$(\arctan t)^n=\sum_{k=0}^{\infty}\biggl[\lim_{t\to0}\frac{\mathrm{d}^k(\arctan t)^n}{\mathrm{d}\,t^k}\biggr]\frac{t^k}{k!}$$

and, by employing (53) and considering $u=u(t)=\arctan t\to0$ as $t\to0$,

$$\begin{aligned}\lim_{t\to0}\frac{\mathrm{d}^k(\arctan t)^n}{\mathrm{d}\,t^k}&=\lim_{t\to0}\sum_{\ell=0}^{k}\frac{\mathrm{d}^\ell u^n}{\mathrm{d}\,u^\ell}\mathrm{B}_{k,\ell}\biggl(\frac{1}{1+t^2},\biggl(\frac{1}{1+t^2}\biggr)',\dots,\biggl(\frac{1}{1+t^2}\biggr)^{(k-\ell)}\biggr)\\&=\sum_{\ell=0}^{k}\lim_{u\to0}\bigl(\langle n\rangle_\ell u^{n-\ell}\bigr)\lim_{t\to0}\mathrm{B}_{k,\ell}\biggl(\frac{1}{1+t^2},\biggl(\frac{1}{1+t^2}\biggr)',\dots,\biggl(\frac{1}{1+t^2}\biggr)^{(k-\ell)}\biggr)\\&=n!\,\mathrm{B}_{k,n}\biggl(\frac{1}{1+t^2}\bigg|_{t=0},\biggl(\frac{1}{1+t^2}\biggr)'\bigg|_{t=0},\biggl(\frac{1}{1+t^2}\biggr)''\bigg|_{t=0},\dots,\biggl(\frac{1}{1+t^2}\biggr)^{(k-n)}\bigg|_{t=0}\biggr)\end{aligned}$$

with the convention $\mathrm{B}_{k,n}=0$ for $n>k$, while, by virtue of (53) and for $\ell\in\mathbb{N}_0$,

$$\biggl(\frac{1}{1+t^2}\biggr)^{(\ell)}\bigg|_{t=0}=\sum_{q=0}^{\ell}\frac{\mathrm{d}^q}{\mathrm{d}\,v^q}\biggl(\frac{1}{1+v}\biggr)\mathrm{B}_{\ell,q}(2t,2,0,\dots,0)$$

$$
\begin{aligned}
&= \lim_{t\to0}\sum_{q=0}^{\ell}\frac{(-1)^q q!}{(1+v)^{q+1}}2^q\frac{1}{2^{\ell-q}}\frac{\ell!}{q!}\binom{q}{\ell-q}t^{2q-\ell}\\
&= \ell!\lim_{t\to0}\sum_{q=0}^{\ell}(-1)^q\binom{q}{\ell-q}(2t)^{2q-\ell}\\
&= \begin{cases}(-1)^p(2p)!, & \ell=2p\\ 0, & \ell=2p+1\end{cases}\\
&= \frac{1+(-1)^\ell}{2}(-1)^{\ell/2}\ell!
\end{aligned}
$$

for $p\in\mathbb{N}_0$, where we used the substitution $v=v(t)=t^2\to0$ as $t\to0$, the identity

$$
\mathrm{B}_{n,k}\bigl(abx_1,ab^2x_2,\dots,ab^{n-k+1}x_{n-k+1}\bigr)=a^kb^n\,\mathrm{B}_{n,k}(x_1,x_2,\dots,x_{n-k+1}) \tag{54}
$$

for $n\ge k\in\mathbb{N}_0$ and $a,b\in\mathbb{C}$, which can be found in [**10**, p. 412] and [**14**, p. 135], and the explicit formula

$$
\mathrm{B}_{n,k}(x,1,0,\dots,0)=\frac{1}{2^{n-k}}\frac{n!}{k!}\binom{k}{n-k}x^{2k-n} \tag{55}
$$

in [**41**, Theorem 5.1], [**50**, Section 3], and [**46**, Section 1.4], with conventions that $\binom{0}{0}=1$ and $\binom{p}{q}=0$ for $q>p\in\mathbb{N}_0$. Accordingly, we acquire

$$
\begin{aligned}
\frac{(\arctan t)^n}{n!}&=\sum_{k=n}^{\infty}\mathrm{B}_{k,n}\biggl(0!,0,-2!,0,4!,\dots,\frac{1+(-1)^{k-n}}{2}(-1)^{(k-n)/2}(k-n)!\biggr)\frac{t^k}{k!}\\
&=\sum_{k=0}^{\infty}\mathrm{B}_{k+n,n}\biggl(0!,0,-2!,0,4!,0,\dots,\frac{1+(-1)^{k}}{2}(-1)^{k/2}k!\biggr)\frac{t^{k+n}}{(k+n)!}.
\end{aligned}
$$

Comparing this result with the series expansion (46), or equivalently with the series expansion (47), and equating coefficients of the terms $\frac{t^{k+n}}{k+n}$ yield

$$
\mathrm{B}_{2k+n,n}\bigl(0!,0,-2!,0,4!,0,-6!,\dots,(-1)^k(2k)!\bigr)=(-1)^k\prod_{m=1}^{n-1}\sum_{\ell_m=0}^{\ell_{m+1}}\frac{1}{2\ell_m+m} \tag{56}
$$

and

$$
\mathrm{B}_{2k+n-1,n}\bigl(0!,0,-2!,0,4!,0,-6!,\dots,(-1)^{k-1}(2k-2)!,0\bigr)=0 \tag{57}
$$

for $k,n\in\mathbb{N}$ with $\ell_n=k$. Since

$$
\begin{aligned}
&\mathrm{B}_{k,n}\bigl((\arctan t)'\big|_{t=0},(\arctan t)''\big|_{t=0},(\arctan t)^{(3)}\big|_{t=0},\dots,(\arctan t)^{(k-n+1)}\big|_{t=0}\bigr)\\
&=\mathrm{B}_{k,n}\left(\frac{1}{1+t^2}\bigg|_{t=0},\left(\frac{1}{1+t^2}\right)'\bigg|_{t=0},\left(\frac{1}{1+t^2}\right)''\bigg|_{t=0},\dots,\left(\frac{1}{1+t^2}\right)^{(k-n)}\bigg|_{t=0}\right)
\end{aligned}
$$

$$= \mathrm{B}_{k,n}\left(0!, 0, -2!, 0, 4!, 0, -6!, \dots, \frac{1+(-1)^{k-n}}{2}(-1)^{(k-n)/2}(k-n)!\right),$$

the identities (56) and (57) can be applied to establish Maclaurin's series expansions for composite functions $f(\arctan t)$, if the mth derivatives of the function f can be explicitly or recursively computed for $m \in \mathbb{N}$.

Remark 12. It is obvious that

$$(\operatorname{arctanh} t)^n = \sum_{k=0}^{\infty}\left[\lim_{t\to0}\frac{\mathrm{d}^k(\operatorname{arctanh} t)^n}{\mathrm{d}\, t^k}\right]\frac{t^k}{k!}$$

and, by employing (53) and considering $u = u(t) = \operatorname{arctanh} t \to 0$ as $t \to 0$,

$$\begin{aligned}
\lim_{t\to0}\frac{\mathrm{d}^k(\operatorname{arctanh} t)^n}{\mathrm{d}\, t^k} &= \lim_{t\to0}\sum_{\ell=0}^{k}\frac{\mathrm{d}^\ell\, u^n}{\mathrm{d}\, u^\ell}\mathrm{B}_{k,\ell}\left(\frac{1}{1-t^2}, \left(\frac{1}{1-t^2}\right)', \dots, \left(\frac{1}{1-t^2}\right)^{(k-\ell)}\right)\\
&= \sum_{\ell=0}^{k}\lim_{u\to0}\left(\langle n\rangle_\ell u^{n-\ell}\right)\lim_{t\to0}\mathrm{B}_{k,\ell}\left(\frac{1}{1-t^2}, \left(\frac{1}{1-t^2}\right)', \dots, \left(\frac{1}{1-t^2}\right)^{(k-\ell)}\right)\\
&= n!\,\mathrm{B}_{k,n}\left(\frac{1}{1-t^2}\bigg|_{t=0}, \left(\frac{1}{1-t^2}\right)'\bigg|_{t=0}, \left(\frac{1}{1-t^2}\right)''\bigg|_{t=0}, \dots, \left(\frac{1}{1-t^2}\right)^{(k-n)}\bigg|_{t=0}\right)
\end{aligned}$$

with the convention $\mathrm{B}_{k,n} = 0$ for $n > k$, while, by virtue of (53) and for $\ell \in \mathbb{N}_0$,

$$\begin{aligned}
\left(\frac{1}{1-t^2}\right)^{(\ell)}\bigg|_{t=0} &= \sum_{q=0}^{\ell}\frac{\mathrm{d}^q}{\mathrm{d}\, v^q}\left(\frac{1}{1-v}\right)\mathrm{B}_{\ell,q}(2t, 2, 0, \dots, 0)\\
&= \lim_{t\to0}\sum_{q=0}^{\ell}\frac{q!}{(1-v)^{q+1}}2^q\frac{1}{2^{\ell-q}}\frac{\ell!}{q!}\binom{q}{\ell-q}t^{2q-\ell}\\
&= \ell!\lim_{t\to0}\sum_{q=0}^{\ell}\binom{q}{\ell-q}(2t)^{2q-\ell}\\
&= \begin{cases}(2p)!, & \ell = 2p\\ 0, & \ell = 2p+1\end{cases}\\
&= \frac{1+(-1)^\ell}{2}\ell!
\end{aligned}$$

for $p \in \mathbb{N}_0$, where we used the substitution $v = v(t) = t^2 \to 0$ as $t \to 0$, the identity (54), and the explicit formula (55). Accordingly, we acquire

$$\begin{aligned}
\frac{(\operatorname{arctanh} t)^n}{n!} &= \sum_{k=n}^{\infty}\mathrm{B}_{k,n}\left(0!, 0, 2!, 0, 4!, 0, 6!\dots, \frac{1+(-1)^{k-n}}{2}(k-n)!\right)\frac{t^k}{k!}\\
&= \sum_{k=0}^{\infty}\mathrm{B}_{k+n,n}\left(0!, 0, 2!, 0, 4!, 0, 6!, \dots, \frac{1+(-1)^k}{2}k!\right)\frac{t^{k+n}}{(k+n)!}.
\end{aligned}$$

Comparing this with the verified guess in (48) and equating coefficients of the terms $\frac{t^{k+n}}{k+n}$ yield

$$\mathrm{B}_{2k+n,n}\big(0!,0,2!,0,4!,0,6!,\dots,(2k)!\big)=\prod_{m=1}^{n-1}\sum_{\ell_m=0}^{\ell_{m+1}}\frac{1}{2\ell_m+m} \tag{58}$$

and

$$\mathrm{B}_{2k+n-1,n}\big(0!,0,2!,0,4!,0,6!\dots,(2k-2)!,0\big)=0 \tag{59}$$

for $k,n\in\mathbb{N}$ with $\ell_n=k$. Since

$$\begin{aligned}
&\mathrm{B}_{k,n}\big((\operatorname{arctanh} t)'\big|_{t=0},(\operatorname{arctanh} t)''\big|_{t=0},\dots,(\operatorname{arctanh} t)^{(k-n+1)}\big|_{t=0}\big)\\
&=\mathrm{B}_{k,n}\left(\frac{1}{1-t^2}\bigg|_{t=0},\left(\frac{1}{1-t^2}\right)'\bigg|_{t=0},\left(\frac{1}{1-t^2}\right)''\bigg|_{t=0},\dots,\left(\frac{1}{1-t^2}\right)^{(k-n)}\bigg|_{t=0}\right)\\
&=\mathrm{B}_{k,n}\left(0!,0,2!,0,4!,0,6!,\dots,\frac{1+(-1)^{k-n}}{2}(k-n)!\right),
\end{aligned}$$

the identities (58) and (59) can be applied to establish Maclaurin's series expansions for composite functions $f(\operatorname{arctanh} t)$, if the mth derivatives of the function f can be explicitly or recursively computed for $m\in\mathbb{N}$.

Can one find out a simpler expression with less multiplicity of sums for the quantity

$$\begin{aligned}
\prod_{m=1}^{n-1}\sum_{\ell_m=0}^{\ell_{m+1}}\frac{1}{2\ell_m+m}&=\sum_{\ell_{n-1}=0}^{k}\frac{1}{2\ell_{n-1}+n-1}\sum_{\ell_{n-2}=0}^{\ell_{n-1}}\frac{1}{2\ell_{n-2}+n-2}\\
&\quad\cdots\sum_{\ell_3=0}^{\ell_4}\frac{1}{2\ell_3+3}\sum_{\ell_2=0}^{\ell_3}\frac{1}{2\ell_2+2}\sum_{\ell_1=0}^{\ell_2}\frac{1}{2\ell_1+1}
\end{aligned}$$

in the brackets of Maclaurin's series expansions (46) and (48)? An anonymous referee suggested to look into the literature about multiple zeta values such as the preprint [**60**].

On 12 February 2022, we came across the paper [**31**]. In this paper, M. Milgram developed a series expansion for the function $\left(\frac{\arctan x}{x}\right)^n$ and obtained some properties of the expansion coefficients.

Remark 13. From the formulas (8) and (9), we can derive

$$\begin{aligned}
\mathrm{e}^{a\arcsin\theta}&=1+a\theta+a^2\frac{\theta^2}{2!}+a\big(a^2+1\big)\frac{\theta^3}{3!}+a^2\big(a^2+2^2\big)\frac{\theta^4}{4!}\\
&\quad+a\big(a^2+1\big)\big(a^2+3^2\big)\frac{\theta^5}{5!}+\cdots,\\
\frac{\mathrm{e}^\theta}{\cos\theta}&=1+\sin\theta+\big(1+1^2\big)\frac{\sin^2\theta}{2!}+\big(1+2^2\big)\frac{\sin^3\theta}{3!}+\cdots,
\end{aligned}$$

and
$$\frac{\mathrm{e}^{a\arcsin\theta}}{\sqrt{1-\theta^2}}=1+a\theta+\left(a^2+1^2\right)\frac{\theta^2}{2!}+a\left(a^2+2^2\right)\frac{\theta^3}{3!}+\cdots.$$

The above three special series expansions without general terms can be found in [**17**, p. 79] and [**25**, pp. 118–119, (642); pp. 154–155, (833); pp. 156–157, (839)] respectively. For more information, please refer to [**3**, pp. 262–263, Proposition 15], [**6**, p. 3], [**15**, p. 308], [**20**, Remark 5.3], and [**26**, pp. 49–50].

Remark 14. Now we quote some texts in [**52**, pp. 124–125] as follows.

> Expanding $\sin(tx)$ and $\cos(tx)$ in powers of $\sin x$, we have
> $$\sin(tx)=t\sum_{n=0}^{\infty}(-1)^n\prod_{k=1}^{n}\left[t^2-(2k-1)^2\right]\frac{\sin^{2n+1}x}{(2n+1)!}$$
> and
> $$\cos(tx)=\sum_{n=0}^{\infty}(-1)^n\prod_{k=0}^{n-1}\left[t^2-(2k)^2\right]\frac{\sin^{2n}x}{(2n)!}$$
> for $|x|<\frac{\pi}{2}$ and all values of t. But
> $$\sin(tx)=\sum_{n=0}^{\infty}(-1)^n\frac{(tx)^{2n+1}}{(2n+1)!}\quad\text{and}\quad\cos(tx)=\sum_{n=0}^{\infty}(-1)^n\frac{(tx)^{2n}}{(2n)!}.$$

These texts recited from [**52**, pp. 124–125] are equivalent to the equality
$$\mathrm{e}^{t\arcsin x}=\sum_{\ell=0}^{\infty}\frac{b_\ell(t)x^\ell}{\ell!}\tag{60}$$
used in [**3**, pp. 262–263, Proposition 15], [**6**, p. 3], [**15**, p. 308], and [**26**, pp. 49–50], where $b_0(t)=1$, $b_1(t)=t$,
$$b_{2\ell}(t)=\prod_{k=0}^{\ell-1}\left[t^2+(2k)^2\right],\quad b_{2\ell+1}(t)=t\prod_{k=1}^{\ell}\left[t^2+(2k-1)^2\right]$$
for $\ell\in\mathbb{N}$. The equality (60) has also been applied in Section 2 in the paper [**20**].

In [**38**, Lemmas 3.1 and 3.2], the quantities $b_{2\ell}(t)$ and $b_{2\ell+1}(t)$ are expanded as finite sums in terms of the first kind Stirling numbers $s(n,k)$.

Remark 15. In [**52**, pp. 124 and 128], Maclaurin's series expansions
$$(\arctan x)^p=\sum_{n=0}^{\infty}(-1)^n x^{2n+p}\prod_{k=1}^{p-1}\left(\sum_{n_k=0}^{n_{k-1}}\frac{1}{2n_{k-1}-2n_k+1}\right)\frac{1}{2n_{p-1}+1},$$
$$(\arcsin x)^p=\sum_{n=0}^{\infty}x^{2n+p}\prod_{k=1}^{p-1}\left[\sum_{n_k=0}^{n_{k-1}}\frac{1}{2^{2(n_{k-1}-n_k)}(2n_{k-1}-2n_k+1)}\right.$$

$$\times \binom{2n_{k-1}-n_k}{n_{k-1}-n_k} \frac{1}{2^{2n_{p-1}}(2n_{p-1}+1)} \binom{2n_{p-1}}{n_{p-1}} \Bigg],$$

and

$$(\operatorname{arcsec} x)^p = (-1)^p \sum_{n=0}^{\infty} \frac{1}{x^{2n+p}} \prod_{k=1}^{p-1} \left[\sum_{n_k=0}^{n_{k-1}} \frac{1}{2^{2(n_{k-1}-n_k)}(2n_{k-1}-2n_k+1)} \right.$$
$$\left. \times \binom{2n_{k-1}-n_k}{n_{k-1}-n_k} \frac{1}{2^{2n_{p-1}}(2n_{p-1}+1)} \binom{2n_{p-1}}{n_{p-1}} \right]$$

were derived, where $n_0 = n$ and $p \in \mathbb{N}$.

Remark 16. The identity (28) is a special of the known identity (16).

The identities in (28) in Corollary 3 are also proved in the proof of Theorem 4.

The quantity $Q(m,k;\alpha)$ in (7) can be equivalently reformulated as

$$\sum_{\ell=0}^{k} \binom{m+\ell}{m} s(m+k, m+\ell) z^\ell, \quad k,m \in \mathbb{N}_0, \quad z \neq 0. \tag{61}$$

The finite sum on the right hand side of the identity (16) is a special case $m = 0$ of the finite sum in (61).

When replacing k by $2k-1$ and taking $z = \frac{2k+m-2}{2}$ in (61), we derive the finite sum on the right hand side of the second identity in (28). The quantity in the bracket on the right hand side of Maclaurin's series expansion (6) is also a special case of the finite sum (61). The identity (50) gives a sum of (61) for taking $m = 1$ and $z = k$ and for replacing k by $2k$.

Does there exist a simpler and general expression for the sum (61), or say, for the quantity $Q(m,k;\alpha)$ in (7)? If yes, Maclaurin's series expansions (6) and (24) in Theorem 1 and Corollary 1, the closed-form formula (29) in Theorem 2, the series representation (30) in Theorem 3, Maclaurin's series expansions (25) in Theorem 4, the series identities (32), (33), and (34) in Theorem 4, and Maclaurin's series expansions (42), (43), and (44) in Corollary 5 would be further simplified.

Remark 17. It is common knowledge that $\arcsin t + \arccos t = \frac{\pi}{2}$ for $|t| < 1$. This means that

$$(\arccos t)^m = \left(\frac{\pi}{2} - \arcsin t\right)^m$$
$$= \sum_{q=0}^{m} (-1)^q \binom{m}{q} \left(\frac{\pi}{2}\right)^{m-q} (\arcsin t)^q$$
$$= \left(\frac{\pi}{2}\right)^m + \sum_{q=1}^{m} (-1)^q \binom{m}{q} \left(\frac{\pi}{2}\right)^{m-q} t^q \left[1 + \sum_{k=1}^{\infty} \frac{(-1)^k}{\binom{q+2k}{q}} Q(q,2k;2) \frac{(2t)^{2k}}{(2k)!}\right]$$
$$= \left(\frac{\pi}{2}\right)^m + \sum_{q=1}^{m} (-1)^q \binom{m}{q} \left(\frac{\pi}{2}\right)^{m-q} t^q$$

$$
\begin{aligned}
&+\sum_{q=1}^{m}\sum_{k=1}^{\infty}\binom{m}{q}\left(\frac{\pi}{2}\right)^{m-q}\frac{(-1)^{q+k}}{\binom{q+2k}{q}}Q(q,2k;2)\frac{2^{2k}t^{q+2k}}{(2k)!}\\
&=\left(\frac{\pi}{2}\right)^{m}+\sum_{q=1}^{m}(-1)^{q}\binom{m}{q}\left(\frac{\pi}{2}\right)^{m-q}t^{q}\\
&+\sum_{k=1}^{\infty}\sum_{q=1}^{m}\binom{m}{q}\left(\frac{\pi}{2}\right)^{m-q}\frac{(-1)^{q+k}}{\binom{q+2k}{q}}Q(q,2k;2)\frac{2^{2k}t^{q+2k}}{(2k)!}\\
&=\left(\frac{\pi}{2}\right)^{m}+\sum_{q=1}^{m}(-1)^{q}\binom{m}{q}\left(\frac{\pi}{2}\right)^{m-q}t^{q}\\
&+\sum_{k=1}^{\infty}(-4)^{k}\sum_{q=1}^{m}(-1)^{q}q!\binom{m}{q}\left(\frac{\pi}{2}\right)^{m-q}Q(q,2k;2)\frac{t^{q+2k}}{(q+2k)!}\\
&=\left(\frac{\pi}{2}\right)^{m}+\sum_{q=1}^{m}(-1)^{q}\binom{m}{q}\left(\frac{\pi}{2}\right)^{m-q}t^{q}\\
&+\sum_{p=3}^{\infty}\left[\sum_{\substack{q,k\in\mathbb{N}\\ q+2k=p}}(-4)^{k}(-1)^{q}q!\binom{m}{q}\left(\frac{\pi}{2}\right)^{m-q}Q(q,p-q;2)\right]\frac{t^{p}}{p!}
\end{aligned}
$$

for $|t|<1$, that is,

$$
\begin{aligned}
(\arccos t)^{m}&=\left(\frac{\pi}{2}\right)^{m}+\sum_{q=1}^{m}(-1)^{q}\binom{m}{q}\left(\frac{\pi}{2}\right)^{m-q}t^{q}\\
&\quad+\sum_{p=3}^{\infty}\left[\sum_{k=1}^{\infty}(-4)^{k}\sum_{q=1}^{p-2k}(-1)^{q}q!\binom{m}{q}\left(\frac{\pi}{2}\right)^{m-q}Q(q,p-q;2)\right]\frac{t^{p}}{p!}
\end{aligned}
\tag{62}
$$

for $|t|<1$, where we used the power series expansion (6) in Theorem 6, used the convention $\binom{u}{v}=0$ for $u<v$, and understood the sum, if the starting index exceeds the finishing index, to be zero.

Substituting the relation $\arccos t=-\mathrm{i}\operatorname{arccosh} t$ into (62), we can derive Maclaurin's series expansion of the inverse hyperbolic cosine $\operatorname{arccosh} t$.

In the papers [**35**, **38**], we will further discover nicer and more beautiful Maclaurin's and Taylor's series expansions of functions related to $(\arccos t)^{\alpha}$ and $(\arcsin t)^{\alpha}$ for $\alpha\in\mathbb{R}$.

Remark 18. Using the relation (27) in (32), (33), (34), and (35) in Theorem 4 deduces

$$
\begin{aligned}
\sum_{\ell=0}^{\infty}(-1)^{\ell}(\ell+1)\frac{[-\mathrm{i}\arcsin(\mathrm{i}\,t)]^{\ell+2}}{(\ell+2)!}&=\frac{1}{2}t^{2}-\frac{1}{3}t^{3}+\frac{1}{4}\sum_{k=3}^{\infty}Q(2,k-1;3)\frac{(2t)^{k+1}}{(k+1)!},\\
\sum_{\ell=0}^{\infty}(-1)^{\ell}(\ell+1)(\ell+2)\frac{[-\mathrm{i}\arcsin(\mathrm{i}\,t)]^{\ell+3}}{(\ell+3)!}&=\frac{1}{3}t^{3}+\frac{1}{8}\sum_{k=3}^{\infty}Q(3,k-2;3)\frac{(2t)^{k+1}}{(k+1)!},
\end{aligned}
$$

$$\sum_{\ell=0}^{\infty}(-1)^{\ell}\binom{\ell+m}{m}\frac{[-\mathrm{i}\arcsin(\mathrm{i}\,t)]^{\ell+m+1}}{(\ell+m+1)!}=\frac{1}{2^{m+1}}\sum_{k=m}^{\infty}Q(m+1,k-m;3)\frac{(2t)^{k+1}}{(k+1)!}$$

for $m \ge 3$, and

$$\mathrm{e}^{-\mathrm{i}\arcsin(\mathrm{i}\,t)}=1+t-t^2\sum_{k=0}^{\infty}\binom{\frac{2k-1}{2}}{2k+1}\frac{(2t)^{2k}}{k+1}, \tag{63}$$

where $Q(m+1,k-m;3)$ is defined by (7).

Replacing t by $\mathrm{i}\,t$ in (63) and reformulating result in

$$\cos(\arcsin t)=1+\sum_{k=0}^{\infty}(-1)^{k}2^{2k}\binom{\frac{2k-1}{2}}{2k+1}\frac{t^{2(k+1)}}{k+1}, \tag{64}$$

where extended binomial coefficient $\binom{z}{w}$ is defined by (13). The expansion (64) is the special case $n=1$ of the relation

$$\cos(n\arcsin z)={}_2F_1\left(\frac{n}{2},-\frac{n}{2};\frac{1}{2};z^2\right),\quad n\in\mathbb{N}, \tag{65}$$

which was listed in the first line on [**18**, p. 1017], where the Gauss hypergeometric function ${}_2F_1(\alpha,\beta;\gamma;z)$ can be defined [**57**, Section 5.9] by

$${}_2F_1(\alpha,\beta;\gamma;z)=\sum_{k=0}^{\infty}\frac{(\alpha)_k(\beta)_k}{(\gamma)_k}\frac{z^k}{k!},\quad |z|<1$$

for complex numbers $\alpha,\beta\in\mathbb{C}\setminus\{0\}$ and $\gamma\in\mathbb{C}\setminus\{0,-1,-2,\dots\}$, where $(\alpha)_k$, $(\beta)_k$, and $(\gamma)_k$ are the Pochhammer symbols defined by (10). Furthermore, the relation (65) is the special case $\alpha=n\in\mathbb{N}$ of Maclaurin's series expansion

$$\cos(\alpha\arcsin x)=\sum_{k=0}^{\infty}\left(\prod_{\ell=1}^{k}\left[4(\ell-1)^2-\alpha^2\right]\right)\frac{x^{2k}}{(2k)!}, \tag{66}$$

which was established in [**38**, Lemma 3.3], where $\alpha\in\mathbb{C}$ and $|x|<1$.

Considering the relation (27) in (42), (43), and (44) in Corollary 5 deduces

$$\Gamma(2,-\mathrm{i}\arcsin(\mathrm{i}\,t))=1-\frac{1}{2}t^2+\frac{1}{3}t^3-\frac{1}{4}\sum_{k=3}^{\infty}Q(2,k-1;3)\frac{(2t)^{k+1}}{(k+1)!},$$

$$\Gamma(3,-\mathrm{i}\arcsin(\mathrm{i}\,t))=2-\frac{1}{3}t^3-\frac{1}{8}\sum_{k=3}^{\infty}Q(3,k-2;3)\frac{(2t)^{k+1}}{(k+1)!},$$

and, for $m\ge 3$,

$$\Gamma(1+m,-\mathrm{i}\arcsin(\mathrm{i}\,t))=m!-\frac{m!}{2^{m+1}}\sum_{k=m}^{\infty}Q(m+1,k-m;3)\frac{(2t)^{k+1}}{(k+1)!}.$$

where $Q(m+1,k-m;3)$ is given by (7) and the incomplete gamma function $\Gamma(a,x)$ is given by (5).

Remark 19. All of Maclaurin's series expansions of positive integer powers of the inverse (hyperbolic) trigonometric functions in this paper can be used to derive infinite series representations of positive integer powers of the circular constant π. For example, taking $t = \frac{1}{2}$ in (6) and simplifying result in

$$\left(\frac{\pi}{3}\right)^m = 1 + m! \sum_{k=1}^{\infty} (-1)^k \frac{Q(m, 2k; 2)}{(m+2k)!}, \quad m \in \mathbb{N}.$$

Remark 20. An anonymous referee pointed out that,

1. in the paper [7], the author showed that Taylor's coefficients of the hyperbolic tangent and cotangent functions are related to the geometric polynomials ω_m and these polynomials appear in the study of the operator $\left(x \frac{\mathrm{d}}{\mathrm{d}\, x}\right)^m$;
2. in [13, Proposition 4], the authors provided Taylor's coefficients of powers of the arcsine function in terms of elementary symmetric polynomials and showed a connection between these coefficients and the multiple zeta values.

Remark 21. This paper is a revised version of the preprint [19], a continuation of the paper [20], and a companion of the articles [35, 37, 38, 48].

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