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ABSTRACT

Two causality conditions that refer only to mass-shell quantities are formulated and their consequences explored. The first condition, called Weak Asymptotic Causality, expresses the requirement that some interaction between the initial particles must occur before the last interaction from which final particles emerge. This condition is shown to imply that if a two-body scattering function is analytic except for singularities in the energy variable at normal thresholds, then a) the physical scattering functions in two adjacent parts of the physical region separated by any normal threshold are parts of a single analytic function, b) the path of continuation joining these two parts bypasses

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the singularity in the upper half-plane of the energy variable, and c) the integral over the physical function can be represented as an integral over a contour that is distorted into the upper-half energy plane (hence not, for example, by a principal-value integral). Singularities possessing finite derivatives of all orders with respect to real variations of the energy are not encompassed by this result.

The second causality condition, called Strong Asymptotic Causality, expresses the requirement that, apart from contributions whose effects fall off faster than any inverse power of Euclidian distance, momentum-energy is carried over macroscopic distances only by stable physical particles. This condition implies that all n-particle scattering functions ($n \geq 4$) are analytic, apart from infinitely differentiable singularities, at physical points not lying on any positive-α Landau surface. Moreover, the scattering functions on the two sides of any such Landau surface are analytically connected by a path that passes around the singularity surface in a well defined manner, which is the same as in perturbation theory. Thus, apart from possible infinitely differentiable singularities, the physical region singularity structure is derived from a mass-shell causality requirement. Several properties of the set $\mathcal{L}^+$ of physical region positive-α Landau surfaces are derived.

I. INTRODUCTION

By a causality requirement we shall mean a requirement that events identified as effects occur later than events identified as their causes. Such requirements have led to important properties of the basic functions of classical electrodynamics,[1] nonrelativistic quantum mechanics,[2] and quantum field theory.[3] The aim of the present work is to formulate causality requirements within a mass-shell S-matrix theory and to derive from them certain properties of the physical-region scattering amplitudes.[4,5]

The procedure is as follows. The momentum-space wave functions representing the initial and final particles of a scattering experiment are chosen to be Schwartz test functions, and the scattering functions are shown to be Schwartz distributions. The mass-shell constraints on these wave functions imply that the space-time wave functions defined by Fourier transformation are solutions of the free-particle Klein-Gordon equation. Consequently the regions over which these space-time functions are nonzero cannot be bounded; these wave functions have appreciable values on cones, called velocity cones, running from the infinite past to the infinite future. It is argued in Section II that these velocity cones can be interpreted as the trajectory regions of the corresponding particles in the sense that the transition amplitude of a reaction will be small unless the velocity cones of appropriate particles intersect. These intersections are interpreted as the locations of the possible particle collisions. It is their space-time ordering that is restricted by the causality conditions.

The space-time wave functions are not strictly confined to their velocity cones, but have "tails" that extend over all space-time. This means that the locations of collisions are not sharply defined. This presents a difficulty that must be surmounted.

In Section III a condition called Weak Asymptotic Causality (WAC) is formulated. This condition expresses the general idea that if a time t can be found such that none of the collisions between initial particles occur at times earlier than t, and none of the collisions from which final particles emerge occur at times later than t, then the corresponding transition amplitude should be small. In other words, the first collision between initial particles should occur no later than the last collision that produces final particles. The WAC condition is formulated so that it refers only to the asymptotic regions long before or long after the relevant collisions take place. Indeed, it is only in these regions that the free-particle wave functions should have physical significance. From the WAC condition we derive the $i\varepsilon$ rule for continuation past any physical region Landau singularity surface of the two-body scattering functions. This weak condition is not strong enough, however, to give the rule for continuation past an arbitrary Landau singularity surface of a general n-particle scattering function.

In Section IV a stronger condition, called Strong Asymptotic Causality (SAC), is formulated. It embodies the idea that energy-momentum is carried over macroscopic distances only by physical particles. More precisely, the probabilities of interactions having energy-momentum transfers that cannot be attributed to physical particles are required to fall off faster than any inverse power of the Euclidian distance, as the distances involved become infinite. The SAC condition is shown to imply that the scattering functions are infinitely differentiable at all physical points not lying on a positive-α Landau surface.

Points that do lie on some positive-α surface are classified as type I points or type II points. Points which lie on only one positive-α Landau surface are included among the type I points. The only known examples of type II points are points at which two initial or two final particle energy-momentum vectors are collinear. The SAC condition is shown to imply that in a neighborhood of a type I point $\bar{K}$ a scattering function can be represented as a sum of a finite number of terms of which the first is infinitely differentiable, while the others are boundary values of holomorphic functions.

Furthermore, these boundary values are themselves infinitely differentiable except on the relevant Landau surfaces. If $\bar{K}$ belongs to only one positive-α Landau surface, then there is only one of these boundary value terms, and the $i\epsilon$-prescription that defines the boundary value agrees with that of perturbation theory. Similar results are derived for type I points at which several positive-α surfaces intersect. No results are obtained for type II points.

The results described above are useful in the following way. In analytic S-matrix theory it is assumed that the only singularities of the scattering functions are those that arise from the unitarity equations. But even granting that the positions of the singularities are known, there is the question of how to continue around them. There is even the prior question of whether the physical scattering functions on the two sides of a singularity passing through the physical region are analytically connected at all. That these two functions can differ is a real possibility. For example, the K-matrix, which also has singularities on the Landau surfaces, is not represented in sectors separated by these surfaces by the same analytic function. This property is a special feature of the scattering matrix. It has usually been assumed that one could accept the results of perturbation theory on this point, and take the scattering function in the various sectors to be parts of a single analytic function, with the rule for continuation around singularities the same as in perturbation theory. The present work provides a physical basis for these assumptions. Infinitely differentiable singularities are not encompassed. Since, however, the singularities generated by the unitarity equations are apparently never infinitely differentiable, this omission is of no practical significance in this context.

As a by-product we obtain a number of useful results concerning the nature of the set $\mathcal{L}^+$ of physical points lying on positive-α Landau surfaces. Let $\mathcal{M}$ be the mass shell. This consists of points in energy-momentum space that satisfy the mass constraints and the conservation laws. Let $\mathcal{M}_0$ be the subset of $\mathcal{M}$ where two (or more) initial or two (or more) final energy-momentum vectors are collinear. Let $\mathcal{L}^+[\mathcal{D}]$ be the Landau surface in $\mathcal{M}$ associated with the Landau diagram $\mathcal{D}$, and let $\mathcal{L}_0^+[\mathcal{D}]$ be the subset of $\mathcal{L}^+[\mathcal{D}]$ that excludes points lying on the $\mathcal{L}^+[\mathcal{D}']$ of any contraction $\mathcal{D}'$ of $\mathcal{D}$. Then $\mathcal{L}^+$ is the union of points lying on the various $\mathcal{L}_0^+[\mathcal{D}]$. Each point $K \notin \mathcal{M}_0$ of $\mathcal{L}_0^+[\mathcal{D}]$ is shown to correspond to a unique (apart from scaling) point in the space of Feynman α's. Each surface $\mathcal{L}_0^+[\mathcal{D}]$ is shown to be an analytic submanifold of $\mathcal{M} - \mathcal{M}_0$ of co-dimension 1. It is shown that the $i\epsilon$ prescriptions associated with a set of intersecting Landau surfaces $\mathcal{L}_0^+[\mathcal{D}_i]$ associated with a set of $\mathcal{D}_i$ that are all contractions of some single $\mathcal{D}$ are necessarily compatible.

II. BASIC FORMALISM

A. Transition Amplitudes

The basic observables in scattering experiments can be considered to be the probability amplitudes for transitions from initial systems of freely moving particles to final systems of freely moving particles. The general mathematical form of these transition amplitudes is dictated in the following way by physical requirements.

Consider an arbitrary reaction involving a total of n initial and final particles. Let the particles be labelled by an index i, $1 \leqslant i \leqslant n$. Each particle is represented by a complex-valued momentum-space wave function ψ_i which, because the particles are freely moving, is a mapping $\psi_i : \mathcal{M}_i \to \underset{\sim}{C}$ from the real manifold

$$\mathcal{M}_i = \{k_i \,|\, k_i^2 \equiv (k_{i0})^2 - \underset{\sim}{k}_i^2 = \mu_i^2,\ \sigma_i k_{i0} > 0\} \qquad (2.1)$$

into the space $\underset{\sim}{C}$ of complex numbers. The vector k_i is the mathematical energy-momentum of the $\underline{i}$th particle and is defined by $k_i = \sigma_i p_i$, where p_i is the physical energy-momentum of the particle, and

$$\sigma_i = \begin{cases} +1 \text{ for final particles,} \\ -1 \text{ for initial particles.} \end{cases} \qquad (2.2)$$

The mass μ_i of each particle is assumed to be nonzero. Other quantum numbers such as spin, isospin, charge, etc., are unimportant in this discussion and are not indicated explicitly. The functions ψ_i can, for the present purposes, be assumed to belong to the spaces $\mathcal{D}(\mathcal{M}_i)$ of functions that have compact support $\text{supp}\,\psi_i \subset \mathcal{M}_i$ and continuous partial derivatives of all orders in $\mathcal{M}_i$.

The transition from the initial system of particles to the final system is represented by a functional $S[\psi_1,\ldots,\psi_n]$ which, when all of the wave functions ψ_i have unit norm

$$\|\psi_i\| = \{(2\pi)^{-3}\int d^4k\ \Theta(\sigma_i k_0)\ \delta(k^2-\mu_i^2)\ |\psi_i(k)|^2\}^{\frac{1}{2}}, \quad (2.3)$$

is a probability amplitude. The functional S is assumed to be linear in the wave functions of the initial particles and antilinear in the wave functions of the final particles. This linearity, together with the probability interpretation of S , implies the inequality

$$|S[\psi_1,\ldots,\psi_n]| \leq \prod_i \|\psi_i\| \quad . \quad (2.4)$$

This inequality in turn implies the continuity of S in each variable ψ_i in the topology induced by the norm (2.3),[6] and hence also in the topology of $\mathcal{D}(\mathcal{M}_i)$.[7] The functional S can, therefore, by virtue of the nuclear theorem,[8] be written $S[\psi_1,\ldots,\psi_n] = S[\psi]$, where ψ is the product wave function

$$\psi(k_1,\ldots,k_n) = \prod_{\text{initial}} \psi_i(k_i) \prod_{\text{final}} \psi_i^*(k_i), \tag{2.5}$$

and $S[\psi]$ is a continuous linear functional (Schwartz distribution) on the space $\mathcal{D}(X\mathcal{M}_i)$ of functions with compact support and continuous partial derivatives of all orders in the product space $X\mathcal{M}_i = \mathcal{M}_1 X \mathcal{M}_2 X \cdots X \mathcal{M}_n$.

Conservation of energy and momentum requires S to be concentrated on the set

$$\mathcal{M} = \{K | K = (k_1,\ldots,k_n) \in X\mathcal{M}_i,\ \Sigma k_i = 0\} . \tag{2.6}$$

The restricted real mass-shell $\mathcal{N}$ is the subset of all points K of $\mathcal{M}$ at which at least two of the vectors k_i are linearly independent. The restriction of S to the set

$$\mathcal{B}(\mathcal{N}) = \{\psi | \psi \in \mathcal{D}(X\mathcal{M}_i),\ (\mathcal{M} \cap \text{supp } \psi) \subset \mathcal{N}\} \tag{2.7}$$

then has the representation[9]

$$S[\psi] = \int dK \psi(K)\, S(K) , \tag{2.8}$$

where $S(K)$ is a Schwartz distribution and

$$dK = (2\pi)^{4-3n}\ \delta(\Sigma k_i)\Pi d^4k_i\ \delta(k_i{}^2-\mu_i{}^2)\ \Theta(\sigma_i k_{i0}) \tag{2.9}$$

is the (Lorentz invariant) volume element of $\mathscr{W}$.

It is convenient to use, instead of $S(K)$, the distribution

$$T(K) = S(K) - S_0(K), \tag{2.10}$$

where $S_0(K)$ is the no-scattering part of the S matrix. Our causality conditions will be formulated in terms of the corresponding functionals $T[\psi]$.

B. Infinite Differentiability

In the following sections the distribution T is sometimes said to be infinitely differentiable, and sometimes holomorphic, at a point $\bar{K}$ of $\mathscr{H}$. These statements are given precise meaning in the following way.

The restricted real mass-shell $\mathscr{H}$ is a subset of the restricted complex mass-shell $\mathscr{W}_c$. The definition of $\mathscr{H}_c$ is analogous to that of $\mathscr{H}$: $\mathscr{M}_c$ is the set defined by

$$\mathscr{M}_c = \{K|k_i{}^2 = \mu_i{}^2,\ \Sigma k_i = 0\}, \tag{2.11}$$

where the components of the vectors k_i are now allowed to assume complex values, and $\mathscr{H}_c$ is the set of all points

$K = (k_1, \ldots, k_n)$ of $\mathcal{M}_c$ at which two or more of the vectors k_i are linearly independent. This set $\mathcal{N}_c$ is a $(3n-4)$-dimensional submanifold of $\mathbb{C}^{4n}$, which means that at every point $\bar{K}\varepsilon\mathcal{N}_c$ there is a (nonunique) local coordinate system.[10] This local coordinate system is defined as a triple $(\Delta_c(\bar{K}), \Pi_{\bar{K}}, D_c(\bar{K}))$ consisting of a neighborhood $\Delta_c(\bar{K})\subset\mathbb{C}^{4n}$ of $\bar{K}$, a polydisc

$$D_c(\bar{K}) = \{z | z\varepsilon\mathbb{C}^{3n-4},\ |z_\lambda - \bar{z}_\lambda| < r_\lambda,\ \bar{z}\varepsilon\mathbb{C}^{3n-4},\ r_\lambda > 0\}, \qquad (2.12)$$

and a nonsingular holomorphic mapping $\Pi_{\bar{K}}: D_c(\bar{K}) \rightarrow \Delta_c(\bar{K})$ which is such that $\bar{K} = \Pi_{\bar{K}}(\bar{z})$ and $\mathcal{N}_c \cap \Delta_c(\bar{K}) = \Pi_{\bar{K}}(D_c(\bar{K}))$. At points $\bar{K}$ of $\mathcal{N}$ this mapping can, and will, be chosen so that $\Delta_c(\bar{K})\cap\mathcal{N} = \Pi_{\bar{K}}(D_c(\bar{K})\cap\mathbb{R}^{3n-4})$.

It is sometimes convenient to choose a local coordinate system in which the local coordinates z_λ are defined by the equations $z_\lambda = U_\lambda \cdot K$, where the $U_\lambda = (u_{\lambda 1}, \ldots, u_{\lambda n})$ are appropriately chosen n-tuples of four-vectors and

$$U_\lambda \cdot K = \sum_{i=1}^{n} u_{\lambda i} \cdot k_i = \sum_{i=1}^{n} \sum_{\nu=0}^{3} g^{\nu\nu}\, u_{\lambda i \nu}\, k_{i\nu}\ . \qquad (2.13)$$

[The metric is $g^{00} = -g^{11} = -g^{22} = -g^{33} = 1$.] Such a coordinate system will be called a simple coordinate system.

Infinite differentiability on $\mathcal{N}$ can now be defined as follows.

<u>Definition 1</u>. Let $F(K)$ be a function defined on some open set $\mathcal{N}\subset\mathcal{H}$. The function $F(K)$ is said to be <u>infinitely differentiable at $\bar{K}\varepsilon\mathcal{N}$</u> if and only if for every choice $(\Delta_c(\bar{K}),\Pi_{\bar{K}},D_c(\bar{K}))$ of local coordinates at $\bar{K}$, the function $F \circ \Pi_{\bar{K}}$ has continuous partial derivatives of all order in some neighborhood $D\subset\Pi_{\bar{K}}^{-1}(\mathcal{N}\cap\Delta_c(\bar{K}))$ of $\bar{z} = \Pi_{\bar{K}}^{-1}(\bar{K})$. If, in addition, the function $F \circ \Pi_{\bar{K}}$ can be represented by a convergent power series in a neighborhood of $\bar{z}$, the function F is said to be <u>holomorphic</u> at $\bar{K}$.

<u>Definition 2</u>. Let $T(K)$ be a Schwartz distribution defined on some open set $\mathcal{N}\subset\mathcal{K}$. The distribution $T(K)$ is said to be infinitely differentiable (holomorphic) at $\bar{K}\varepsilon\mathcal{N}$ if on some neighborhood $\mathcal{N}'\subset\mathcal{N}$ of $\bar{K}$ there is defined an infinitely differentiable (holomorphic) function $F(K)$ that satisfies the equation

$$\int dK\ \psi(K)\ [T(K) - F(K)] = 0 \qquad (2.14)$$

for all wave functions ψ in

$$B(\mathcal{N}') = \{\psi|\psi\ \varepsilon\ \mathcal{D}(\mathcal{K}\mathcal{M}_1),\ (\mathcal{M}\cap\text{supp}\ \psi)\subset\mathcal{N}'\}\ . \qquad (2.15)$$

Because the different possible local coordinate systems are holomorphically equivalent,[11] the conditions of the definitions are satisfied for all choices of local coordinate systems if they are satisfied for any particular choice.

C. Space-time Wave Functions

A preliminary problem of this paper is to develop some kind of space-time picture of a scattering process. To this end we introduce space-time wave functions

$$\tilde{\psi}_i(x) = (2\pi)^{-3} \int d^4k \; \delta(k^2-\mu_i^2) \; \Theta(\sigma_i k_0) \; e^{-i\sigma_i k\cdot x} \; \psi_i(k). \qquad (2.16)$$

These functions $\tilde{\psi}_i(x)$ have the important property[12] that, for every positive integer N, the equation

$$\lim_{\tau\to\infty} \tau^N \tilde{\psi}_i(\hat{x}\tau) = 0 \qquad (2.17)$$

is satisfied uniformly in $\hat{x}$ on compact subsets of the complement of

$$\hat{V}(\psi_i) = \{\hat{x} \,|\, \hat{x} = kt, \; k\varepsilon \text{ supp } \psi_i, \; t \text{ real}\}. \qquad (2.18)$$

This property entails that for any fixed positive numbers ε, N, and δ there exists a τ_0 such that for all $\tau>\tau_0$ one has $|\tilde{\psi}_i(\hat{x}\tau)| < (|\hat{x}|\tau)^{-N}\delta$ for all $\hat{x}$ in the complement of the set

$$\hat{V}_\varepsilon(\psi_i) \equiv \{\hat{x} \,|\, \hat{x} = kt, \; |k-k'| \leqslant \varepsilon, k'\varepsilon \text{ supp } \psi_i, \; t \text{ real}\} \cup \{\hat{x} \,|\, |\hat{x}| \leqslant \varepsilon\}. \qquad (2.19)$$

[The norm $|x|$ of any four-vector x is the Euclidian norm $|x| = (\Sigma x_\nu^2)^{\frac{1}{2}}$.] The rapid uniform collapse of $\tilde{\psi}_i(\hat{x}\tau)$ into $\hat{V}_\varepsilon(\psi_i)$ as $\tau\to\infty$ suggests that the $\underline{i}$th particle may in some limiting sense be regarded as confined to

$$V_\varepsilon(\psi_i,\tau) = \{x|x = \hat{x}\tau,\ \hat{x}\varepsilon\hat{V}_\varepsilon(\psi_i)\} . \qquad (2.20)$$

This suggestion is supported by the following consideration. Let the various wave functions $\tilde{\psi}_i$ be displaced by the respective amounts $u_i\tau$. The displaced momentum-space wave functions are $\psi_i(k)\exp(i\ \sigma_i k\cdot u_i\tau)$, and the corresponding transition amplitude is denoted by $T[\psi;U\tau]$. Thus, for product wave functions ψ in $\mathcal{B}(\mathcal{H})$, the amplitude $T[\psi;U\tau]$ has the representation

$$T[\psi;U\tau] = \int dK\ e^{-iU\cdot K\tau}\ \psi(K)\ T(K). \qquad (2.21)$$

If $T(K)$ is essentially constant in the (perhaps very small) support of ψ, the approximation

$$T[\psi;U\tau]\approx\lambda\int d^4x \prod_{\text{initial}} \tilde{\psi}_i(x-u_i\tau) \prod_{\text{final}} \tilde{\psi}_i^*(x-u_i\tau) \qquad (2.22a)$$

$$= \lambda\int d^4(\hat{x}\tau) \prod_{\text{initial}} \tilde{\psi}_i((\hat{x}-u_i)\tau) \prod_{\text{final}} \tilde{\psi}_i^*((\hat{x}-u_i)\tau) \qquad (2.22b)$$

can be made. If an $\varepsilon>0$ can be found such that no point lies simultaneously in all of the displaced cones

$$\hat{V}_{\varepsilon}(\psi_i;u_i) = \{\hat{x}|\hat{x}-u_i \; \varepsilon \; \hat{V}_{\varepsilon}(\psi_i)\}, \tag{2.23}$$

then equations (2.17) and (2.22) imply that

$$\lim_{\tau\to\infty} \tau^N \; T[\psi;U\tau] = 0 \tag{2.24}$$

for all positive integers N. [Henceforth, the notation $f(\tau) \Rightarrow 0$ will indicate the rapid decrease (2.24) of any function $f(\tau)$.] That is, if the intersection of all the sets $\hat{V}_{\varepsilon}(\psi_i;u_i)$ is empty, then the probability that a reaction of the corresponding particles takes place decreases rapidly as τ becomes infinite.

This result provides a justification for considering the particles to be mainly confined to the space-time regions where the corresponding wave functions $\tilde{\psi}_i$ are not small. It also suggests that the image under $\hat{x}\to x = \hat{x}\tau$ of the region of intersection of the displaced cones $\hat{V}_{\varepsilon}(\psi_i;u_i)$ should be interpretable as the location of the "collision" of the corresponding particles, in the limit $\tau\to\infty$. This idea has been discussed in detail in Ref. 5, and shown to be completely in accord with the nature of the one-particle exchange contribution to a scattering process.

This interpretation of overlap regions as the locations of the corresponding collisions is the basis of the present work. These collisions constitute the "events" of S-matrix theory, and causality conditions place restrictions on their space-time ordering.

III. WEAK ASYMPTOTIC CAUSALITY (WAC)

A. Formulation of WAC

If the particle trajectories (i.e. the displaced velocity cones) are such that all possible collisions involving two or more initial particles occur later than all possible collisions from which two or more final particle can emerge, the reaction is considered to be acausal and the corresponding transition amplitude is required to be small. This requirement is made precise in the following way. Let ψ be a product wave function, and let $T[\psi]$ be the corresponding transition amplitude. Let the particles represented by ψ be displaced by amounts $u_i\tau$. The displaced particles are represented by the wave functions $\psi_i(k)\,\exp(i\sigma_i k\cdot u_i\tau)$, and the transition amplitude corresponding to them is denoted by $T[\psi;U\tau]$. For any fixed time $\hat{t}$ and positive number ε define the two sets

$$\hat{D}^{\pm}(\hat{t},\varepsilon) = \{\hat{x}\,|\,\pm(\hat{x}_0 - \hat{t}) \geqslant -\varepsilon\}. \qquad (3.1)$$

Finally, let $\mathcal{a}(\hat{t},\varepsilon,\psi)$ be the set of all n-particle displacements $U = (u_1,u_2,\ldots,u_n)$ such that (a) the Euclidean distance between points of $\hat{V}_\varepsilon(\psi_i;u_i)\cap\hat{D}^-(\hat{t},\varepsilon)$ and points of $\hat{V}_\varepsilon(\psi_j;u_j)\cap\hat{D}^-(\hat{t},\varepsilon)$ has a lower bound $d^-_{ij}>0$ for all pairs $(i\neq j)$ of initial particles, and (b) the distance between points of $\hat{V}_\varepsilon(\psi_i;u_i)\cap\hat{D}^+(\hat{t},\varepsilon)$ and points of $\hat{V}_\varepsilon(\psi_j;u_j)\cap\hat{D}^+(\hat{t},\varepsilon)$ has a lower bound $d^+_{ij}>0$ for all pairs

$(i \neq j)$ of final particles. A set $\mathcal{A}(\hat{t},\varepsilon,\psi)$ is called a set of acausal displacements. The weak causality condition is as follows:

Weak Asymptotic Causality (WAC). For any fixed product wave function ψ in $\mathcal{B}(\mathcal{H})$, fixed time $\hat{t}$, and fixed positive number ε, the condition $T[\psi;U\tau] \to 0$ is satisfied uniformly in U on every compact subset of the set $\mathcal{A}(\hat{t},\varepsilon,\psi)$ of acausal displacements.

This causality condition is justified in Appendix A by proving that it holds in nonrelativistic quantum mechanics and in all classical models with finite range interactions.

The WAC condition is also plausible within the framework of relativistic theories. If the set of particle displacements U belongs to $\mathcal{A}(\hat{t},\varepsilon,\psi)$, then the displaced velocity cones

$$V_{\varepsilon}(\psi_i;u_i,\tau) \equiv \{x = \hat{x}\tau \,|\, \hat{x}\varepsilon\hat{V}_{\varepsilon}(\psi_i;u_i)\} \qquad (3.2)$$

of the initial particles become increasingly far apart, as τ becomes infinite, for all times $x_0 < \hat{t}\tau + \varepsilon\tau$, and the displaced velocity cones of the final particles become increasingly far apart for all times $x_0 > \hat{t}\tau - \varepsilon\tau$. But if the initial particles become increasingly far apart in $x_0 < \hat{t}\tau + \varepsilon\tau$, then the state generated near $x_0 = \hat{t}\tau$ by the initial particles should be represented with increasing precision, as $\tau\to\infty$, by the displaced initial free-particle state. Similarly, the state near $x_0 = \hat{t}\tau$ that develops into the final free-particle state should be represented with increasing precision by the displaced final free-particle state. (See Figure 1.) Therefore, both these states near $x_0 = \hat{t}\tau$ are represented with increasing precision by the corresponding free-particle states, and the transition amplitude $T[\psi;U\tau]$ should approach its no-scattering value. This value is zero since the no-scattering part has been subtracted from T.

According to this argument, the amplitude $T[\psi;U\tau]$ would be expected to vanish as τ becomes infinite. But should it decrease faster than every inverse power of τ? This property means that for any fixed N, no matter how large, the amplitude decreases faster than τ^{-N}. Now, the overlap integrals

$$0^{\pm}_{ij}(U,\tau) = \int_{D^{\pm}(\hat{t},\varepsilon,\tau)} d^4x \, |\tilde{\psi}_i(x - u_i\tau) \, \tilde{\psi}_j(x - u_j\tau)| , \qquad (3.3)$$

where

$$D^{\pm}(\hat{t},\varepsilon,\tau) \equiv \{x = \hat{x}\tau \,|\, \hat{x}\varepsilon\hat{D}^{\pm}(\hat{t},\varepsilon)\} , \qquad (3.4)$$

should provide a measure of the probability that interactions take place in $D^{\pm}(\hat{t},\varepsilon,\tau)$. If U belongs to $\mathcal{A}(\hat{t},\varepsilon,\psi)$, then (a) $0^{-}_{ij}(U,\tau) \Rightarrow 0$ for all pairs $(i \neq j)$ of initial particles, and (b) $0^{+}_{ij}(U,\tau) \Rightarrow 0$ for all pairs $(i \neq j)$ of final particles.[13] The overlap integrals therefore decrease faster than, say, τ^{-N^N}. If the propagation of dynamical effects is itself causal, at least up to terms that fall off faster than any inverse power of (Euclidian) distance, the fact that the initial and final overlaps fall off at a very large rate (τ^{-N^N}) should insure that the transition amplitude falls off at least at a relatively slow rate (τ^{-N}).

The discussion of the previous two paragraphs is based on the idea of a development of a system in time. It does not, however, require a fundamental quantity that represents the "state" of a system at an instant of time. As τ becomes infinite, the duration of the strip $\epsilon\tau \geqslant (x_0 - \hat{t}\tau) \geqslant -\epsilon\tau$ over which the initial and final particle states are compared becomes infinite. Therefore, the notion of a "state" of a system needs to become precise only when the time interval to which it refers becomes infinite. This is in accord with the general S-matrix philosophy.

B. Consequences of WAC

The weak asymptotic causality condition does not permit a complete specification of the singularity structure of $T(K)$, but it does have some useful consequences. Suppose that $\mathcal{N}$ is a connected open set in $\mathcal{H}$ and that the set $\mathcal{L}^+$ of points lying on positive-α Landau surfaces passes through $\mathcal{N}$. Suppose also that $T(K)$ is holomorphic on $\mathcal{N} - \mathcal{L}^+$. It is then of interest to know whether the functions that represent $T(K)$

in the various regions of $\mathcal{N}-\mathcal{L}^+$ are holomorphic continuations of each other, and if so, to know the path that connects them.

In Section V it is shown that if $\mathcal{N}$ is sufficiently small and if each point K of the set $\mathcal{N}\cap\mathcal{L}^+$ is generated by exactly one positive-α Landau diagram $\mathcal{D}$,[14] apart from diagrams that differ from $\mathcal{D}$ by overall translations and scalings, then

$$\mathcal{N}\cap\mathcal{L}^+ = \{K|K\varepsilon\mathcal{N},\Lambda(K) = 0\} \tag{3.5}$$

where $\Lambda(K)$ is a real analytic function defined in a full 4n-dimensional neighborhood of $\mathcal{N}$. The gradient $\nabla\Lambda(K) = (u_1,\ldots,u_n)$, where $u_{i\nu} = \partial\Lambda/\partial k_i^{\nu}$, is well defined and is nonzero in $\mathcal{N}$. This result motivates the following theorem.

Theorem 1. Suppose the following four conditions are satisfied.

(a) A real analytic function $\Lambda(K)$ is defined in a full neighborhood of a neighborhood $\mathcal{N}\subset\mathcal{H}$ of $\bar{K}\varepsilon\mathcal{H}$.

(b) There is a local coordinate system $(\Delta_c^{'}(\bar{K}),\Pi_{\bar{K}}^{'},D_c^{'}(\bar{K}))$ with $z_1 = \Lambda(K)$[15], such that the distribution $T = T \circ \Pi_{\bar{K}}^{'}$

is a distribution in z_1 that is infinitely smooth in the variables $(z_2,\dots,z_{3n-4})$. That is, for any test function $\psi(z)$ with support in $D_c'(\bar{K})$ the amplitude $T[\psi]$ has the representation

$$T[\psi] = \int dz\ F(z)\frac{d^m}{dz_1{}^m}\,[J(z)\psi(z)], \tag{3.6}$$

where m is an integer, $J(z)$ is the Jacobian appropriate to the transformation $\Pi'_{\bar{K}}$, and $F(z)$ is continuous in z_1 and has continuous derivatives of all orders in $(z_2,\dots,z_{3n-4})$.

(c) For some fixed time $\hat{t}$, some fixed $\varepsilon>0$, and some fixed product wave function $\emptyset$ in $\mathcal{B}(\mathcal{N})$, with $\emptyset(\bar{K}) \neq 0$, the set $\mathcal{A}(\hat{t},\varepsilon,\emptyset)$ contains $-\nabla\Lambda(\bar{K})$.

(d) The WAC condition is valid.

Let $(\Delta_c(\bar{K}),\Pi_{\bar{K}},D_c(\bar{K}))$ be any simple coordinate system at $\bar{K}$. Then for any α, $0<\alpha<1$, there exists a real neighborhood $\mathcal{N}'\subset(\mathcal{N}\cap\Delta_c(\bar{K})\cap \text{supp}\ \emptyset)$ of $\bar{K}$ such that the restriction of the functional T to $\mathcal{B}(\mathcal{N}')$ can be written in the form

$$T[\psi] = \lim_{\substack{|\delta|\to 0\\ \delta\in C^+(\alpha)}} \int dK\psi(K)[T^0(K) + T^1(K'(K,\delta))], \tag{3.7}$$

where

$$K'(K,\delta) \equiv \Pi_{\overline{K}}(\Pi_{\overline{K}}^{-1}(K) + i\delta) \tag{3.8}$$

and

$$C^{+}(\alpha) = \{\delta \mid \delta \in \underline{R}^{3n-4},\ (\delta,\gamma) > |\delta|\ |\gamma|\ \alpha\}. \tag{3.9}$$

The vector γ in (3.9) is nonzero and is given by

$$\gamma_\lambda = \frac{\partial \Lambda \circ \Pi_{\overline{K}}}{\partial z_\lambda}(\bar{z}), \quad (1 \leqslant \lambda \leqslant 3n-4) \tag{3.10}$$

where $\Pi_{\overline{K}}(\bar{z}) = \overline{K}$. The function $T^{0}(K)$ is infinitely differentiable on $\mathcal{N}'$, and the function $T^{1}(K)$ is holomorphic (has a power series expansion in local coordinates[10]) on

$$\mathcal{E}_\alpha = \{K \mid K \in \mathcal{N}_c \cap \Delta_c(\overline{K}),\ \mathrm{Im}[\Pi_{\overline{K}}^{-1}(K)] \in C^{+}(\alpha)\}. \tag{3.11}$$

This theorem is proved in Appendix C.

The specific form of the domain $\mathcal{E}_\alpha$ of Theorem 1 depends on the particular choice of simple coordinate system. A variation of Theorem 1 that does not refer to a particular simple coordinate system is the following Theorem 1A, which is also proved in Appendix C.

<u>Theorem 1A</u>. Suppose the assumptions of Theorem 1 are satisfied. For any $\varepsilon > 0$ define

$$C_\varepsilon^{\,+}(\bar{K}) = \{K|\mathrm{Im}\{[\nabla\Lambda(\bar{K}) + U]\cdot K\}>0 \text{ for all } U\varepsilon R_\varepsilon\}, \qquad (3.12)$$

where

$$R_\varepsilon = \{U|U = (u_1,\ldots,u_n), \|U\| = [\sum_{i\nu} u_{i\nu}^{\,2}]^{1/2} \leqslant \varepsilon\}, \qquad (3.13)$$

the components $u_{i\nu}$ being real. Then for any $\varepsilon>0$ there exists a complex neighborhood $\mathcal{N}_\varepsilon \subset \mathcal{H}_c$ of $\bar{K}$ such that the restriction of T to $\mathcal{B}(\mathcal{H}\cap\mathcal{N}_\varepsilon)$ has the form

$$T[\psi] = \lim_{s\to 0} \int dK\psi(K)[T^0(K) + T^1(K'(K,s))], \qquad (3.14)$$

where $K'(K,s)$ is any function uniformly continuous in $K\varepsilon\mathcal{N}_\varepsilon\cap\mathcal{H}$ and s, $0<s<1$, such that: $K'(K,s)$ is infinitely differentiable on $\mathcal{N}_\varepsilon\cap\mathcal{H}$ and all derivatives are continuous in both K and s; $K'(K,0) = K$ for all $K\varepsilon[\mathcal{N}_\varepsilon\cap\mathcal{H}]$; and $K'(K,s)\varepsilon[\mathcal{N}_\varepsilon \cap C_\varepsilon^+(\bar{K})]$ for all $s>0$. The function $T^0(K)$ is infinitely differentiable on $\mathcal{N}_\varepsilon\cap\mathcal{H}$ and $T^1(K)$ is holomorphic on $\mathcal{N}_\varepsilon\cap C_\varepsilon^{\,+}(\bar{K})$.

The content of Theorem 1A is this: at points K sufficiently near $\bar{K}$, the functional $T[\psi]$ is represented by a function that is, apart from infinitely differentiable singularities, holomorphic in a domain that is essentially the upper half-plane of the variable $\sigma(K;\bar{K}) = \nabla\Lambda(\bar{K})\cdot K$.

This theorem is applicable, for example, to the case of two particle scattering $[1 + 2 \to 3 + 4]$. The only (positive-α) Landau surfaces in the physical region are those corresponding to normal thresholds in $s = (k_3 + k_4)^2$. These surfaces are given by functions Λ of the form $\Lambda(K) = (k_3 + k_4)^2 - M^2$. Thus the displacement $\nabla\Lambda(\bar{K})$ has the form

$$\nabla\Lambda(\bar{K}) = (0,0,u,u), \tag{3.15}$$

where $u = 2(\bar{k}_3 + \bar{k}_4)$. This displacement vector simply shifts the two final particles, 3 and 4, by twice the total energy-momentum vector of the reaction, as is illustrated in Figure 2. If $\bar{k}_1$ and $\bar{k}_2$ are not collinear, and if $\bar{k}_3$ and $\bar{k}_4$ are not collinear, then it is clear from the figure that for any product wave function ϕ with sufficiently small compact support centered at $\bar{K}$, there exists a $\hat{t}$ and an ε for which $-\nabla\Lambda(\bar{K})$ belongs to $\mathcal{A}(\hat{t},\varepsilon,\phi)$. Indeed, because u is positive timelike (k_3 and k_4 are positive timelike), the displacement $-\nabla\Lambda(\bar{K})$ moves the regions of intersection of the final particle velocity cones to a position earlier than that of the initial particles cones. Thus condition (c) can be satisfied for any value of $\hat{t}$ lying between these two regions, for some sufficiently small ε. If $\tilde{T}(z)$ is analytic in the variables other than $z_1 = \Lambda(K)$, and if WAC is valid, then all the conditions of the theorem are met. The function $T^1(K)$ is then holomorphic in what is essentially the upper half-plane of the variable $\sigma(K;\bar{K})$. This upper half plane is to lowest order in $(K - \bar{K})$ the upper half plane of the variable s, so $T^1(K)$ is holomorphic in the intersection of a neighborhood of $\bar{K}$ with what is essentially the upper half plane of the variable s.

Another application of Theorem 1 is to the pole contribution to the three particle scattering amplitude. If in the vicinity of the pole at $\Lambda(K) = 0$ the amplitude is assumed to have the form $T(K) = R(K)D[\Lambda(K)] + H(K)$, where $R(K)$ and $H(K)$ are holomorphic and $D[\Lambda]$ is a distribution that is holomorphic for $\Lambda \neq 0$, then the conditions of the theorem on the structure of $T(K)$ are satisfied. The function $\Lambda(K)$ is given by $\Lambda(K) = (k_3 + k_4 + k_6)^2 - M^2$, and the displacement $\nabla\Lambda(\bar{K})$ is, therefore,

$$\nabla\Lambda(\bar{K}) = (0,0,u,0,u,u), \tag{3.16}$$

where $u = 2(\bar{k}_3 + \bar{k}_4 + \bar{k}_6)$. The result of this displacement is shown in Figure 3. Suppose now that none of the initial particle momenta are collinear and none of the final particle momenta are collinear. Then inspection of Figure 3 shows that for wave functions ϕ with sufficiently small compact support centered at $\bar{K}$, there exists a $\hat{t}$ and an ε for which $-\nabla\Lambda(\bar{K})$ belongs to $\mathcal{Q}(\hat{t},\varepsilon,\phi)$. Theorem 1 again prescribes a path of continuation of T^1 which involves infinitesimal detours into the upper half plane of $\sigma(K;\bar{K})$.

The WAC condition does not give the $i\varepsilon$ prescriptions for normal thresholds of all types of reactions. For example, if the case just considered were modified by adding one external line at each vertex in such a way that each subreaction involved two initial and two final particles, then the conditions of the theorem could not be satisfied. Indeed the conditions of the theorem provide, in such a case, no distinction between the two collisions that allows one to identify one collision as the cause and the other as the effect; the two vertices are completely equivalent so far as weak causality is concerned.

The two vertices are, of course, not completely equivalent. Positive energy is generally carried into one and out of the other by the external particles. This provides the necessary distinction between cause and effect, because energy-momentum is always transferred over macroscopic distances in a way such that positive energy flows forward in time. To proceed further, this energy balance consideration must be incorporated into the causality condition.

The WAC condition can be augmented by an energy balance condition so as to give the $i\varepsilon$ prescriptions for all normal thresholds. Rather than dwelling on this point, we shall pass directly to the logical extension of this idea. Transmission of energy and momentum over macroscopic distances is, as far as we know, associated not only with the forward transmission of positive energy, but with transmission of just those amounts of energy and momentum that can be carried by physical particles. A formulation of this idea is given in the next section.

IV. STRONG ASYMPTOTIC CAUSALITY (SAC)

A. Formulation of SAC

The condition of strong asymptotic causality (SAC) is a formulation of the notion that momentum-energy is transmitted over macroscopic distances only by stable physical particles: if a reaction requires a transfer of energy-momentum that cannot be carried by stable physical particles, then SAC requires the probability of that reaction to fall off faster than any inverse power of the lower bound on the Euclidian distances over which such transfers must carry.

The central idea in the formulation of this requirement is that particle collisions are located in the intersections of the trajectory regions (i.e., displaced velocity cones) of the corresponding wave functions. From a collision involving two or more initial particles certain other stable physical particles may emerge. The momenta of these new particles must be consistent with conservation laws, and their trajectory regions must originate in the collision region where they are produced. These new trajectory regions may intersect other trajectory regions, defining new collision regions from which additional particles may emerge. In this fashion a causal network of collision regions connected by physical particle trajectories can be built up. (See Figure 4).

In order to formulate this idea more precisely the following definitions are introduced.

Definition 3. A causal space-time diagram $\mathcal{D}$ is a triple $\mathcal{D} = (V,L,\varepsilon)$ consisting of a set $V = (v_1,\ldots,v_m)$ of space-time points (vertices), a set $L = (L_1,\ldots,L_s)$ of directed line segments of space-time points, and a matrix ε of structure constants. The following properties hold:

(a) Each line segment L_j has the representation

$$L_j = \{x \mid x = t\ell_j^{+} + (1-t)\,\ell_j^{-}, \quad 0 \leq t \leq 1.\}, \tag{4.1}$$

where the endpoints ℓ_j^{σ} are spacetime points.

(b) The set V is the intersection of the end points:

$$V = \{x \mid x = \ell_i^{\sigma} = \ell_j^{\sigma'} \text{ for some } \sigma,\sigma' \text{ and } i \neq j\}. \tag{4.2}$$

Lines intersect effectively only at end points.

(c) The structure constants $\varepsilon_{jr} (1 \leq j \leq s,\ 1 \leq r \leq m)$ are defined by

$$\varepsilon_{jr} = \begin{cases} +1 & \text{if } v_r = \ell_j^{+}, \\ -1 & \text{if } v_r = \ell_j^{-}, \\ 0 & \text{otherwise.} \end{cases}$$

(d) Each line segment L_j is associated with a freely moving physical particle of nonzero mass μ_j and momentum-energy p_j. The real momentum-energy vector p_j satisfies $p_j^o > 0$ and $p_j^2 = \mu_j^2$, and is related to L_j by

$$\Delta_j \equiv \ell_j^+ - \ell_j^- = \alpha_j p_j, \qquad (4.4)$$

where α_j is some positive real number.

(e) Momentum is conserved at each vertex:

$$\sum_j p_j \, \varepsilon_{jr} = 0, \text{ (all } r). \qquad (4.5)$$

(Any other additively conserved quantum number must obey a similar conservation law.)

(f) Each v_r satisfies (4.2) with $\sigma = \sigma' = +1$ and also with $\sigma = \sigma' = -1$. (This condition can be imposed by virtue of the stability condition on the masses of physical particles).

The line segments of $\mathcal{D}$ are divided into two classes: internal and external. A line segment is internal if the set V contains both of its endpoints. Otherwise it is external. The vertices are similarly classified: a vertex is external if it is the end point of at least one external line. Otherwise it is internal. A $\mathcal{D}$ with no internal lines is called trivial.

<u>Definition 4</u>. Let $\psi = \Pi\psi_i$ be a product wave function.

An n-particle displacement $U = (u_1,\ldots,u_n)$ belongs to the set $\mathcal{C}(\psi)$, and is called <u>causal with respect to ψ</u>, if and only if for each $\varepsilon>0$ there exists a causal space-time diagram $\mathcal{D}_\varepsilon$ such that: (a) the diagram $\mathcal{D}_\varepsilon$ has n external lines that are associated (in the sense of Definition 3) in a one-to-one fashion with the n initial and final particles represented by ψ. In particular, the physical momentum-energy vectors associated with the external lines are $p_i = \sigma_i k_i$, where $K = (k_1,\ldots,k_n)$ belongs to the support of ψ; (b) the vertex of $\mathcal{D}_\varepsilon$ that contains the endpoint of the <u>i</u>th external line, is contained in $\hat{V}_\varepsilon(\psi_i;u_i)$.

The sets of displacements that are not causal with respect to ψ are <u>acausal</u> with respect to ψ:

$$\mathcal{A}(\psi) = \{U|U \not\in \mathcal{C}(\psi)\} . \tag{4.6}$$

The strong asymptotic causality condition analogous to WAC would be the requirement that for any fixed product wave function $\psi\varepsilon\mathcal{B}(\mathcal{H})$ the relation $T[\psi;U\tau] \Rightarrow 0$ be satisfied uniformly on compact subsets of $\mathcal{A}(\psi)$.

We shall, however, deal directly with the connected part $T_c[\psi]$ of $T[\psi]$. Only the connected causal space-time diagrams $\mathcal{D}$ should be relevant to $T_c[\psi]$. [A connected diagram is one for which the point set $\bigcup L_j$ is connected.] Let $\mathcal{C}_c(\psi)$ be the subset of $\mathcal{C}(\psi)$ which is formed by requiring also that the space-time diagram $\mathcal{D}$ of Definition 3 be connected. The corresponding acausal set is

$$\mathcal{A}_c(\psi) = \{U | U \not\in \mathcal{C}_c(\psi)\}. \qquad (4.7)$$

The SAC condition is then defined as follows:

<u>Strong Asymptotic Causality</u>. For any fixed product wave function $\psi \varepsilon \mathcal{B}(\mathcal{H})$ the condition $T_c[\psi; U\tau] \nrightarrow 0$ is satisfied uniformly on compact subsets of $\mathcal{A}_c(\psi)$.

B. Consequences of SAC

Consider displacements of the form

$$U_0(K) = (a + t_1 k_1,\ a + t_2 k_2, \ldots, a + t_n k_n), \qquad (4.8)$$

where $K = (k_1, \ldots, k_n)$ is any point of $\text{supp}\ \psi$, a is any real four-vector, and the t_i are real scalars. If the momenta of the external lines of a diagram $\mathcal{D}$ are given by K, and the positions of these lines are specified by a set of displacements from a common origin of the form $U_0(K)$, then the external lines of $\mathcal{D}$ all pass through a common point. The set $\mathcal{C}_0(\psi)$ of all displacements of the form (4.8) is then immediately seen to be a subset of $\mathcal{C}_c(\psi)$.

The sets $\mathcal{C}_c(K)$ and $\mathcal{C}_c(K)$ are defined to be the sets obtained by replacing $\text{supp}\ \psi$ by K in the foregoing definitions.

Let $\mathcal{L}^+$ be the set of all points $K \in \mathcal{M}$ for which the set $\mathcal{C}_c(K) - \mathcal{C}_0(K)$ is nonempty. The set $\mathcal{L}^+$ is characterized by the following theorem.

Theorem 2. The set $\mathcal{L}^+$ is the union of all positive-α Landau surfaces that are associated with connected Landau diagrams.

Proof. The positive-α Landau loop equations associated with a diagram $\mathcal{D}$ are precisely the statement that the set of vectors $\Delta_j \equiv \alpha_j p_j$ fit together to form a nontrivial causal diagram $\mathcal{D}$. The conservation law constraints and mass-shell conditions are demanded both by the Landau equations and by the existence of $\mathcal{D}$. Thus, the statement that there exists a nontrivial connected causal diagram $\mathcal{D}$ satisfying $K(\mathcal{D}) = \bar{K}$, where $K(\mathcal{D})$ is the set of energy-momentum vectors associated with external lines of $\mathcal{D}$, is equivalent to the statement that the Landau equations associated with diagram $\mathcal{D}$ have a positive-α solution at $\bar{K}$.[16] At a point $\bar{K} \varepsilon (\mathcal{M} - \mathcal{M}_0)$, where $\mathcal{M}_0$ is the subset of the mass-shell $\mathcal{M}$ in which two or more initial particle energy-momenta are collinear or two or more final particle energy-momenta are collinear, the existence of a nontrivial connected causal diagram $\mathcal{D}$, satisfying $K(\mathcal{D}) = \bar{K}$ is equivalent to the fact that $\mathcal{C}_c(\bar{K})$-$\mathcal{C}_0(\bar{K})$ is nonempty. This is because the trivial connected causal diagrams $\mathcal{D}$ satisfying $K(\mathcal{D}) = \bar{K}$ come only from $\mathcal{C}_0(\bar{K})$ and each nontrivial one is given by some U in $\mathcal{C}_c(\bar{K})$ that is not in $\mathcal{C}_0(\bar{K})$. At points $\bar{K}$ in $\mathcal{M}_0$ the set

$\mathcal{C}_c(\bar{K}) - \mathcal{C}_0(\bar{K})$ is nonempty. [See Section V, paragraph 2.] But all points $\bar{K}\varepsilon\mathcal{M}_0$ clearly lie on some positive-α Landau surface. This completes the proof.

This geometric interpretation of the Landau equations has been emphasized by Coleman and Norton.[16] We use it continually. In particular, the set of points lying on positive-α Landau surfaces is regarded as precisely the set of points K at which $K = K[\mathcal{D}(K)]$ for some causal nontrivial $\mathcal{D} = \mathcal{D}(K)$.

We consider only connected diagrams, and by a Landau surface always mean a Landau surface associated with a nontrivial connected causal diagram.

A first consequence of SAC is Theorem 3.

Theorem 3. SAC implies that the scattering function $T_c(K)$ is infinitely differentiable at all points of $\mathcal{H} - \mathcal{L}^+$.

The proof is given in Appendix D. Theorems 2 and 3 combine to say that the singularities of $T_c(K)$ (or more precisely, the points at which $T_c(K)$ is not infinitely differentiable) are confined to the positive-α Landau surfaces.

We next turn to points that lie on $\mathcal{L}^+$. Let $\bar{K}$ be a point of $\mathcal{L}^+$. Let $\mathcal{U} = \{U_1, \ldots, U_{3n-4}\}$ be any set of (3n-4) n-particle displacements that define a simple local coordinate system at $\bar{K}$ through the equations $z_\lambda = U_\lambda \cdot K$. Define the set

$$\Gamma(\mathcal{U}) = \{U | U = \Sigma t_\lambda U_\lambda, |t| = 1\}. \tag{4.9}$$

[The norm $|t|$ is the Euclidean norm of $t = (t_1, .., t_{3n-4})$.] A <u>product neighborhood</u> $\mathcal{N}$ is a neighborhood such that for some product wave function χ, supp $\chi = \overline{\mathcal{N}}$. For any product neighborhood $\mathcal{N}$ define the set

$$\Gamma_c(\mathcal{U};\mathcal{N}) = \overline{\{U | U \varepsilon \mathcal{C}_c(\chi) \cap \Gamma(\mathcal{U})\}}, \tag{4.10}$$

where supp $\chi = \overline{\mathcal{N}}$ and the bar over the right-hand side indicates closure.

<u>Definition 5</u>. A point $\bar{K}$ of $\mathcal{L}^+$ is of type I if and only if for every set $\mathcal{U}$ that can be used to define a simple coordinate system $(\Delta_c(\bar{K}), \Pi_{\bar{K}}, D_c(\bar{K}))$ at $\bar{K}$ there exists a product neighborhood $\mathcal{N}$ of $\bar{K}$, $(\mathcal{N} \cap \mathcal{H}) \subset (\mathcal{H} \cap \Delta_c(\bar{K}))$, such that: (a) the set $\Gamma_c(\mathcal{U};\mathcal{N})$ is contained in a finite number of closed disjoint subsets $\Gamma_c^{\,j}(\mathcal{U};\mathcal{N})$; and (b) each of these sets $\Gamma_c^{\,j}(\mathcal{U};\mathcal{N})$ can be contained in a corresponding set of the form

$$\Gamma^+(\mathcal{U};e_j) = \{U | U = \Sigma t_\lambda U_\lambda,\ |t| = 1,\ (t, e_j) > 0\}, \tag{4.11}$$

where e_j is some vector in $\underset{\sim}{R}^{3n-4}$. A point $\bar{K} \epsilon \mathcal{L}^+$ is of type II if it is not of type I.

The set $\mathcal{M}_0 \subset \mathcal{L}^+$ of points $K = (k_1,\ldots,k_n)$ of $\mathcal{M}$ at which two initial or two final particle momenta k_i are collinear consists entirely of type II points. No other type II points are known. The problem of showing that various points $\bar{K} \varepsilon \mathcal{L}^+ - \mathcal{W}_0$ are of type I is considered in the next section.

The structure of $T_c(K)$ near type I points is intimately related to the geometric structure of the set $\Gamma_c(\mathcal{U};\mathcal{n})$. Let Ω be the unit sphere

$$\Omega = \{t \mid t \;\varepsilon\; \underset{\sim}{R}^{3n-4},\; |t| = 1\} , \tag{4.12}$$

and let

$$\Omega_c^{\;j}(\mathcal{U};\mathcal{n}) = \{t \mid t \;\varepsilon\; \Omega,\; (\Sigma t_\lambda U_\lambda) \varepsilon\; \Gamma_c^{\;j}(\mathcal{U};\mathcal{n})\}. \tag{4.13}$$

Because the various closed sets $\Gamma_c^{\;j}$ are mutually disjoint, the corresponding compact sets $\Omega_c^{\;j}$ also have this property. It is therefore possible to construct open neighborhoods $\omega_j \subset \Omega$ of the sets $\Omega_c^{\;j}$ that have disjoint closures $\bar{\omega}_j$. Moreover, because of condition (b) of Definition 5, the neighborhoods ω_j can be constructed so that the polar cones

$$C^+(\bar{\omega}_j) = \{\delta \mid \delta \;\varepsilon\; \underset{\sim}{R}^{3n-4},\; (\delta',\delta)>0 \text{ for all } \delta' \varepsilon\; \bar{\omega}_j\} \tag{4.14}$$

are nonempty. Finally, let

$$\mathcal{L}_j^+(\mathcal{N}) = \{K | K \in \mathcal{N},\ \Gamma_c^j(\mathcal{U};\mathcal{N}) \cap \mathcal{C}_c(K) \text{ is nonempty}\}. \qquad (4.15)$$

The structure of $T_c(K)$ at type I points is then given by the following theorem, which is proved in Appendix D.

Theorem 4. Let $\bar{K}\varepsilon\mathcal{L}^+$ be a type I point. Let $(\Delta_c(\bar{K}),\Pi_{\bar{K}},D_c(\bar{K}))$ be any simple coordinate system with local coordinates $z_\lambda = U_\lambda \cdot K$. Let $\mathcal{U} = \{U_1,\ldots,U_{3n-4}\}$, and let $\mathcal{N}$ be some product neighborhood of $\bar{K}$ that satisfies the conditions of Definition 5. Let ω_j be the neighborhoods of the $\Omega_c^j(\mathcal{U};\mathcal{N})$ defined in the preceding paragraph. Finally, let the SAC condition be valid. Then there exists a neighborhood $\mathcal{N}_1$ of $\bar{K}$, $\mathcal{N}_1 \subset (\mathcal{N}\cap\mathcal{H})$, such that the restriction of $T_c[\psi]$ to $\mathcal{B}(\mathcal{N}_1)$ has the representation

$$T_c[\psi] = \int dK\psi(K)\ T_c^0(K) + \sum_j \lim_{\substack{|\delta|\to 0 \\ \delta\varepsilon C^+(\bar{\omega}_j)}} \int dK\psi(K)\ T_c^j(K'(K,\delta)). \qquad (4.16)$$

The summation runs over the indices that label the $\Gamma_c^j(\mathcal{U};\mathcal{N})$, and the quantity $K'(K,\delta)$ is defined by

$$K'(K,\delta) = \Pi_{\bar{K}}(\Pi_{\bar{K}}^{-1}(K) + i\delta). \qquad (4.17)$$

The function $T_c^0(K)$ is infinitely differentiable on

$\mathcal{N}_1$, and the functions $T_c^j(K)$ are holomorphic on the sets

$$\mathcal{E}_j = \{K \mid K \in \mathcal{H}_c \cap \Delta_c(\bar{K}),\ \mathrm{Im}\ \Pi_{\bar{K}}^{-1}(K)\ \varepsilon\ C^+(\bar{\omega}_j)\}. \qquad (4.18)$$

Moreover, each limit function

$$T_c^j(K) = \lim_{\substack{|\delta| \to 0 \\ \delta \varepsilon C^+(\bar{\omega}_j)}} T_c^j(K'(K,\delta)) \qquad (4.19)$$

exists and is infinitely differentiable on $\mathcal{N}_1 - \mathcal{L}_j^+(\mathcal{N}_1)$.

Thus, aside from an infinitely differentiable background term, the amplitude $T_c(K)$ can be represented at type I points as the sum of a finite number of terms, each with its own $i\varepsilon$ prescription.[17]

V. CAUSAL DISPLACEMENTS AS GRADIENTS TO LANDAU SURFACES

In order to apply Theorem 4 at a point $\bar{K}\varepsilon\mathcal{L}^+$, one must establish that $\bar{K}$ is of type I. This is done by exploiting the very close connection between the causal displacement vectors U at $\bar{K}$ and the normal vectors to the various Landau surfaces that pass through $\bar{K}$. For example, when $\bar{K}$ belongs to only one positive-α Landau surface $\mathcal{L}^+[\mathcal{D}]$, there is essentially only one causal displacement U at $\bar{K}$, and this displacement can be identified with the normal to $\mathcal{L}^+[\mathcal{D}]$. The continuity of the normal then implies that $\bar{K}$ is of type I. This result, and a number of related ones, are contained in the theorems that follow.

First we note that all points of $\mathcal{M}_0$ are type II points. [Recall that $\mathcal{M}_0$ is the set of all points $K = (k_1,\ldots,k_n)$ of $\mathcal{M}$ at which two initial or two final particle momenta k_i are collinear.] This result is seen as follows. Let $\bar{K}\varepsilon\mathcal{M}_0$, and let $\bar{k}_1$ and $\bar{k}_2$ be collinear initial particle momenta. [Similar arguments hold for collinear final particle momenta.] Then, for every product wave function ψ that does not vanish at $\bar{K}$, every $\varepsilon>0$, and every U of the form $U = (u,0,\ldots,0)$, the various displaced velocity cones $V_\varepsilon(\psi_i,u_i)$ always intersect in a way that allows the conditions of Definition 4 to be satisfied with a diagram $\mathcal{D}$ of the type illustrated in

Figure 5. Thus, for any $\mathcal{U} = \{U_1,\ldots,U_{3n-4}\}$ that defines a simple coordinate system at $\bar{K}$, and for any product neighborhood $\mathcal{N}$ of $\bar{K}$, it is always possible to find a connected path in $\Gamma_c(\mathcal{U};\mathcal{N})$ that connects $\bar{U} = (\bar{u},0,\ldots,0)\varepsilon\Gamma_c(\mathcal{U};\mathcal{N})$ with $-\bar{U}\varepsilon\Gamma_c(\mathcal{U};\mathcal{N})$. For this reason condition (b) of Definition 5 cannot be satisfied.

To classify points $\bar{K}\,\varepsilon\,\mathcal{W}$ that do not lie in $\mathcal{M}_0$, some additional notation is introduced. The symbol $\bar{\mathcal{D}}$ represents a fixed causal space-time diagram. The symbol $V(\bar{\mathcal{D}}) = (v_1(\bar{\mathcal{D}}),\ldots,v_m(\bar{\mathcal{D}}))$ represents the set of space-time vectors that give the positions of the vertices of $\bar{\mathcal{D}}$. The symbol $K(\bar{\mathcal{D}}) = (k_1(\bar{\mathcal{D}}),\ldots,k_n(\bar{\mathcal{D}}))$ represents the set of mathematical momenta associated with the external lines of $\bar{\mathcal{D}}$.

<u>Definition 6</u>. A diagram $\mathcal{D}$ is <u>similar</u> to a diagram $\bar{\mathcal{D}}$ if and only if its lines and vertices can be labeled so that $\mathcal{D}$ and $\bar{\mathcal{D}}$ have the same matrix ε of structure constants ε_{jr}, and the same types of particles associated with corresponding lines. The set of causal diagrams similar to $\bar{\mathcal{D}}$ is denoted by $[\bar{\mathcal{D}}]$.

This definition of $\mathcal{D}\varepsilon[\bar{\mathcal{D}}]$ does not require $V(\mathcal{D})$ to coincide with $V(\bar{\mathcal{D}})$, nor $K(\mathcal{D})$ to coincide with $K(\bar{\mathcal{D}})$. It does require each line of $\bar{\mathcal{D}}$ to have a positive timelike image in any diagram $\mathcal{D}\,\varepsilon[\bar{\mathcal{D}}]$. Moreover, any line $\bar{L}_i$ of $\bar{\mathcal{D}}$ and its image L_i in $\mathcal{D}\,\varepsilon[\bar{\mathcal{D}}]$ must be associated with the same type of particle.

Definition 7. A contraction $\mathcal{D}'$ of $\bar{\mathcal{D}}$ is a nontrivial diagram lying on the boundary of $[\bar{\mathcal{D}}]$ that is formed by shrinking to points some, but not all, of the internal lines of $\bar{\mathcal{D}}$. The notation $\mathcal{D}' \subset \bar{\mathcal{D}}$ means that $\mathcal{D}'$ is a contraction of $\bar{\mathcal{D}}$.

Definition 8. The positive-α Landau surface $\mathcal{L}^+[\bar{\mathcal{D}}]$ is the set of points K such that $K = K(\bar{\mathcal{D}})$ for some $\mathcal{D} \varepsilon [\bar{\mathcal{D}}]$:

$$\mathcal{L}^+[\bar{\mathcal{D}}] = \{K | K = K(\mathcal{D}),\ \mathcal{D} \varepsilon [\bar{\mathcal{D}}]\}. \tag{5.1}$$

The restricted positive-α Landau surface $\mathcal{L}_0^+[\bar{\mathcal{D}}]$ is the set of points K of $\mathcal{L}^+[\bar{\mathcal{D}}]$ that do not lie on $\mathcal{L}^+[\mathcal{D}']$ for any contraction $\mathcal{D}'$ of $\bar{\mathcal{D}}$:

$$\mathcal{L}_0^+[\bar{\mathcal{D}}] = \mathcal{L}^+[\bar{\mathcal{D}}] - \bigcup_{\mathcal{D}' \subset \bar{\mathcal{D}}} \mathcal{L}^+[\mathcal{D}'] \ . \tag{5.2}$$

It is clear from Definition 8 that the set $\mathcal{L}^+$ is the union of the restricted positive-α surfaces $\mathcal{L}_0^+$.

The restricted surfaces $\mathcal{L}_0^+$ are of interest because of their relatively simple topology:

Theorem 5. If $\bar{\mathcal{D}}$ is any fixed nontrivial connected causal diagram, and if $\bar{K} \varepsilon (\mathcal{L}_0^+[\bar{\mathcal{D}}] - \mathcal{M}_0)$, then there exists a neighborhood $\mathcal{N} \subset (\mathcal{H} - \mathcal{M}_0)$ of $\bar{K}$ in which $\mathcal{L}_0^+[\bar{\mathcal{D}}]$ is an analytic submanifold of codimension 1.[10]

This theorem, which is proved in Appendix E, means in particular that nonmanifold points such as acnodes and cusps[18] cannot lie on $\mathcal{L}^+ - \mathcal{M}_0$. The set $\mathcal{L}^+ - \mathcal{M}_0$ is the union of manifolds of codimension 1 in $\mathcal{K}$.

By virtue of theorem 5 the normal vector to a surface $\mathcal{L}_0^+[\bar{\mathcal{D}}]$ is well defined (to within a scale factor) at each point $K \notin \mathcal{M}_0$ of that surface. The content of part (b) of the next theorem is that this normal vector (appropriately scaled) is the n particle displacement $U = (u_1,\ldots,u_n)$ that generates (by displacing lines originally passing through some common origin) the positions of the external lines of any diagram $\mathcal{D}$ that satisfies $K(\mathcal{D}) = K$. [Henceforth, the phrase "U generates $\mathcal{D}$" will mean that $U = (u_1,\ldots,u_n)$ generates, by displacements $u_1,\ldots,u_n$ of lines originally passing through the origin, the positions of the external lines of $\mathcal{D}$.]

Theorem 6. Let $\bar{\mathcal{D}}$ be any fixed nontrivial connected causal space-time diagram, and let $\bar{K} \in \mathcal{K}$ be a point of $\mathcal{L}_0^+[\bar{\mathcal{D}}]$. Then there is a full 4n-dimensional neighborhood $\mathcal{N}(\bar{K})$ of $\bar{K}$ and a real analytic function $\Lambda(K)$, holomorphic in K over $\mathcal{N}(\bar{K})$, such that

a) The gradient $\nabla\Lambda(K)$ is nonzero at each point of $\mathcal{N}(\bar{K})$ and

$$\mathcal{L}_0^+[\bar{\mathcal{D}}] \cap \mathcal{N}(\bar{K}) = \{K \mid K \in \mathcal{K} \cap \mathcal{N}(\bar{K}),\ \Lambda(K) = 0\}. \tag{5.3}$$

(b) If $K(\mathcal{D}) \varepsilon \mathcal{L}_0^+[\bar{\mathcal{D}}] \cap \mathcal{N}(\bar{K})$ for some $\mathcal{D} \varepsilon [\bar{\mathcal{D}}]$, and if $U = (u_1, \ldots, u_n)$ is a set of n displacements that generates the diagram $\mathcal{D}$, then U must have the form

$$U = \lambda \nabla \Lambda(K(\mathcal{D})) + U_0(K(\mathcal{D})), \qquad (5.4a)$$

where $\lambda > 0$ and $U_0(K(\mathcal{D}))$ is of the form (4.8). In other words,

$$u_i^{\nu} = \frac{\partial \Lambda}{\partial k_{i\nu}} + a^{\nu} + t_i k_i^{\nu}, \qquad (5.4b)$$

where λ, a^{ν}, and t_i are real constants that depend only on the indicated indices, and λ is strictly positive.

If two surfaces $\mathcal{L}_0^+[\mathcal{D}_1]$ and $\mathcal{L}_0^+[\mathcal{D}_2]$ coincide in some neighborhood of $\bar{K} \varepsilon (\mathcal{L}^+ - \mathcal{M}_0)$, the two surfaces cannot be oriented in opposite ways. This follows from Theorem 7.

Theorem 7. Let $\mathcal{D}_1$ and $\mathcal{D}_2$ be two fixed nontrivial connected causal space-time diagrams, and let $\bar{K} \varepsilon (\mathcal{L}^+ - \mathcal{M}_0)$ belong to both $\mathcal{L}_0^+[\mathcal{D}_1]$ and $\mathcal{L}_0^+[\mathcal{D}_2]$. Let the corresponding real analytic functions from Theorem 6(a) be

$\Lambda_1(K)$ and $\Lambda_2(K)$. If $\mathcal{L}_0^+[\mathcal{D}_1]$ and $\mathcal{L}_0^+[\mathcal{D}_2]$ coincide in some neighborhood $\mathcal{N} \subset \mathcal{H}$ of $\bar{K}$, then $\nabla\Lambda_1(\bar{K}) = \lambda\nabla\Lambda_2(\bar{K}) + U_0(\bar{K})$, where $\lambda>0$ and $U_0(\bar{K})$ is of the form (4.8).

A proof of Theorem 7 is given in Appendix F.

At points $\bar{K} \varepsilon W$ not in $\mathcal{M}_0$, displacements $U_0(\bar{K})$ of the form (4.8) produce no essential changes in a diagram $\mathcal{D}$. Their only effects are a common translation of all external lines of $\mathcal{D}$ and displacements of these lines along themselves. The parameter λ fixes the scale of the diagram. Thus, part (b) of Theorem 6 says that if $K = K(\mathcal{D})$, where $\mathcal{D}\varepsilon[\bar{\mathcal{D}}]$ and $K\varepsilon\mathcal{L}_0^+[\bar{\mathcal{D}}]$, then the positions of the external lines of $\mathcal{D}$ are ~~essentially~~ obtained (essentially uniquely) by regarding the various components of $\nabla\Lambda(K)$ as the displacements of the corresponding external lines of $\mathcal{D}$. Theorem 7 says that the sense of the causal direction along $\nabla\Lambda(K)$ is an <u>intrinsic</u> <u>feature</u> of the surface $\mathcal{L}_0^+[\bar{\mathcal{D}}]$; this sense does not depend on the particular class of similar diagrams $[\bar{\mathcal{D}}]$ that might be used to define the given surface $\mathcal{L}_0^+[\bar{\mathcal{D}}]$.

To classify a point $\bar{K}\varepsilon(\mathcal{L}^+-\mathcal{M}_0)$ it is necessary to determine the complete set of displacements U that generate diagrams $\mathcal{D}$ that satisfy $K(\mathcal{D}) = \bar{K}$. The following two theorems give the structure of these sets. The first theorem is special; the second is general.

<u>Theorem 8</u>. Let $\bar{\mathcal{D}}$ be a fixed nontrivial connected causal space-time diagram, and let $K(\bar{\mathcal{D}})=\bar{K}$ be a point of

$\mathcal{W}$. If $\bar{K} = K(\mathcal{D})$, where $\mathcal{D}$ belongs either to $[\bar{\mathcal{D}}]$ or to $[\mathcal{D}']$ for some contraction $\mathcal{D}'$ of $\bar{\mathcal{D}}$, then any displacement U that generates $\mathcal{D}$ is of the form

$$U = \sum_g \lambda_g \nabla \Lambda_g(\bar{K}) + U_0(\bar{K}), \tag{5.5}$$

where $\lambda_g \geq 0$ for all g, and U_0 is of the form (4.8). The (finite) sum in (5.5) runs over the indices g that label diagrams $\mathcal{D}_g \subset \bar{\mathcal{D}}$ or $\mathcal{D}_g = \bar{\mathcal{D}}$ for which $\bar{K} \epsilon \mathcal{L}_0^+[\mathcal{D}_g]$. This result is proved in Appendix E.

Theorem 9. Let $\bar{K}$ belong to $\mathcal{L}^+ - \mathcal{M}_0$. Let I be a minimal set of indices g such that any restricted surface $\mathcal{L}_0^+[\mathcal{D}\ ]$ that contains $\bar{K}$ coincides near $\bar{K}$ with one of the surfaces $\mathcal{L}_0^+[\mathcal{D}_g]$ for some $g \epsilon I$ [The set I is known to be finite.[19]] If $K = K(\bar{\mathcal{D}})$ for some connected causal space-time diagram $\bar{\mathcal{D}}$, then any displacement U that generates $\bar{\mathcal{D}}$ is of the form

$$U = \sum_{g \epsilon I} \lambda_g \nabla \Lambda_g(\bar{K}) + U_0(\bar{K}), \tag{5.6}$$

where $\lambda_g \geq 0$ for all g and $U_0(\bar{K})$ is of the form (4.8) Theorem 9 is a trivial consequence of Theorems 7 and 8. The characterization (5.6) of the displacements that generate diagrams $\mathcal{D}$ for which $\bar{K} = K(\mathcal{D})$ will be used to show that almost all points of $\mathcal{L}^+ - \mathcal{M}_0$ are of type I.

To show that a point $\bar{K}$ of $\mathcal{L}^+ - \mathcal{M}_0$ is of type I it is not necessary to consider the sets $\Gamma_c(\mathcal{U};\mathcal{N})$ for all sets $\mathcal{U}$ that define simple local coordinate systems at $\bar{K}$ or for product neighborhoods $\mathcal{N}$ of $\bar{K}$. It is sufficient to consider instead the sets

$$\Gamma_c(\mathcal{U};\bar{K}) = \overline{\{U|U\varepsilon\mathcal{C}_c(\bar{K})\cap\Gamma(\mathcal{U})\}} \qquad (5.7)$$

for any one (fixed) set $\bar{\mathcal{U}}$.

Theorem 10. Let $\bar{\mathcal{U}} = \{\bar{U}_1,\ldots,\bar{U}_{3n-4}\}$ define a simple coordinate system at $\bar{K}\varepsilon(\mathcal{L}^+ - \mathcal{M}_0)$. Then the point $\bar{K}$ is of type I if and only if $\Gamma_c(\bar{\mathcal{U}};\bar{K})$ can be covered by a finite number of disjoint closed subsets $\Gamma_c{}^j(\bar{\mathcal{U}};\bar{K})$ of $\Gamma(\bar{\mathcal{U}})$, each of which can be contained in a corresponding set of the form (4.11). Theorem 4 remains true if the ω_j are taken to be open neighborhoods (with disjoint closures) of the corresponding sets $\Omega_c{}^j(\bar{\mathcal{U}};\bar{K})$.

This theorem is proved in Appendix F.

Theorem 10 shows that the structure of $\mathcal{C}_c(\bar{K})$ determines whether a point $\bar{K}\varepsilon\mathcal{L}^+ - \mathcal{M}_0$ is of type I. To determine the structure of $\mathcal{C}_c(\bar{K})$ at these points we use the following theorem, which is proved in Appendix F.

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Theorem 11. If $\bar{K}$ belongs to $\mathcal{L}^+ - \mathcal{M}_0$, then the set $\mathcal{C}_c(\bar{K})$ consists of all displacements U that generate connected causal diagrams $\mathcal{D}$ that satisfy $\bar{K} = K(\mathcal{D})$.

Combining Theorems 9, 10 and 11, we obtain the following theorem.

Theorem 12. Let $\bar{K}$ be a point of $\mathcal{L}^+ - \mathcal{M}_0$. Let I be a minimal set of indices g such that any restricted surface $\mathcal{L}_0^+[\mathcal{D}]$ that contains $\bar{K}$ coincides near $\bar{K}$ with one of the restricted surfaces $\mathcal{L}_0^+[\mathcal{D}_g]$ for some $g \in I$. If the vectors $\nabla\Lambda_g(\bar{K})$ and the (4n-dimensional) vectors $F_\rho (1 \leqslant \rho \leqslant n+4)$ defined by

$$(F_\rho)_r{}^\mu = \delta_{\rho r} \bar{k}_r{}^\mu, \qquad (1 \leqslant \rho \leqslant n), \tag{5.8 a}$$

and

$$(F_\rho)_r{}^\mu = \delta_\rho{}^{\mu+n+1}. \qquad (n+1 \leqslant \rho \leqslant n+4) \tag{5.8 b}$$

are linearly independent, then, the point $\bar{K}$ is of type I. Furthermore, the representation of $T_c(K)$ in Theorem 4 has only one boundary value term at $\bar{K}$.

The proof is trivial. The vectors F_ρ form a basis for $\mathcal{C}_0(\bar{K})$. If the vectors $\nabla\Lambda_g(\bar{K})$ and F_ρ are linearly independent, there exists a set $\mathcal{U} = \{U_1, \ldots, U_{3n-4}\}$ that contains all the $\nabla\Lambda_g(\bar{K}), g \in I$, and defines a simple coordinate system at $\bar{K}$ (Appendix B). The set $\mathcal{C}_c(\bar{K}) \cap \Gamma(\mathcal{U})$ is then trivially contained in a <u>single</u> set $\Gamma^+(\mathcal{U},e)$ of the form (4.11). This implies that $\bar{K}$ is of type I and that only one boundary value term is required in the representation (4.16) of $T_c(K)$.[20]

Theorem 12 is applicable, in particular, to the case where $\bar{K}$ belongs to only one surface $\mathcal{L}_0^+[\bar{\mathcal{D}}]$:

<u>Corollary</u>. If only one surface $\mathcal{L}_0^+[\mathcal{D}]$ passes through $\bar{K}\epsilon(\mathcal{L}^+-\mathcal{M}_0)$, the point $\bar{K}$ is of type I. Moreover, only one boundary value term is needed in the representation (4.16) of $T_c(K)$.

In the situation described in the corollary only one boundary value term $T_c^1(K)$ is needed in Theorem 4. By taking the neighborhood $\mathcal{N}$ of Theorem 4 small enough, the region of holomorphy of $T_c^1(K)$ can be expanded to include any given point in $\Delta_c(\bar{K})\cap\mathcal{H}_c$ in the upper half-plane of $\sigma(K;\bar{K}) = \nabla\Lambda(\bar{K})\cdot K$. The argument is similar to that in Theorem 1A and will not be repeated.

The corollary includes, of course, the special case in which $\mathcal{L}^+$ consists near $\bar{K}$ of a single restricted surface $\mathcal{L}_0^+[\bar{\mathcal{D}}]$.[21] It also includes more complicated cases. For example, a point $\bar{K}\epsilon(\mathcal{L}^+-\mathcal{M}_0)$ that lies on the edge of the surface $\mathcal{L}_0^+[\mathcal{D}_\tau]$ of the triangle diagram $\mathcal{D}_\tau$ does not lie on $\mathcal{L}_0^+[\mathcal{D}_\tau]$. It lies on the surface $\mathcal{L}_0^+[\mathcal{D}]$ of a contraction $\mathcal{D}$ of $\mathcal{D}_\tau$. If these two surfaces are the only parts of $\mathcal{L}^+$ that penetrate some neighborhood of $\bar{K}$, then the corollary applies.

The hypothesis of the corollary is satisfied at almost all points $\bar{K}$ of $\mathcal{L}^+ - \mathcal{M}_0$. This is a consequence of the fact that only a finite number of distinct surfaces $\mathcal{L}_0^+[\bar{\mathcal{D}}]$ intersect any bounded neighborhood $\mathcal{N} \subset \mathcal{H}$ of $\bar{K}$.[19] The union of their intersections is therefore of zero measure in $\mathcal{N} \cap \mathcal{L}^+$, and the complement of that union contains almost all points of $\mathcal{N} \cap \mathcal{L}^+$. That is, in any bounded open set $\mathcal{N}$ of $\mathcal{H}$, the set of points $\bar{K} \varepsilon (\mathcal{L}^+ - \mathcal{M}_0)$ which lie on only one surface $\mathcal{L}_0^+$ contains almost all points of $\mathcal{N} \cap (\mathcal{L}^+ - \mathcal{M}_0)$.

A second consequence of Theorems 9, 10 and 11 is that if all the surfaces $\mathcal{L}_0^+[\mathcal{D}]$ that pass through $\bar{K}$ come from diagrams $\mathcal{D}$ that are contractions of the same fixed diagram $\bar{\mathcal{D}}$, then $\bar{K}$ is of type I:[22]

<u>Theorem 13</u>. A point $\bar{K} \in (\mathcal{L}^+ - \mathcal{M}_0)$ is of type I if there is a nontrivial connected causal space-time diagram $\bar{\mathcal{D}}$ such that the diagrams $\mathcal{D}_g$ of Theorem 9 are all contractions of $\bar{\mathcal{D}}$. In such a circumstance only one boundary value term is needed in the representation of Theorem 4 of $T_c(K)$ at $\bar{K}$.

The proof is given in Appendix E.

It is not known if all points of $\mathcal{L}^+ - \mathcal{M}_0$ are of type I. Any counterexample would have to lie on at least four different surfaces $\mathcal{L}_0^+[\mathcal{D}_g]$. Two of these $\mathcal{D}_g$ would have to be contractions of some diagram $\mathcal{D}_1$ and two would have to be contractions of some other diagram $\mathcal{D}_2$. But all four $\mathcal{D}_g$ could not be contractions of any single diagram. We have not succeeded in finding such a case. In any event such points would be rare, and in a sense accidental, because their existence requires the intersection of surfaces $\mathcal{L}_0^+[\mathcal{D}_g]$ corresponding to contractions of one diagram $\mathcal{D}_1$ to intersect the intersection of surfaces $\mathcal{L}_0^+[\mathcal{D}_g]$ corresponding to contractions of another "unrelated" diagram $\mathcal{D}_2$. Two unrelated diagrams are diagrams that are not both contractions of any single diagram. It seems probable that singularities associated with unrelated diagrams will be additive and hence independent. A proof should emerge from the study of discontinuity formulas. That, however, is a subject in itself.

VI. AN EXAMPLE

As a concrete example of the analysis of Section V, we consider the Landau surface $\mathcal{L}^+[\mathcal{D}]$ for the butterfly diagram of Figure 6. Since at any point $\bar{K}\varepsilon\mathcal{L}^+[\mathcal{D}]$ it is possible to contract either one of the triangles to a point, the surface $\mathcal{L}^+[\mathcal{D}]$ is nothing but the intersection of the Landau surfaces $\mathcal{L}^+[\mathcal{D}_1]$ and $\mathcal{L}^+[\mathcal{D}_2]$ for the two triangle diagrams $\mathcal{D}_1$ and $\mathcal{D}_2$ that make up $\mathcal{D}$. (See Figure 7.) If $\mathcal{L}^+[\mathcal{D}]$ is the only part of $\mathcal{L}^+$ to penetrate a neighborhood of $\bar{K}\varepsilon(\mathcal{L}^+-\mathcal{M}_0)$, this example forms a nontrivial example of the situation described by Theorems 12 and 13.

Consider first the surface $\mathcal{L}^+[\mathcal{D}_1]$ corresponding to the triangle diagram $\mathcal{D}_1$. The important variables are the two subenergies, $\sigma_1 = (k_1+k_2)^2$ and $\sigma_3 = (k_3+k_4)^2$, and the momentum transfer $\sigma_2 = (k_5+k_6+k_7+k_8)^2$. In terms of the variables

$$x_i = \frac{\sigma_i-\mu_j^2-\mu_k^2}{2\mu_j\mu_k} \quad , \tag{6.1}$$

where the μ_j are the masses of the internal particles, and (ijk) is a permutation of (123), the Landau surface $\mathcal{L}\ [\mathcal{D}_1]$ is given by zeros of the real analytic function[24]

$$\Lambda_1(K) = 1-x_1^2-x_2^2-x_3^2-2x_1x_2x_3 \ . \tag{6.2}$$

The physical region for the reaction corresponding to $\mathcal{D}_1$ is contained in a volume defined by $x_1 \geq 1$, $x_2 \leq -1$, and $x_3 \geq 1$. The two surfaces $x_1 = 1$ and $x_3 = 1$ are the Landau surfaces associated with diagrams that are contractions of $\mathcal{D}_1$.

If x_2 is held fixed at some value $\bar{x}_2 < -1$, the surface $\mathcal{L}\ [\mathcal{D}_1]$ becomes the familiar curve in the $x_1 x_3$ plane shown in Figure 8. The gradient $\nabla\Lambda_1$ is

$$\nabla\Lambda_1(\bar{K}) = (u,u,v,v,w,w,w,w), \tag{6.3a}$$

where

$$u = -2(\mu_2\mu_3)^{-1}(x_1 + x_2x_3)(\bar{k}_1 + \bar{k}_2) , \tag{6.3b}$$

$$v = -2(\mu_1\mu_2)^{-1}(x_3 + x_1x_2)(\bar{k}_3 + \bar{k}_4) , \tag{6.3c}$$

$$w = -2(\mu_1\mu_3)^{-1}(x_2 + x_1x_3)(\bar{k}_5 + \bar{k}_6 + \bar{k}_7 + \bar{k}_8) . \tag{6.3d}$$

Inspection of (6.3) shows that $\nabla\Lambda_1(\bar{K})$ and the vectors F_i, $1 \leq i \leq n+4$, defined in (5.8) are linearly independent at points $\bar{K}$ which do not lie in $\mathcal{M}_0$. The surface $\mathcal{L}\ [\mathcal{D}_1]$ is therefore a submanifold of $\mathcal{H}$ of codimension 1 at $\bar{K} \in \mathcal{W} - \mathcal{M}_0$

According to Theorem 6, the vector $\nabla\Lambda_1(\bar{K})$ generates the diagram $\mathcal{D}_1$ at $\bar{K}$. For this to be true it is first necessary that the four-vectors $\Delta_3 = w-u$ and $\Delta_1 = v-w$

be positive timelike. [The other vector $\Delta_2 = v-u$ is then automatically positive timelike.] For this to be true the quantities $\Delta_1{}^2$, $\Delta_3{}^2$, $-\Delta_3\cdot(\bar{k}_1+\bar{k}_2)$, and $\Delta_1\cdot(\bar{k}_3+\bar{k}_4)$ must all be positive. Some algebra yields the following equations:

$$\Delta_1{}^2 = [x_2(x_3{}^2-1)^{1/2} \pm x_3(x_2{}^2-1)^{1/2}]^2\,[F_\pm(x_2,x_3)]^2, \tag{6.4a}$$

$$\Delta_3{}^2 = [x_3{}^2-1]\,[F_\pm(x_2,x_3)]^2, \tag{6.4b}$$

$$-\Delta_3\cdot(\bar{k}_1+\bar{k}_2) = (\mu_3+x_1\mu_2)(x_3{}^2-1)^{1/2}F_\pm(x_2,x_3), \tag{6.4c}$$

$$\Delta_1\cdot(\bar{k}_3+\bar{k}_4) = (\mu_1+x_3\mu_2)[x_2(x_3{}^2-1)^{1/2}\pm x_3(x_2{}^2-1)^{1/2}]\,F_\pm(x_2,x_3), \tag{6.4d}$$

where

$$F_\pm(x_2,x_3) = (x_3{}^2-1)^{1/2}(\mu_3{}^{-1}+x_2\mu_1{}^{-1}) \pm (x_2{}^2-1)^{1/2}(\mu_2{}^{-1}+x_3\mu_1{}^{-1}). \tag{6.4e}$$

The upper (+) sign refers to the (+) branch of the curve of Figure 8, and the (−) sign refers to the (−) branch of the curve. It is clear from (6.4a) and (6.4b) that both Δ_1 and Δ_3 are timelike unless $F_\pm$ vanishes. For both to be positive timelike the expressions in (6.4c) and (6.4d) must be positive. A necessary condition for them even to have the same sign is that $x_2(x_3{}^2-1)^{1/2} \pm x_3(x_2{}^2-1)^{1/2}$ be positive. This can only happen on the segment AB of the curve of Figure 8.

It is then a simple matter to show that F_+ is positive on AB. F_- need not be considered since AB is on the (+) part of the curve. Thus, Δ_3 and Δ_1 are positive time-like on AB.[25] It can also be shown that the momenta p_i of the internal lines of $\mathcal{D}_1$ are related to the Δ_i through the equations $\Delta_i = \alpha_i p_i$, where $\alpha_i = \mu_i^{-1}(\Delta_i^2)$. The lengthy algebra needed to show this is straightforward, but not instructive, and is omitted. Thus, the displacement $\nabla\Lambda_1$ generates a diagram $\mathcal{D}_1'$ which is similar to $\mathcal{D}_1$.

The displacement $\nabla\Lambda_1$ generates the trivial diagram when F_- vanishes. There is at most one such point for fixed x_2, and it corresponds to the second type Landau singularity[26] given by $\sigma_1^{1/2} = \sigma_3^{1/2} \pm \sigma_2^{1/2}$. Because they do not lie on $\mathcal{L}^+$, the function $T_c(K)$ is infinitely differentiable at such points.

Inspection of (6.3) also shows that as K approaches endpoints of the segment AB, the displacement $\bar{\nabla}\Lambda_1(K)$ changes continuously into the gradient of the Landau surface of the appropriate contracted diagram. This continuous behavior is implicit in Theorem 10. It means that the iε-prescriptions for the leading surface $\mathcal{L}_0^+[\mathcal{D}_1]$ and the surfaces $\mathcal{L}_0^+[\mathcal{D}_1']$, $\mathcal{D}_1' \subset \mathcal{D}_1$, are compatible (the corollary of Theorem 12).

A similar analysis can be applied to the diagram $\mathcal{D}_2$ (Figure 7). The function Λ_2 is given by

$$\Lambda_2(K) = 1 - y_1^2 - y_2^2 - y_3^2 - 2y_1y_2y_3 \ , \tag{6.5a}$$

where

$$y_1 = \frac{(k_5 + k_6)^2 - \mu_5^2 - \mu_6^2}{2\mu_5\mu_6} \ , \tag{6.5b}$$

$$y_2 = \frac{(k_5 + k_6 + k_7 + k_8)^2 - \mu_4^2 - \mu_6^2}{2\mu_4\mu_6} \ , \tag{6.5c}$$

$$y_3 = \frac{(k_7 + k_8)^2 - \mu_4^2 - \mu_5^2}{2\mu_4\mu_5} \ . \tag{6.5d}$$

The gradient $\nabla\Lambda_2(\bar{K})$ is given by

$$\nabla\Lambda_2(\bar{K}) = (0,0,0,0,w_1,w_1,w_2,w_2), \tag{6.6a}$$

where

$$w_1 = -\frac{(y_1 + y_2y_3)}{\mu_5\mu_6}(\bar{k}_5 + \bar{k}_6) - \frac{(y_2 + y_1y_3)}{\mu_4\mu_6}(\bar{k}_5 + \bar{k}_6 + \bar{k}_7 + \bar{k}_8), \tag{6.6b}$$

$$w_2 = -\frac{(y_3 + y_1y_2)}{\mu_4\mu_5}(\bar{k}_7 + \bar{k}_8) - \frac{(y_2 + y_1y_3)}{\mu_4\mu_6}(\bar{k}_5 + \bar{k}_6 + \bar{k}_7 + \bar{k}_8). \tag{6.6c}$$

Note that if $\nabla\Lambda_1$ and $\nabla\Lambda_2$ generate $\mathcal{D}_1$ and $\mathcal{D}_2$, then displacements of the form $\lambda_1\nabla\Lambda_1 + \lambda_2\nabla\Lambda_2$, with λ_1 and λ_2 positive, generate diagrams in $[\mathcal{D}]$, where $\mathcal{D}$ is the butterfly diagram with contractions $\mathcal{D}_1$ and $\mathcal{D}_2$. This is the result given in Theorem 8.

At any point $\bar{K}\varepsilon(\mathcal{L}_0{}^+[\mathcal{D}_1]\cap\mathcal{L}_0{}^+[\mathcal{D}_2])$, the vectors $\nabla\Lambda_1(\bar{K})$ and $\nabla\Lambda_2(\bar{K})$ and the vectors F_i, $1\leq i\leq n+4$, defined in (5.8) are linearly independent. The surface $\mathcal{L}^+[\mathcal{D}] = \mathcal{L}_0{}^+[\mathcal{D}_1]\cap\mathcal{L}_0{}^+[\mathcal{D}_2]$ is therefore a submanifold of $\mathcal{H}-\mathcal{H}_0$ of codimension 2. The hypotheses of Theorems 12 and 13 are satisfied, so the point $\bar{K}$ is of type I.

VII. SUMMARY

In this work the relationship between continuity properties of scattering functions in the physical region and macroscopic space-time phenomena has been examined. It was shown how singularities on Landau surfaces can be regarded as caused by processes in which the transfer of energy-momentum is carried by physical particles.

The algebraic equivalence of the Landau equations and corresponding space-time diagrams was emphasized earlier by Norton and Coleman.[16] The present work extends the algebraic result of Norton and Coleman by showing (in the course of proving Theorem 1) that if the scattering functions are infinitely differentiable except on the Landau surfaces, then either the space-time collision regions must be ordered so that the momentum-energy can be carried from the initial particles to the final particles by means of physical particles, or the transition amplitude drops off faster than any inverse power of a scale parameter. We also obtain the more difficult converse: if transition amplitudes fall off faster than any inverse power of the scaling parameter when the space-time collision regions are not causally connected via physical particles, then the scattering functions must be infinitely differentiable except on the Landau surfaces. Moreover, apart from infinitely differentiable singularities, the

$i\epsilon$-prescriptions associated with the Landau surfaces coincide with those of perturbation theory.

APPENDIX A

The proof that the weak asymptotic causality condition is valid in nonrelativistic quantum mechanics is based on an inequality of Brenig and Haag.[27] Let $\emptyset_t^{\,o}$ be the state at time t that would develop from an asymptotic initial particle state $\emptyset$ if there were no interactions between the particles, and let $\emptyset_t$ be the corresponding state if there are interactions. Similarly, let $\psi_t^{\,o}$ be the state at time t that would develop into the asymptotic final particle state ψ if there were no interactions, and let ψ_t be the corresponding state if there are interactions. Then the transition amplitude $\langle\psi|T|\emptyset\rangle$ can be written

$$\langle\psi|T|\emptyset\rangle = \langle\psi_t|\emptyset_t\rangle - \langle\psi_t^{\,o}|\emptyset_t^{\,o}\rangle \, , \qquad \text{(A.1)}$$

where t is any arbitrary time. From (A.1) follows the inequality

$$|\langle\psi|T|\emptyset\rangle| < \|\psi_t - \psi_t^{\,o}\| \, \|\emptyset_t - \emptyset_t^{\,o}\| + \|\psi_t - \psi_t^{\,o}\| + \|\emptyset_t - \emptyset_t^{\,o}\| \, . \qquad \text{(A.2)}$$

The norm $\|\cdot\|$ in (A.2) is defined for all functions $f(\underset{\sim}{x}_1,\ldots,\underset{\sim}{x}_m,t)$ by

$$\|f_t\| = \langle f_t|f_t\rangle^{1/2} \equiv \left\{ \int d\underset{\sim}{x}_1 \ldots d\underset{\sim}{x}_m \, |f(\underset{\sim}{x}_1,\ldots,\underset{\sim}{x}_m,t)|^2 \right\}^{1/2} , \qquad \text{(A.3)}$$

and it is assumed that ϕ_t° and ψ_t° have unit norms. The quantities $\|\psi_t - \psi_t^{\circ}\|$ and $\|\phi_t - \phi_t^{\circ}\|$ are bounded by the inequalities[27]

$$\|\phi_t - \phi_t^{\circ}\| \leqslant \int_{-\infty}^{t} dt' \, \|V \, \phi_{t'}^{\circ}\| \tag{A.4a}$$

and

$$\|\psi_t - \psi_t^{\circ}\| \leqslant \int_{t}^{\infty} dt' \, \|V \, \psi_{t'}^{\circ}\| \, . \tag{A.4b}$$

where V is the interaction Hamiltonian.

Let the asymptotic initial and final particles now be displaced by amounts $u_j\tau$, and let the displaced initial and final particles be represented by $\phi^{U\tau}$ and $\psi^{U\tau}$. For these displaced particles the inequality (A.2) leads to the following inequality:

$$|T[\phi\psi;U\tau]| \leqslant F[\phi;U,\tau] \; G[\psi;U,\tau] + F[\phi;U,\tau] + G[\psi;U,\tau] , \tag{A.5}$$

where

$$F[\phi;U,\tau] = \int_{-\infty}^{(\hat{t}+\varepsilon)\tau} dt' \, \|V \, \phi_{t'}^{U\tau,\circ}\| \, , \tag{A.6a}$$

and

$$G[\psi;U,\tau] = \int_{(\hat{t}-\varepsilon)\tau}^{\infty} dt' \, \|V\, \psi_{t'}{}^{U\tau,o}\|. \qquad \text{(A.6b)}$$

Here $\hat{t}$ is any arbitrary time, the number ε is positive, and the scale parameter τ is greater than 1.

If the potential V has a finite range R, the integrals which define $\|V\, \emptyset_t{}^{U\tau,o}\|$ and $\|V\, \psi_t{}^{U\tau,o}\|$ are restricted to the domain

$$A(R) = \{(\underset{\sim}{x}_1,\ldots,\underset{\sim}{x}_m)\,|\,|\underset{\sim}{x}_i - \underset{\sim}{x}_j| \leqslant R \text{ for all } i \text{ and } j\}. \qquad \text{(A.7)}$$

[Here it is assumed that there are m particles in all stages of the reaction; no creation or annihilation of particles is allowed.] Thus the quantity $\|V\, \emptyset_t{}^{U\tau,o}\|$ has the form

$$\|V\, \emptyset_{\hat{t}'\tau}{}^{U\tau,o}\| = \left\{\tau^{3m} \int_{A(R\tau^{-1})} d\hat{\underset{\sim}{x}}_1 \ldots d\hat{\underset{\sim}{x}}_m \, |V(\hat{\underset{\sim}{x}}_1\tau,\ldots,\hat{\underset{\sim}{x}}_m\tau) \prod_{\text{initial}} \tilde{\psi}_i([\hat{\underset{\sim}{x}}_i - \underset{\sim}{u}_i]\tau, [\hat{t}' - u_{io}]\tau)|^2\right\}^{1/2}. \qquad \text{(A.8)}$$

Now the wave functions $\tilde{\psi}_i([\hat{x} - u_i]\tau)$, considered as function of $\hat{x}$, collapse uniformly into the cones $\hat{V}_\varepsilon(\psi_i;u_i)$ as τ becomes infinite. Consequently, if U belongs to $\mathcal{A}(\hat{t},\varepsilon,\emptyset\psi)$ so that the initial-particle cones $\hat{V}_\varepsilon(\psi_i;u_i)$ are well separated before $(\hat{t} + \varepsilon)$, then for some sufficiently large $\hat{\tau}$ the product wave function in (A.8) is of rapid decrease in τ (and $|\hat{t}'|$) uniformly in $(\hat{\underset{\sim}{x}}'_1,\ldots,\hat{\underset{\sim}{x}}_m,\hat{t}')$ for $(\hat{\underset{\sim}{x}}_1,\ldots,\hat{\underset{\sim}{x}}_m)$ in $A(R\hat{\tau}^{-1})$ and $\hat{t}' < (\hat{t}+\varepsilon)$. Thus if V is bounded (or even merely integrable), the function $F[\emptyset;U,\tau]$ [and by similar arguments $G[\psi;U,\tau]$] is of rapid decrease when U belongs to $\mathcal{A}(\hat{t},\varepsilon,\emptyset\psi)$. Then the inequality (A.5) implies that the weak asymptotic causality condition is satisfied for any given U in $\mathcal{A}(\hat{t},\varepsilon,\emptyset\psi)$.

To extend the analysis to compact sets Γ of $\mathcal{A}(\hat{t},\varepsilon,\emptyset\psi)$ it is only necessary to observe that the velocity cones $\hat{V}_\varepsilon(\psi_i,u_i)$ never come closer in the appropriate regions $\hat{x}_0 \leqslant (\hat{t}+\varepsilon)$ and $\hat{x}_0 \geqslant (\hat{t}-\varepsilon)$ than some distance $\delta(\Gamma)$. The number $\hat{\tau}$ is chosen so that $R \ll \delta(\Gamma)\hat{\tau}$, and the analysis proceeds as before. This insures that the WAC condition is satisfied uniformly on compact subsets of $\mathcal{A}(\hat{t},\varepsilon,\emptyset\psi)$. These arguments can be extended also to the case of potentials that have decreasing exponential bounds at large r.

The same ideas can be formulated in a classical theory by considering a statistical ensemble of classical experiments in which the momentum-space probability functions $P_i(\underset{\sim}{k})$ of the initial and final particles have small compact support, and in which the spatial distributions $P_i(\underset{\sim}{x},t)$ at time $t = 0$ fall off faster than any power of $|\underset{\sim}{x}|^{-1}$.

Let $V_i(x)$ be the velocity cone that corresponds to the support of $P_i(\underset{\sim}{k})$ and that has its tip at $x = (x_0,\underset{\sim}{x})$:

$$V_i(x) = \{x' \mid x' - x = \lambda(\omega_i(\mathbf{k}),\mathbf{k}) \ \sout{\text{for some}}, \ \lambda \varepsilon \mathbf{R} \\ \sout{\text{and}}, \ \mathbf{k} \varepsilon \text{ supp } P_i\}. \tag{A.9}$$

Here $\omega_i(\mathbf{k})$ is $(\mathbf{k}^2 + m_i^{\,2})^{1/2}$. Furthermore, let

$$V_i(x;r) = \bigcup_{|\mathbf{x}' - \mathbf{x}| \leqslant r} V_i(x_0,\mathbf{x}'). \tag{A.10}$$

Now, if the trajectory of the ith freely moving particle passes through a point $x' = (x_0,\mathbf{x}')$ for which $|\mathbf{x}' - \mathbf{x}| < r$ then the trajectory must lie entirely in $V_i(x;r)$. This means that the fraction of the trajectories in the statistical ensemble for which particle i remains always inside $V_i(x;r)$ is just

$$\overline{P}_i(x;r) = \int_{|\mathbf{x}' - \mathbf{x}| \leqslant r} dx' \ P_i(\mathbf{x}',x_0). \tag{A.11}$$

The rapid fall off of $P_i(\mathbf{x},0)$ for large $|\mathbf{x}|$ implies that

$$D_i(r\tau) \equiv 1 - \overline{P}_i(0;r\tau) \tag{A.12}$$

goes rapidly to zero as τ becomes infinite: $D_i(r\tau) \Rightarrow 0$.

The stipulation in the weak asymptotic causality condition is (essentially) that the displaced velocity cones of the initial particles do not intersect for $t \leqslant \epsilon\tau$ and that the displaced velocity cones of the final particles do not intersect for $t \geqslant -\epsilon\tau$. The condition that the displaced cones do not intersect in these regions means that when $\tau = 1$ the minimum (Euclidian) distance between the cones in the regions $\hat{t} \leqslant \epsilon$ and $\hat{t} \geqslant -\epsilon$ is nonzero. If D_0 is this minimum distance, the minimum distance when τ is arbitrary is $D_0\tau$, which becomes infinite as τ becomes infinite.

Since the displaced cones $V_i(u_i\tau)$ have a minimum spatial separation $D_0\tau$ in the appropriate regions $\pm t \leqslant \epsilon\tau$, they can be replaced by slightly larger regions $V_i(u_i\tau;r\tau)$ that have a minimum spatial separation $d_0\tau > 0$.

Let the initial and final particles of the classical treatment be subjected to the displacements $u_i\tau$. The corresponding displaced spatial distributions $P_i^{U\tau}(\underset{\sim}{x},t)$ are given by

$$P_i^{U\tau}(\underset{\sim}{x},t) = P_i(\underset{\sim}{x} - \underset{\sim}{u}_i\tau,\ t - u_{i0}\tau). \qquad (A.13)$$

Thus, the probability that the freely moving particle i remains always inside $V_i(u_i\tau;r\tau)$ is

$$\bar{P}_i^{U\tau}(u\tau\ ;r\tau) = \int_{|\underset{\sim}{x}'-\underset{\sim}{u}_i\tau|<r\tau} dx'\ P_i^{U\tau}(\underset{\sim}{x}',u_{i0}\tau) = \bar{P}_i(0;r\tau). \qquad (A.14)$$

The probability that every particle i remains inside its displaced region $V_i(u\tau,r\tau)$ is $\prod_i \overline{P}_i(0;r\tau)$. This number rapidly approaches unity as τ becomes infinite.

Let us suppose that the interaction between the particles has a finite range R, in the sense that a set of particles do not interact unless the distance between some pair of them becomes less than R. But for sufficiently large τ the distance $d_0\tau$ of closest approach of the regions $V_i(u_i\tau;r\tau)$ is greater than R. Thus for this value of τ there will be no interaction between initial particles in the region $t \leqslant \varepsilon\tau$ for those members of the ensemble for which each initial particle is in its region $V_i(u_i\tau,r\tau)$. The fraction of the members for which these conditions are realized (simultaneously for all particles) rapidly approaches unity. Consequently, the probability that the initial particles interact in $t \leqslant \varepsilon\tau$ rapidly approaches zero as τ becomes infinite. Similarly, the probability that the final particles interact in $t \geqslant -\varepsilon\tau$ rapidly approaches zero as τ becomes infinite.

The fact that the fraction of members of the ensemble that have reactions in $t < 0$ decreases rapidly as becomes infinite means that the difference between the classical joint probability function

$$P_{in}^{U\tau}(\underset{\sim}{X},\underset{\sim}{V},t) = P_{in}^{U\tau}(\underset{\sim}{x}_1,\ldots,\underset{\sim}{x}_m;\underset{\sim}{v}_1,\ldots,\underset{\sim}{v}_m;t) \qquad (A.15)$$

and its unperturbed value

$$P_{in}^{U\tau,o}(\underset{\sim}{X},\underset{\sim}{V},t) = \prod_{\text{initial}} P_i^{U\tau,o}(\underset{\sim}{x}_i,\underset{\sim}{v}_i;t) \tag{A.16}$$

must, when integrated, become small as τ becomes large:

$$\int d\underset{\sim}{X}\, d\underset{\sim}{V}\, |P_{in}^{U\tau}(\underset{\sim}{X},\underset{\sim}{V},0) - P_{in}^{U\tau,o}(\underset{\sim}{X},\underset{\sim}{V},0)| \rightarrow 0. \tag{A.17a}$$

Similarly, we must have

$$\int d\underset{\sim}{X}\, d\underset{\sim}{V}\, |P_{out}^{U\tau}(\underset{\sim}{X},\underset{\sim}{V},0) - P_{out}^{U\tau,o}(\underset{\sim}{X},\underset{\sim}{V},0)| \rightarrow 0. \tag{A.17b}$$

The classical expression for the overlap probability is

$$T = \int d\underset{\sim}{X}\, d\underset{\sim}{V}\, \text{Min}\, \{P_{in}^{U\tau}(\underset{\sim}{X},\underset{\sim}{V},0),\, P_{out}^{U\tau}(\underset{\sim}{X},\underset{\sim}{V},0)\}. \tag{A.18}$$

This gives the fraction of the members of the "in" ensemble that can occur as members of the "out" ensemble, or conversely. (If in a certain "bin" the in ensemble has n_1 members and the out ensemble has n_2 members, the minimum of n_1 and n_2 is the maximum number of members common to both ensembles.) It follows from (A.17) that T differs from its unperturbed value

$$T^o = \int d\underset{\sim}{X}\, d\underset{\sim}{V}\, \text{Min}\, \{P_{in}^{U\tau,o}(\underset{\sim}{X},\underset{\sim}{V},0),\, P_{out}^{U\tau,o}(\underset{\sim}{X},\underset{\sim}{V},0)\}, \tag{A.19}$$

by a term that goes rapidly to zero as τ becomes infinite.

Thus, for a fixed U in $\mathcal{A}(0,\varepsilon,\psi)$, the weak asymptotic causality condition is valid in a classical model with finite range interactions. The analysis is extended to compact sets Γ of $\mathcal{A}(0,\varepsilon,\psi)$ in the same way as in the quantum mechanical case.

APPENDIX B

By way of establishing notation, we give a constructive proof of the following well-known proposition: the restricted complex mass-shell $\mathcal{H}_c$ is a (3n-4)-dimensional analytic submanifold of $\underset{\sim}{C}^{4n}$.

Proof. Let the n-tuples $K = (k_1, \ldots, k_n)$ of complex momentum vectors be associated with points $z = (z_1, \ldots, z_{4n})$ of $\underset{\sim}{C}^{4n}$ through the equations

$$z_{4i+\mu-3} = k_{i\mu}, \quad (1 \leqslant i \leqslant n, \; 0 \leqslant \mu \leqslant 3). \tag{B.1}$$

Then the set $\mathcal{M}_c$ can be written as

$$\mathcal{M}_c = \{z \mid z \varepsilon \underset{\sim}{C}^{4n}, \; f_1(z) = \cdots = f_{n+4}(z) = 0\}, \tag{B.2}$$

where the functions $f_i(z)$ are defined by

$$f_i(z) = \tfrac{1}{2} \sum_{\mu=0}^{3} g^{\mu\mu} (z_{4i+\mu-3})^2 - \tfrac{1}{2} m_i^{\,2}, \quad (1 \leqslant i \leqslant n), \tag{B.3a}$$

and by

$$f_i(z) = g^{i-n-1, i-n-1} \sum_{j=1}^{n} z_{4j+i-n-4}, \quad (n+1 \leqslant i \leqslant n+4). \tag{B.3b}$$

(The metric is $g^{oo} = -g^{11} = -g^{22} = -g^{33} = 1$.)

Consider the Jacobian matrix

$$J_{ij}(z) = \frac{\partial f_i}{\partial z_j}(z). \tag{B.4}$$

Explicit computation shows that the set of points of $\mathcal{M}_c$ where rank J is less than n+4 is exactly $\mathcal{M}_c-\mathcal{K}_c$. Therefore, at every point $\bar{K}$ (or $\bar{z}$) of $\mathcal{K}_c$ a (nonsingular) set of coordinates for $\underset{\sim}{C}^{4n}$ can be defined by

$$F_i(z) = f_i(z), \qquad (1\leq i\leq n+4), \tag{B.5a}$$

and

$$F_i(z) = \sum_{j=1}^{4n} E_{ij} z_j, \quad (n+5\leq i\leq 4n). \tag{B.5b}$$

The (3n-4) fixed real vectors

$$E_i = (E_{i1},\ldots,E_{i4n}) \tag{B.6}$$

appearing in (B.5b) are any vectors which, together with the n+4 vectors

$$E_i(\bar{z}) = \left(\frac{\partial f_i}{\partial z_i}(\bar{z}),\ldots,\frac{\partial f_i}{\partial z_{4n}}(\bar{z})\right), \quad (1\leq i\leq n+4), \tag{B.7}$$

form a set of linearly independent vectors. The functions F_i define a coordinate system in a sufficiently small neighborhood $\Delta_c(\bar{K}) \subset \underset{\sim}{C}^{4n}$ of $\bar{K} = K(\bar{z})$. It follows[28] from (B.2) that the set $\Delta_c(\bar{K}) \cap \mathcal{H}_c$ is a (3n-4)-dimensional analytic submanifold of $\underset{\sim}{C}^{4n}$. Since this construction can be made for any point $\bar{K} \varepsilon \mathcal{H}_c$, the proposition is proved.

Remark. The mapping $F: \underset{\sim}{C}^{4n} \to \underset{\sim}{C}^{4n}$ defined by (B.5), and hence also its inverse, is real analytic. It follows that the mapping $\Pi_{\bar{K}}$ introduced in Section IIB is also real analytic.

Remark. The vectors E_k for $n+5 \leqslant k \leqslant 4n$ can be associated with n-particle displacement vectors $U_\lambda = (u_{\lambda 1}, \ldots, u_{\lambda n})$, $1 \leqslant \lambda \leqslant 3n-4$, in the following way:

$$u_{\lambda j}{}^{\mu} = E_{\lambda+n+4,4j+\mu-3}, \quad (1 \leqslant j \leqslant n,\ 0 \leqslant \mu \leqslant 3). \qquad \text{(B.8)}$$

The local coordinates (B.5b) of $\mathcal{H}_c$ then become

$$\bar{F}_\lambda = U_\lambda \cdot K = \sum_{j,\mu} u_{\lambda j}{}^{\mu}\, k_{j\mu}, \quad (1 \leqslant \lambda \leqslant 3n-4), \qquad \text{(B.9)}$$

where the bar indicates the relabeling of indices. Thus, the local coordinate system constructed in the proof of the proposition is a "simple" coordinate system. [See Equation (2.13).]

Remark. For any point $\bar{K} \varepsilon \mathcal{H}$ the set of 3n-4 linearly independent vectors U_λ defined above provides a unique decomposition of any displacement vector U into the sum

$$U = \sum_{\lambda=1}^{3n-4} t_\lambda U_\lambda + U_0(\bar{K}), \qquad \text{(B.10a)}$$

where

$$U_{0j}{}^{\mu}(\bar{K}) = \bar{t}_j \bar{k}_j{}^{\mu} + a^{\mu}, \qquad (1 \leq j \leq n), \qquad \text{(B.10b)}$$

is a causal displacement for any $\emptyset$ such that supp $\emptyset$ contains $\bar{K}$. The displacement $U_0(\bar{K})$ displaces each particle of the set specified by $\bar{K}$ along its own trajectory, and gives a single overall displacement to all particles. Thus $U_0(\bar{K})$ is a member of the causal set $\mathcal{C}_0(\bar{K})$ defined below Equation (4.8).

The z_j defined above are simply components of the vectors k_i. In the rest of the paper the z's denote the (3n-4) variables of a real local coordinate system.

Notice that $U_0(\bar{K})$ belongs to the null space of the matrix $\partial K/\partial z$. That is,

$$U_0[K(\bar{z})] \cdot \frac{\partial K(\bar{z})}{\partial z_\lambda} \equiv \sum_{j\mu} U_{0j}{}^{\mu} \frac{\partial \bar{k}_{j\mu}}{\partial z_\lambda} = 0, \quad (1 \leq \lambda \leq 3n-4). \qquad \text{(B.11)}$$

This follows from the restrictions on K imposed by (B.2). Moreover, at any point of $\mathcal{H}$ all vectors in the null space of $\partial K(\bar{z})/\partial z$ are of the form $U_0[K(\bar{z})]$, since this null space has dimension n+4.

APPENDIX C

A. Proof of Theorem 1.

Let $\mathcal{A}(\hat{t},\varepsilon,\phi)$ be the set described in assumption (c). This set contains $-\nabla\Lambda(\bar{K})$, and is thus nonempty. It is in fact open in the topology induced by the Euclidean norm

$$\|U - U'\| = \{\sum_{i,\nu} |u_{i\nu} - u'_{i\nu}|^2\}^{\frac{1}{2}}. \tag{C.1}$$

To see this, define for any neighborhood N of any displacement $\bar{U}$ in $\mathcal{A}(\hat{t},\varepsilon,\phi)$ the set

$$\hat{V}_\varepsilon(\phi_i,N) = \bigcup_{U\varepsilon N} \hat{V}_\varepsilon(\phi_i;u_i). \tag{C.2}$$

Every two initial-particle cones $\hat{V}_\varepsilon(\phi_i;\bar{u}_i)$ and $\hat{V}_\varepsilon(\phi_j;\bar{u}_j)$ are separated by some finite (Euclidian) distance d_0 in $\hat{D}^-(\hat{t},\varepsilon)$. Therefore the sets $\hat{V}_\varepsilon(\phi_i,N)$ and $\hat{V}_\varepsilon(\phi_j,N)$ are separated in $\hat{D}^-(t,\varepsilon)$ by a distance $d'_0 \geqslant (d_0-2\Delta)$, where Δ is the diameter of N. If Δ is chosen small enough, then the distance $d_0-2\Delta$ is positive, and the sets $\hat{V}_\varepsilon(\phi_i,N)$ and $\hat{V}_\varepsilon(\phi_j,N)$ are disjoint in $\hat{D}^-(\hat{t},\varepsilon)$. Similar arguments hold for each pair of initial particles and each pair of final particles. Thus every U in some neighborhood of $\bar{U}$

belongs to $\mathcal{A}(\hat{t},\varepsilon,\emptyset)$. Since $\bar{U}$ is an arbitrary point of $\mathcal{A}(\hat{t},\varepsilon,\emptyset)$, this set is open.

According to hypothesis, the displacement $V = -\nabla\Lambda(\bar{K})$ belongs to $\mathcal{A}(\hat{t},\varepsilon,\emptyset)$. Since $\mathcal{A}(\hat{t},\varepsilon,\emptyset)$ is open there exists a neighborhood N of V with compact closure $\bar{N}$ contained in $\mathcal{A}(\hat{t},\varepsilon,\emptyset)$. The WAC condition then requires that $T[\emptyset;U\tau] \rightarrow 0$ uniformly on $\bar{N}$. The symbols N and $\bar{N}$ hereafter designate these two sets.

If the relation

$$\text{supp } \psi \subset \text{supp } \emptyset \tag{C.3}$$

is true, the relation

$$\mathcal{A}(\hat{t},\varepsilon,\emptyset) \subset \mathcal{A}(\hat{t},\varepsilon,\psi) \tag{C.4}$$

is also true. Thus it follows from WAC that the rapid decrease $T[\psi;U\tau] \rightarrow 0$ is obtained uniformly on $\bar{N}$ for any fixed product wave function ψ in $\mathcal{B}(\mathcal{H})$ with support satisfying (C.3).

Let $\mathcal{U} = \{U_1,\ldots,U_{3n-4}\}$ be any set of n-particle displacements that define a simple coordinate system $(\Delta_c(\bar{K}),\Pi_{\bar{K}},D_c(\bar{K}))$ with local coordinates $F_\lambda = U_\lambda \cdot K$, and let

$$\Gamma(\mathcal{U}) = \{U | U = \Sigma t_\lambda U_\lambda,\ t = (t_1,\ldots,t_{3n-4}) \in \Omega\}, \tag{C.5}$$

where Ω is the unit sphere in $\underset{\sim}{R}^{3n-4}$. Then, the set $\Gamma(\mathcal{U})$ is contained in the union of the finite number of open sets $\Gamma_\rho^{\pm}(\mathcal{U})$ constructed as follows. It is shown in Appendix B that the displacement $\nabla\Lambda(\bar{K})$ can be written as

$$\nabla\Lambda(\bar{K}) = \nabla_1\Lambda(\bar{K}) + U_0(\bar{K}) \;, \tag{C.6}$$

where $U_0(\bar{K})$ is a causal displacement belonging to $\mathcal{C}_o(\bar{K})$, and

$$\nabla_1\Lambda(\bar{K}) = \Sigma\gamma_\lambda U_\lambda . \tag{C.7}$$

The vector $\gamma = (\gamma_1,\ldots,\gamma_{3n-4})$ must be nonzero, since otherwise $-\nabla\Lambda(\bar{K})$ would not belong to $\mathcal{A}(\hat{t},\varepsilon,\emptyset)$. Let the normalization of $\Lambda(\bar{K})$ be such that $\gamma \equiv e_1$ is a vector of Ω, and let $e_2,\ldots,e_{3n-4}$ be any 3n-4 other vectors in Ω which, together with e_1, form an orthonormal basis for $\underset{\sim}{R}^{3n-4}$. For any α, $0<\alpha<1$, a finite open covering of Ω is given by the sets

$$\Omega_1^{\pm} = \{t \,|\, t \in \Omega, \pm (t,e_1) > (1 - \alpha^2)^{1/2}\}, \tag{C.8a}$$

and

$$\Omega_\rho^{\pm} = \{t \,|\, t \in \Omega, \pm (t,e_\rho) > \beta(r-1)^{-1/2}\}, \; (2 \leqslant \rho \leqslant 3n-4 = r), \tag{C.8b}$$

where $\alpha>\beta>0$, and (t,t') is the usual inner product $(t,t') = \Sigma t_\lambda t'_\lambda$ of $\underset{\sim}{R}^{3n-4}$. The set $\Gamma(\mathcal{U})$ is thus covered by the open sets

$$\Gamma_\rho^{\pm}(\mathcal{U}) = \{U|U = \Sigma t_\lambda U_\lambda, \ t \ \varepsilon \ \Omega_\rho^{\pm}\}. \tag{C.9}$$

The crucial step is to show that for any $0<\alpha<1$ there is some (real) neighborhood $\mathcal{N}_o \subset (\mathcal{N} \cap \Delta'_c(\bar{K}) \cap \Delta_c(\bar{K}) \cap \text{supp } \emptyset)$ of $\bar{K}$, such that for any fixed product wave function ψ in $\mathcal{B}(\mathcal{N}_o)$ the transition amplitude $T[\psi;U\tau]$ is of rapid decrease $(T[\psi;U\tau] \nrightarrow 0)$ uniformly on

$$\Gamma_0(\mathcal{U}) = \Gamma(\mathcal{U}) - \Gamma_1^{+}(\mathcal{U}). \tag{C.10}$$

Since

$$\Gamma_0(\mathcal{U}) \subset \Big[\bigcup_{\rho\geq 2} \bar{\Gamma}_\rho^{+}(\mathcal{U})\Big] \cup \Big[\bigcup_{\rho\geq 1} \bar{\Gamma}_\rho^{-}(\mathcal{U})\Big], \tag{C.11}$$

it is sufficient to prove the uniform rapid decrease on the closed sets $\bar{\Gamma}_1^{-}(\mathcal{U})$ and $\bar{\Gamma}_\rho^{\pm}(\mathcal{U})$, $(\rho\geq 2)$.

For any fixed product $\psi \ \varepsilon \ \mathcal{B}(\mathcal{H})$ satisfying (C.3) the uniform rapid decrease of $T[\psi;U\tau]$ on $\bar{\Gamma}_1^{-}(\mathcal{U})$ is a consequence of the WAC condition, provided α is small enough so that $\bar{\Gamma}_1^{-}(\mathcal{U}) \subset [N \cap \Gamma(\mathcal{U})]$. [If the original α is not small enough then a smaller one can be used.] To use this fact let $\mathcal{O} \subset \text{supp } \emptyset$ be an open set with the property

$\mathcal{O} \cap \mathcal{M}^c(\mathcal{N} \cap \Delta'_c(\bar{K}) \cap \Delta_c(\bar{K}))$, and let $\mathcal{O}' \subset \mathcal{O}$ be an open neighborhood of $\bar{K}$ with the property $\mathcal{O}' \subset \text{supp}\ \psi' \subset \mathcal{O}$, where ψ' is a product wave function. Let $\mathcal{N}_1$ be the intersection $\mathcal{N}_1 = \mathcal{O}' \cap \mathcal{N}$ of $\mathcal{O}'$ and $\mathcal{N}$. Finally, let $\chi \in \mathcal{B}(\mathcal{W})$ be a product wave function that is unity in supp ψ' and zero outside supp $\emptyset$. Then for any product ψ in $\mathcal{B}(\mathcal{N}_1)$, the wave function $\bar{\psi} \equiv \psi\chi$ satisfies $T[\psi - \bar{\psi}; U\tau] \equiv 0$. Since $\bar{\psi}$ is a product wave function in $\mathcal{B}(\mathcal{W})$ that satisfies (C.3), $T[\bar{\psi}; U\tau]$ is of rapid decrease uniformly on $\bar{\Gamma}_1^-(\mathcal{U})$. Thus $T[\psi; U\tau]$ also has this property.

The uniform rapid decrease on the other sets $\bar{\Gamma}_\rho^\pm, \rho \geq 2$ is a consequence of the smoothness requirement on $T(K)$. Let $z = (z_1, \ldots, z_{3n-4})$ be the local coordinates for which $T(z)$ is smooth in the variables $(z_2, \ldots, z_{3n-4})$. Let U be some displacement in $\Gamma_\sigma^\pm$ and let $h_\sigma(U)$ be the coordinate transformation defined by

$$\zeta_1 = z_1, \tag{C.12a}$$

$$\zeta_\sigma = U \cdot K(z), \tag{C.12b}$$

$$\zeta_\rho = \sum_\lambda e_{\rho\lambda}\ U_\lambda \cdot K(z), \quad (2 \leq \rho \leq 3n-4, \rho \neq \sigma), \tag{C.12c}$$

where the vectors $e_\rho = (e_{\rho 1}, \ldots, e_{\rho, 3n-4})$ are the orthonormal basis vectors used in (C.8), and the U_λ are as in (C.5). Define

$$\phi_\rho(z) = \sum_\lambda e_{\rho\lambda} U_\lambda \cdot K(z) = V_\rho \cdot K(z), \quad (1 \leqslant \rho \leqslant 3n-4), \tag{C.13}$$

and let $Q(z)$ be the determinant of the square matrix $Q_{\rho\lambda} = \partial\phi_\rho/\partial z_\lambda$. Finally, for any K in $\mathcal{N}_1$ (with $\mathcal{N}_1$ taken sufficiently small) write

$$\nabla\Lambda(K) = \Sigma\ g_\rho(K)\ V_\rho + U_0(K)\ , \tag{C.14}$$

where $U_0(K)$ belongs to $\mathcal{C}_0(K)$. The functions $g_\rho(K)$ are continuous, and $g_\rho(\bar{K}) = \delta_{\rho 1}$. Using the readily verified relation $U_0 \cdot \partial K/\partial z = 0$, one finds by explicit calculation that the Jacobian $H_\sigma(z,U)$ of the transformation $h_\sigma(U)$ is

$$H_\sigma(z,U) = Q(z)\ (g_1(z)\ X_\sigma - g_\sigma(z)\ X_1), \tag{C.15}$$

where $U = \Sigma\ X_\rho V_\rho$. Thus if U belongs to $\Gamma_\sigma^{\pm}$, then the Jacobian does not vanish on the set

$$D_\sigma(\bar{K}) = \{z | K(z) \in \mathcal{N}_1 \equiv \mathcal{N}_1(\bar{K}),\ |Q(z)| > \varepsilon,$$
$$|g_\sigma(z)\ g_1^{-1}(z)| < \beta(r-1)^{-1/2}\}. \tag{C.16}$$

[The open sets $D_\sigma(\bar{K})$ always contain $\bar{z}$ and hence are nonempty, for all $\sigma \geqslant 2$.] Therefore if z belongs to D_σ and U belongs to $\Gamma_\sigma^{\pm}$ then the holomorphic transformation

$h_\sigma(U)$ can be inverted, giving the $z_\lambda = z_\lambda(\zeta,X)$ as holomorphic functions of ζ and X. Then because $z_1(\zeta,X)$ is simply ζ_1, the smoothness of $T(z)$ in the variables $(z_2,\ldots,z_{3n-4})$ implies the smoothness of $T'(\zeta,X) = T(z(\zeta,X))$ in the variables $(\zeta_2,\ldots,\zeta_{3n-4},X)$ when ζ belongs to $h_\sigma(U)D_\sigma$ and U belongs to $\Gamma_\sigma{}^{\pm}$. The proof of this is deferred to the end (Lemma 1).

Let $\mathcal{N}_\sigma = \Pi'_K(D_\sigma)$. Then for U in $\Gamma_\sigma{}^{\pm}$ and ψ in $\mathcal{B}(\mathcal{N}_\sigma)$ the amplitude $T[\psi;U\tau]$ can be written

$$T[\psi;U\tau] = \int d\zeta_\sigma\, e^{-i\zeta_\sigma\tau}\; f(\zeta_\sigma,X), \qquad (C.17)$$

where

$$f(\zeta_\sigma,X) = \int d\zeta_1\ldots d\zeta_{\sigma-1}\, d\zeta_{\sigma+1}\ldots d\zeta_{3n-4}\, H_\sigma{}'(\zeta,X)\; J'(\zeta,X)\, \bar{\Psi}'(\zeta,X)\, T'(\zeta,X) \qquad (C.18)$$

is a distribution in ζ_σ that depends on X. The function J is the holomorphic Jacobian associated with the local coordinate system $(\Delta'_c(\bar{K}),\Pi'_{\bar{K}},D'_c(\bar{K}))$. The holomorphy of $H'_\sigma(\zeta,X) = H_\sigma(z(\zeta,X))$ and $J'(\zeta,X) = J(z(\zeta,X))$, and the smoothness of T and $\bar{\Psi} = \psi \circ \Pi'_K$, imply the infinite differentiability of f in ζ_σ and in X for all $U\varepsilon\bar{\Gamma}_\sigma{}^{\pm}$. (See Lemma 1). The function f must also have compact support since the function $\bar{\Psi}'(\zeta,X) = \bar{\Psi}(z(\zeta,X))$ does. It follows therefore, for all U in $\bar{\Gamma}_\sigma{}^{\pm}$, that all derivatives $\partial^n f/\partial\zeta_\sigma{}^n$ are absolutely summable and hence that the integrals

$$I_n(X) = \int d\zeta_\sigma \left|\frac{\partial^n f}{\partial \zeta_\sigma{}^n}(\zeta_\sigma, X)\right| \qquad (C.19)$$

are bounded for U in $\bar{\Gamma}_\sigma{}^{\pm}$. Equation (C.17) then implies[29] that $T[\psi;U\tau] \Rightarrow 0$ uniformly on $\bar{\Gamma}_\sigma{}^{\pm}$. Since the index σ was arbitrary, the amplitude $T[\psi;U\tau] \Rightarrow 0$, uniformly on the set $\Gamma_0(\mathcal{U})$ defined in (C.10) for all product wave functions ψ in $\mathcal{B}(\mathcal{N}_o)$, where $\mathcal{N}_o \equiv \cap \mathcal{N}_\sigma$ is open in $\mathcal{H}$ and contains $\bar{K}$.

To complete the proof let $\mathcal{N}' \subset X\mathcal{M}_i$ be a neighborhood of $\bar{K}$, and let $\bar{\mathcal{N}}' \cap \mathcal{H}$ be a subset of $\mathcal{N}_o$. Let χ be a product wave function in $\mathcal{B}(\mathcal{N}_o)$ with unit value on $\mathcal{N}'$. Then, for any ψ in $\mathcal{B}(\mathcal{N}' \cap \mathcal{H})$, one has

$$T[\psi] = T[\psi\chi]. \qquad (C.20)$$

If the notation $\tilde{T}(t) \equiv T[\chi;\Sigma t_\lambda U_\lambda]$ is introduced, the amplitude $T[\psi]$ can be written in the form of the convolution[30]

$$T[\psi] = \int dt\, \tilde{\psi}(-t)\, \tilde{T}(t), \qquad (C.21)$$

where

$$\tilde{\psi}(t) = (2\pi)^{-(3n-4)} \int dz\, e^{-i(z,t)}\, (\psi \circ \Pi_K)(z). \qquad (C.22)$$

The z_λ in (C.22) are the local coordinates $U_\lambda \cdot K$. Define

$$\Theta(\Omega_1{}^+;t) = \begin{cases} 1 & \text{if } t = 0 \text{ or } t|t|^{-1} \equiv \hat{t} \in \bar{\Omega}_1{}^+, \\ 0 & \text{otherwise.} \end{cases} \tag{C.23}$$

Equation (C.21) can then be rewritten

$$T[\psi] = \int dt\tilde{\psi}(-t)\left[\tilde{T}^o(t) + \tilde{T}^1(t)\right], \tag{C.24}$$

where

$$\tilde{T}^o(t) = \left[1 - \Theta(\Omega_1{}^+,t)\right]\tilde{T}(t) \tag{C.25}$$

and

$$\tilde{T}^1(t) = \Theta(\Omega_1{}^+,t)\,\tilde{T}(t). \tag{C.26}$$

The results of the preceding paragraph show that $\tilde{T}^o(t) \Rightarrow 0$ uniformly in $t|t|^{-1}$ as $|t|\to\infty$. Therefore, the function $\tilde{T}^o(t)$ has an infinitely differentiable Fourier transform $\mathcal{T}^o(z)$, and[31]

$$\int dt\tilde{\psi}(-t)\,\tilde{T}^o(t) = \int dz(\psi \circ \Pi_{\bar{K}})(z)\,\mathcal{T}^o(z). \tag{C.27}$$

Let $J(z)$ be the Jacobian appropriate to the local coordinates $(\Delta_c(\bar{K}),\Pi_{\bar{K}},D_c(\bar{K}))$. Define $T_J{}^o(z) \equiv J^{-1}(z)\,\mathcal{T}^o(z)$, and let $T^{o}(K) \equiv T_J{}^{o}(\Pi_{\bar{K}}{}^{-1}(K))$. Then (C.27) becomes

$$\int dt\tilde{\psi}(-t)\, \tilde{T}^{o}(t) = \int dK\psi(K)\, T^{o}(K), \qquad (C.28)$$

where $T^{o}(K)$ is infinitely differentiable on $\mathcal{N}'\cap\mathcal{H}$.

The function $\tilde{T}^{1}(t)$ is not necessarily of rapid decrease when $|t|\to\infty$, but it has at most polynomial growth.[32] Hence, the function $\exp[-(\delta,t)]\,\tilde{T}^{1}(t)$ is of rapid decrease when δ belongs to

$$C^{+} = \{\delta\,|\,(\delta,\hat{t}) > 0 \text{ for all } \hat{t}\in\bar{\Omega}_1{}^{+}\}, \qquad (C.29)$$

and it has a Fourier transform $\mathcal{T}^{1}(z)$ that is holomorphic for $\mathrm{Im}\,z$ in C^{+}.[33] If $T_J{}^{1}(z) \equiv J^{-1}(z)\,\mathcal{T}^{1}(z)$ is introduced for z in

$$E^{+} = \{z\,|\,z\in\Delta_c(\bar{K}),\ \mathrm{Im}\,z\in C^{+}\}, \qquad (C.30)$$

the second term in (C.24) becomes,[31] after simple manipulation,

$$\int dt\tilde{\psi}(-t)\, \tilde{T}^{1}(t) = \lim_{\substack{|\delta|\to 0\\ \delta\in C^{+}}} \int dz(\psi \circ \Pi_{\bar{K}})(z)\, J(z)\, T_J{}^{1}(z+i\delta). \qquad (C.31)$$

From this it follows that

$$\int dt\tilde{\psi}(-t)\, \tilde{T}^{1}(t) = \lim_{\substack{|\delta|\to 0\\ \delta\in C^{+}}} \int dK\psi(K)\, T^{1}(K'(\kappa,\delta)), \qquad (C.32)$$

where $T^1(K) = T_J^{\ 1}(\Pi_K^{\ -1}(K))$ and

$$K'(K, \delta) = \Pi_{\overline{K}}(\Pi_{\overline{K}}^{\ -1}(K) + i\delta). \tag{C.33}$$

This completes the proof.

Lemma 1. Suppose $T[\psi]$ has the representation

$$T[\psi] = \int_S dz \left[\frac{d^m}{dz_1^{\ m}} J\psi(z)\right] F(z) \tag{C.34}$$

where S is some domain, and $F(z)$ is a function that is continuous in z_1 and has continuous partial derivatives of all orders in the variables $(z_2,\ldots,z_n)$. Let $h:S\to S'$ be some nonsingular holomorphic mapping from $S \subset \underset{\sim}{R}^n$ onto $S' \subset \underset{\sim}{R}^n$ such that $z_1' \equiv h_1(z) = z_1$. Then there is a function $G(z')$, $z' = h(z)$, which is continuous in z_1' and has continuous partial derivatives of all orders in $(z_2',\ldots,z_{3n-4}')$, such that

$$T[\psi] = \int_{S'} dz' \left[\frac{d^m}{dz_1'^{\,m}} J\psi(h^{-1}(z'))\right] G(z'). \tag{C.35}$$

Proof. Under the mapping h, the operator $d^m/dz_1^{\ m}$ transforms into a differential operator

$$D = \sum_{p'} \sum_{p=1}^{m} h_{pp'}(z') \, D_p^{\ p'} \frac{d^p}{dz_1'^{\,p}} , \tag{C.36}$$

where the $h_{pp'}$ are holomorphic functions and the $D_p^{p'}$ are derivative monomials in the variables $(z_2', \ldots, z_{3n-4}')$. The quantity $T[\psi]$ then has the form

$$T[\psi] = \int_{S'} dz' \; H'(z')\left[DJ\psi(h^{-1}(z'))\right] F(h^{-1}(z')). \qquad (C.37)$$

The function $F'(z') \equiv F(h^{-1}(z'))$ also has the property that it is continuous in the first variable z_1' and C^∞ in the other variables $(z_2', \ldots, z_{3n-4}')$. The function $H'(z')$ is the holomorphic Jacobian for the transformation h. For each p and p' the derivatives $D_p^{p'}$ can be transferred (through partial integrations) to the functions $H'h_{pp'}F'$. This transforms (C.37) into the form

$$T[\psi] = \int_{S'} dz' \sum_{p=1}^{m} G_p(z') \frac{d^p}{dz_1'^p} J\psi(h^{-1}(z')). \qquad (C.38)$$

The functions $G_p(z')$ also have the property that they are continuous in the first variable z_1' and C^∞ in the others. Through further partial integrations the derivatives $d^p/dz_1'^p$ can all be transformed into derivatives $d^m/dz_1'^m$, yielding

$$T[\psi] = \int_{S'} dz' \left[\frac{d^m}{dz_1'^m} J\psi(h^{-1}(z'))\right] \sum_p G_p'(z'). \qquad (C.39)$$

The function $G = \sum_p G_p'$ is the function required by the Lemma.

Lemma 2. Equation (C.21) is valid:

$$T[\psi] = T[\chi\ \psi] = \int dt\tilde{\psi}(-t)\ \tilde{T}(t). \qquad (C.40)$$

Proof. Since χ belongs to $\mathcal{D}$, the functional $T[\chi\ \psi] = F[\psi]$ is a continuous linear functional on the space $\mathcal{E}$ of functions ψ that possess continuous partial derivatives of all orders. (Note that the support of ψ is not restricted here.) This is because $\chi\ \psi$ belongs to $\mathcal{D}$ for every ψ in $\mathcal{E}$. The functional F then belongs to $\mathcal{E}'$, and the result[34] of Bremmermann is directly applicable, yielding (C.40) for all $\psi \varepsilon \mathcal{D}$ that satisfy (C.20).

B. Proof of Theorem 1A.

The proof consists of two parts. The first is a demonstration that the number α, the simple coordinate system of Theorem 1, the number ε, and the set $\mathcal{N}_\varepsilon$ can be chosen so that $[\mathcal{N}_\varepsilon \cap C_\varepsilon^+(\bar{K})] \subset \mathcal{E}_\alpha$. The second consists of the necessary generalization of the way the limit (3.7) is taken.

Choose a simple coordinate system $(\Delta_c(\bar{K}), \Pi_{\bar{K}}, D_c(\bar{K}))$ in which $z_1 = \sigma(K;\bar{K}) = \nabla\Lambda(\bar{K})\cdot K$. Such a choice is, of course, possible only if $\nabla\Lambda(\bar{K})$ does not belong to $\mathcal{C}_o(\bar{K})$. [See (B.11).] But if $\nabla\Lambda(\bar{K})$ belonged to $\mathcal{C}_o(\bar{K})$ the various displaced velocity cones $\hat{V}(\phi_i, -\partial_i\Lambda(\bar{K}))$ would have a common point and assumption (c) of Theorem 1 could not be satisfied. Thus, coordinates with $z_1 = \sigma$ can be chosen.

Let $\mathcal{N}'' \subset [\Delta_c(\bar{K}) \cap \mathcal{H}_c]$ be a complex neighborhood of $\bar{K}$ such that its closure $\bar{\mathcal{N}}''$ is also contained in $\Delta_c(\bar{K}) \cap \mathcal{H}_c$ and the set $\Pi_{\bar{K}}^{-1}(\bar{\mathcal{N}}'')$ is convex. Then for $K \varepsilon [\mathcal{N}'' \cap C_\varepsilon^{+}(\bar{K})]$ one has

$$\text{Im} z_1(K) \geqslant \sup_{U \varepsilon R_\varepsilon} \text{Im}(U \cdot K) = \varepsilon \|\text{Im } K\|. \tag{C.41}$$

Since $\Pi_{\bar{K}}^{-1}(\bar{\mathcal{N}}'')$ is convex, there is some $A>0$ such that for $K \varepsilon \bar{\mathcal{N}}''$

$$\|\text{Im } K\| \geqslant |\text{Im} z| A \tag{C.42}$$

where $z = \Pi_{\bar{K}}^{-1}(K)$. For since the mapping $\Pi_{\bar{K}}$ is holomorphic, the functions $f_{i\nu}(x,y) \equiv \text{Im} k_{i\nu}(x+iy)$ have derivatives of all orders for $(x+iy)$ in $\bar{\mathcal{N}}''$ and can therefore be expanded about $y = 0$ by using the Taylor formula (with remainder)[35]:

$$f_{i\nu}(x,y) = f_{i\nu}(x,0) + \sum_\lambda y_\lambda \frac{\partial f_{i\nu}}{\partial y_\lambda}(x,ty), \tag{C.43}$$

where t, $0 \leqslant t \leqslant 1$, is some number that depends in general on

y. Since $k_{i\nu}(z)$ is real when z is real, $f_{i\nu}(x,0) = 0$. We can therefore write

$$\|\mathrm{Im}\ K\| = |y|\ \Delta(x,y), \tag{C.44}$$

where

$$\Delta(x,y) = |y|^{-1}\left\{\sum_{i\nu} \left|\sum_{\lambda} y_\lambda \frac{\partial f_i}{\partial y_\lambda}(x,ty)\right|^2\right\}^{\frac{1}{2}}. \tag{C.45}$$

Consider now

$$A = \inf_{\substack{(x+iy)\epsilon\Pi_{\bar{K}}^{-1}(\mathcal{R}'') \\ y\neq 0}} \Delta(x,y). \tag{C.46}$$

If $A = 0$, there must be a sequence of points (x_n,y_n), with $y_n\neq 0$, such that

$$\lim_{n\to\infty} \Delta(x_n,y_n) = 0. \tag{C.47}$$

Moreover, because $\mathcal{R}''$ is closed, the sequence (x_n,y_n) approaches a limit $(\bar{x},\bar{y})$ with $(\bar{x}+i\bar{y})\epsilon\ \Pi_{\bar{K}}^{-1}(\mathcal{R}'')$. If $\bar{y}\neq 0$, then $\|\mathrm{Im}\ K(\bar{x}+i\bar{y})\| = 0$. But for a simple coordinate system the vanishing of the imaginary part of $K(\bar{x}+i\bar{y})$ implies $\bar{y}=0$. This precludes the case $\bar{y}\neq 0$. To discuss the case where $\bar{y}=0$, we first define $w_n = y_n|y_n|^{-1}$. The sequence w_n, suitably restricted to a subsequence, is convergent to some $\bar{w}$ with unit norm. The continuity of the derivatives $\partial f_{i\nu}/\partial y_\lambda$ further implies that

$$A = \lim_{n\to\infty} \Delta(x_n, y_n) = \left\{ \sum_{i\nu} \left| \sum_\lambda \bar{w}_\lambda \frac{\partial f_{i\nu}}{\partial y_\lambda} (\bar{x}, 0) \right|^2 \right\}^{1/2}. \quad (C.48)$$

If $A = 0$, the equations

$$\sum_\lambda \bar{w}_\lambda \frac{\partial f_{i\nu}}{\partial y_\lambda} (\bar{x}, 0) = 0 \quad (C.49)$$

must be satisfied for all i and ν. Because the real analyticity of the $k_{i\nu}(z)$ implies

$$\frac{\partial k_{i\nu}}{\partial z_\lambda} (\bar{x}) = \frac{\partial f_{i\nu}}{\partial y_\lambda} (\bar{x}, 0), \quad (C.50)$$

the equations (C.49) state that the vectors $V_\lambda = (v_{i\lambda}, \ldots, v_{n\lambda})$, with $v_{i\lambda,\nu} = \partial k_{i\nu}/\partial z_\lambda$, are linearly dependent. These vectors V_λ form the rows of the Jacobian matrix of the mapping $\Pi_{\bar{K}}(z)$. Since this Jacobian has maximal rank in $D_c(\bar{K})$, the rows cannot be linearly dependent. This contradiction implies that A cannot be zero. Consequently A is greater than zero.

For any $\varepsilon>0$ one can find an $0<\alpha<1$ such that $\alpha<\varepsilon A$. Then (C.41) and (C.42) imply

$$\text{Im } z_1 > |\text{Im } z|\alpha. \quad (C.51)$$

This implies that z is in $\Pi_{\bar{K}}^{-1}(\mathcal{E}_\alpha)$. Thus, $[\mathcal{N}''\cap C_\varepsilon^{+}(\bar{K})]\subset\mathcal{E}_\alpha$ and the first part of the theorem is proved.

It is clear that $\mathcal{N}''$ can be chosen so that $[\mathcal{N}''\cap\mathcal{K}]\subset\mathcal{N}'$ where $\mathcal{N}'$ is the neighborhood of Theorem 1. Theorem 1 therefore implies Theorem 1A, provided the manner of taking the limit (3.7) can be converted to that of (3.14). Let $z_\lambda(x,s)$, $1\leq\lambda\leq 3n-4$, be any uniformly continuous functions of $x\varepsilon D(\bar{K}) \equiv D_c(\bar{K})\cap \underset{\sim}{R}^{3n-4}$ and s, $0\leq s\leq 1$, which have the following three properties: (a) partial derivatives (with respect to x) of all orders exist and are continuous in both x and s; (b) $z(x,0) = x$ for all x; (c) $z(x,s)$ belongs to $\Pi_{\bar{K}}^{-1}(\mathcal{E}_\alpha)$ for all $\dot{x}$ and $s>0$. We want to show that

$$T[\psi] = \lim_{s\to 0} \int dK\psi(K)[\, T^{o}(K) + T^{1}(K'(K,s))], \qquad (C.52)$$

where $T^{o}(K)$ and $T^{1}(K)$ are the functions of Theorem 1 and

$$K'(K,s) = \Pi_{\bar{K}}(z(\Pi_{\bar{K}}^{-1}(K),s)) \qquad (C.53)$$

Since $[\mathcal{N}''\cap C_\varepsilon^{+}(\bar{K})]\subset\mathcal{E}_\alpha$, all paths $K'(K,s)$ of the type allowed by the theorem are of the type (C.53). Thus a proof of (C.52) proves also Theorem 1A.

The relevant term in (C.52) is the one involving $T^{1}(K'(K,s))$. In terms of local coordinates it can be written

$$I(s) \equiv \int dK\psi(K)\ T^1(K'(K,s)) = \int dx J(x)\psi(x) T_J^1(z(x,s)). \qquad (C.54)$$

The function $T_J{}^1(z)$ was defined in the proof of Theorem 1 as $J^{-1}(z)T^1(z)$, where $T^1(z)$ is defined for $\operatorname{Im} z \varepsilon C^+(\alpha)$ and $z \varepsilon D_c(\bar{K})$ as

$$T^1(z) = (2\pi)^{-(3n-4)} \int dt e^{+i(z,t)} \tilde{T}^1(t) \qquad (C.55)$$

For $s>0$ and $x \varepsilon D(\bar{K})$ the quantity $\operatorname{Im}(z(x,s),t)$ is bounded from below by $\eta(s)|t|>0$, for all (real) $t \neq 0$ in the support of $\tilde{T}^1(t)$. This lower bound is a consequence of the continuity of $z(x,s)$ and the assumption that $z(x,s) \varepsilon \Pi_{\bar{K}}{}^{-1}(\mathcal{E}_\alpha)$ for $s>0$. The integral (C.55) therefore converges uniformly in z and the integrations in (C.54) can be interchanged:[36]

$$I(s) = \int dt \tilde{T}^1(t)\ \Phi(-t,s), \qquad (C.56)$$

where

$$\Phi(t,s) = (2\pi)^{-(3n-4)} \int dx J(x) J^{-1}(z(x,s))\psi(x) e^{-i(z(x,s),t)}. \qquad (C.57a)$$

The next step is to show that the integral (C.56) converges uniformly in s in some strip $0 \leqslant s \leqslant s_o$, where $0 < s_o \leqslant 1$. The function $\tilde{T}^1(t)$ is continuous and of at most polynomial growth, so there is some integer N such that $\tilde{T}^1(t)(1+|t|^{2N})^{-1}$ is absolutely summable. On the other hand, the function $(1+|t|^{2N})\Phi(-t,s)$ is bounded in both t and s for t in the support of $\tilde{T}^1(t)$ and s in some strip $0\leqslant s\leqslant s_o$. To see this, consider the functions $z_\lambda(x,s)$ as a mapping ζ from $D(\bar{K})$ into $D_c(\bar{K})$ for each s. Let $W(x,s)$ be the Jacobian of ζ. Assumption (a) about $z(x,s)$ implies that $W(x,s)$ is continuous in both x and s, and assumption (b) implies that $W(x,0) = 1$ for all x. It follows that there exists s_o, $0<s_o\leqslant 1$, such that $W(x,s)$ does not vanish on any product set of the form $P \times I$, where $I = \{s|0\leqslant s\leqslant s_o\}$ and P is any compact subset of $D(\bar{K})$. For any $s\varepsilon I$, therefore, the mapping ζ can be inverted on P. Since supp ψ is a compact subset of $D(\bar{K})$, this result can be applied to (C.57), yielding

$$\Phi(t,s) = (2\pi)^{-(3n-4)} \int_{\Gamma(s)} dzJ(x(z))J^{-1}(z)W^{-1}(z)\psi(x(z))e^{-i(z,t)}. \tag{C.57b}$$

The contours $\Gamma(s)$ in (C.57b) are the images under ζ of $D(\bar{K})\cap$ supp ψ for various values of s. The sets $\Gamma(s)$ are compact for all $s\varepsilon I$. Consider now the function $(1+|t|^{2N})$ $\Phi(-t,s)$:

$$(1+|t|^{2N})\Phi(-t,s) = (2\pi)^{-(3n-4)} \int_{\Gamma(s)} dzJ(x(z))J^{-1}(z)W^{-1}(z)\psi(x(z)) \left[1+(-1)^N\left(\sum_\lambda \frac{\partial^2}{\partial z_\lambda^{\,2}}\right)^N\right]e^{i(z,t)}. \tag{C.58}$$

Partial integrations of (C.58) yield

$$(1+|t|^{2N})\Phi(-t,s) = (2\pi)^{-(3n-4)} \int_{\Gamma(s)} dzW^{-1}(z)F_N(z)e^{i(z,t)}, \quad (C.59)$$

where

$$F_N(z) = W(z)\Big[1+(-1)^N(\sum_\lambda \frac{\partial^2}{\partial z_\lambda{}^2})^N\Big]J(x(z))J^{-1}(z)W^{-1}(z)\psi(x(z)). (C.60)$$

Equation (C.59) can be rewritten:

$$(1+|t|^{2N})\Phi(-t,s) = (2\pi)^{-3n-4}\int dxF_N(z(x,s))e^{i(z,t)}. \quad (C.61)$$

The continuity of the mapping ζ in both x and s, and the continuity of the functions J, W, and ψ ensure the boundedness of $F_N(z(x,s))$ on $D(\bar{K}) \times I$. The boundedness of $e^{i(z,t)}$ for all t in the support of $\tilde{T}^1(t)$ is ensured by the fact that z is either real $(s=0)$ or in $\Pi_{\bar{K}}{}^{-1}(\mathcal{E}_\alpha)$. Thus, the function $(1+|t|^{2N})\Phi(-t,s)$ is bounded in both t and s, with t in the support of $\tilde{T}^1(t)$ and $0\leqslant s\leqslant s_o$, and the integral (C.56) converges uniformly.[37]

The order of the limit $s\to 0$ and the integration over t can therefore be interchanged:

$$\lim_{s\to 0} I(s) = \int dtT^1(t)\ \Phi(-t,0). \quad (C.62)$$

Because $\Phi(-t,0)$ is just $\tilde{\psi}(-t)$, equation (C.62) is

$$\lim_{s\to 0} I(s) = \int dt\, \tilde{T}^1(t)\ \tilde{\psi}(-t). \tag{C.63}$$

Since it was shown in the proof of Theorem 1 that

$$\lim_{\substack{|\delta|\to 0 \\ \delta\varepsilon C^+(\alpha)}} \int dK\psi(K)\ T^1[K'(K,\delta)] = \int dt\ \tilde{T}^1(t)\ \tilde{\psi}(-t), \tag{C.64}$$

the proof is complete.

APPENDIX D

A. Proof of Theorem 3.

Consider an arbitrary point $\bar{K}$ of $\mathcal{W} - \mathcal{L}^+$, and let $\mathcal{U} = \{U_1, \cdots, U_{3n-4}\}$ be a set of linearly independent displacements that define a simple coordinate system $(\Delta_c(\bar{K}), \Pi_{\bar{K}}, D_c(\bar{K}))$ at $\bar{K}$. Because $\mathcal{W} - \mathcal{L}^+$ is open,[19] there exists a (product) neighborhood $\mathcal{N}_1' \subset X\mathcal{M}_i$, $(\bar{\mathcal{N}}_1' \cap \mathcal{M}) \subset (\Delta_c(\bar{K}) \cap [\mathcal{W} - \mathcal{L}^+])$, of $\bar{K}$ such that $\bar{\mathcal{N}}_1'$ is the support of some product wave function X'. Because $\mathcal{U}$ defines a set of local coordinates at $\bar{K}$, the set $\mathcal{N}_1'$ can be chosen small enough so that the set $\Gamma(\mathcal{U})$ defined in (4.9) has an empty intersection with $\mathcal{G}_0(X')$. [See (B.10).]

Consider next a product X satisfying supp $X \equiv \bar{\eta}_1 \subset \mathcal{O} \subset \mathcal{N}_1'$, where $\mathcal{O}$ is open. Then

$$\mathcal{G}_c(X) \subset \mathcal{G}_0(X') \ . \tag{D.1}$$

To prove (D.1) assume the converse: suppose there is a U in $\mathcal{G}_c(X)$ that is not in $\mathcal{G}_0(X')$. Because the points of $\bar{\eta}' \cap \mathcal{M}$ lie in $\mathcal{W} - \mathcal{L}^+$ and hence in $\mathcal{M} - \mathcal{M}_0$, we can assume that no two initial k_i are collinear in $\bar{\eta}'$ and no two final k_i are collinear in $\bar{\eta}'$. Then, because U is not in $\mathcal{G}_0(X')$, one can find some $\epsilon > 0$ such that the sets $\hat{V}_\epsilon(X_i, u_i)$ of Definition 4 have no common point. But then the diagram $\mathcal{D}_\epsilon$ required by Definition 4, and the fact that U is in $\mathcal{G}_c(X)$, must be a nontrivial diagram. This diagram $\mathcal{D}_\epsilon$ belongs to $\mathcal{G}_c(K) - \mathcal{G}_0(K)$ for some K in $\bar{\eta}' \cap \mathcal{M}$. But then this K lies on $\mathcal{L}^+$, contrary to the definition of $\bar{\eta}'$. This contradiction proves (D.1).

Because $\Gamma(\mathcal{U})$ does not intersect $\mathcal{B}_0(X)$, it does not intersect $\mathcal{B}_c(X)$, and is therefore a (compact) subset of $\mathcal{Q}_c(X)$.

Since $\Gamma(\mathcal{U})$ is a compact subset of $\mathcal{Q}_c(X)$, the SAC condition implies that $\tilde{T}_c(t) = T_c[X;\Sigma t_\lambda U_\lambda] \Rightarrow 0$ uniformly in $t|t|^{-1}$ as $|t| \to \infty$.

Let the product wave function X have unit value on the closure $\bar{\mathcal{N}}_2$ of some neighborhood $\mathcal{N}_2 \subset \mathcal{H}$ of $\bar{K}$, where $\bar{\mathcal{N}}_2$ is a subset of $\mathcal{N}_1 \cap \mathcal{H}$. If ψ belongs to $\mathcal{B}(\mathcal{N}_2)$, then $T_c[\psi] = T_c[\psi\,\chi]$. This relation can be rewritten[30]

$$T_c[\psi] = \int dt \tilde{\psi}(-t)\ \tilde{T}_c(t), \tag{D.2}$$

where

$$\tilde{\psi}(t) = (2\pi)^{-(3n-4)} \int dz\ e^{-i(z,t)} (\psi \circ \pi_{\bar{K}})\ (z), \tag{D.3}$$

is defined just as in Appendix C. Since $\tilde{T}_c(t)$ is of rapid decrease uniformly in $t|t|^{-1}$ when $|t|$ becomes infinite, it has an infinitely differentiable Fourier transform $T_c(z)$.[38] Moreover the convolution theorem[31] can be used to convert (D.2) to

$$T_c[\psi] = \int dz (\psi \circ \pi_{\bar{K}})(z)\ T_c(z). \tag{D.4}$$

Let $J(z)$ be the weight function (Jacobian) appropriate to the mapping $\Pi_{\bar{K}}$, and let $T_{cJ}(z) = J^{-1}(z)\ T_c(z)$. Finally, let $T_c(K) = T_{cJ}(\Pi_{\bar{K}}^{-1}(K))$. This function is infinitely differentiable on $\mathcal{N}_2$, and

$$T_c[\psi] = \int dK \psi(K)\ T_c(K) \tag{D.5}$$

for every wave function in $\mathcal{B}(\mathcal{N}_2)$. The distribution $T_c(K)$ is therefore infinitely differentiable on $\mathcal{N}_2$ and hence at $\bar{K}$.

B. Proof of Theorem 4.

Let $\mathcal{N} \subset [\mathcal{H} \cap \Delta_c(\bar{K})]$ be a neighborhood which satisfies the conditions of Definition 5. Then, any neighborhood $\mathcal{N}_1$ of $\bar{K}$ fulfills the conditions of the theorem if its closure $\bar{\mathcal{N}}_1$ is contained in $\mathcal{N}$.

To prove this let χ be a product wave function in $\mathcal{B}(\mathcal{N})$ with unit value on $\bar{\mathcal{N}}_1$, and let $T_c[\chi;U(t)] \equiv \tilde{T}_c(t)$ for any displacement U of the form $U(t) = \Sigma\, t_\lambda U_\lambda$. Being the Fourier transform of a distribution with compact support, $\tilde{T}_c(t)$ is infinitely differentiable.[32] If ψ is any wave function in $\mathcal{B}(\mathcal{N}_1)$, the transition amplitude $T_c[\psi]$ can be written[30]

$$T_c[\chi\ \psi] = T_c[\psi] = \int dt\, \tilde{\psi}(-t)\, \tilde{T}_c(t), \tag{D.6}$$

where $\tilde{\psi}$ is defined in (D.3). The domain of integration is broken up in the following way. Let

$$\omega_0 = \Omega - \bigcup_{i \geq 1} \bar{\omega}_i, \tag{D.7}$$

and for all $i \geq 0$ let

$$C(\omega_i) = \{t \mid t \neq 0,\ t|t|^{-1} \in \omega_i\}. \tag{D.8}$$

Define the step functions

$$\Theta_i(t) = \begin{cases} 1 & \text{if} \quad t \in C(\omega_i), \\ 0 & \text{if} \quad t \notin \bar{C}(\omega_i), \end{cases} \tag{D.9}$$

and adjust the (finite) values on the boundaries of the $C(\omega_i)$ so that

$$\sum_{i \geqslant 0} \Theta_i(t) = 1 \tag{D.10}$$

for all t. Equation (D.6) then becomes

$$T[\psi] = \sum_{i \geqslant 0} \int dt \tilde{\psi}(-t)\ \tilde{T}^i(t), \tag{D.11}$$

where subscripts c are now dropped and

$$\tilde{T}^i(t) = \Theta_i(t)\ \tilde{T}(t). \tag{D.12}$$

Consider first the term $\tilde{T}^o(t)$. Because the set $\bar{\omega}_0$ corresponds to a closed subset of $\Gamma(\mathcal{U}) - \Gamma_c(\mathcal{U};\mathcal{N})$, the SAC condition implies that, as $|t|$ increases, the function $\tilde{T}^o(t)$ is of rapid decrease, $\tilde{T}^o(t) \Rrightarrow 0$, uniformly in $t|t|^{-1}$ for t in $\bar{C}(\omega_0)$. Since $\tilde{T}^o(t)$ vanishes for $t \notin \bar{C}(\omega_0)$, the restriction that t belong to $\bar{C}(\omega_0)$ can be removed: $\tilde{T}^o(t) \nrightarrow 0$ as $|t|\to\infty$. This means that $\tilde{T}^o(t)$ has an infinitely differentiable Fourier transform $\mathbf{T}^o(z)$, and the first term in (D.11) can be written[31]

$$\int dt\,\tilde{\psi}(-t)\,\tilde{T}^o(t) = \int dz(\psi \circ \pi_{\bar{K}})(z)\,\mathbf{T}^o(z). \qquad \text{(D.13)}$$

Let $J(z)$ be the weight function appropriate to the mapping $\Pi_{\bar{K}}$, and let $T_J{}^o(z) = J^{-1}(z)\,\mathbf{T}^o(z)$. The definition $T^o(K) = T_J{}^o(\Pi_{\bar{K}}{}^{-1}(K))$ allows (D.13) to be written in the desired form

$$\int dt\,\tilde{\psi}(-t)\,\tilde{T}^o(t) = \int dK\psi(K)\,T^o(K), \qquad \text{(D.14)}$$

where $T^o(K)$ is infinitely differentiable on $\mathcal{H}_1$.

The functions $\tilde{T}^i(t)$, $i\geqslant 1$, which vanish for $t \notin \bar{C}(\omega_i)$, are of at most polynomial growth as $|t|$ becomes infinite. Any exponential damps this polynomial growth, and hence the functions $\exp\{-(\delta,t)\}\tilde{T}^i(t)$ are of exponential decrease as $|t|\to\infty$ uniformly in $t|t|^{-1}$ for any $\delta\varepsilon C^+(\bar{\omega}_i)$. The function $\exp\{-(\delta,t)\}\tilde{T}^i(t)$ has a Fourier transform $\mathbf{T}^i(x+i\delta)$ that is holomorphic for $\delta\varepsilon C^+(\bar{\omega}_i)$.[33] It is evident that

$$\int dt\,\tilde{\psi}(-t)\,\tilde{T}^i(t) = \lim_{\substack{|\delta|\to 0\\ \delta\varepsilon C^+(\bar{\omega}_i)}} \int dt\,\tilde{\psi}(-t)\,e^{-(\delta,t)}\,\tilde{\mathbf{T}}^i(t). \qquad \text{(D.15)}$$

The convolution theorem[31] can be used to write (D.15) as

$$\int dt\,\tilde{\psi}(-t)\,\tilde{T}^i(t) = \lim_{\substack{|\delta|\to 0 \\ \delta\in C^+(\bar{\omega}_i)}} \int dx\,(\psi \circ \pi_K)(x)\,T^i(x+i\delta). \tag{D.16}$$

Define $T_J{}^i(z) = J^{-1}(z)\,T^i(z)$ for z in the set

$$E_i = \{z|z \in D_c(\bar{K}),\ \mathrm{Im}\ z \in C^+(\bar{\omega}_i)\}, \tag{D.17}$$

and define $T^i(K) = T_J{}^i(\Pi_{\bar{K}}{}^{-1}(K))$. Then, (D.16) takes the form

$$\int dt\,\tilde{\psi}(-t)\,\tilde{T}^i(t) = \lim_{\substack{|\delta|\to 0 \\ \delta\in C^+(\bar{\omega}_i)}} \int dK\,\psi(K)\,T^i(K'(K,\delta)), \tag{D.18}$$

where

$$K'(K,\delta) = \Pi_{\bar{K}}(\Pi_{\bar{K}}{}^{-1}(K) + i\delta). \tag{D.19}$$

The function $T^i(K)$ is holomorphic on $\mathcal{E}_i = \Pi_{\bar{K}}(E_i)$. Equations (D.14) and (D.18) combine to yield the desired representation (4.16).

It remains to show that if $\bar{K}$ is any point in $\mathcal{N}_1 - \mathcal{L}_i^+$, then the limit function (4.19) exists and is infinitely differentiable at $\bar{K}$. By virtue of (D.18) the function

$T^i(K)$ exists as a distribution. It is only necessary to show that it is infinitely differentiable. Let $\mathcal{N}_2$ be a neighborhood of $\bar{K}$ with closure contained in $\mathcal{N}_1 - \mathcal{L}_i^+$, and let ψ be a product wave function in $\mathcal{B}(\mathcal{N}_2)$. Then it follows from the results just derived that $T[\psi;U]$ has the formal representation.

$$T[\psi;U(t)] \equiv \tilde{T}(\psi,t) = \int dz J(z)\psi(z)e^{-i(z,t)} \{\sum_j T_T^j(z)\}, \tag{D.20a}$$

$$= \int dt' \tilde{\psi}(t-t') \Sigma \quad \tilde{T}^i(t'), \tag{D.20b}$$

where [redacted], $U(t) = \Sigma\, t_\lambda U_\lambda$, and the various $T_T^j(z)$ are distributions. Because $\mathcal{N}_2$ contains no points of $\mathcal{L}_i^+$ the displacements in $\Gamma_c{}^i(\mathcal{U};\mathcal{N}_1)$ belong to $\mathcal{A}_c(\psi)$. The image of $\Gamma_c{}^i(\mathcal{U};\mathcal{N}_1)$ in Ω is in $\bar{\omega}_i$. Since the sets $\bar{\omega}_j (j>0)$ are disjoint, there is a neighborhood ω_i' of $\bar{\omega}_i$ with closure $\bar{\omega}_i'$ that does not intersect any of the sets $\bar{\omega}_j$ with $j \neq 0,i$. The set $\bar{\omega}_i'$ is therefore the image in Ω of a set $\bar{\Gamma}_c{}'^i(\mathcal{U};\mathcal{N}_1) \subset \Gamma(\mathcal{U})$ that is a subset of $\mathcal{A}_c(\psi)$. The SAC condition then requires that $\tilde{T}(\psi,\hat{t}\tau)$ be of rapid decrease, $\tilde{T}(\psi,\hat{t}\tau) \Rightarrow 0$, as $\tau \to \infty$, uniformly in $\hat{t} = t|t|^{-1}$ for $\hat{t}$ in $\bar{\omega}_i'$ This requirement is also satisfied by the first $(j=0)$ contribution to (D.20) since $T^0(z)$ is infinitely differentiable. (See D.13.). For $j \neq 0,i$ the set $\bar{\omega}_i'$ is a subset of $\Omega - \bar{\omega}_j$. According to the lemma (Lemma 3) proved below, the contributions $j \neq 0,i$ to (D.20) must, therefore, also be of rapid decrease uniformly on $\bar{\omega}_i'$.

Therefore, the ith term of (D.20) is also of rapid decrease uniformly on $\bar{\omega}_i'$. But by virtue of Lemma 3 the ith term must be of rapid decrease uniformly also on the complement of ω_i'. Thus for all t we have

$$\lim_{|t|\to\infty} |t|^N \int dz J(z)\psi(z)\, T_J^i(z)\, e^{-i(z,t)} = 0 \qquad \text{(D.21)}$$

for all integers N. This implies that $T_J^i(z)$ is infinitely differentiable in the interior of the support of ψ.[38] Since ψ can be chosen to be nonzero at $\bar{z}$, the function $T_J^i(z)$ must be infinitely differentiable at $\bar{z}$. Thus $T^i(K)$ is, by definition, infinitely differentiable at $\bar{K}$.

This completes the proof.

Lemma 3. Let ω be an open subset of Ω, and let $\bar{\omega}'$ be a closed subset of $\Omega - \bar{\omega}$. Define

$$G(t) = \int_{\bar{C}(\omega)} dt' \tilde{\psi}(t-t')\, \tilde{T}(\chi,t'), \qquad \text{(D.22)}$$

where $\tilde{T}(\chi,t)$ and $\tilde{\psi}(t)$ are defined in the proof of Theorem 4. Then, for every integer N, the limit

$$\lim_{\tau\to\infty} \tau^N G(\hat{t}\tau) = 0 \qquad \text{(D.23)}$$

is obtained uniformly in $\hat{t} = t|t|^{-1}$ on $\bar{\omega}'$.

Proof. The function $\tilde{T}(\chi;t)$ is of at most polynomial growth as $|t|\to\infty$.[32] There is, therefore, an integer p for which $(1+|t|^p)^{-1}\,\tilde{T}(\chi;t)$ is bounded. Let A be that bound. Then

$$|G(\hat{t}\tau)| \leqslant A \int_{\bar{C}(\omega)} dt'(1+|t'|^p)|\tilde{\psi}(\hat{t}\tau-t')|. \qquad (D.24)$$

$$\leqslant A \int_{\bar{C}(\omega)} dt'(1+|t'|^p)\; C_q|\hat{t}\tau-t'|^{-q} \;. \qquad (D.25)$$

The τ dependence of the right-hand side of (D.25) can be explicitly exhibited:

$$|G(\hat{t}\tau)| \leqslant A\; C_q\; \tau^{3n-4-q}\; \{A_o(\hat{t}) \pm \tau^p\; A_p(\hat{t})\}, \qquad (D.26)$$

where

$$A_r(\hat{t}) = \int_{\bar{C}(\omega)} dt'|t'|^r\; |\hat{t}-t'|^{-q},\; (r = 0,p). \qquad (D.27)$$

Since $\bar{\omega}'\subset\Omega-\bar{\omega}$, the magnitude $|t-t'|$ is bounded from below by a positive number when $\hat{t}$ is restricted to $\bar{\omega}'$. It follows that the integrals $A_r(\hat{t})$ are bounded on $\bar{\omega}'$ if q is chosen large enough. In fact, if N is any positive integer, the number q can be chosen large enough that τ^{N+1} times the right-hand side of (D.26) is (uniformly) bounded on $\bar{\omega}'$. It follows that $G(\hat{t}\tau)$ satisfies (D.23)

APPENDIX E

A. Proof of Theorem 5.

Let $\bar{\mathcal{D}}$ be a fixed nontrivial connected causal space-time diagram, and let $\bar{K}$ belong to $(\mathcal{L}_0^+[\bar{\mathcal{D}}]-\mathcal{M}_0)$. According to Theorem 6, the set $\mathcal{L}_0^+[\bar{\mathcal{D}}]$ is locally the set of zeros (on $\mathcal{H}$) of a real analytic function $\Lambda(K)$ that has nonzero gradient near $\bar{K}$. This would immediately imply that $\mathcal{L}_0^+[\bar{\mathcal{D}}]$ is an analytic submanifold of $\mathcal{H}$ of codimension 1 at $\bar{K}$ were it not for the possibility that the gradient of Λ with respect to local coordinates might vanish, even though $\nabla\Lambda(\bar{K})$ does not. To rule out this possibility, let $\bar{\Lambda}(z) = \Lambda(K(z))$, where z is a set of local coordinates at $\bar{K}$ for $\mathcal{W}$. Then,

$$\frac{\partial\bar{\Lambda}}{\partial z_\lambda} = \frac{\partial\Lambda}{\partial K}\;\frac{\partial K}{\partial z_\lambda} \equiv \sum_{i,\mu} \frac{\partial\Lambda}{\partial k_{i\mu}}\;\frac{\partial k_{i\mu}}{\partial z_\lambda}\;, \quad (1\leqslant\lambda\leqslant 3n-4). \qquad \text{(E.1)}$$

Now, any vector $\nabla\Lambda \equiv \partial\Lambda/\partial K$ that causes (E.1) to vanish for all λ is of the form $U_0(K)$. [See (B.11).] But if $\nabla\Lambda$ were of this form, the displacements U which generate diagrams $\mathcal{D} \varepsilon [\bar{\mathcal{D}}]$, $\bar{K} = K(\mathcal{D})$, would also be of this form. Hence, because $\bar{K}$ is not in $\mathcal{M}_0$, the diagrams $\mathcal{D} \varepsilon [\bar{\mathcal{D}}]$, $\bar{K} = K(\mathcal{D})$, would be trivial. This is contrary to hypothesis. Thus, $\partial\bar{\Lambda}/\partial z_\lambda$ is nonzero and the surface $\mathcal{L}_0^+[\bar{\mathcal{D}}]$ is an analytic submanifold of $\mathcal{H}$ of codimension 1 at $\bar{K}$.

B. Proof of Theorem 6.

Let $\bar{\mathscr{D}}$ be a fixed nontrivial connected causal space-time diagram with n external lines and m vertices, and let $\mathscr{D} \varepsilon [\bar{\mathscr{D}}]$. Let

$$Q_r[K(\mathscr{D})] \equiv \sum_j |\varepsilon_{jr}| \; k_j(\mathscr{D}) \tag{E.2}$$

be the sum of the mathematical energy-momentum vectors k_j of the external lines attached to vertex r of $\mathscr{D} \varepsilon [\bar{\mathscr{D}}]$. Energy-momentum conservation at vertex r then gives

$$Q_r[K(\mathscr{D})] = F_r[V(\mathscr{D})], \tag{E.3}$$

where

$$F_r[V] = \Sigma'_j \; \mu_j \; \varepsilon_{jr} \; \|\Delta_j(V)\|^{-1} \Delta_j(V). \tag{E.4}$$

The primed sum extends only over internal lines. The vectors $\Delta_j(V)$ are defined by

$$\Delta_j(V) = \Sigma_r \; \varepsilon_{jr} \; v_r, \tag{E.5}$$

and the quantity

$$\|\Delta_j(V)\| = [\Delta_j(V)\cdot\Delta_j(V)]^{\frac{1}{2}} \tag{E.6}$$

is a Lorentz length. [The $\|\Delta_j[V(\mathcal{D})]\|^2$ are all strictly positive for $\mathcal{D}\varepsilon[\bar{\mathcal{D}}]$, by definition.] The μ_j and ε_{jr} are the masses and structure constants of $\bar{\mathcal{D}}$. Equation (E.3) is obtained by first expressing $Q_r[K(\mathcal{D})]$ in terms of the momentum-energies associated with the internal lines incident upon vertex r, and then using the identity $\|\Delta_j\| \equiv \|\alpha_j p_j\| = \alpha_j \mu_j$ to eliminate α_j.

In terms of the quantities just defined the positive-α Landau surface $\mathcal{L}^+[\bar{\mathcal{D}}]$ is the intersection of the mass shell $\mathcal{M}$ with the set

$$\mathcal{S}[\bar{\mathcal{D}}] = \{K = (k_1,\ldots,k_n)\,|\,Q_r(K) = F_r[V;\bar{\mathcal{D}}],\ V\varepsilon\Omega^+\}, \qquad \text{(E.7)}$$

where the argument $\bar{\mathcal{D}}$ in $F_r[V;\bar{\mathcal{D}}] = F_r(V)$ emphasizes the dependence of the ε_{jr} and μ_j in $F_r(V)$ upon $\bar{\mathcal{D}}$, and

$$\Omega^+ \equiv \{V\,|\,\Delta_j(V) \text{ are positive timelike}\}. \qquad \text{(E.8)}$$

If $\bar{V}\varepsilon\Omega^+$ and $\bar{K}\varepsilon\mathcal{L}^+[\bar{\mathcal{D}}]$ satisfy $Q_r(\bar{K}) = F_r(\bar{V})$, then the set

$$\Omega^+(\bar{K}) = \{V\,|\,V\varepsilon\Omega^+,\ Q_r(\bar{K}) = F_r(V)\}, \qquad \text{(E.9)}$$

consists precisely of those points V which satisfy

$$\Delta_j(V) = \lambda_j\ \Delta_j(\bar{V}), \quad \text{(All internal lines } j), \qquad \text{(E.10)}$$

where the λ_j are strictly positive scalars. For if V satisfies (E.10), it clearly belongs to $\Omega^+(\bar{K})$. [See (E.4).] Conversely, if V belongs to $\Omega^+(\bar{K})$, the vectors

$$D_r(V) = F_r(V) - F_r(\bar{V}) \tag{E.11}$$

must vanish. This gives

$$\sum_r v_r \cdot D_r(V) = \sum_j{}' \mu_j \{\|\Delta_j(V)\| - \|\Delta_j(\bar{V})\|^{-1} \Delta_j(V)\cdot\Delta_j(\bar{V})\} = 0. \tag{E.12}$$

Each term in the braces is nonpositive, hence each must vanish. This implies (E.10).

Condition (E.10) is essentially the condition that V belong to the null space of the Jacobian matrix $H(\bar{V})$ defined by

$$H_{r\mu,s\nu}(\bar{V}) = \frac{\partial F_{r\mu}}{\partial v_s{}^{\nu}}(\bar{V}) \tag{E.13a}$$

$$= \sum_{kj}{}' \varepsilon_{jr}\varepsilon_{js}\mu_j \|\Delta_j(\bar{V})\|^{-3}\{\|\Delta_j(\bar{V})\|^2 g_{\mu\nu} - \Delta_{j\mu}(\bar{V})\, \Delta_{j\nu}(\bar{V})\}. \tag{E.13b}$$

The null space of $H(\bar{V})$ consists of all m-tuples $W = (w_1,\ldots,w_m)$ of four-vectors for which the equations

$$\sum_{r\mu} w_r{}^{\mu} H_{r\mu,s\nu}(\bar{V}) = 0 \tag{E.14}$$

are satisfied for all s and ν. It is evident from (E.13b) that all vectors that satisfy (E.10) belong to this null space. Hence, the set $\Omega^+(\bar{K})$ is contained in the null space of $H(\bar{V})$. Conversely, any vector V in the null space of $H(\bar{V})$ must satisfy (E.10), without the restriction to positive λ_j. For if (E.14) is true, the equation

$$\sum_{r\mu,s\nu} w_r^{\ \mu} w_s^{\ \nu} H_{r\mu,s\nu}(\bar{V}) = 0 = \sum_j{}' \mu_j \|\Delta_j(\bar{V})\|^{-3}\Big\{\|\Delta_j(\bar{V})\|^2 \cdot \|\Delta_j(W)\|^2 - \left[\Delta_j(\bar{V})\cdot\Delta_j(W)\right]^2\Big\} \tag{E.15}$$

is also true. Since each term in the braces is nonpositive, each must vanish. This implies (E.10), without the restriction to positive λ_j.

Explicit computation shows that any linear combination of the vectors $\bar{V}$ and $E_\rho (0 \leqslant \rho \leqslant 3)$, where E_ρ is a 4m-dimensional vector with components

$$(E_\rho)_r^{\ \mu} = \delta_\rho^{\ \mu}, \tag{E.16}$$

belongs to the null space of $H(\bar{V})$. Since $\Delta_j(E_\rho) = 0$ for all ρ and j, the vectors $\bar{V}\varepsilon\Omega^+(\bar{K})$ and E_ρ must be linearly independent. The dimension $N(\bar{V})$ of the null space of $H(\bar{V})$ must therefore be at least five: $N(\bar{V}) \geqslant 5$.

On the other hand, $N(\bar{V})$ cannot be greater than 5. For suppose it were. Then there would exist some $\bar{W}$, linearly independent of the vectors $\bar{V}$ and E_ρ, such that $\bar{W}$, $\bar{V}$ and the E_ρ, and hence also any linear combination of them, belong to the null space of $H(\bar{V})$. Consider the identity

$$\Delta_j(\bar{W} + \alpha\bar{V}) \equiv \lambda_j(\alpha)\, \Delta_j(\bar{V}) = (\bar{\lambda}_j + \nabla)\, \Delta_j(\bar{V}), \tag{E.17}$$

where the $\bar{\lambda}_j$ are defined by

$$\Delta_j(\bar{W}) = \bar{\lambda}_j \Delta_j(\bar{V}). \tag{E.18}$$

The number α can obviously be chosen so that $\lambda_j(\alpha) \geqslant 0$ for all j and $\lambda_j(\alpha) = 0$ for some j. Let $\{\alpha_\lambda\}$ be a sequence $\alpha_\lambda \to \alpha$, with $\alpha_\lambda > \alpha$, and introduce

$$W_\lambda = \bar{W} + \alpha_\lambda \bar{V}, \tag{E.19}$$

The vectors W_λ belong to $\Omega^+(\bar{K})$ and they converge to a limit, $W' = \bar{W} + \alpha\bar{V}$, which is not zero since $\bar{W}$ is linearly independent of $\bar{V}$. The set of four-vectors W' defines a diagram $\mathcal{D}'$ which is a contraction of $\bar{\mathcal{D}}$. The diagram $\mathcal{D}'$ cannot be a trivial diagram because the trivial diagrams are generated only by linear combinations of the E_ρ, and W' cannot be one of these because of the linear independence of the $\bar{W}$, $\bar{V}$ and E_ρ. The function $F'_s(V)$ corresponding to the vertex s of $\mathcal{D}'$ is simply the sum of the functions $F_r(V)$ corresponding to those vertices r of $\bar{\mathcal{D}}$ that unite to form vertex s in the contraction of $\bar{\mathscr{D}}$ that gives $\mathscr{D}'$. The function $Q'_s(K)$ corresponding to the vertex s of $\mathscr{D}'$ is formed in the same way from the $Q_r(K)$ of $\bar{\mathscr{D}}$. Thus we obtain

$$Q'_s(\bar{K}) = F'_s(W_\lambda) \qquad \text{(E.20)}$$

for each value of λ. Since the F'_s do not depend on those internal lines of $\bar{\mathscr{D}}$ that are contracted in forming $\mathscr{D}'$, the limit can be taken: $Q'_s(\bar{K}) = F'_s(W')$. But then $\bar{K}$ belongs to $\mathscr{L}^+[\mathscr{D}']$. This contradicts the assumption of the theorem. Thus the quantity $N(\bar{V})$ cannot be greater than 5. But then $N(\bar{V})$ is exactly 5, and the matrix $H(\bar{V})$ has rank

$$R(\bar{V}) \equiv 4m - N(\bar{V}) = 4m - 5 \equiv R. \qquad \text{(E.21)}$$

The knowledge that $N(\bar{V}) = 5$ is itself useful. It says that all vectors W in the null space of $H(\bar{V})$ are of the form

$$W = \lambda\bar{V} + \sum_\rho a^\rho E_\rho. \qquad \text{(E.22)}$$

Thus all $V\varepsilon\Omega^+(K)$ are of this form. Variations of the scalars a_ρ simply translate the entire diagram, and variations of λ merely change the scaling of the diagram. Thus (E.22) tells us that there is essentially only one diagram $\mathcal{D}$ from the set $[\bar{\mathcal{D}}]$ that satisfies $K(\mathcal{D}) = \bar{K}$.

The vectors Q_r satisfy the four conditions $\sum_r Q_r^{\mu} = 0$, $0 \leqslant \mu \leqslant 3$. Thus we may consider the reduced space in which one of the four-vectors Q_r is eliminated. Similarly one of the four-vectors v_r is eliminated by requiring $\sum_r v_r^{\nu} = 0$. Since the eliminated rows and columns are linear combinations of the remaining ones the reduced $4(m-1)$-dimensional matrix H still has rank $R = 4m-5$.

Following the procedure of Goursat[39] one can now construct a function $\Phi(Q)$ of the remaining $(m-1)$ Q's that is real analytic at $Q = \bar{Q} \equiv Q(\bar{K})$, has a nonvanishing gradient $\nabla\Phi(Q)$ at $Q = \bar{Q}$, and which vanishes on the set

$$\mathcal{R}(\Omega') = \{Q = (Q_1,\ldots,Q_m) | Q_r = F_r(V),\ V\varepsilon\Omega'\}, \qquad \text{(E.23)}$$

for some neighborhood $\Omega' \subset \Omega^+$ of $\bar{V}$. The construction of $\Phi(Q)$ goes as follows. Since the rank of the reduced H, which we will call $\tilde{H}$, is just one less than the maximum possible rank $4(m-1)$, one may, by virtue of the implicit function theorem, arrange the Q_r^{μ} and the v_r^{μ} so that the first $R = 4m-5$ of the v_r^{μ} (called x_i's) can be expressed as real analytic functions $\bar{x}_i(X_1,\ldots,X_R,t) = \bar{x}_i(X,t)$ of the first R of the Q_r's (called X_j's) and the final v_r^{μ} (called t). These expressions $\bar{x}_i(X,t)$ for the v_r^{μ}'s are then inserted into the expression for the final Q_r^{μ} (called T). This gives

$$T = \tilde{T}(\bar{x}_1(X,t),\ldots,\bar{x}_R(X,t),t) = \bar{T}[X,t] \qquad \text{(E.24)}$$

Differentiation of (E.24) gives

$$\frac{\partial \bar{T}}{\partial t} = \sum_{i=1}^{R} \frac{\partial \tilde{T}}{\partial \bar{x}_i} \frac{\partial \bar{x}_i}{\partial t} + \frac{\partial \tilde{T}}{\partial t} . \qquad \text{(E.25)}$$

Similarly, one has

$$\tilde{X}_j(\bar{x}_1(X,t),\ldots,\bar{x}_R(X,t),t) = \bar{X}_j[X,t] \equiv X_j , \qquad \text{(E.26)}$$

which upon differentiation gives

$$\frac{\partial \bar{X}_j}{\partial t} = 0 = \sum_{i=1}^{R} \frac{\partial \tilde{X}_j}{\partial \bar{x}_i} \frac{\partial \bar{x}_i}{\partial t} + \frac{\partial \tilde{X}_j}{\partial t} , \ (1 \leqslant j \leqslant R). \qquad \text{(E.27)}$$

Equations (E.25) and (E.27) can be combined and simplified by writing $T = \bar{X}_0$ and $t = \bar{x}_0$ and by recognizing that the matrix $\partial \tilde{X}_j / \partial x_i$ is just $\tilde{H}_{ji}$:

$$\frac{\partial \bar{X}_j}{\partial t} = \sum_{i=0}^{R} \tilde{H}_{ji} \frac{\partial \bar{x}_i}{\partial t} , \ (0 \leqslant j \leqslant R). \qquad \text{(E.28)}$$

Multiplication by the matrix C of cofactors of $\tilde{H}$ yields

$$\sum_j C_{ij} \frac{\partial \bar{X}_j}{\partial t} = (\det \tilde{H}) \frac{\partial \bar{x}_i}{\partial t} . \qquad \text{(E.29)}$$

This equation, when combined with (E.27) and the fact that $\det \tilde{H} = 0$, yields

$$C_{oo} \frac{\partial \bar{T}}{\partial t} = 0. \tag{E.30}$$

But C_{oo} is the cofactor (minor) of $\tilde{H}$ that was chosen to be nonzero. There is, therefore, a full neighborhood of the image $(\bar{X},\bar{t})$ of $\bar{V}$ in which

$$\frac{\partial \bar{T}}{\partial t} = 0. \tag{E.31}$$

This implies that $\bar{T}$ is independent of t:

$$T = \bar{T}[X]. \tag{E.32}$$

Since the X_j and T are just the $Q_r{}^{\mu}$, equation (E.32) can be rewritten

$$T - \bar{T}[X] \equiv \Phi(Q) = 0. \tag{E.33}$$

This defines the real analytic function $\Phi(Q)$. It is evident from (E.33) that $\nabla\Phi(Q)$ is nonzero at $\bar{Q}$. The neighborhood Ω' of $\bar{V}$ is chosen small enough that C_{oo} is nonzero and $\bar{T}[X]$ is single-valued and holomorphic on the image $\mathcal{R}(\Omega')$ of Ω'.

We now show that there exists a 4m-dimensional neighborhood hood $\mathcal{N}(\bar{Q})$ of $\bar{Q}$ such that

$$\mathcal{R}(\Omega^+)\cap\mathcal{N}(\bar{Q}) = \{Q|Q\varepsilon\mathcal{N}(\bar{Q}),\Phi(Q) = 0\}\cap\mathcal{E} , \qquad (E.34)$$

where

$$\mathcal{E} = \{Q|Q = (Q_1,\ldots,Q_m), \ \Sigma\ Q_j = 0\}. \qquad (E.35)$$

The fact that $\mathcal{R}(\Omega^+)$ is confined to $\mathcal{E}$ follows immediately from (E.3) and (E.4) by explicit computation. The nontrivial content of (E.34) is that, subject to this restriction, the zeros of Φ exactly coincide with $\mathcal{R}(\Omega^+)$ in some neighborhood of $\bar{Q}$.

The construction of the function Φ ensures that it vanishes on $\mathcal{R}(\Omega')$:

$$\mathcal{R}(\Omega')\subset\{Q|\Phi(Q) = 0\}\cap\mathcal{E} . \qquad (E.36)$$

To show (E.34) we first show that a neighborhood $\mathcal{N}'(\bar{Q})$ of $\bar{Q}$ can be chosen so that

$$\mathcal{R}(\Omega^+)\cap\mathcal{N}'(\bar{Q})\subset\mathcal{R}(\Omega'). \qquad (E.37)$$

Suppose this were not true. Then one could find a sequence of points $Q(\lambda)\to\bar{Q}$ such that, for each value of λ, $Q(\lambda)$ is in $\mathcal{R}(\Omega^+)$ but not in $\mathcal{R}(\Omega')$. Each of these points $Q(\lambda)$ is generated by a corresponding point $V(\lambda)\varepsilon\Omega^+$, which can be required to satisfy $\Sigma\, v_r(\lambda) = 0$ and $\Sigma' \| \Delta_j(V(\lambda) \| = 1$. [The value of the mapping F of (E.4) is insensitive to such restrictions.] The points $V(\lambda)$ are then confined to a bounded region of V space. For if this were not true, the Euclidean norms of the difference vectors $\Delta_j(V(\lambda))$ would have to be unbounded for some j. This cannot be reconciled with the required boundedness of both their Lorentz norms and the energy components of all the $Q(\lambda)$.

Since the $V(\lambda)$ are confined to a bounded region the infinite sequence of $V(\lambda)$ must have a subsequence that has a limiting point $V(\infty)$. If this limit point were in Ω^+ then the continuity of $F(V)$ would ensure that the image (under F) of $V(\infty)$ would be $\bar{Q}$. This would require that $V(\infty)$ have the form (E.22). The normalization and translation conditions would then ensure that $V(\infty) \equiv \bar{V}$. This is not possible since the $V(\lambda)$ must all lie outside the neighborhood Ω' of $\bar{V}$. Thus $V(\infty)$ cannot be an element of Ω^+.

The only other possibility is that some of the $\| \Delta_j(V(\infty)) \|$ are zero. The corresponding vectors $\Delta_j(V(\infty))$ must then also be zero. For if $\| \Delta_j(V(\lambda)) \| \to 0$ but $\Delta_j(V(\lambda)) \neq 0$, then the energy parts of some of the Q_r are forced to become infinite, which contradicts the requirement $Q_r(\lambda) \to \bar{Q}_r$. Thus certain of the vectors $\Delta_j(V(\infty))$ must be zero. Not all can be zero because of the condition $\Sigma' \| \Delta_j(V(\lambda)) \| = 1$. Thus, after appropriate scaling, overall translation and specification of the individual external momenta incident on each vertex, the diagram corresponding to $V(\infty)$ is a contraction $\mathcal{D}'$ of $\bar{\mathcal{D}}$. Equation (E.20) again yields a violation of our original hypothesis that $\bar{K}\varepsilon\mathcal{L}_0^+[\bar{\mathcal{D}}]$. Thus none of the $\Delta_j(V(\infty))$ can vanish.

All alternatives having been ruled out, equation (E.37) is established. It follows from (E.36) that there exists a 4m-dimensional neighborhood $\mathcal{N}'(\bar{Q})$ of $\bar{Q}$ such that

$$\mathcal{R}(\Omega^+)\cap\mathcal{N}'(\bar{Q})\subset\{Q|Q\varepsilon\mathcal{N}'(\bar{Q}),\ \Phi(Q) = 0\}\cap\mathcal{E}\ . \tag{E.38}$$

This result is half of (E.34).

To complete the proof of (E.34) we construct a 4m-dimensional neighborhood $\mathcal{N}''(\bar{Q})$ of $\bar{Q}$ such that

$$\{Q|Q\varepsilon\mathcal{N}''(\bar{Q}),\ \Phi(Q) = 0\}\cap\mathcal{E}\subset\mathcal{R}(\Omega^+)\cap\mathcal{N}''(\bar{Q}). \tag{E.39}$$

Then (E.34) is satisfied with $\mathcal{N}(\bar{Q}) = \mathcal{N}'(\bar{Q})\cap\mathcal{N}''(\bar{Q})$.

To prove (E.39) consider the equations

$$x_i = \bar{x}_i(X,t), \tag{E.40}$$

where the functions on the right are those appearing in (E.24). Combining (E.40) with the condition $\Sigma v_r = 0$, one obtains a system of equations

$$v_r^{\mu} = \bar{v}_r^{\mu}(X,t) \tag{E.41}$$

that gives all the v_r^μ as functions of the X_j $(1 \leq j \leq R)$, and t, where t is just one of the v_r^{μ}'s. Let $\bar{X}$ be the projection of $\bar{Q}$ onto X-space, and let $\bar{t}$ be the value of t such that $\bar{V} \equiv V(\bar{X},\bar{t})$ is the point of Ω' that satisfies $\bar{Q} = F(\bar{V})$ and $\Sigma'\Delta_j(\bar{V}) = 1$. Because of the non-singular nature of the mapping (E.41) there are neighborhoods $\mathcal{N}_X$ and $\mathcal{N}_t$ of $\bar{X}$ and $\bar{t}$ such that the image [under (E.41)] of $\mathcal{N}_X \times \mathcal{N}_t$ is contained in Ω'. Moreover, (E.26) and (E.33) imply that if the projection $X(Q)$ of Q onto X-space belongs to $\mathcal{N}_X$, if $Q \varepsilon \mathcal{E}$, and if $\Phi(Q) = 0$, then $Q = F[\bar{V}(X(Q),t)]$ for any $t \varepsilon \mathcal{N}_t$. Thus, every point of $\{\Phi(Q) = 0\}$ and $\mathcal{E}$ that satisfies $X(Q) \varepsilon \mathcal{N}_X$ is generated by some point V in Ω'. Taking $\mathcal{N}''(\bar{Q})$ to be the set $\{Q | X(Q) \varepsilon \mathcal{N}_X\}$, we have (E.39). Thus (E.34) is proved.

The proof of Theorem 6 is completed by transforming the preceding results from Q-space into K-space. Thus one defines

$$\Lambda(K) = \Phi(Q(K)) \qquad \text{(E.42)}$$

and lets $\mathcal{N}(\bar{K})$ be any K-space neighborhood of $\bar{K}$ with image [under (E.2)] in Q-space contained in $\mathcal{N}(\bar{Q})$. Since $\mathcal{S}[\bar{\mathcal{D}}]$ is the K-space image of $\mathcal{R}(\Omega^+)$, (E.34) becomes (5.3). [All points of $\mathcal{L}^+ \cap \mathcal{N}$ belong to $\mathcal{M}$, and hence also to $\mathcal{K}$ if $\mathcal{N}(\bar{K}) = \mathcal{N}$ is a small enough neighborhood of $\bar{K} \varepsilon \mathcal{K}$.]

If K is a point of $\mathcal{L}^+[\bar{\mathcal{D}}] \cap \mathcal{N}(\bar{K})$ then the point $Q(K)$ lies in $\mathcal{R}(\Omega^+) \cap \mathcal{N}(\bar{Q})$. Hence, by virtue of (E.37), $Q(K)$ lies in $\mathcal{R}(\Omega')$. Thus there is a $V(K)$ in Ω' such that $F[V(K)] = Q(K)$. For all points in Ω' we have $C_{oo} \neq 0$. Thus the rank, $R[V(K)]$ of $H(V(K))$ is $4m-5$. The arguments that gave (E.22) show also that any vector in the null space of $H(V(K))$ is of the form

$$W = \lambda V(K) + \sum_{\rho} a^{\rho} E_{\rho}. \tag{E.43}$$

However, the gradient $\nabla\Phi(Q)$ at $Q = Q(K)$ belongs to the null space of $H(V(K))$, as is seen from

$$\frac{\partial \Phi[F(V)]}{\partial v_s^{\nu}} = \sum_{r\mu} \frac{\partial \Phi(F_{r\mu})}{\partial F_{r\mu}} \frac{\partial F_{r\mu}(V)}{\partial v_s^{\nu}}, \tag{E.44a}$$

$$= \sum_{r\mu} \frac{\partial \Phi(Q_{r\mu})}{\partial Q_{r\mu}} H_{r\mu,s\nu}(V) \tag{E.44b}$$

$$= \sum_{r\mu} (\nabla \Phi)_r^{\mu} H_{r\mu,s\nu}(V) \equiv 0. \tag{E.44c}$$

Since $\nabla \Phi$ is nonzero we may rewrite (E.43) (using new λ and a^{ρ}) as $v_r^{\mu}(K) = \lambda\nabla\Phi[Q(K)]_r^{\mu} + \sum_{\rho} a^{\rho}(E_{\rho})_r^{\mu}$ or, more briefly, as

$$V(K) = \lambda\nabla\Phi + \sum a^{\rho}E_{\rho}, \tag{E.45}$$

where the sign of Φ is chosen so that λ is positive.

The positions of the vertices $v_r(K)$ determine the positions of the lines of the corresponding diagram. In particular the position of the external line L_i is generated by the displacement

$$u_i = \lambda(\nabla\ \Phi)_{r(i)} + \Sigma\ a^\rho (E_\rho)_{r(i)} \qquad \text{(E.46)}$$

where $r(i)$ labels the vertex to which L_i is connected. The general displacement that generates this position of L_i is obtained by adding an arbitrary translation of this line along itself:

$$u_i{}^\mu = \lambda(\nabla\ \Phi)_{r(i)}{}^\mu + \Sigma\ a^\rho (E_\rho)_{r(i)}{}^\mu + t_i k_i{}^\mu \qquad \text{(E.47)}$$

The E_ρ is independent of r [See (E.16).] and can be considered a set of vectors over i, rather than r. Since $Q_{r(i)}$ is a sum of terms containing k_i we can write

$$\frac{\partial\Phi}{\partial k_{i\mu}} = \Sigma\ \frac{\partial\Phi}{\partial Q_{r\nu}}\ \frac{\partial Q_{r\nu}}{\partial k_{i\mu}} = \frac{\partial\Phi}{\partial Q_{r(i)}{}^\mu} \equiv (\nabla\ \Phi)_{r(i)}{}^\mu . \qquad \text{(E.48)}$$

Substitution of (E.48) into (E.47) then gives

$$u_i = \lambda\ \frac{\partial\Phi[Q(K)]}{\partial k_{i\mu}} + \Sigma\ a^\rho\ \delta_\rho{}^\mu + t_i k_i{}^\mu , \qquad \text{(E.49)}$$

which is just (5.4).

C. Proof of Theorem 8.

Let $\bar{V} = V(\bar{\mathcal{D}})$. It was shown below (E.14) that the null space of $H(\bar{V})$ contains $\bar{\Omega}^+(\bar{K})$, the closure of the set $\Omega^+(\bar{K})$. The set $\bar{\Omega}^+(\bar{K})$ contains the vectors $V = V(\mathcal{D})$ for all diagrams $\mathcal{D}$ that satisfy $\bar{K} = K(\mathcal{D})$ and belong either to $[\bar{\mathcal{D}}]$ or to $[\mathcal{D}']$ for some $\mathcal{D}' \subset \bar{\mathcal{D}}$. Hence, the null space of $H(\bar{V})$ contains all points V that correspond to the diagrams $\mathcal{D}$ of the theorem.

Let the vectors E_ρ defined by (E.16) together with the vectors of the set $\{V_1,\ldots,V_p\}$, where $p = N(\bar{V})-4$, be a basis for the null space of $H(\bar{V})$. Thus, any vector W of this null space has a unique representation

$$W = \sum_{i=1}^{p} \lambda_i V_i + \sum_{\rho=0}^{3} a^\rho E_\rho . \tag{E.50}$$

Because of (E.10), the vector W must satisfy the equations

$$\Delta_j(W) = \alpha_j(W)\, \Delta_j(\bar{V}) = (\sum_i \lambda_i X_{ij})\Delta_j(\bar{V}), \tag{E.51}$$

where the X_{ij} are defined by

$$\Delta_j(V_i) = X_{ij}\Delta_j(\bar{V}) \tag{E.52}$$

Because $\bar{\mathcal{D}}$ is connected, the condition $\Delta_j(W) = 0$ for all j implies $\lambda_i = 0$ for all i. This in turn implies the linear independence of the vectors $\underset{\sim}{R}_i = (X_{i1}, X_{i2}, \ldots)$.

These vectors R_i form a basis for the space of vectors $\underset{\sim}{\alpha} = (\alpha_1, \alpha_2, \ldots)$ appearing in (E.51). Through (E.51) the vector $\underset{\sim}{\alpha}(W)$ specifies W up to an overall translation $\Sigma a^\rho E_\rho$.

In terms of $\underset{\sim}{\alpha}$ vectors the set $\bar{\Omega}^+(\bar{K})$ has the following description. For any W in $\bar{\Omega}^+(\bar{K})$ the vector $\underset{\sim}{\alpha}(W)$ is a linear combination, $\underset{\sim}{\alpha}(W) = \Sigma\lambda_i(W)\underset{\sim}{R}_i$, of the $\underset{\sim}{R}_i$. The vector W is in $\bar{\Omega}^+(\bar{K})$ if and only if the vectors $\underset{\sim}{\lambda}(W) = (\lambda_1(W), \ldots, \lambda_p(W))$ and $\underset{\sim}{C}_j = (X_{1j}, \ldots, X_{pj})$ satisfy $\underset{\sim}{\lambda}(W)\cdot\underset{\sim}{C}_j \geqslant 0$ for all j. [The index j labels the internal lines of $\bar{\mathcal{D}}$.] From this description it is clear that $\bar{\Omega}^+(\bar{K})$ is convex and starlike [$W \epsilon \bar{\Omega}^+(\bar{K})$ implies $\lambda W \epsilon \bar{\Omega}^+(\bar{K})$ for all $\lambda \geqslant 0$.]

Consider a nonzero vector $\underset{\sim}{\alpha}(W)$ corresponding to a point W of $\bar{\Omega}^+(\bar{K})$. If all other points W' of $\bar{\Omega}^+(\bar{K})$ give an $\underset{\sim}{\alpha}(W')$ proportional to $\underset{\sim}{\alpha}(W)$ then $p = 1$ and the dimension of the null space of $H(\bar{V})$ is five. In this case no contraction $\mathcal{D}' \subset \mathcal{D}$ can give point $K(\mathcal{D}') = \bar{K}$ and Theorem 6 gives the desired result. Alternatively, if there is a $\underset{\sim}{\alpha}(W')$ not proportional to $\underset{\sim}{\alpha}(W)$, then let P be the plane through the origin that contains both $\underset{\sim}{\alpha}(W)$ and $\underset{\sim}{\alpha}(W')$. The intersection of P with the image A^+ in $\underset{\sim}{\alpha}$-space of $\bar{\Omega}^+(\bar{K})$ is two dimensional, convex, and starlike. The boundaries of $P \cap A^+$ are therefore two half-lines originating at the origin which, because A^+ contains no vector $\underset{\sim}{\alpha}$ with any negative components, must intersect in an angle less than π. Let $\underset{\sim}{\alpha}(W_1)$ and $\underset{\sim}{\alpha}(W_2)$ be vectors in A^+ that define these two boundary rays. In terms of these vectors, the original vector $\underset{\sim}{\alpha}(W)$ has the representation

$$\underset{\sim}{\alpha}(W) = y_1\underset{\sim}{\alpha}(W_1) + y_2\underset{\sim}{\alpha}(W_2), \tag{E.53}$$

where y_1 and y_2 are strictly positive.

Because $\underset{\sim}{\alpha}(W_1)$ and $\underset{\sim}{\alpha}(W_2)$ lie in the boundary of $P \cap A^+$, the corresponding vectors $\underset{\sim}{\lambda}(W_1)$ and $\underset{\sim}{\lambda}(W_2)$ must be orthogonal to some of the vectors $\underset{\sim}{C}_i$. The vector $\underset{\sim}{\lambda}(W_1)$ is orthogonal to $\underset{\sim}{C}_i$, $i \varepsilon I_1$, and the vector $\underset{\sim}{\lambda}(W_2)$ is orthogonal to $\underset{\sim}{C}_i$, $i \varepsilon I_2$.

There are two alternatives for the vector $\underset{\sim}{\alpha}(W_1)$. The first is that $\underset{\sim}{\alpha}(W_1)$ and its positive multiples are the only vectors in A^+ for which the corresponding vectors $\underset{\sim}{\lambda}$ are orthogonal to the vectors $\underset{\sim}{C}_i$, $i \varepsilon I_1$. The second is that there is some second linearly independent vector $\underset{\sim}{\alpha}(W_1')$ in A^+ with $\underset{\sim}{\lambda}(W_1')$ orthogonal to the vectors $\underset{\sim}{C}_i$, $i \varepsilon I_1$.

In the first case the vector W_1 satisfies $W_1 = V(\mathcal{D}_1)$, where $\bar{K} = K(\mathcal{D}_1)$ and the diagram $\mathcal{D}_1$ cannot be further contracted at $\bar{K}$. Thus, the point $\bar{K}$ belongs to $\mathcal{L}_0^+[\mathcal{D}_1]$, and W_1 must have the form (E.45). Equation (E.53) allows one to write

$$W = \lambda_1 \nabla\Phi + y_2 W_2 + \Sigma a^\rho E_\rho, \tag{E.54}$$

where λ_1 is positive.

In the second case the analysis just peformed on $\underset{\sim}{\alpha}(W)$ is applied to $\underset{\sim}{\alpha}(W_1)$. The plane P_1 corresponding to P contains $\underset{\sim}{\alpha}(W_1')$ and $\underset{\sim}{\alpha}(W_1)$. The intersection $P_1 \cap A^+$ has boundary rays defined by $\underset{\sim}{\alpha}(W_{11})$ and $\underset{\sim}{\alpha}(W_{12})$ such that the corresponding vectors $\underset{\sim}{\lambda}(W_{11})$ and $\underset{\sim}{\lambda}(W_{12})$ are both orthogonal not only to the vectors $\underset{\sim}{C}_i$, $i \varepsilon I_1$, but to some additional $\underset{\sim}{C}_i$ as well. In terms of these new vectors $\underset{\sim}{\alpha}(W_{11})$ and $\underset{\sim}{\alpha}(W_{12})$ the vector $\underset{\sim}{\alpha}(W_1)$ can be written

$$\underset{\sim}{\alpha}(W_1) = y_{11}\underset{\sim}{\alpha}(W_{11}) + y_{12}\underset{\sim}{\alpha}(W_{12}), \tag{E.55}$$

where y_{11} and y_{12} are strictly positive.

The entire analysis is then repeated with $\underset{\sim}{\alpha}(W_{11})$, $\underset{\sim}{\alpha}(W_{12})$ and $\underset{\sim}{\alpha}(W_2)$, etc. At each stage at least one new $\underset{\sim}{C}_i$ is added to the previous set of $\underset{\sim}{C}_i$'s. Since the number of $\underset{\sim}{C}_i$'s is finite, the procedure must terminate. At that stage all the vectors into which W is decomposed are associated with diagrams that have no further contractions. Thus we obtain

$$W = \Sigma\lambda_g \nabla\Phi_g + \Sigma a^\rho E_\rho, \tag{E.56}$$

where $\lambda_g \geqslant 0$ and the sum runs over those diagrams $\mathcal{D}_g \subset \bar{\mathcal{D}}$ or $\mathcal{D}_g = \bar{\mathcal{D}}$ that satisfy $\bar{K} = K(\mathcal{D}_g)$, but which have no contractions that do.

The arguments following (E.42) complete the proof.

D. Proof of Theorem 13.

Because of theorems 10 and 11 it is sufficient to show that the set (5.5) is convex, apart from vectors of the form $U_0(\overline{K})$. In particular, we wish to show that the simultaneous equations

$$\overline{U} = \Sigma \lambda_g \nabla \Lambda_g(\overline{K}) + U_0(\overline{K}) \qquad (\lambda_g \geqslant 0) \tag{E.57a}$$

and

$$-\overline{U} = \Sigma \lambda_g' \nabla \Lambda_g(\overline{K}) + U_0'(\overline{K}) \qquad (\lambda_g' \geqslant 0) \tag{E.57b}$$

imply that $\overline{U} = U_0(\overline{K}) = -U_0'(\overline{K})$. Adding (E.57a) to (E.57b) we obtain

$$W = \sum_g (\lambda_g + \lambda_g') \nabla \Lambda_g(\overline{K}) \tag{E.58}$$

where $W = -U_0(\overline{K}) - U_0'(\overline{K})$. Define [see (E.42)]

$$V = \sum_g (\lambda_g + \lambda_g') \nabla \mathcal{Q}_g(\overline{Q}). \tag{E.59}$$

This V gives the positions of all the vertices of a diagram with external lines specified by W. Because all the $\mathcal{D}_g$ are contractions of $\mathcal{D}$, we have

$$\begin{aligned} \Delta_j(V) &= \sum_g (\lambda_g + \lambda_g') \Delta_j(\nabla \mathcal{Q}_g) \\ &= \sum_g (\lambda_g + \lambda_g') X_{gj} \Delta_j(V(\mathcal{D})) \end{aligned} \tag{E.60}$$

Because the λ_g, λ_g', X_{gj} and $\Delta_j(V(\mathcal{D}))$ are all nonnegative, so are the $\Delta_j(V)$. But the positions of the external lines of V are given by

$W = -U_0(\overline{K}) - U_0'(\overline{K})$. Therefore V must be a trivial diagram, since for $\overline{K} \notin \mathcal{M}_0$ no nontrivial connected causal diagram can have its external lines coincident with those of a trivial diagram. But if V is trivial then (E.60) implies that $\lambda_g + \lambda_g'$ is zero, for all g. Thus λ_g and λ_g' vanish separately and $U = U_0(\overline{K}) = -U_0'(\overline{K})$.

To complete the proof the $\mathcal{U} = (U_1, \cdots U_{3n-4})$ of Theorem 10 is chosen to contain a subset S of the set of vectors $\nabla\Lambda_g(\overline{K})$ such that S together with the n+4 vectors of $U_0(\overline{K})$ are a set of linearly independent vectors that span a space that contains all of the vectors $\nabla \Lambda_g(\overline{K})$. The set of vectors of the form $\Sigma' \lambda_g \nabla \Lambda_g(\overline{K})$ with $\lambda_g \geq 0$ and $\Sigma\lambda_g \neq 0$, where Σ' is over S, is a convex set by the argument given above. Then Theorem 11 insures that $\Gamma_c(U;\overline{K})$ is contained in a single set of the form (4.11), and Theorem 10 completes the proof.

APPENDIX F

A. Proof of Theorem 7.

The first step of the proof is to show that $\nabla\Lambda_1(\bar{K}) = \lambda\nabla\Lambda_2(\bar{K}) + U_o(\bar{K})$, where λ is real and $U_o(\bar{K})$ is of the form (4.8). The more difficult second step is to show that the number λ must be positive.

Since $\mathcal{L}_0^+[\mathcal{D}_2]$ is a submanifold of codimension 1 in $\mathcal{H}$ at $\bar{K}$, there exists a local coordinate system $(\Delta_c(\bar{K}), \Pi_{\bar{K}}, D_c(\bar{K}))$ at $\bar{K}$ such that $z_1 = \Lambda_2(K)$. [See (E.1)] The fact that $\mathcal{L}_0^+[\mathcal{D}_1]$ and $\mathcal{L}_0^+[\mathcal{D}_2]$ coincide in some neighborhood $\mathcal{N}$ of $\bar{K}$ means that in some neighborhood $N \subset \Pi_{\bar{K}}^{-1}(\mathcal{N} \cap \Delta_c(\bar{K}))$ of $\bar{z} = \Pi_{\bar{K}}^{-1}(\bar{K})$, the function $\bar{\Lambda}_1(z) \equiv \Lambda_1(K(z))$ vanishes whenever $z_1 = 0$. That is, in some neighborhood $N' \subset N$ of $\bar{z}$ the real analytic function $\bar{\Lambda}_1(z)$ has a power series expansion

$$\bar{\Lambda}_1(z) = \sum_{m=1}^{\infty} a_m(z_2, \ldots, z_{3n-4}) z_1^m, \qquad \text{(F.1)}$$

where the a_m are real analytic functions. Explicit computation then shows that

$$\frac{\partial\bar{\Lambda}_1}{\partial z_j}(\bar{z}) = \lambda \frac{\partial\bar{\Lambda}_2}{\partial z_j}(\bar{z}), \qquad \text{(F.2)}$$

where λ is some real number. Since $\Lambda_1(K)$ and $\Lambda_2(K)$ are real analytic functions of K, we have

$$\frac{\partial \Lambda_g}{\partial z_j} = (\nabla \Lambda_g) \frac{\partial K}{\partial z_j} , \; (g = 1,2). \tag{F.3}$$

[See (E.1).] Equations (F.2) and (F.3) combine to yield

$$[\nabla \Lambda_1(\bar{K}) - \lambda \nabla \Lambda_2(\bar{K})] \frac{\partial K}{\partial z_j} (\bar{z}) = 0 \tag{F.4}$$

for all j. Since the only vectors that are annihilated by the matrix $\partial K/\partial z$ have the form (4.8) [see (B.11)], equation (F.4) implies that $\nabla \Lambda_1(\bar{K}) = \lambda \nabla \Lambda_2(\bar{K}) + U_o(\bar{K})$, where λ is real and $U_o(\bar{K})$ has the form (4.8).

We first examine the case where λ is strictly positive; the other case $(\lambda<0)$ will then be easy to rule out.

For each value of g [$g=1,2$] equation (E.42) gives

$$\frac{\partial \bar{\Lambda}_g(z)}{\partial z_\lambda} = \sum_{r\nu i\mu} \frac{\partial \Phi_g(Q[K(z)])}{\partial Q_{r\nu}} \frac{\partial Q_{r\nu}}{\partial k_{i\mu}} \frac{\partial k_{i\mu}}{\partial z_\lambda} \tag{F.5a}$$

$$= \sum_{i\mu} \frac{\partial \Phi_g}{\partial Q_{i\mu}} \frac{\partial k_{i\mu}}{\partial z_\lambda} \tag{F.5b}$$

where Q_i is the vertex momentum Q_r that depends on k_i. According to (B.11) the left side of this equation determines $\partial \Phi_g/\partial Q_{i\mu}$, apart from vectors of the form $U_o[K(z)]$. Then, in view of (E.45), $\nabla\bar{\Lambda}_g$ determines the positions of the external vertices of the diagrams $\mathcal{D}_g$ apart from scalings, overall translations, and translations of the position of the vertex v_i that is connected to L_i along L_i. The L_i are here considered to be complete lines, not just line segments.

It is useful to introduce diagrams $\mathcal{D}_1(z)$ and $\mathcal{D}_2(z)$ that differ from the original diagrams $\mathcal{D}_1$ and $\mathcal{D}_2$ by scaling and choice of origin. The fact that $\mathcal{D}_1$ is a non-trivial connected causal diagram ensures that there is a pair of vertices, v_I and v_F, such that v_I is connected to two initial lines, v_F is connected to two final lines, and v_F is in the positive light cone of v_I. Let the position and scale of $\mathcal{D}_1(z)$ be fixed by placing v_I at the origin and requiring that $|v_I-v_F| = 1$. According to the results established above, the external lines whose intersection defines v_I in $\mathcal{D}_1$ must also cross in $\mathcal{D}_2$, and similarly for v_F. Thus the position and scale of $\mathcal{D}_2(z)$ can also be fixed by placing v_I at the origin and normalizing so that $|v_I-v_F| = 1$. [In effect, λ is normalized to unity.]

Diagrams constructed according to the rule just described will be called <u>adjusted</u> diagrams. The result (E.45) is also applicable to them. In particular, equation (E.45) implies that for a sufficiently small neighborhood $\mathcal{N}(\bar{z})$ of $\bar{z}$ that does not intersect the Landau surfaces for any contractions of $\mathcal{D}_g$, each point z of $\mathcal{N}(\bar{z})\subset\{\bar{\Lambda}_g(z) = 0\}$ corresponds to a unique adjusted diagram $\mathcal{D}_g(z) \varepsilon [\mathcal{D}_g(\bar{z})]$. This is because all ambiguities of translation and scaling have been removed.

The vertex of $\mathcal{D}_g(z)$ that is connected to the external line L_i is called $v_{gi}(z)$, and the line parallel to k_i passing through $v_{gi}(z)$ is called $L_{gi}(z)$.

The arguments given above show that for each i, $L_{1i}(z)$ coincides with $L_{2i}(z)$, but they do not show that $v_{1i}(z)$ coincides with $v_{2i}(z)$; these two points could be different points of $L_i(z)$. The main part of the proof consists in showing that the vertices $v_{1i}(z)$ and $v_{2i}(z)$ do in fact coincide if either is connected to two different initial lines (including L_i) or two different final lines (including L_i).

Let L_i be an initial particle line. If both v_{1i} and v_{2i} are connected to the same additional initial external line $L_j (i \neq j)$, then $v_{1i}(\bar{z})$ and $v_{2i}(\bar{z})$ must coincide. For since $\bar{K}$ does not lie on $\mathfrak{M}_0$, the two different initial lines intersect in at most one point. More generally, suppose that v_{1i} is connected to the two initial lines L_i and $L_j (i \neq j)$, and that v_{2i} is connected to the two initial lines L_i and $L_k (i \neq k,\ j \neq k)$. Then again $v_{1i}(\bar{z})$ and $v_{2i}(\bar{z})$ must coincide. For a small rotation of the two intersecting lines L_i and L_j about the axis $k_i(\bar{z}) + k_j(\bar{z})$ through $v_{1i}(\bar{z})$ gives a nearby point z' of $\{\Lambda_1(z) = 0\}$. This is because the sum $k_i + k_j$ is not changed. The new point z' must belong also to $\{\Lambda_2(z) = 0\}$. Thus there must be a point $v_{2i}(z') = v_{2k}(z')$. But then $L_i(z')$ must intersect the line

$L_k(z') = L_k(\bar{z})$. This can be true for several z' near $\bar{z}$ only if $L_k(\bar{z})$ passes through the point $v_{1i}(\bar{z})$. This implies that the point $v_{2i}(\bar{z})$ must coincide with $v_{1i}(\bar{z})$.

We now show that this result $(v_{1i}(\bar{z}) = v_{2i}(\bar{z}))$ also holds provided only that the vertex v_{1i} is contained in two initial lines L_i and $L_j (i \neq j)$. For every z in some neighborhood of $\bar{z}$ the line $L_i(z)$ contains both $v_{1i}(z)$ and $v_{2i}(z)$. The point $v_{1i}(z)$ may or may not be a vertex of $\mathcal{D}_2(z)$. In either case one can construct a causal diagram $\mathcal{D}_3(z)$ containing $v_{1i}(z)$ as an external vertex and with external lines coinciding with those of $\mathcal{D}_2(z)$. One simply regards the part of $L_i(z)$ lying between $v_{1i}(z)$ and $v_{2i}(z)$ as an internal line of $\mathcal{D}_3(z)$, and similarly for all lines L_k that in diagram $\mathcal{D}_1$ are connected to v_{1i}. All the conditions for a causal diagram are satisfied by these diagrams $\mathcal{D}_3(z) \varepsilon [\mathcal{D}_3(\bar{z})]$. Since the external lines $L_i(z)$ of $\mathcal{D}_3(z)$ are the same as those of $\mathcal{D}_2(z)$, we see that in some neighborhood N of $\bar{z}$ the surface $\mathcal{L}^+[\mathcal{D}_3(\bar{z})]$ contains the surface $\{z|\bar{\Lambda}_2(z) = 0\}$.

We will now show that $\bar{K}$ belongs to $\mathcal{L}_0^+[\mathcal{D}_3(\bar{z})]$. Suppose that this is not true, and that $\bar{K}$ belongs to $\mathcal{L}^+[\mathcal{D}'_3(\bar{z})]$, where $\mathcal{D}'_3(\bar{z}) \subset \mathcal{D}_3(\bar{z})$. According to the arguments of Appendix E [see (E.10)] the internal lines of $\mathcal{D}'_3(\bar{z})$ must be parallel to the corresponding internal lines of $\mathcal{D}_3(\bar{z})$. But then there would be a diagram $\mathcal{D}'_2(\bar{z})$ contained in $\mathcal{D}'_3(\bar{z})$ that would have the same external lines as $\mathcal{D}_2(\bar{z})$.

This diagram $\mathcal{D}'_2(\bar{z})$ would be either an element of $[\mathcal{D}_2(\bar{z})]$ that has the same external lines as $\mathcal{D}_2(\bar{z})$, or a contraction of such a diagram. The conditions of the theorem ensure that no contraction of $\mathcal{D}_2(\bar{z})$ has the same external lines as $\mathcal{D}_2(\bar{z})$. And (E.45) shows that the only element of $[\mathcal{D}_2(\bar{z})]$ that has the same external lines as $\mathcal{D}_2(\bar{z})$ is $\mathcal{D}_2(\bar{z})$ itself. This would make $\mathcal{D}'_2(\bar{z})$ identical to $\mathcal{D}_2(\bar{z})$. But then the contraction $\mathcal{D}'_3(\bar{z})$ of $\mathcal{D}_3(\bar{z})$ would be identical to $\mathcal{D}_3(\bar{z})$, which is not possible. It follows that $\bar{K}$ belongs to $\mathcal{L}_0^+[\mathcal{D}_3(\bar{z})]$.

The surface $\mathcal{L}_0^+[\mathcal{D}_3(\bar{z})]$ is a submanifold of $\mathcal{H}$ of codimension 1 in a neighborhood of $\bar{K}$. In the space of local coordinates z let $\mathcal{L}_0^+[\mathcal{D}_3(\bar{z})]$ be represented by $\{\bar{\Lambda}_3(z) = 0\}$. Since $\mathcal{L}_0^+[\mathcal{D}_3(\bar{z})]$ contains $\mathcal{L}_0^+[\mathcal{D}_2(\bar{z})]$ in some neighborhood of $\bar{K}$, and since both are submanifolds of $\mathcal{H}$ of codimension 1, it follows from the arguments leading to (F.1) that the two surfaces are identical in some neighborhood of $\bar{K}$.

A rotation of the lines $L_i(z)$ and $L_j(z)$ which intersect at $v_{1i}(z)$ about the axis $k_i(z) + k_j(z)$ takes one to a nearby point on $\{\bar{\Lambda}_1(z) = 0\}$, and hence on $\{\bar{\Lambda}_3(z) = 0\}$. The vertices of the unique corresponding diagram $\mathcal{D}_3(z)$ must be the same as those of $\mathcal{D}_3(\bar{z})$, since the positions of the vertices depend only on the $Q[K(z)]$, by virtue of (E.45), and these remain unaltered. However, the vertex of $\mathcal{D}_3(z)$ at $v_{2i}(z)$ will not coincide with the vertex of $\mathcal{D}_3(\bar{z})$ at $v_{2i}(\bar{z})$ for arbitrary rotations unless $v_{2i}(\bar{z}) = v_{1i}(\bar{z})$. This is the desired result.

Since $\mathcal{D}_1$ and $\mathcal{D}_2$ are interchangeable, the above result shows that $v_{1i}(\bar{z})$ and $v_{2i}(\bar{z})$ must coincide if either is connected to two different initial lines.

Similar arguments hold for vertices connected to final lines.

The preceding result is useful in the following way. For any $v_I'(\bar{z})$ that is connected to two initial lines there is a v_F' connected to two final lines that lies in the positive light cone of v_I' or coincides with v_I'. Thus the original v_I and v_F can be chosen to satisfy the additional conditions that there is no v_F' positive timelike relative to v_F, and there is no v_I' negative timelike relative to v_I. Then The total external momentum Q_F at v_F must be positive timelike, and Q_I at v_I must be negative timelike. This is because the internal lines connected to v_F must all terminate at v_F, and the internal lines connected to v_I must all originate at v_I.

The above discussion refers to the case in which the signs of $\nabla\bar{\Lambda}_1$ and $\nabla\bar{\Lambda}_2$ are the same. If these signs were opposite, the external lines and vertices of $\mathcal{D}_2(\bar{z})$ would be obtained from those of $\mathcal{D}_1(\bar{z})$ by reflection through the origin. But this clearly cannot give a causal diagram, for the vertex $\hat{v}_F (\equiv -v_F)$ of $\mathcal{D}_2(\bar{z})$ would have no vertices $\hat{v}_I' (\equiv -v_I')$ of $\mathcal{D}_2(\bar{z})$ lying negative timelike to it. No internal lines could terminate on it and Q_F could not be positive timelike, contrary to fact. Thus the gradients $\nabla\Lambda_1$ and $\nabla\Lambda_2$ must have the same sign.

B. Proof of Theorem 10.

The first step is to show that if the conditions of Definition 5 are met for any particular set $\bar{\mathcal{U}} = \{\bar{U}_1,\ldots,\bar{U}_{3n-4}\}$ that defines a simple coordinate system at $\bar{K}$, they are met for all such sets. To see this consider an n-particle displacement U. According to (B.10) it has the unique representation

$$U = \sum_{\lambda=1}^{3n-4} \mathbf{t}_\lambda \bar{U}_\lambda + U_0(\bar{K}) \qquad (F.6)$$

where $U_0(\bar{K})$ is linearly independent of the vectors in the set $\bar{\mathcal{U}}$. Provided $\mathbf{t} = (\mathbf{t}_1,\ldots,\mathbf{t}_{3n-4})$ is not zero, the projection of U onto $\Gamma(\bar{\mathcal{U}})$ is $\bar{V} = \Sigma\hat{\mathbf{t}}_\lambda \bar{U}_\lambda$, where $\hat{\mathbf{t}}_\lambda = \Sigma \mathbf{t}_\lambda |\mathbf{t}|^{-1}$. Since (F.6) is valid for any displacement U, it is valid in particular for the members of any set $\mathcal{U} = \{U_1,\ldots,U_{3n-4}\}$ that define a simple coordinate system at $\bar{K}$. For these U_γ the equation (F.6) becomes

$$U_\gamma = \sum_\lambda \mathbf{t}_{\gamma\lambda} \bar{U}_\lambda + U_{0,\gamma}(\bar{K}). \qquad (F.7)$$

Finally, since any displacement U has a unique representation of the type (F.6) with the $\bar{U}_\lambda$ replaced by the U_γ, we have

$$U = \sum_{\gamma} \beta_{\gamma} U_{\gamma} + U_0'(\bar{K}) = \sum_{\lambda} (\sum_{\gamma} \beta_{\gamma} t_{\gamma\lambda}) \bar{U}_{\lambda} + U_0 (\bar{K}). \qquad (F.8)$$

That is,

$$t_{\lambda} = \sum_{\gamma} \beta_{\gamma} t_{\gamma\lambda}. \qquad (F.9)$$

Because both $\mathcal{U}$ and $\bar{\mathcal{U}}$ define simple coordinate systems at $\bar{K}$, the matrix M of coefficients $t_{\gamma\lambda}$ is nonsingular. Therefore, the vector t, which defines the projection of U onto $\Gamma(\bar{\mathcal{U}})$, also uniquely defines the projection of U onto $\Gamma(\mathcal{U})$. Thus, the sets $\Gamma_c^j(\bar{\mathcal{U}};\mathcal{N})$ of Definition 5 are isomorphic to the corresponding sets $\Gamma_c^j(\mathcal{U};\mathcal{N})$ for any other choice of the set $\mathcal{U}$. Moreover, if e is some vector in $\underset{\sim}{R}^{3n-4}$, the equation (F.9) yields $(t,e) = (\beta,Me)$. Hence, if the projection of U onto $\Gamma(\bar{\mathcal{U}})$ is in $\Gamma^+(\bar{\mathcal{U}},e)$, the projection of U onto $\Gamma(\mathcal{U})$ is in $\Gamma^+(\mathcal{U},Me)$. This proves the statement that if the conditions of Definition 5 are met for any particular set $\bar{\mathcal{U}} = \{\bar{U}_1,\ldots,\bar{U}_{3n-4}\}$ that defines a simple coordinate system at $\bar{K}$, they are met for all such sets.

Next we prove the following Lemma 1:

For any $\delta>0$ one can find a product neighborhood $\mathcal{N}$ of any $\bar{K} \in \mathcal{M} - \mathcal{M}_0$ such that

$$\Gamma_c(\mathcal{U};\mathcal{N}) \subset \Gamma_c(\mathcal{U};\bar{K},\delta), \qquad (F.10)$$

where

$$\Gamma_c(\mathcal{U};\bar{K},\delta) = \{U|U + \Delta\varepsilon\, \mathcal{C}_c(\bar{K})\cap\Gamma(\mathcal{U});\Delta = \Sigma d_\lambda U_\lambda, [\Sigma d_\lambda{}^2]^{\frac{1}{2}} \leqslant \delta\}. \quad (F.11)$$

To prove this we first express $\Gamma_c(\mathcal{U};\mathcal{N})$ in a different way. Let $\hat{V}_\varepsilon(k_i';u_i)$ be the set obtained from $\hat{V}_\varepsilon(\psi_i;u_i)$ by replacing supp ψ_i by k_i'. Let $C(K)$ be the set of connected causal diagrams $\mathcal{D}$ that satisfy $K = K(\mathcal{D})$, $\Sigma v_r(\mathcal{D}) = 0$, and $\Sigma'\|\Delta_j(V(\mathcal{D}))\| = 1$. Define

$$\Gamma_c'(\mathcal{U};K,K',\varepsilon) = \{U'|U' = \Sigma t_\lambda' U_\lambda,\ v_{r(i)}(\mathcal{D})\varepsilon\hat{V}_\varepsilon(k_i';u_i'+ a), \quad (F.12)$$
$$\mathcal{D}\varepsilon C(K)\},$$

where $v_{r(i)}(\mathcal{D})$ is the vertex of $\mathcal{D}$ that is connected to the external line L_i, and a is a real number giving an overall translation. Define

$$\Gamma_c(\mathcal{U};K,K',\varepsilon) = \{U|U\varepsilon\Gamma(\mathcal{U}),\ \beta U = U'\varepsilon\Gamma_c'(\mathcal{U};K,K',\varepsilon),\beta > 0\}. \quad (F.13)$$

Finally, define

$$\Gamma_c(\mathcal{U};\mathcal{N},\varepsilon) = \{U|U\varepsilon\Gamma_c(\mathcal{U};K,K',\varepsilon),\ K \text{ and } K' \text{ in} \mathcal{N}\}. \quad (F.14)$$

Then for sufficiently small product $\mathcal{N}$ of $\bar{K}\varepsilon\mathcal{M}-\mathcal{M}_0$ we have

$$\Gamma_c(\mathcal{U};\mathcal{N}) \subset \bigcap_{\epsilon>0} \Gamma_c(\mathcal{U};\mathcal{N},\epsilon). \tag{F.15}$$

To prove (F.15), assume it is false. Then for some $\bar{\epsilon} > 0$ there is some U in $\Gamma_c(\mathcal{U};\mathcal{N})$ that is not in $\Gamma_c(\mathcal{U};\mathcal{N},\bar{\epsilon})$. Since this U is in $\Gamma_c(\mathcal{U};\mathcal{N})$ one can, for each $\epsilon > 0$, find a $\mathcal{D}_\epsilon$ that satisfies the conditions of Definition 4, with this U and with supp $\psi = \bar{\mathcal{N}}$. Thus a sequence of $\mathcal{D}_\epsilon$ can be constructed for any sequence $\epsilon_i \to 0$. The norms $N(\mathcal{D}_{\epsilon_i}) = \Sigma' \,||\Delta_j(V(\mathcal{D}_{\epsilon_i}))||$ either approach zero or they do not. If they do not, then the normalized $\bar{\epsilon}_i \equiv \epsilon_i/N(\mathcal{D}\epsilon_i)$ must reach values less than $\bar{\epsilon}$. But then $\Gamma_c(\mathcal{U};\mathcal{N},\bar{\epsilon})$ would contain U, contrary to assumption. Thus the norms $N(\mathcal{D}_{\epsilon_i})$ must approach zero. This means the diagrams $\mathcal{D}_{\epsilon_i}$ approach trivial diagrams. But for sufficiently small $\bar{\mathcal{N}} \equiv$ supp ψ about $\bar{K} \in \mathcal{M} - \mathcal{M}_0$ the $U \in \Gamma(\mathcal{U})$ cannot satisfy the conditions of Definition 4 with any trivial (or nearly trivial) $\mathcal{D}_\epsilon$ for ϵ smaller than some $\epsilon_1 > 0$, because the various $V_\epsilon(\psi_i;u_i)$ can have no common point, (or nearly common point), in these circumstances. This rules out the possibility that the norms $N(\mathcal{D}_{\epsilon_i})$ approach zero, and hence proves (F.15).

Because of (F.15) it is sufficient for the proof of Lemma 1 to prove <u>Lemma 1'</u>: For any $\delta > 0$ one can find a product neighborhood $\mathcal{N}$ of any $\bar{K} \in \mathcal{M} - \mathcal{M}_0$ such that

$$\bigcap_{\epsilon>0} \Gamma_c(\mathcal{U};\mathcal{N},\epsilon) \subset \Gamma_c(\mathcal{U};\bar{K},\delta) \tag{F.10'}$$

To prove Lemma 1', assume it is false. Then there must be some $\delta > 0$ such for any product neighborhood $\mathcal{N}$ of $\bar{K}$ there is some $U(\mathcal{N})$ that belongs to $\Gamma_c(\mathcal{U};\mathcal{N},\epsilon)$ for all $\epsilon > 0$, but does not belong to

$\Gamma_c(\mathcal{U};\bar{K},\delta)$. Thus for any sequence $\{\epsilon_s, \mathcal{N}_s\}$, $s = 1,2,\cdots$, with $\epsilon_s \to 0$ and $\mathcal{N}_s \to \bar{K}$ there is a sequence of U_s such that

$$U_s \in \Gamma_c(\mathcal{U};\mathcal{N}_s,\epsilon_s) \tag{F.16a}$$

and

$$U_s \notin \Gamma_c(\mathcal{U};\bar{K},\delta). \tag{F.16b}$$

Each U_s satisfying (F.16a) corresponds to a $U'_s = U/\beta$ that generates a diagram $\mathcal{D}_s$ satisfying

$$v_{r(i)}(\mathcal{D}_s) \in \hat{V}_{\epsilon_s}(k'_{is};u_{is} + a_s) \tag{F.17a}$$

and

$$\mathcal{D}_s \in C(K_s), \tag{F.17b}$$

where $K_s \to \bar{K}$ and $K'_s \to \bar{K}$.

It has been shown elsewhere[19] that the number of different positive-α Landau surfaces that pass through any bounded region is finite. The infinite sequence $\mathcal{D}_s$ must therefore be divided between a finite number of classes $[\mathcal{D}]$, at least one of which must have an infinite number of the diagrams $\mathcal{D}_s$. Let this class be denoted by $[\mathcal{D}_1]$, and let the $\mathcal{D}_s$ not in $[\mathcal{D}_1]$ be disregarded. The sets $V(\mathcal{D}_s)$ are confined to a bounded region and must have at least one accumulation point $\bar{V} = V(\bar{\mathcal{D}})$. The argument for this was given in Appendix E below (E.37). The arguments of Appendix E [see(E.20)] also show that $\bar{K} = K(\bar{\mathcal{D}})$.

If we can show that the sequence $\{U_s\}$ has an accumulation point $\bar{U}$ in $\mathcal{C}_\kappa(\bar{K}) \cap \Gamma(\mathcal{U})$, we shall have established a contradiction with (F.16b), and shall therefore have proved Lemma 1'.

Each displacement U_s corresponds to a unique displacement U_s' in $\Gamma_c'(\mathcal{U};K_s,K_s',\epsilon_s)$. The points U_s' have a unique limit point $\overline{U}'$ defined by the condition

$$\overline{U}' \in \{U' \mid U' = \Sigma t_\lambda' U_\lambda, \overline{v}_{r(i)} \in \hat{V}_0(\overline{k}_i;u_i' + a)\}. \tag{F.18}$$

The fact that the $\overline{U}'$ defined by (F.18) is unique follows from (B.10), since the various U' that satisfy the second condition in (F.18) differ by vectors of the form $U_0(\overline{K})$. The U_s' of (F.16a) satisfy, according to (F.14) and (F.12), the condition

$$U_s' \in \{U' \mid U' = \Sigma t_\lambda' \; U_\lambda, \; v^s_{r(i)} \equiv v_{r(i)}(\mathcal{D}_s) \in \hat{V}_\epsilon(k_i';u_i + a)$$
$$k' \in \overline{\eta}_s\} \tag{F.19}$$

The continuity properties of the set on the right of (F.19) ensure that the U_s' approach the $\overline{U}'$ of (F.18).

If the $\overline{U}'$ is nonzero then the $U_s = U_s'/\beta_s$ must approach the limit $\overline{U} = \overline{U}'/\overline{\beta}$ where $\overline{\beta}^2 = \Sigma \, t_\lambda^2(\overline{U}')$ is nonzero. This $\overline{U}$ would lie in $\mathcal{C}_c(\overline{K}) \cap \Gamma(U)$, thus contradicting (F.16b). Thus the proof will be completed by showing that $\overline{U}'$ is nonzero.

To see that the vector $\overline{U}'$ is different from zero notice first that, because $\overline{\mathcal{D}}$ is nontrivial, the earliest vertex $\overline{v}_I$ must be definitely earlier than the latest vertex $\overline{v}_F$. By virtue of the stability requirement, the initial vertex $\overline{v}_I$ must be connected to at least two initial lines and the final vertex $\overline{v}_F$ must be connected to two final lines. Because $\overline{K}$ does not belong to $\mathcal{M}_0$ the initial lines connected to $\overline{v}_I$ meet only at $\overline{v}_I$ and the final lines connected to $\overline{v}_F$ meet only at $\overline{v}_F$. Such a configuration does not allow the $\bar{U}'$ of (F.18) to be zero, since $\bar{U}' = 0$ means that all the external lines pass through a common point [See(B.10)]. This completes the proof of Lemma 1' and by virtue of (F.15), the proof of Lemma 1.

Given Lemma 1 and the result proved just before it, the proof of Theorem 10 is trivial.

C. Proof of Theorem 11.

Let $\tilde{\mathcal{C}}_c(\bar{K})$ be the set of U that generate $\mathcal{D}$ that satisfy $K(\mathcal{D}) = \bar{K}$. What must be shown is that for each $\bar{K} \in \mathcal{M} - \mathcal{M}_0$

$$\mathcal{C}_c(\bar{K}) = \tilde{\mathcal{C}}_c(\bar{K}) \qquad \text{(F.20)}$$

It is obvious that $\tilde{\mathcal{C}}_c(\bar{K}) \subset \mathcal{C}_c(\bar{K})$ and that $\mathcal{C}_0(\bar{K}) \subset \tilde{\mathcal{C}}_c(\bar{K})$. What must be shown is that for each $\bar{K} \in \mathcal{M} - \mathcal{M}_0$

$$\mathcal{C}_c(\bar{K}) - \mathcal{C}_0(\bar{K}) \subset \tilde{\mathcal{C}}_c(\bar{K}) \qquad \text{(F.21)}$$

To prove this, first define

$$\mathcal{C}_c'(\bar{K},\epsilon) = \{U' \mid V_{r(i)}(\mathcal{D}) \in \hat{V}_\epsilon(\bar{R}_i; u_i' + a),\ \mathcal{D} \in C(K)\}$$

where $C(K)$ is defined above (F.12). And define

$$\mathcal{C}_c(\bar{K},\epsilon) = \{U \mid \beta U = U' \in \mathcal{C}_c'(\bar{K},\epsilon),\ \ \beta > 0\} \qquad \text{(F.22)}$$

Then for $\bar{K} \in \mathcal{M} - \mathcal{M}_0$ we have

$$\mathcal{C}_c(\bar{K}) - \mathcal{C}_0(\bar{K}) \subset \bigcap_{\epsilon > 0} \mathcal{C}_c(\bar{K},\epsilon) \qquad \text{(F.23)}$$

The proof of (F.23) is the same as the proof of (F.15), except for the obvious substitutions. It remains only to show that

$$\bigcap_{\epsilon>0} \mathcal{C}_c(\bar{K},\epsilon) \subset \tilde{\mathcal{C}}_c(\bar{K}). \qquad \text{(F.24)}$$

The proof of this is similar to the proof of Lemma 1'. If (F.24) were not true then there would be some $U \notin \tilde{\mathcal{C}}_c(\bar{K})$ that belongs to each $\mathcal{C}_c(\bar{K},\epsilon)$ on the left. Thus for each $\epsilon > 0$ there would be a $\mathcal{D}_\epsilon \in C(\bar{K})$ such that the conditions of Definition 4 can be satisfied with this U, and with supp ψ replaced by $\bar{K}$. A sequence $\epsilon_s \to 0$ gives then a corresponding sequence $\mathcal{D}_s \in C(\bar{K})$. As in Lemma 1 the $V(\mathcal{D}_s)$ must accumulate at a $\bar{V}$ that corresponds to a $\bar{\mathcal{D}}$ that satisfies $K(\bar{\mathcal{D}}) = \bar{K}$. But then U would belong to $\tilde{\mathcal{C}}_c(\bar{K})$. This contradiction proves (F.24), and hence also the theorem.

FOOTNOTES AND REFERENCES

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10. The notation is essentially that of R.C. Gunning and H. Rossi, Analytic Functions of Several Complex Variables, (Prentice-Hall, 1965). See also Appendix B.

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13. This is an easily proved consequence of (2.17) and the intersection properties of the sets $\hat{V}_\varepsilon(\psi_i;u_i)$.

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15. The necessary condition that $\bar{\Lambda}(z) \equiv \Lambda \circ \Pi'_{\bar{K}}(z)$ has non-vanishing gradient follows from condition(c) of the theorem. This result is contained in the proof of the theorem. See also Section V.

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17. This property is referred to as the "general iε rule". See H.P. Stapp, "Crossing, Hermitian Analyticity and the Connection Between Spin and Statistics", Lawrence Radiation Laboratory Report UCRL-16816, April 1966.

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19. H.P. Stapp, "Finiteness of the Number of Positive-Landau Surfaces in Bounded Portions of the Physical Region", Lawrence Radiation Laboratory Report UCRL-17219, October 1966.

20. Points described by Theorem 12 are called almost-simple points in Ref. 17.

21. Such points are called simple points in Ref. 17.

22. These are the simply multiplicative points of Ref. 17.

23. See, for example, Ref. 14.

24. Ref. 14, p.62

25. This agrees with the perturbation theory result that AB is the positive-α part of the curve. See Ref. 14, p.64 .

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29. H. Bremermann, op. cit., p. 85.

30. Lemma 2 of Appendix C.

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34. H. Bremermann, op. cit., p. 166. The lemma is problem 6 on page 166. The proof is a straightforward application of the technique of Riemann sums found on page 49.

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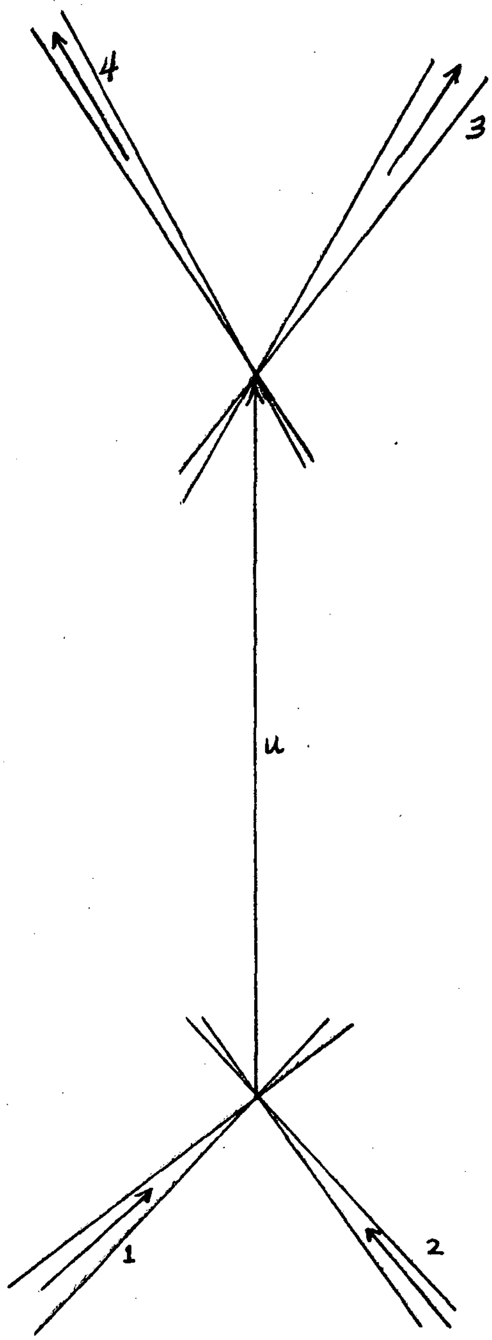

Figure 2

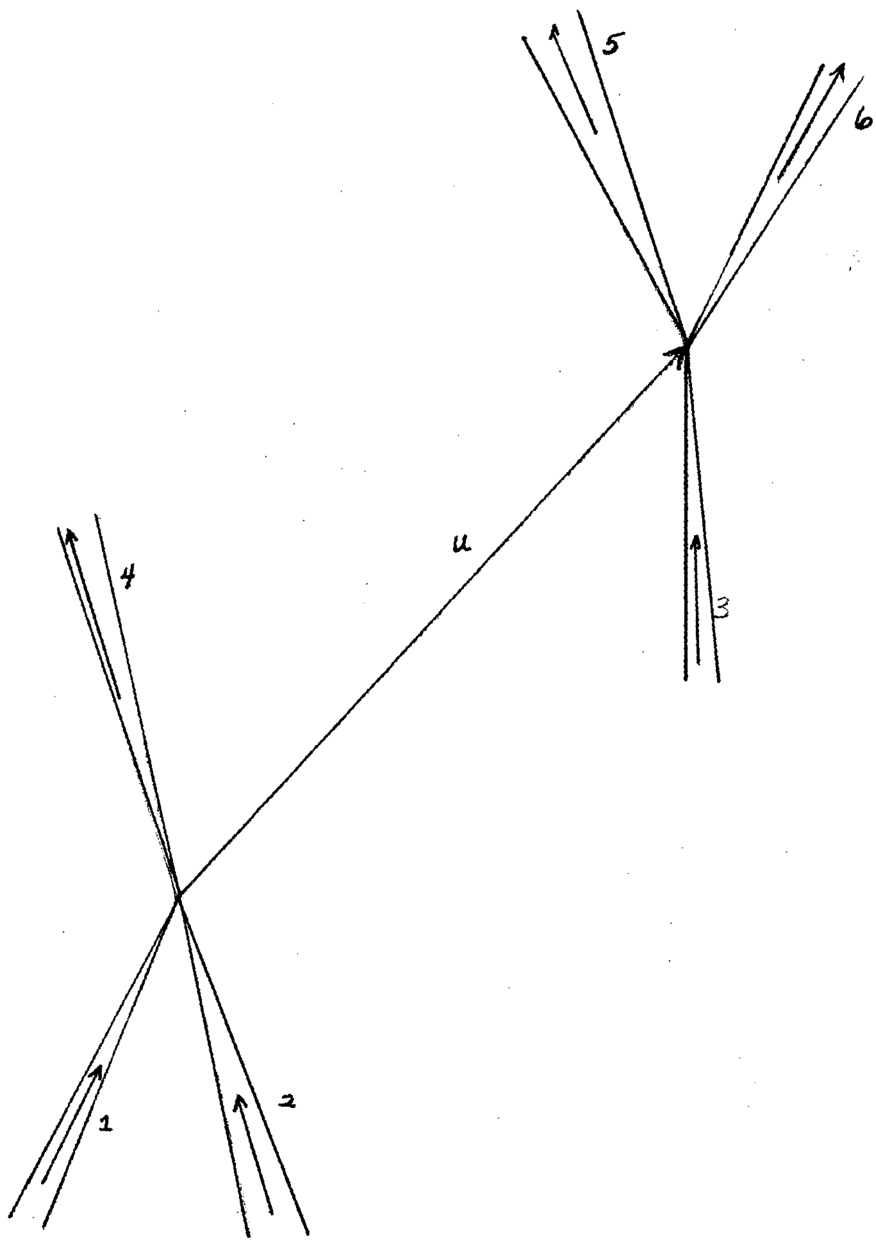

Figure 3

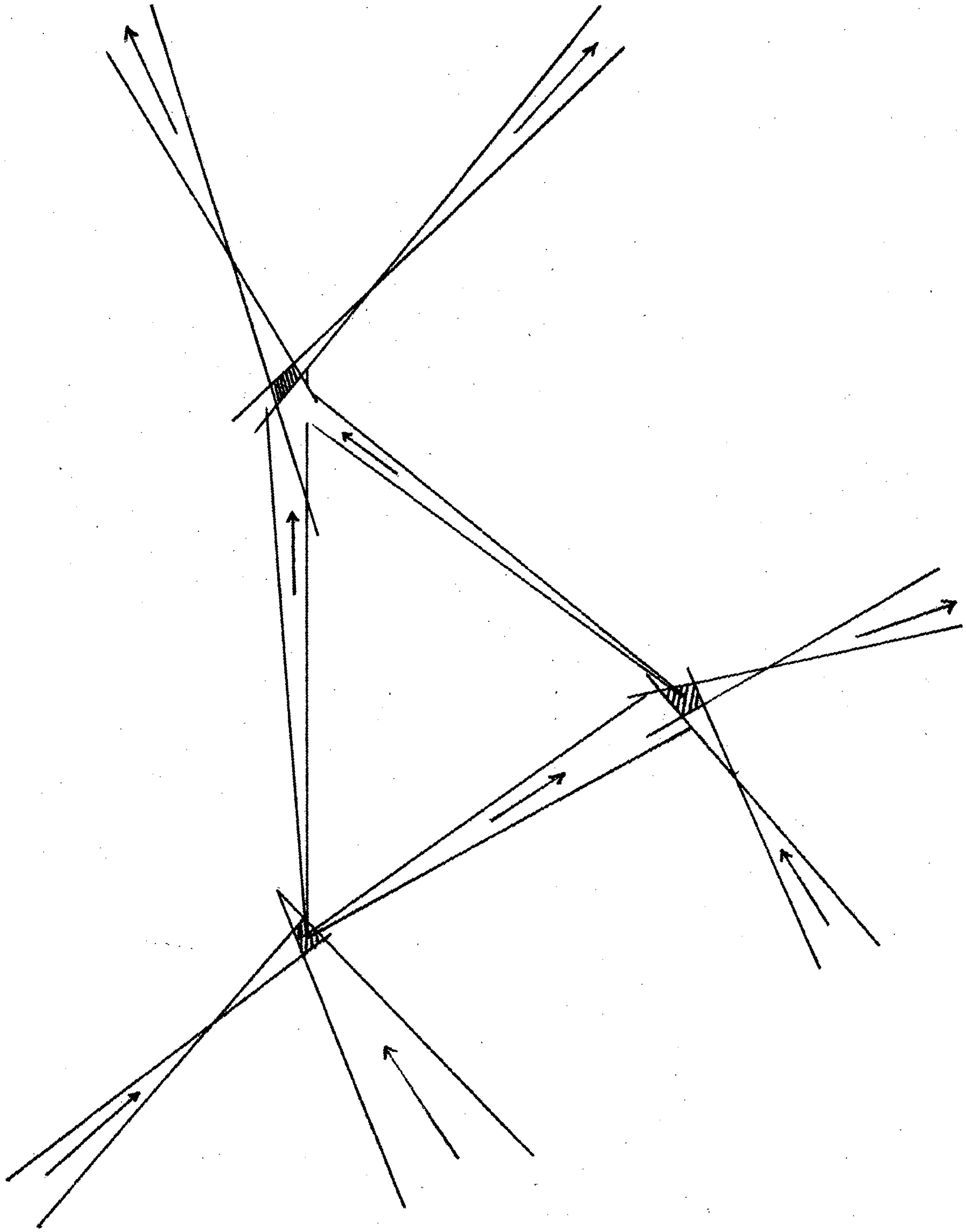

Figure 4

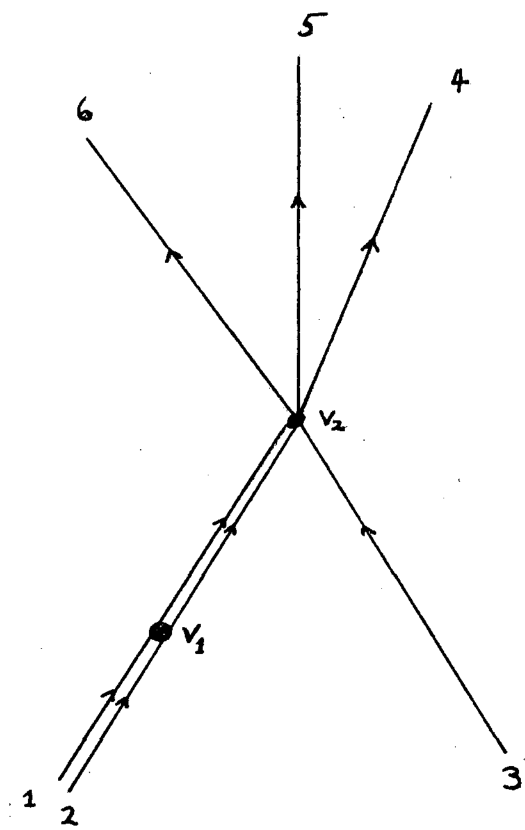

Figure 5

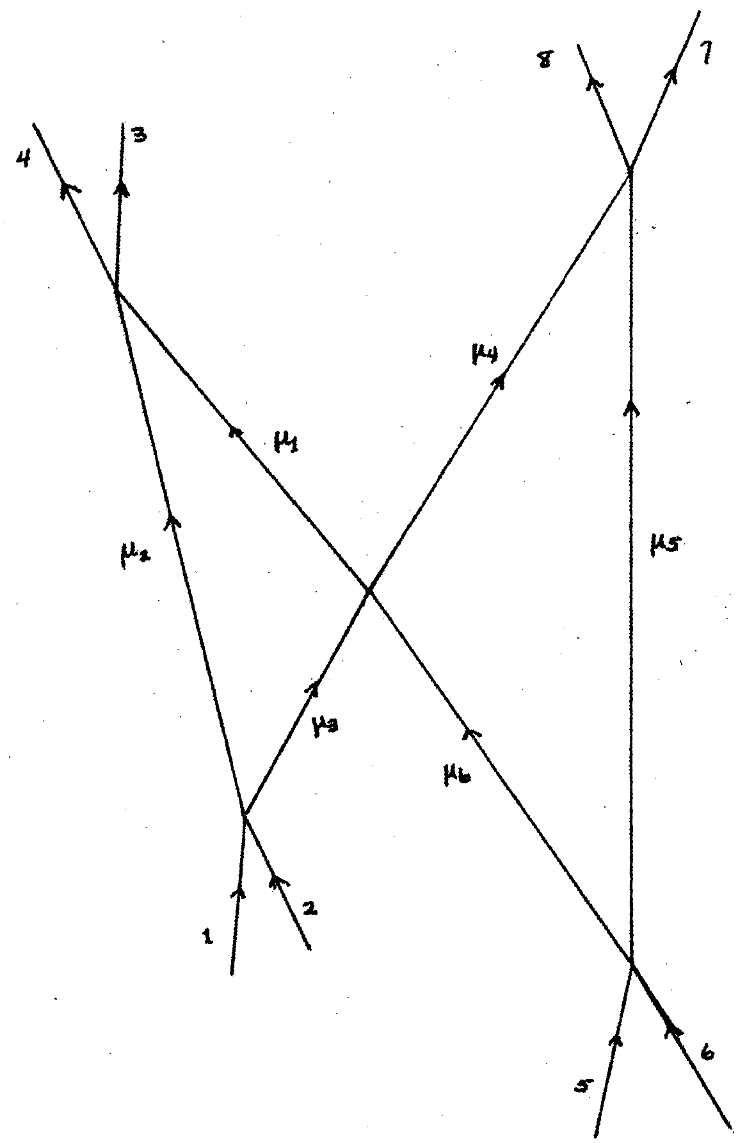

Figure 6

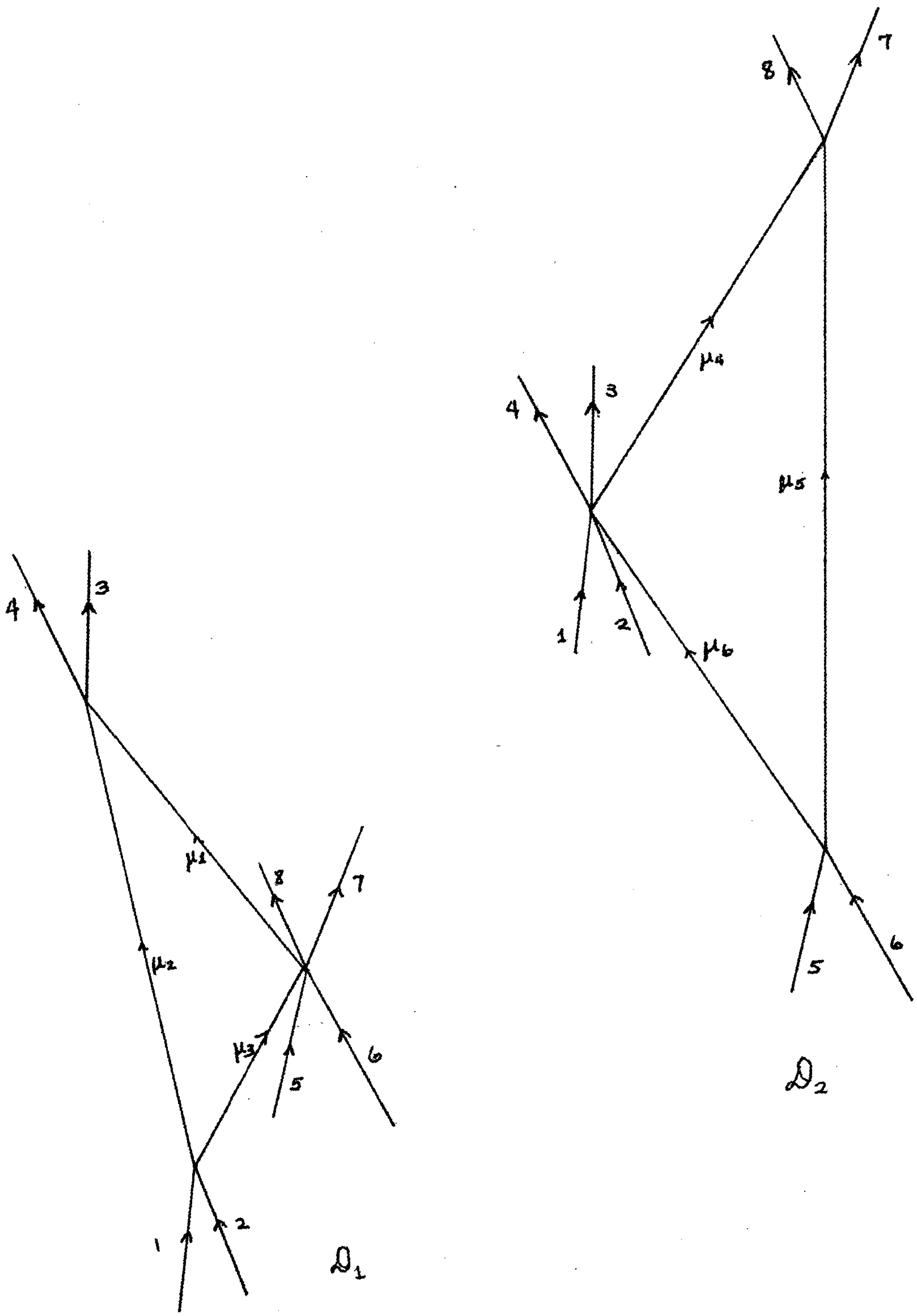

Figure 7

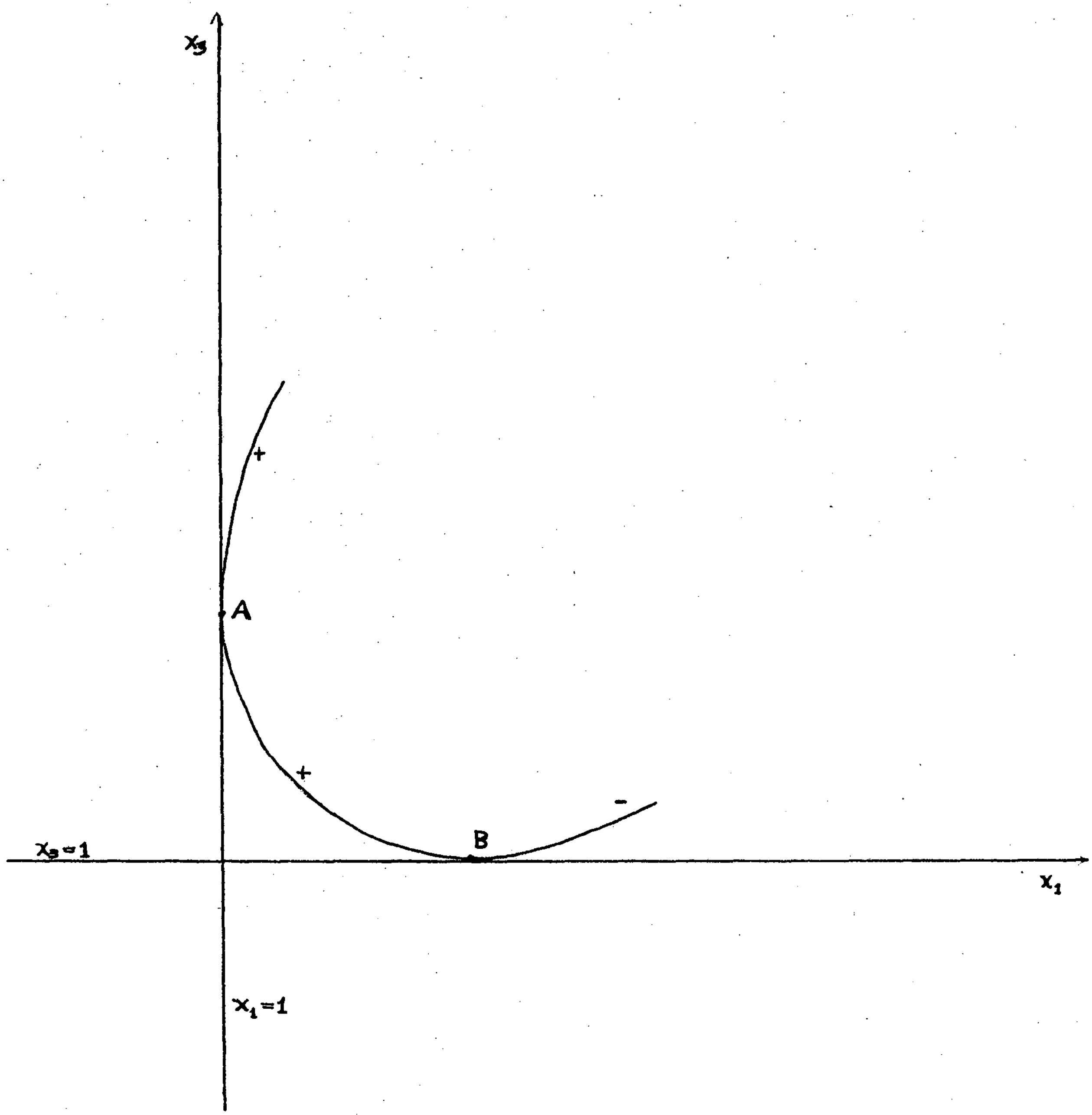

Figure 8