

MACROSCOPIC MODELS FOR SUPERCONDUCTIVITY*

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Abstract. This paper reviews the derivation of some macroscopic models for superconductivity and also some of the mathematical challenges posed by these models. The paper begins by exploring certain analogies between phase changes in superconductors and those in solidification and melting. However, it is soon found that there are severe limitations on the range of validity of these analogies and outside this range many interesting open questions can be posed about the solutions to the macroscopic models.

Key words. superconductivity, Ginzburg–Landau equations

AMS(MOS) subject classification. 82D55

1. Introduction. The aim of this paper is to review some macroscopic models for the composition and electromagnetic behavior of solids that can change their phase from *normal to superconducting*.¹ This transformation is popularly associated with a transition temperature T_c , but, in fact, it occurs across a curve $H_0 = H_c(T)$ in the plane of the applied temperature and the magnitude of the applied magnetic field H_0 as in Fig. 1; T_c is the largest temperature at which superconductivity is possible, which is when $H_0 = 0$. The coupling between the thermal and electromagnetic effects principally takes place through a latent heat release at the phase change and because of ohmic heating, and both these effects are usually small. Hence we can make a good mathematical model by assuming isothermal conditions and considering what happens when H_0 is close to H_c , with the temperature only appearing parametrically.

The observation that normal materials can change into superconductors as H_0 is decreased through H_c was made by Onnes [1]; the other key ingredient for a mathematical model was noted by Meissner and Ochsenfeld [2], namely, that magnetic flux is completely expelled from any region that is in the superconducting phase.

The simplest configuration in which to describe these phenomena is that of a cylindrical wire, with cross section Ω (Fig. 2(a)), placed in an axial magnetic field $(0, 0, H_0)$ [3]. Here and throughout the paper we assume that Maxwell's equations hold everywhere, with the displacement current being negligible. Thus the electric and magnetic fields \mathbf{E} , \mathbf{H} , the current density \mathbf{j} , and the charge density ρ satisfy

$$\begin{aligned} \operatorname{div} \mathbf{E} &= \frac{\rho}{\epsilon}, & \operatorname{div} \mathbf{H} &= 0, \\ \operatorname{curl} \mathbf{H} &= \mathbf{j}, & \operatorname{curl} \mathbf{E} + \mu \frac{\partial \mathbf{H}}{\partial t} &= 0 \end{aligned}$$

where the permeability μ and permittivity ϵ are assumed constant. When the wire is entirely normal we assume Ohm's law

$$\mathbf{j} = \sigma \mathbf{E},$$

where σ is the constant electrical conductivity.

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¹After the completion of this manuscript, we became aware of the related accounts [44], [45] both of which have similar objectives to ours and provide expositions which are helpful to applied mathematicians.

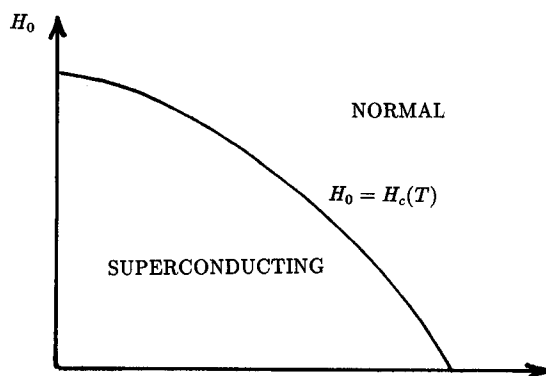
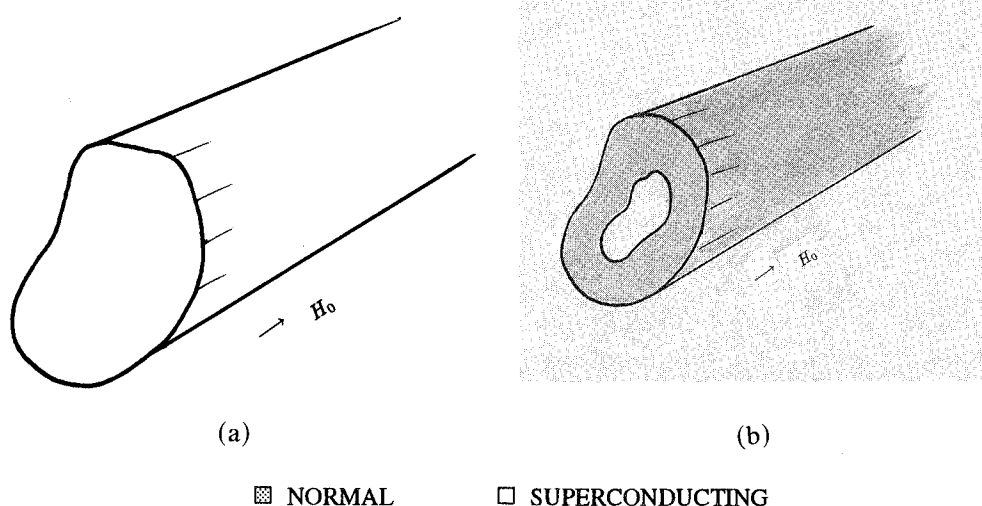


FIG. 1. Critical field as a function of temperature.

FIG. 2. Evolution of a superconducting wire (a) $H_0 < H_c$, (b) $H_0 > H_c$.

We nondimensionalize these equations by setting

$$\mathbf{H} = H_e \mathbf{H}', \quad \mathbf{E} = \frac{H_e}{\sigma \ell} \mathbf{E}', \quad \rho = \frac{\varepsilon H_e}{\sigma \ell^2} \rho',$$

$$\mathbf{j} = \frac{H_e}{\ell} \mathbf{j}', \quad \mathbf{x} = \ell \mathbf{x}', \quad t = \mu \sigma \ell^2 t',$$

where ℓ is a typical length scale for the sample and H_e is a typical value of the external magnetic field. Dropping the primes, so that H_0 and H_c are henceforth dimensionless, yields the dimensionless system

$$(1a-d) \quad \operatorname{div} \mathbf{E} = \rho, \quad \operatorname{div} \mathbf{H} = 0, \quad \operatorname{curl} \mathbf{H} = \mathbf{j}, \quad \operatorname{curl} \mathbf{E} + \frac{\partial \mathbf{H}}{\partial t} = 0,$$

with

$$(2) \quad \mathbf{j} = \mathbf{E}$$

when the wire is entirely normal. For the moment we shall only consider situations in which $\rho = 0$, but in §3 we shall be forced to abandon charge neutrality.

We now seek a solution $\mathbf{H} = (0, 0, H_3(x, y, t))$, $\mathbf{E} = (E_1(x, y, t), E_2(x, y, t), 0)$. All of (1), (2) can be satisfied if

$$(3) \quad \frac{\partial H_3}{\partial t} = \Delta H_3$$

with $H_3 = H_0$ on $\partial\Omega$, together with a suitable initial condition. However, when the wire is entirely superconducting, the solution of (1), (2) would be $\mathbf{H} = \mathbf{0} = \mathbf{E}$ everywhere in Ω , assuming that the Meissner effect were strictly true. If this were so, \mathbf{H} would be discontinuous on $\partial\Omega$, which would suggest a superconducting current sheet in $\partial\Omega$, perpendicular to the z -axis.

In modelling the evolution of one of these configurations, we follow [3] and consider the so-called *intermediate state*, defined to be one in which both normal and superconducting regions are present simultaneously as in Fig. 2(b). We begin by assuming a configuration in which part of the wire is normal and is separated from the remaining superconducting region by a smooth cylinder Γ . This assumption is questionable, and we shall return to it later, but the configuration might be achieved by taking H_0 to be a suitable function of t which increases (or decreases) through H_c . Thus, in the normal region, (1)–(3) hold, and we expect

$$(4) \quad H_3 \downarrow H_c$$

as we approach Γ . Hence we expect the superconducting current in Γ to be perpendicular to the z -axis and to have magnitude H_c . Now by writing (1d) as

$$\frac{\partial H_3}{\partial t} + \operatorname{div}(E_2, -E_1, 0) = 0$$

and noting that \mathbf{E} and H_3 vanish in the superconducting region, we derive the jump condition across Γ that its normal velocity v_n towards the superconducting region satisfies

$$[H_3]_S^N v_n = H_c v_n = (E_2 n_1 - E_1 n_2)_N,$$

where, for definiteness, we take $\mathbf{n} = (n_1, n_2, 0)$ to be the outward normal vector to the normal region. Finally, since $\mathbf{E} = \mathbf{j} = \operatorname{curl} \mathbf{H}$ in the normal region, we assert that

$$(5) \quad \frac{\partial H_3}{\partial n} = -H_c v_n$$

as Γ is approached from the normal region.

This model (3)–(5) for the intermediate state is convenient in that it just involves \mathbf{H} and not \mathbf{E} , although we note that it has no nontrivial steady-state solutions (as we shall

see, this is not the case for the fully three-dimensional generalization). More importantly, it is nothing more than a one-phase “Stefan” model [4], which is itself the simplest macroscopic model that could be written down for an evolving phase boundary in the classical theory of melting or solidification. In its simplest dimensionless two-phase form, the Stefan model is

$$(6) \quad \frac{\partial T}{\partial t} = \Delta T, \quad T \neq T_m,$$

where T is the temperature and T_m is the melting temperature, together with an energy balance for the velocity v_n of the phase boundary in the form

$$(7) \quad \left[\frac{\partial T}{\partial n} \right]_{\text{Solid}}^{\text{Liquid}} = -Lv_n,$$

where L is the latent heat. When T_m is constant, and this model is supplemented by suitable initial and boundary conditions, it is known to be well posed just as long as neither superheating nor supercooling occurs, i.e., $T_{\text{solid}} < T_m, T_{\text{liquid}} > T_m$ [5]. However, when either of these conditions is violated, the model appears to be ill-posed and thus needs to be regularized [6]. The most popular way of doing this is by writing

$$(8) \quad T_m = -\sigma/R - \beta v_n,$$

where $1/R$ is the mean curvature of the surface $T = T_m$ with a suitable sign, and σ and β are positive constants. The mathematical effects of (8) are not well understood, although some well-posedness results are beginning to appear ([7], [8]), but (8) is often accepted as covering many practical cases of unstable crystal growth.

The layout of the rest of this paper will be strongly influenced by the analogy between models for solidification and models for superconductivity. A particularly useful link is provided by the so-called “phase field” regularization of (6)–(8) [9], whereby the phase boundary $T = T_m$ is smoothed by introducing an “order parameter” $F \in (-1, 1)$ such that (6) is replaced by

$$(9) \quad \frac{\partial T}{\partial t} + \frac{L}{2} \frac{\partial F}{\partial t} = \Delta T.$$

The order parameter represents the mass fraction of material to have changed phase, say from liquid ($F = 1$) to solid ($F = -1$). We then append a “Ginzburg–Landau” equation for F , obtained by relating the evolution of F to the variational derivative of a suitably chosen free energy functional, in the form

$$(10) \quad \alpha \frac{\partial F}{\partial t} = \alpha \xi^2 \Delta F + \frac{1}{2a} (F - F^3) + 2T,$$

together with suitable initial and boundary conditions; α, ξ , and a are all constants. Although (9), (10) have been studied extensively, they are difficult to analyze, and most of the evidence for their well-posedness comes from numerical simulation. However, their most intriguing feature from the viewpoint of superconductivity modelling is their ability

to reduce formally to the classical and regularized Stefan models (6)–(8) as $a, \alpha, \xi \rightarrow 0$.² When these parameters are small, the structure comprises liquid and solid regions separated by a thin transition layer in which F and T are smoothly varying “travelling wave” solutions of self-consistent local approximations to (9), (10).

We shall now begin our discussion of macroscopic superconductivity modelling, starting with free boundary models analogous to (3)–(5) and then proceeding to models in which the phase boundary is smoothed as in (9), (10). In §2 we shall write down the generalization of (3)–(5) to a three-dimensional superconductor undergoing a phase change. This will take the form of a “vectorial” Stefan problem, albeit a very different one from the alloy solidification vector equation, as described say in [10]. Nonetheless, this vectorial Stefan model will be shown sometimes to have instabilities that are similar to those which cause ill-posedness in the classical Stefan model (6), (7) in superheated or supercooled situations. This means that the model is only capable of describing certain superconductor configurations. In particular, for intermediate states when both phases are present simultaneously, we shall only expect well-posedness when the normal region is expanding and the superconducting region is contracting. In these circumstances the model predicts the evolution of a smooth boundary Γ separating the two phases. We shall see in §5 that a further constraint is also necessary for (3)–(5) to be applicable; this will restrict the use of these equations to what will later be called Type I superconductors in which the normal region is expanding.

In order to understand other Type I configurations, and what will later be called Type II superconductors, we must consider the behavior near the phase boundary more carefully. This problem was attacked by London [11], who proposed that the superconducting region should not be modelled simply by writing $\mathbf{H} = \mathbf{0}$, but rather that there should be a distributed superconducting current \mathbf{j} in that region, such that

$$(11a) \quad \mathbf{j}_s \propto \mathbf{A},$$

and hence

$$(11b) \quad \frac{\partial \mathbf{j}_s}{\partial t} \propto \mathbf{E}, \quad \text{curl } \mathbf{j}_s \propto \mathbf{H},$$

where \mathbf{A} is a suitable magnetic vector potential which, in the superconducting region, is only appreciable near Γ . We shall see in §3 that this kind of model can be written down more systematically by again using a statistical Ginzburg–Landau theory. For Type I superconductors, this theory will stand in relation to the vectorial Stefan model as does the phase field theory to the scalar Stefan model for solidification. However, it will also permit us to analyze the more commonly occurring Type II superconductors in which, when both normal and superconducting phases are simultaneously present, there is a “filament” morphology for the normal phase. Such a state is called a *mixed state*.

In §4 we shall study some implications of the Ginzburg–Landau theory and in particular the asymptotic limits in which it reduces to the relatively simple Type I model of §2. The more intricate Type II configuration does not yet seem to be amenable to such a reduction. Indeed the very problem of demarcating between Type I and Type II behavior can be tackled in a variety of ways, which we shall discuss in §§4–6, but many of the details remain obscure.

Finally, in §6, we shall present some conjectures and open questions concerning the solutions of the various models in certain interesting situations.

²It is even possible formally to retrieve the so-called Cahn–Hilliard model for solid/solid phase transitions from (9), (10) in a suitable limit.

2. Free boundary models. In (3)–(5) we have already seen a simple configuration that permits a Stefan free boundary model to be formulated. To generalize this model to three dimensions, we begin by assuming that the material extends to infinity in all directions, and that a uniform field is applied there. Later we shall consider more general boundary conditions. As in the introduction, we assume that $\mathbf{H} = \mathbf{0}$ in the superconducting phase and that Ohm's law applies in the normal phase, so that

$$(12a) \quad \frac{\partial \mathbf{H}}{\partial t} = -\text{curl curl } \mathbf{H},$$

i.e.,

$$(12b) \quad \frac{\partial \mathbf{H}}{\partial t} = \Delta \mathbf{H}$$

since

$$(13) \quad \text{div } \mathbf{H} = 0$$

there. The generalization of (4) is

$$(14) \quad |\mathbf{H}| \downarrow H_c$$

as the phase boundary Γ , which now has curvature in two directions, is approached from the normal region.

We write (1d) in the form

$$\frac{\partial \mathbf{H}}{\partial t} + \text{div} \begin{pmatrix} 0 & -E_3 & E_2 \\ E_3 & 0 & -E_1 \\ -E_2 & E_1 & 0 \end{pmatrix} = \mathbf{0}$$

and apply the divergence theorem to obtain

$$[\mathbf{E} \wedge \mathbf{n}]_S^N = -v_n [\mathbf{H}]_S^N.$$

However, $\mathbf{E} = \text{curl } \mathbf{H}$ in the normal region and $\mathbf{E} = \mathbf{H} = \mathbf{0}$ in the superconducting region. Hence, approaching Γ from the normal region, we find that

$$(15) \quad \text{curl } \mathbf{H} \wedge \mathbf{n} = -v_n \mathbf{H};$$

this condition was written down in [12].

We can make some elementary observations about (12)–(15).

(i) It is a consequence of (12a) and (15) that (13) holds; this fact is proved in Appendix 1.

(ii) Unlike the situation for the scalar case in an aligned external field (3)–(5), non-trivial steady states are now possible in which $\mathbf{H} = \nabla \alpha$, $\Delta \alpha = 0$ with $\partial \alpha / \partial n = 0$, $|\nabla \alpha| = H_c$ on Γ . (Since $\text{div } \mathbf{H} = 0$, $\mathbf{H} \cdot \mathbf{n} = 0$ on Γ by the divergence theorem.) Thus the scalar

potential α can be identified with that in an ideal fluid flow around a constant pressure cavity Γ [13].

(iii) Whether the problem is steady or unsteady Γ , \mathbf{H} is tangential and of constant modulus on Γ . Hence if Γ is finite, then it is topologically equivalent to a torus, by the Gauss–Bonnet theorem.

(iv) Similarity solutions can be found, as in the Stefan problem, when there is circular symmetry and \mathbf{H} is either aligned (as in (3)–(5)) or azimuthal [14].

(v) The local stability of steady and unsteady solutions to (12)–(15) can be studied by methods similar to those employed for the Stefan problem [14]. We may consider a locally planar phase boundary Γ , whose equation is $x = Vt$, with the normal region being in $x > Vt$, and we seek perturbations to the exact solution

$$(16) \quad \mathbf{H} = \mathbf{H}^0 = (0, H_c e^{-V(x-Vt)}, 0), \quad x > Vt.$$

We have the option of considering boundary perturbations perpendicular or parallel to \mathbf{H}^0 .

In the first case, we have a situation described by the model (3)–(5), and hence the classical analysis for the Stefan problem carries over to show that the solution is stable to all wavelengths when the normal region is expanding ($V < 0$), but unstable to all sufficiently short wavelengths when it is contracting ($V > 0$).

In the second case we can write Γ as

$$x = Vt + \epsilon e^{\sigma t} \sin ny, \quad n > 0, \quad \epsilon \ll 1,$$

and

$$(17) \quad \mathbf{H} = \mathbf{H}^0 + \epsilon(H_1, H_2, H_3),$$

where $H_i(x, y, t)$ satisfy the diffusion equation. The free boundary conditions (13), (14) now enable us both to find a unique H_2 that decays spatially as we move away from the free boundary, and also to find a dispersion relation for σ which gives stability for all wavenumbers n . The field components H_1 and H_3 can be computed subsequently.

These formal arguments lead us to believe that the status of the vectorial Stefan model (12)–(15) is analogous to that of the classical Stefan model in that it is only likely to be well posed when the normal region is expanding: some preliminary results are given in [15]. When the normal region is contracting, the model needs to be regularized, and the Stefan analogy suggests that this might be done either by introducing higher order derivatives into the free boundary condition (as in (8)) or by smoothing the phase boundary altogether (as in (9), (10)). The former is difficult to do in the absence of further uncontroversial physical evidence, but a first step in the direction of the latter was taken by London [11] who proposed that the superconducting phase should be endowed with a structure given by (11). Since $\mathbf{j} = \text{curl}^2 \mathbf{A}$, this would imply a boundary layer structure for the vector potential in the superconducting region when the constant of proportionality in (11a) is large; \mathbf{H} would no longer need to be discontinuous on the phase boundary Γ .³ However, it is possible to formulate a model in which \mathbf{H} is perfectly smooth everywhere in the specimen, and this is what we shall do in the next section.

³However, \mathbf{H} might still be discontinuous at a boundary between a superconducting region and foreign material.

3. Ginzburg–Landau models. It might appear at first sight that modelling the transition layer between normal and superconducting regions would require quantum mechanical considerations as in the “BCS” theory [16]. Fortunately, in the steady state, a phenomenological theory can be written down quite easily without such quantum modelling. It is even more satisfying that this phenomenological Ginzburg–Landau theory can be derived as a formal limit of the BCS theory [17].

We first need to define real vector and scalar potentials \mathbf{A} and ϕ such that, from (1),

$$(18) \quad \mathbf{H} = \text{curl } \mathbf{A}, \quad \mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} = -\nabla \phi;$$

\mathbf{A} is unique up to the addition of a gradient, and, once \mathbf{A} is given, ϕ is unique up to the addition of a function of t , and we shall give their precise specifications shortly.

We next need the idea that macroscopic superconductivity models must incorporate the long-range interactions between superconducting electrons and hence, unlike the phase transitions described by the phase field model (9), (10), they must be described by a *complex* order parameter

$$(19) \quad \Psi = f e^{i\chi},$$

where f and χ are real and f is positive. Here f^2 measures the number density of superconducting electrons during the transformation from $f = 0$ (normal) to $f = 1$ (superconducting).

3.1. Steady state. In [18], Ginzburg and Landau consider only the steady case with $\mathbf{E} = \mathbf{0}$, $\phi = 0$ as would be the case, for example, when no current enters or leaves the device. They proceed by expanding the free energy as a power series in $|\Psi|^2$, together with the all-important term proportional to $|\nabla \Psi|^2$. The addition of a term of this form penalizes variations of the order parameter Ψ ; in the phase-field for solid/solid transformations such a term can be thought of as representing the surface energy. However, we have yet to take into account the interaction between the magnetic field and the electric current associated with the presence of a gradient in Ψ , which is most conveniently done in terms of \mathbf{A} rather than \mathbf{H} . This interaction must be such that the free energy density is gauge-invariant in the sense that, if \mathbf{A} is replaced by $\mathbf{A} + \nabla \omega$, then ω can be added to χ to make the resulting free energy density independent of ω . We must, therefore, append a term proportional to $i\mathbf{A}\Psi$ to $\nabla \Psi$ and thus, when we add the magnetic field energy $\gamma^2 |\mathbf{H}|^2$, we obtain an expanded and truncated free energy density

$$-|\Psi|^2 + \frac{1}{2}|\Psi|^4 + \gamma^2 |\mathbf{H}|^2 + |\xi \nabla \Psi - i\gamma \lambda^{-1} \mathbf{A} \Psi|^2.$$

Here $\gamma = H_e / \sqrt{2} H_c$, where H_c is the critical magnetic field strength; λ and ξ are material constants independent of H_e , which will shortly be seen to be dimensionless length scales for variations in \mathbf{H} and Ψ , respectively; when $\lambda \ll \xi$, the field boundary layer thickness is much less than the order parameter boundary layer thickness and vice versa. The ratio of these “penetration depths” is $\kappa = \lambda / \xi$, called the *Ginzburg–Landau parameter*, and we shall henceforth assume it is $O(1)$. In dimensional terms, the penetration depths are often about $1 \mu\text{m}$. For completeness the free energy is given in dimensional form in Appendix 2.

We note that the final term in the energy density may be written

$$\xi^2 |\nabla f|^2 + f^2 |\gamma \lambda^{-1} \mathbf{A} - \xi \nabla \chi|^2.$$

The first term represents the previously mentioned surface energy, and the second term can be interpreted as a gauge invariant “kinetic energy density” associated with the superconducting currents (we shall see that the superconducting current density is given by $\mathbf{j} = -(f^2/\lambda^2)(\mathbf{A} - (\lambda\xi/\gamma)\nabla\chi)$).

When we now minimize the total free energy with respect to the conjugate of Ψ , denoted by Ψ^* , and \mathbf{A} , we obtain the following dimensionless coupled equations for Ψ and \mathbf{A} :

$$(20) \quad (\xi \nabla - i\gamma \lambda^{-1} \mathbf{A})^2 \Psi + \Psi(1 - |\Psi|^2) = 0,$$

$$(21) \quad -\lambda^2 \text{curl}^2 \mathbf{A} = \frac{i\xi\lambda}{2\gamma} (\Psi^* \nabla \Psi - \Psi \nabla \Psi^*) + |\Psi|^2 \mathbf{A}.$$

Here (20) can be thought of as a nonlinear Schrödinger equation in which ξ gives the length scale for variations in Ψ , and the corresponding natural boundary condition for Ψ at the surface of a specimen in vacuo is

$$(21a) \quad \mathbf{n} \cdot (\xi \nabla - i\gamma \lambda^{-1} \mathbf{A}) \Psi = 0,$$

where \mathbf{n} is the normal to the surface.⁴ From (21) it can be seen that \mathbf{A} varies on the length scale λ . Also, conditions on \mathbf{A} at the specimen surface will be that \mathbf{A} and $\mathbf{n} \wedge \text{curl } \mathbf{A}$ are continuous, as is usual in classical electromagnetism. There it is also conventional to choose the gauge for \mathbf{A} such that

$$\text{div } \mathbf{A} = 0,$$

for reasons of convenience in calculations and for proving well-posedness; moreover, with this gauge and when \mathbf{A} vanishes at infinity, \mathbf{A} has the convenient integral representation $\mathbf{A}(\mathbf{x}) = (4\pi)^{-1} \int \mathbf{j}(\mathbf{x}') d\mathbf{x}' / |\mathbf{x} - \mathbf{x}'|$. We shall also make this choice here, but, more importantly, for $f \neq 0$ we can rewrite the equations in terms of real variables by introducing the new vector potential

$$(22) \quad \mathbf{Q} = \mathbf{A} - \gamma^{-1} \lambda \xi \nabla \chi$$

so that $\text{div } \mathbf{Q} = -\gamma^{-1} \lambda \xi \Delta \chi$. Then we obtain coupled equations for just f and \mathbf{Q} :

$$(23) \quad \xi^2 \Delta f = f^3 - f + \gamma^2 \lambda^{-2} f |\mathbf{Q}|^2,$$

⁴This boundary condition has been shown by de Gennes [19] to be modified to $\mathbf{n} \cdot (\xi \nabla - i\gamma \lambda^{-1} \mathbf{A}) \Psi = -\Psi/b$ at a boundary with another material; b is very large for insulators and very small for magnetic materials, with nonmagnetic metals lying in between.

$$(24) \quad -\lambda^2 \text{curl}^2 \mathbf{Q} = f^2 \mathbf{Q}.$$

The equation

$$\text{div}(f^2 \mathbf{Q}) = 0$$

is also a consequence of (20), (21), but it is now a trivial deduction from (24). We note that, from (1c),

$$(25a,b) \quad \mathbf{j} = \text{curl}^2 \mathbf{A} = -\frac{1}{\lambda^2} f^2 \mathbf{Q},$$

so that (23), (24) permit not only the calculation of f and \mathbf{Q} everywhere, but also the spatially distributed and (in the steady state) divergence-free superconducting current. Note that this formula differs from (11a) by the factor f^2 . We also note that the vector potential that determines this current from (25b) is not in general divergence-free, but that (25a) can always be used to determine the current irrespective of the gauge.

The natural boundary condition (21a) becomes

$$\mathbf{n} \cdot \nabla f = 0, \quad \mathbf{n} \cdot (f \mathbf{Q}) = 0,$$

and when we impose the condition that $\mathbf{n} \wedge \mathbf{Q}$ and $\mathbf{n} \wedge \text{curl} \mathbf{Q}$ are continuous at the boundary of the specimen, we expect to have enough boundary conditions for f and \mathbf{Q} just as long as $f > 0$. However, when f vanishes at some point or in some simply connected region, we must return to (20), (21) and prescribe the change in χ around the set where $f = 0$.

This can be illustrated by looking for a solution of the form

$$\begin{aligned} \Psi &= f(r) e^{in\theta}, \\ \mathbf{Q} &= Q(r) \mathbf{e}_\theta, \end{aligned}$$

where n is an integer. Such a solution is motivated by a local expansion near a zero of f . We find that f and Q are solutions of

$$\begin{aligned} -\frac{\xi^2}{r} \frac{d}{dr} \left(r \frac{df}{dr} \right) + \frac{\gamma^2}{\lambda^2} f Q^2 &= f - f^3, \\ \lambda^2 \frac{d}{dr} \left(\frac{1}{r} \frac{d}{dr} (rQ) \right) &= f^2 Q, \end{aligned}$$

with

$$f \rightarrow 1, Q \rightarrow 0 \quad \text{as } r \rightarrow \infty,$$

$$Q \sim -\frac{\xi \lambda n}{\gamma r}, f \text{ bounded} \quad \text{as } r \rightarrow 0,$$

and Berger and Chen [20] have shown the existence of a solution for f and Q for all n , which necessarily has $f(0) = 0$. The superconducting current is given by

$$\mathbf{j}_s = -\frac{1}{\lambda^2} f^2 Q \mathbf{e}_\theta,$$

and this solution represents a superconducting current “vortex.” The axial flux through the vortex is

$$\int \int \mathbf{H} \cdot d\mathbf{S} = \frac{2\pi\xi\lambda n}{\gamma};$$

thus the flux is quantized and a single “fluxon” has magnitude $2\pi\xi\lambda/\gamma$ in our scaled units.

We shall return to these vortex solutions, which are of considerable importance in what we shall later term Type II superconductors, in §5.

3.2. Evolution model. It is not as easy to make the above model time-dependent as it is, say, with the phase field model, because of the coupling with Maxwell’s equations. Fortunately an alternative approach is available, namely, that of averaging the microscopic BCS theory [16]. The procedure requires that the temperature T is close to T_c . It is described in [21] and results in the equations

$$(26) \quad -\alpha\xi^2 \frac{\partial f}{\partial t} + \xi^2 \Delta f = f^3 - f + \gamma^2 \lambda^{-2} f |\mathbf{Q}|^2,$$

$$(27) \quad -\lambda^2 \text{curl}^2 \mathbf{Q} = f^2 \mathbf{Q} + \lambda^2 \left(\frac{\partial \mathbf{Q}}{\partial t} + \nabla \Phi \right),$$

$$(28) \quad \alpha f^2 \Phi + \text{div}(f^2 \mathbf{Q}) = 0,$$

where α is a dimensionless relaxation time, λ and ξ are as before, and

$$\Phi = \phi + \frac{\xi\lambda}{\gamma} \frac{\partial \chi}{\partial t}.$$

In the steady state, and in the absence of an electric field, so that $\phi = 0$, these equations reduce to (23), (24) and thereby give independent support to the earlier phenomenological theory of Ginzburg and Landau. For completeness, (26)–(28) are written down in dimensional form in Appendix 3.

We expect that when suitable initial conditions are given for f , Φ , and \mathbf{Q} , together with appropriate boundary conditions, then, with the usual proviso that $f > 0$, (26)–(28) constitute a well-posed problem for f , Φ , and \mathbf{Q} from which the physical fields \mathbf{E} , \mathbf{H} can be computed using (18). These fields will then automatically satisfy (1b–d) but we note that in this unsteady case, the charge density need not vanish. It is given by

$$\text{div } \mathbf{E} = -\text{div} \left(\frac{\partial \mathbf{Q}}{\partial t} + \nabla \Phi \right) = \lambda^{-2} \text{div } f^2 \mathbf{Q} = -\lambda^{-2} \alpha f^2 \Phi.$$

Moreover, the current \mathbf{j} , which now has both superconducting and normal components, is given by $\mathbf{j} = -(1/\lambda^2)f^2\mathbf{Q} + \mathbf{E}$.

Having written down equations somewhat analogous to the phase field model, our next task is to try to relate these equations to the free boundary model (12)–(15) by taking suitable asymptotic limits in α , λ , and ξ .

4. Asymptotic solutions of the Ginzburg–Landau model: reduction to a free boundary problem. We now proceed to try to relate the above models (23), (24) and (26)–(28) to the free boundary models of §2 in the same way that it was possible to relate the phase field model to the Stefan problem. As well as obviating the need for any quantum mechanical considerations, it would be slightly easier to carry out this procedure just in the steady state first; however, the extra complication caused by the time derivatives in (26)–(28) is not great, and we shall proceed directly to the evolution case. As with the phase field model, we initially anticipate that the solution structure will comprise normal and superconducting regions separated by a transition layer; however, we shall soon see that this expectation will often not be warranted.

We seek limiting behavior as $\lambda, \xi \rightarrow 0$ with $\kappa = \lambda/\xi = O(1)$. Away from any transition regions, we look for asymptotic expansions of the form

$$\begin{aligned}
 \Phi &\sim \Phi^{(0)} + \lambda\Phi^{(1)} + \dots, \\
 \mathbf{Q} &\sim \mathbf{Q}^{(0)} + \lambda\mathbf{Q}^{(1)} + \dots, \\
 \mathbf{H} &\sim \mathbf{H}^{(0)} + \lambda\mathbf{H}^{(1)} + \dots, \\
 f &\sim f^{(0)} + \lambda f^{(1)} + \dots.
 \end{aligned}
 \tag{29}$$

When we assume smooth variations for Φ , \mathbf{Q} , and f , we soon find that either $f^{(0)} = 0$ or $\mathbf{Q}^{(0)} = \mathbf{0}$; such variations will correspond to “outer” normal and superconducting regions respectively. Proceeding to second order in these outer expansions, we find that in the normal region, $f^{(1)} = 0$, and hence

$$\frac{\partial \mathbf{Q}^{(0)}}{\partial t} + \nabla \Phi^{(0)} = -\text{curl}^2 \mathbf{Q}^{(0)}, \quad \mathbf{H}^{(0)} = \text{curl} \mathbf{Q}^{(0)},$$

so that

$$\frac{\partial \mathbf{H}^{(0)}}{\partial t} = \Delta \mathbf{H}^{(0)} \tag{30}$$

as in (12a). Indeed, this equation holds to arbitrary order in λ in the normal region as does the equation $\mathbf{Q}^{(0)} = \mathbf{0}$ in the superconducting region; the perturbations to these equations are exponentially small in λ .

It remains to consider a local analysis in a transition layer between these two regions. When local coordinates parallel and perpendicular to such a transition region are introduced, as in [14], the lowest-order equations correspond to a one-dimensional travelling wave. If z is the normal coordinate, scaled with λ , we may take $\mathbf{Q} = (0, -Q, 0)$ so that \mathbf{H} is tangential to the transition region to lowest order and of magnitude Q' , where $' = d/dz$. To lowest order, the order parameter and potential satisfy

$$(31) \quad f'' = \kappa^2(f^3 - f + \gamma^2 f Q^2),$$

$$(32) \quad Q'' = f^2 Q.$$

Now if the solutions of these equations are to match with the previously derived outer normal and superconducting regions, we require

$$(33) \quad Q \rightarrow 0, f \rightarrow 1 \quad \text{as } z \rightarrow -\infty \text{ (superconducting),}$$

$$(34) \quad f \rightarrow 0 \quad \text{as } z \rightarrow +\infty \text{ (normal).}$$

By integrating (31) and (32) once, it is now a simple matter to deduce⁵ that

$$(35) \quad Q' \rightarrow \frac{1}{\gamma\sqrt{2}} \quad \text{as } z \rightarrow +\infty.$$

By matching, this means that, as we approach the phase boundary from the normal region, the magnitude of $\mathbf{H}^{(0)}$, as given by (30), must tend to $1/\gamma\sqrt{2}$ (in addition to $\mathbf{H}^{(0)}$ being tangential to the phase boundary). Thus, the magnitude of the dimensional magnetic field

$$(36) \quad H_e |\mathbf{H}^{(0)}| \sim \frac{H_e}{\gamma\sqrt{2}} = H_c$$

to lowest order in λ . The functions f, Q' and $j = Q''$ are shown schematically in Fig. 3.

Beyond this level, the asymptotic analysis becomes even more intricate, but, by proceeding to $\mathbf{Q}^{(1)}, \mathbf{H}^{(1)}$ in the transition region and using the Fredholm alternative, we find that in our local coordinate system

$$(37) \quad \lim_{z \rightarrow +\infty} \left\{ \frac{\partial H_i^{(1)}}{\partial z} + H_i^{(0)}/R_i^{(0)} \right\} = -v_n^{(0)} \lim_{z \rightarrow +\infty} H_i^{(0)},$$

where, to lowest-order, $v_n^{(0)}$ is the normal velocity of the phase boundary, $(H_1^{(0)}, H_2^{(0)})$ is the component of \mathbf{H} tangential to this boundary and $1/R_i^{(0)}$ are the phase boundary curvatures in the $H_1^{(0)}, H_2^{(0)}$ directions with appropriate signs. By matching with the normal region in a suitable curvilinear coordinate system, (37) can be seen to imply (14).

We have thus retrieved our vector Stefan model (12)–(15) in general circumstances, assuming only that $\kappa = O(1)$. Our first surprise comes when we compute $|\mathbf{H}|$ in the transition region to second order; we find that the second-order matching condition for the outer normal field is that

⁵For a rigorous demonstration of this result, see [22].

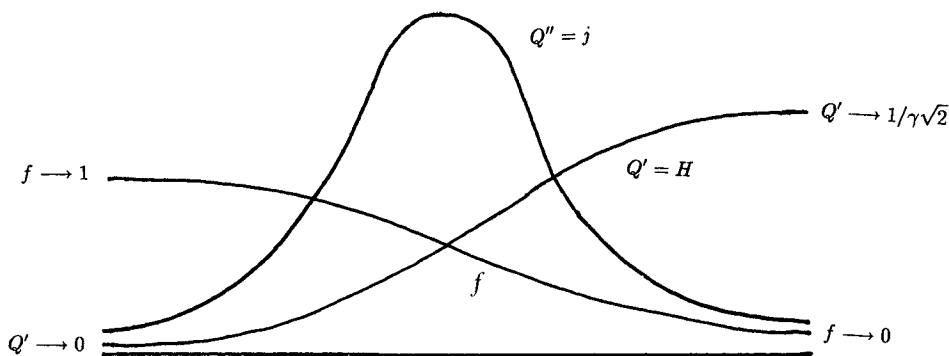


FIG. 3. Solution near a phase boundary.

$$(38) \quad |\mathbf{H}| \rightarrow \frac{1}{\gamma\sqrt{2}} - \left\{ \beta v_n^{(0)} + \delta(1/R_1^{(0)} + 1/R_2^{(0)}) \right\} \lambda.$$

As expected, there is a correction proportional to the velocity and the mean curvature of the phase boundary, the constants β and δ being determined by κ and the structure of the solutions to (31)–(34). The remarkable fact is that these constants are only both positive when $\kappa < 1/\sqrt{2}$, and, when $\kappa > 1/\sqrt{2}$, at least one of them is negative. We recall that when the corresponding constants σ and β in the limiting phase field boundary condition (8) were both positive, they stabilized what would have been an ill-posed problem in their absence. Here it seems that the bracketed terms in (38) are *only* stabilizing when $\kappa < 1/\sqrt{2}$. However, we should not read too much into this analogy because the Ginzburg–Landau model only allows the possibility of stabilizing terms to enter (38) at *first* order; in the corresponding phase field analysis [9], the fact that the order parameter F changes from -1 to $+1$ rather than 0 to $+1$ means that an asymptotic limit can be taken in which stabilization occurs at *lowest* order.

5. Superconductor classification and type II superconductors. The preceding section has attempted to give a microscopic basis for the free boundary models described in §2, but, in view of (38), the sharp-interface theory is only satisfactory mathematically in the case $\kappa < 1/\sqrt{2}$; hence it is only in this case that we can justify the use of the model (12)–(15) to describe the incomplete phase diagram in Fig. 4. It predicts that, for materials satisfying this criterion, called *Type I* superconductors, the change of phase occurs by means of phase boundaries propagating through the material in ways analogous to, say, solid/liquid phase boundaries. Moreover, it predicts that these phase boundaries will be prone to instability in cases when the normal region contracts, in the same way that the freezing of supercooled liquids is unstable.

These predictions for Type I superconductors are more or less in agreement with observation [23]–[25], the only exception being that the phase diagram in Fig. 4 does not indicate the hysteresis which is observed in such materials. The experimental observations that indicate that some new mechanism operates when $\kappa > 1/\sqrt{2}$ are shown schematically in Fig. 5(a), (b). For such materials, called *Type II* superconductors, the superconducting phase does not switch off abruptly as H_0 increases through some critical value (which was H_c for Type I materials and is now called H_{c1}), but rather shrinks

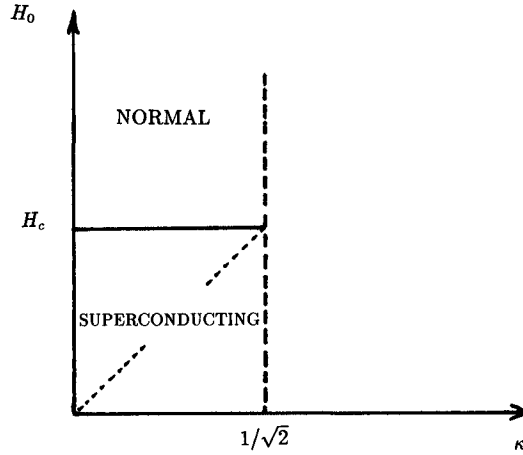


FIG. 4. Phase diagram for a Type I Superconductor.

until H_0 reaches a new value H_{c2} . Moreover, some small “surface” superconductivity is observed for $H_{c2} < H_0 < H_{c3}$; the reason for this terminology will become apparent shortly.

Our task is now to explain the hysteresis in Fig. 5a and, more importantly, how phase changes occur when ξ and λ are small in Type II superconductors. The first mathematical clue to this behavior can be discerned if we consider small, steady perturbations about the exact solution of (23), (24) that represent purely normal material in a field perpendicular to the z -direction, namely,

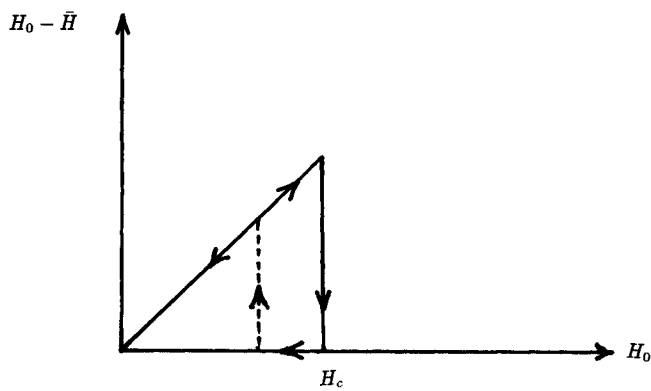
$$(39) \quad f = 0, \quad \mathbf{Q} = (0, -(H_0 z + Q_0), 0), \quad Q_0, H_0 = \text{const.}$$

When f depends on z alone (i.e., $\chi = 0$, $\Psi = f(z)$ in (20)), the linearized equation for f , namely,

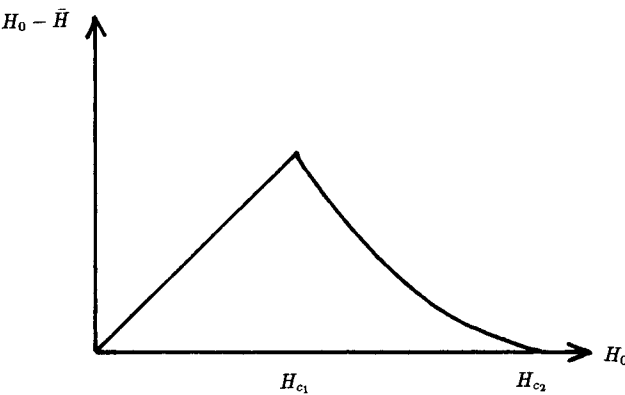
$$(40) \quad \xi^2 f'' + \left(1 - \left(\frac{\gamma}{\lambda}(H_0 z + Q_0)\right)^2\right) f = 0,$$

is easy to solve for a variety of boundary conditions. For a film of thickness $2d$, we impose the natural boundary conditions

$$f'(\pm d) = 0$$



(a) $\kappa < 1/\sqrt{2}$: Type I materials (with hysteresis)



(b) $\kappa > 1/\sqrt{2}$: Type II materials (no hysteresis)

FIG. 5. Type I vs. Type II Superconductors.

and zero perturbation for \mathbf{H} at $z = \pm d$. As H_0 is decreased, a nonzero solution first exists for $d = \infty$ when $H_0 = \kappa/\gamma$ and f is proportional to $e^{-\xi^2 z^2/2}$. The amplitude is obtained in [26], [27] from a weakly nonlinear analysis of (31), (35) but the important result is that the value of H_0 at which a localized planar region of weak superconductivity first appears is H_{c2} , where

$$H_{c2} \geq H_c \simeq \frac{1}{\gamma\sqrt{2}}$$

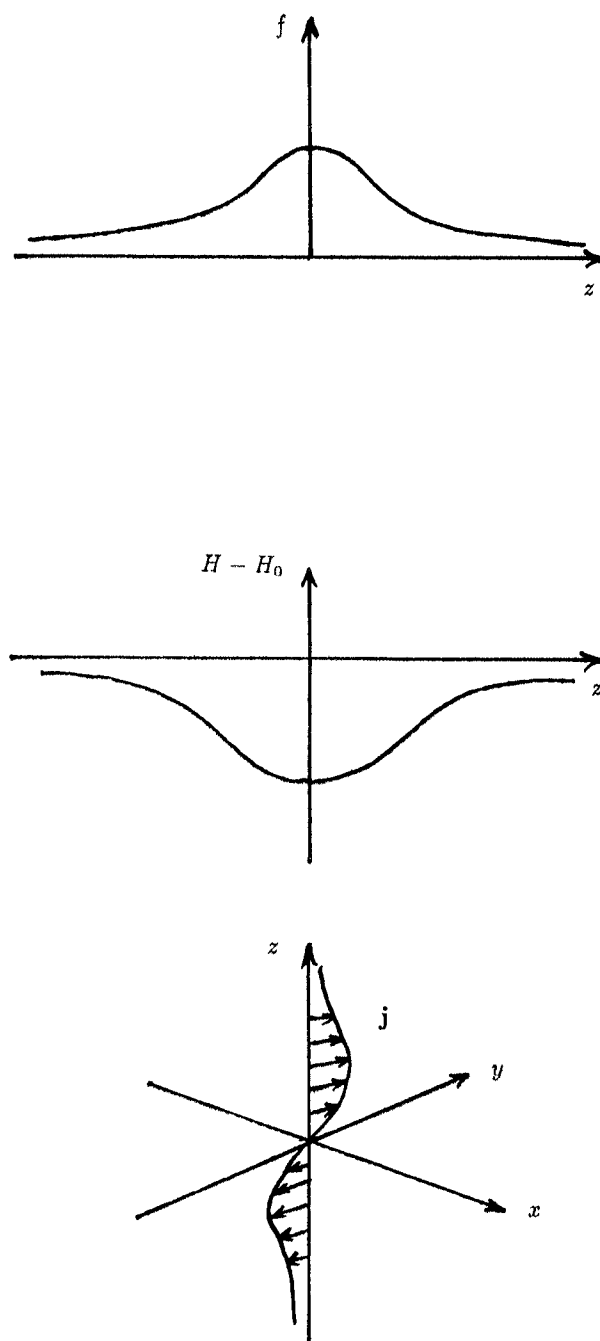


FIG. 6. One-dimensional solution bifurcating from the normal state.

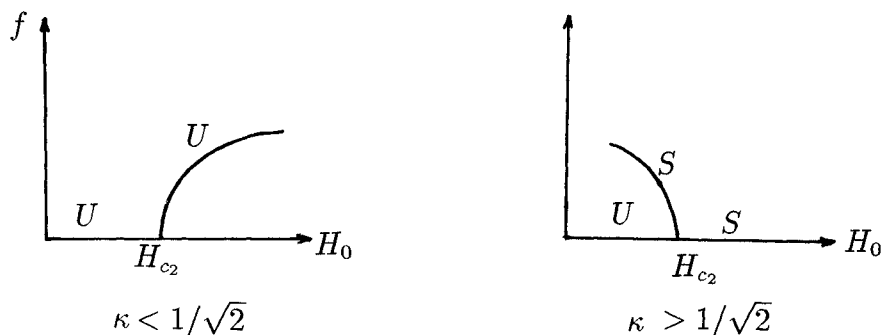


FIG. 7. Response diagrams for Type I and Type II Superconductors.

according as $\kappa \gtrless 1/\sqrt{2}$. The results of the weakly nonlinear analysis give the current flow and bifurcation diagrams shown in Figs. 6 and 7.

This approach can be adapted to consider finite slabs of material and other boundary conditions. We shall not give any details here but merely note that the field at which superconductivity first appears is even larger than κ/γ for finite slabs, and that by considering boundary layers near the surface of slabs of size much greater than ξ , the theory can predict the existence of the previously-mentioned third critical field H_{c3} [28]. However, this third critical field only exists in configurations in which the imposed magnetic field has a component parallel to the boundary of the material.

These ideas go some way towards explaining the phenomena in Fig. 5, and, in particular, the gross features of Fig. 5b and the hysteresis in Fig. 5a, because the bifurcation at H_{c2} in a Type I superconductor can now be thought of as an analogy of nucleation in supercooled freezing. However, we have not yet elucidated the structure of the phases in Type II superconductors when $H_{c1} < H_0 < H_{c2}$. Some indication of this can be obtained by considering two-dimensional perturbations to (39). One possibility would be to write $\Psi = e^{imy} f(z)$ where m and f are real, but, from (25), this would simply give a superconducting current of the form $\mathbf{j} = -(f^2/\lambda^2)(0, H_0 z + Q_0 - m\xi\lambda/\gamma, 0)$, i.e., a shift in the origin of z of size $m\xi\lambda/H_0\gamma$. However, in the case when $Q_0 = 0$, we see that $e^{-imy} f(-z)$ is also a solution of the linearized equations, and when we write

$$\Psi = e^{imy} f(z) + e^{-imy} f(-z)$$

(so that $f \neq |\Psi|$), it is easy to see that χ , the phase of Ψ , varies by 2π as we go around any zero of Ψ at $z = 0, y = (n + \frac{1}{2})\pi/m$. Hence the associated superconducting current is

$$\mathbf{j} = -\frac{|\Psi|^2}{\lambda^2} \left(0, H_0 z - \frac{\xi\lambda}{\gamma} \nabla\chi, 0 \right) \simeq \left(0, \frac{|\Psi|^2 \xi}{\lambda\gamma} \nabla\chi, 0 \right)$$

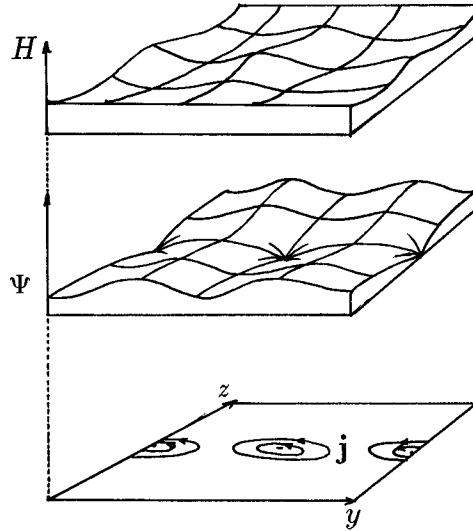


FIG. 8. Two-dimensional solution bifurcating from the normal solution. As H_0 decreases further the variations in f and H become more pronounced and resemble Fig. 9.

and is nearly azimuthal, as is the vortex solution of §3.1 (see Fig. 8). The situation in Fig. 8 is dramatically different from Fig. 6 because $|\Psi|$ vanishes at the “quantized” vortices. It is the basis of the work in [26], [29], where solutions periodic both in x and y as in Fig. 9 are sought for H_0 slightly less than H_{c2} . The zeros of Ψ are found to lie on the points of a lattice with the phase of Ψ varying by 2π around each zero. In [30] a hexagonal lattice is shown to have the lowest free energy; there seems to be no prescription for the vortex spacing, but it must be such that the vortices and their surrounding normal material can transmit the whole of the applied magnetic field. As the field H_0 is lowered further it is conjectured that the lattice points move further apart with f tending closer to unity in the regions between, so that the solution resembles a lattice of “normal filaments,” each surrounded by a superconducting current vortex, embedded in a superconducting matrix (see Fig. 9). When the distance between filaments is large compared to the penetration depths λ, ξ (i.e., when interaction between the filaments is negligible), we expect each filament to resemble the vortex solution given in §3.

As the separation between filaments increases, filaments will migrate to the boundary of the specimen and be lost. As H_0 reaches the critical value H_{c1} , the last filament disappears and the material becomes wholly superconducting. The value of H_{c1} has been estimated in [29] using an energy argument.

Of course the wholly superconducting state will still have a boundary layer at the surface of the material in which H decreases from its surface value to its value of zero in the bulk of the material. Such a solution was given in [18].

We are now in a position to complete the response diagram Fig. 4; this is done in Fig. 10. For Type II superconductors the state in which $H_{c1} < H_0 < H_{c2}$ and in which the phase morphology is that of normal cores embedded in a superconducting matrix, with the magnetic field and superconducting current being axial and azimuthal respectively, is called Abrikosov’s *mixed state*. We emphasize that the response to variations in H_0 for Type II materials is reversible, with the normal cores growing and multiplying as H_0 increases from H_{c1} , but for Type I materials it is not.

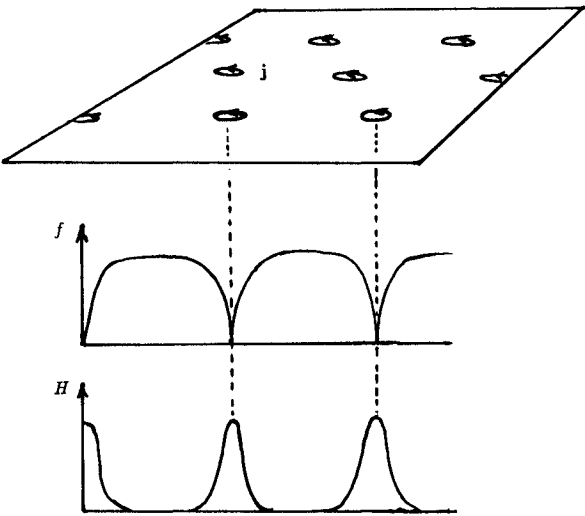


FIG. 9. Mixed state near $H_0 = H_{c1}$.

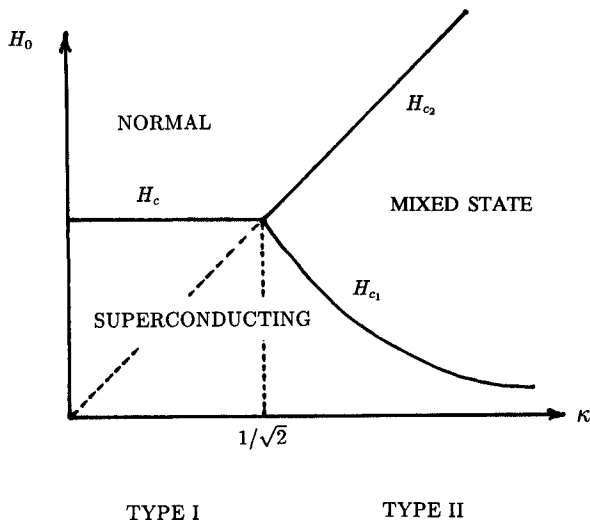


FIG. 10. Completion of Fig. 4.

6. Open questions.

6.1. “Melting” of the mixed state. Perhaps the most important technological question concerns the modelling of Type II materials in the mixed state. When κ is large we may be able to simplify (26)–(28) by exploiting the fact that the normal filaments are sufficiently thin to be represented as distributions in the field equation for \mathbf{H} . Since $f \simeq 1$, this equation reduces to the London model

(41)
$$\lambda^2 \Delta \mathbf{H} - \mathbf{H} = \mathbf{0}$$

away from the filaments, and, indeed, this simplification has been justified in [20]. The model must be closed by relating the strength and velocity of the filaments to \mathbf{H} , and

this has been attempted in [31] by assigning a force and mobility to each filament, using energy arguments similar to those used in dislocation dynamics in metal plasticity. It seems that vortex filaments in superconductors have almost as much freedom to move as do vortices in an inviscid liquid except that they can be pinned at defects in the underlying lattice.

Bulk motion, or “melting,” of the filaments in the presence of defects is the principal phenomenon to be modelled; when large numbers of filaments are present, one possibility would be a homogenization of the distributions on the right-hand side of the London model (41).

6.2. Behavior at $\kappa = 1/\sqrt{2}$. Clearly there is singular bifurcation behavior in the vicinity of this critical value of κ , and it would be of interest to see how a material with such a composition arranges its morphology and reversibility. It may be a coincidence, but the Ginzburg–Landau equations have several other special properties at $\kappa = 1/\sqrt{2}$.

Firstly, in one dimension with $f = f(z)$ and $\mathbf{Q} = (0, Q(z), 0)$, the free energy density

$$\xi^2(f')^2 + \frac{\gamma^2}{\lambda^2} f^2 Q^2 + \frac{(f^2 - 1)^2}{2} + \gamma^2 (Q')^2$$

may be written as

$$\begin{aligned} & \left(\xi f' + \frac{\gamma}{\lambda} f Q \right)^2 + \left(\gamma Q' + \frac{f^2 - 1}{2\kappa} \right)^2 \\ & + \frac{1}{2} (f^2 - 1)^2 \left(1 - \frac{1}{2\kappa^2} \right) \\ & + \frac{\gamma}{\kappa} [Q(1 - f^2)]'. \end{aligned}$$

Hence, when $\kappa = 1/\sqrt{2}$, the equations exhibit “self-duality symmetry” [32], in that the free energy can be written as an integral of a sum of squares of just two first-order operators together with an exact differential. In this case solutions of the second-order equations (31), (32) are given by solutions of the first-order equations⁶

$$\xi f' + \frac{\gamma}{\lambda} f Q = 0,$$

$$\gamma Q' + \frac{f^2 - 1}{2\kappa} = 0.$$

This reduction depends on the application of compatible boundary conditions, which is automatic when the transition layer conditions (33), (34) are assumed. Similar solutions may be found in two dimensions with the magnetic field perpendicular to the plane of interest. In this case, with $\mathbf{Q} = (Q_1(x, y), Q_2(x, y), 0)$ solutions of (23), (24) are now given by setting each of three squares in the free energy equal to zero, yielding the first-order equations

⁶Note that the exact differential in the free energy density integrates to give a constant in the free energy that depends on the boundary conditions and may be infinite.

$$\begin{aligned}\xi \frac{\partial f}{\partial x} + \frac{\gamma}{\lambda} f Q_2 &= 0, \\ \xi \frac{\partial f}{\partial y} - \frac{\gamma}{\lambda} f Q_1 &= 0, \\ \gamma \left(\frac{\partial Q_2}{\partial x} - \frac{\partial Q_1}{\partial y} \right) + \frac{(f^2 - 1)}{2\kappa} &= 0.\end{aligned}$$

Hence, when $f \neq 0$, $w = \log f^2$ satisfies the inhomogeneous Liouville equation

$$\lambda^2 \Delta w + 1 - e^w = 0,$$

but again this reduction requires the application of compatible boundary conditions.

Secondly, [33] has shown that the force between two superconducting current vortices is attractive for $\kappa < 1/\sqrt{2}$ and repulsive for $\kappa > 1/\sqrt{2}$; for $\kappa = 1/\sqrt{2}$, there is no force between vortices. In this latter case multivortex solutions to the steady-state Ginzburg–Landau equations exist, and [34] has shown that the solutions containing N quanta of flux may be parametrized by the points of the plane where f vanishes together with their vortex numbers. Thus such a solution can be thought of as a superposition of N vortices, although the details of their interaction remain an interesting open problem; some numerical calculations are reported in [35].

Because of the nature of the forces between vortices when $\kappa < 1/\sqrt{2}$ and $\kappa > 1/\sqrt{2}$, we conjecture that in these cases the only steady vortex solutions to the equations are for an isolated vortex, or an infinite number of vortices as in the Abrikosov vortex lattice; we expect the latter configuration to be stable for hexagonal arrays of identical unit vortex-number vortices when $\kappa > 1/\sqrt{2}$, and the former when $\kappa < 1/\sqrt{2}$, even for higher vortex numbers [36].

6.3. Type I morphologies. Although the modelling of Type I materials seems to be easier than that of Type II materials, interesting questions remain concerning even quite simple configurations. Consider, for example, what happens when a superconducting wire carries a prescribed current I down its surface. As I is increased, the corresponding azimuthal magnetic field will increase until it reaches H_c , when the surface of the wire must become normal. If a normal sheath were then to form around the surface of the wire, leaving a superconducting core, the superconducting current would lie on the surface of this core. This increased current density there would then cause an increased field, meaning that the sheath would contract inwards. However, if the whole wire became normal, the current would be uniformly distributed over its cross section, leading to a reduced current density, with a magnetic field everywhere below H_c ! The wire can thus be neither wholly normal nor wholly superconducting, and the intermediate state may have a noncylindrical morphology or even be unsteady; several proposals have been made for this [37]–[39]. A similar argument can be applied to the problem in Fig. 1 in the case when the wire is initially superconducting, and an applied *transverse* field of magnitude H_0 is increased.

6.4. Thermal effects. As mentioned in the introduction, thermal effects are not usually significant, at least for Type I superconductors with T not too near T_c ; in other words, the thermal response is assumed instantaneous, and T only enters as a parameter. When the thermal response is less rapid, thermal effects can be incorporated into

the free boundary model (12)–(15) by allowing H_c in (13) to depend on T as in Fig. 1, and appending equations for T on either side of the free boundary, together with Stefan-type conditions on the boundary itself. This has been done in one space dimension in [40]. Heat is generated via Ohmic heating in the normal phase, and θ , the dimensionless temperature difference from T_c scaled with T_c , satisfies

$$(42) \quad \Delta\theta = \beta \frac{\partial\theta}{\partial t}$$

in the superconducting region, and

$$(43) \quad \begin{aligned} \Delta\theta &= \beta \frac{\partial\theta}{\partial t} - \eta |\mathbf{j}_N|^2 \\ &= \beta \frac{\partial\theta}{\partial t} - \eta |\operatorname{curl} \mathbf{H}|^2 \end{aligned}$$

in the normal region. On the interface between the two regions,

$$(44) \quad [\theta]_N^S = 0$$

$$(45) \quad \left[\frac{\partial\theta}{\partial n} \right]_S^N = -L(\theta)v_n.$$

Here $L(\theta)$ is the (dimensionless) latent heat associated with the change of phase; $\beta = \rho c / \mu \sigma k$ and $\eta = H_e^2 / \sigma k T_c$ are dimensionless parameters measuring the ratios of thermal to electromagnetic timescales and ohmic heating to thermal conduction respectively (the density ρ , specific heat c , and thermal conductivity k are assumed constant). In some respects this model resembles the one-phase alloy solidification problem, with \mathbf{H} playing the role of the impurity concentration. We also note that it bears a superficial resemblance to the “thermistor” problem [41] with a step-function conductivity, but a closer examination shows that the interface conditions for the two problems are quite different.

Of more interest, however, is the interaction between thermal and electromagnetic effects for a Type II superconductor. In particular “hot spots” generated near vortex filaments may have a significant effect on their motion. This situation can be modelled by a generalization of the Ginzburg–Landau equations in which temperature appears as a variable rather than a parameter. As shown in Appendix 3, the resulting time-dependent Ginzburg–Landau equations are

$$(46) \quad -\alpha \tilde{\xi}^2 \frac{\partial f}{\partial t} + \tilde{\xi}^2 \Delta f = f^3 + \theta f + \frac{\tilde{\gamma}^2}{\tilde{\lambda}^2} f |\mathbf{Q}|^2,$$

$$(47) \quad -\tilde{\lambda}^2 (\operatorname{curl})^2 \mathbf{Q} = f^2 \mathbf{Q} + \tilde{\lambda}^2 \left(\frac{\partial \mathbf{Q}}{\partial t} + \nabla \Phi \right),$$

$$(48) \quad \alpha f^2 \Phi + \operatorname{div}(f^2 \mathbf{Q}) = 0,$$

together with a temperature equation of the form

$$(49) \quad \Delta \theta = \beta \frac{\partial \theta}{\partial t} - \eta |\mathbf{j}_N|^2 - L(\theta) \frac{\partial(f^2)}{\partial t}.$$

We note that θ appears as a coefficient in (46), and that, where previously ξ , λ , and γ depended on θ (see Appendix 2), their equivalents $\tilde{\xi}$, $\tilde{\lambda}$, and $\tilde{\gamma}$ are temperature-independent (see Appendix 3). If this temperature-independent scaling had not been made, as $T \uparrow T_c$ in (26)–(28) we would have had $\xi, \lambda \rightarrow \infty$ as $(T_c - T)^{-1/2}$ and $\gamma \rightarrow \infty$ as $T_c - T$. We also remark that $\theta < 0$ in the transition layer and in the superconducting region.

It is now possible to derive the temperature-dependent free boundary problem (42)–(45) as a formal asymptotic limit of (46)–(49) as $\tilde{\lambda}$ and $\tilde{\xi} \rightarrow 0$ as in §4, but we will not give the details here (see [42]). Also we note that the steady version of (46)–(48) can be used to study bifurcation behavior similar to Fig. 7 by varying the applied temperature rather than H_0 (see [42], [43]).

Appendix 1. Here we show that $\operatorname{div} \mathbf{H} = 0$ everywhere is a consequence of (12a) and (15). We denote the normal region by Ω_N , and assume that it is entirely bounded by the free boundary Γ (the result holds even when some portion of the normal region meets a fixed boundary of the sample; see [42]). For ease of notation we denote $\operatorname{div} \mathbf{H}$ by u .

We first note that

$$\operatorname{div}(u\mathbf{H}) = \mathbf{H} \cdot \nabla u + u^2.$$

Hence

$$\begin{aligned} \int_{\Omega_N} (\mathbf{H} \cdot \nabla u + u^2) dV &= \int_{\Omega_N} \operatorname{div}(u\mathbf{H}) dV \\ &= \int_{\Gamma} u\mathbf{H} \cdot \mathbf{n} dS \\ &= 0, \end{aligned}$$

since $\mathbf{H} \cdot \mathbf{n} = 0$ on Γ . Hence

$$\int_{\Omega_N} u^2 dV = - \int_{\Omega_N} \mathbf{H} \cdot \nabla u dV.$$

Differentiating this equation with respect to t we have

$$\begin{aligned} \frac{d}{dt} \int_{\Omega_N} u^2 dV &= - \frac{d}{dt} \int_{\Omega_N} \mathbf{H} \cdot \nabla u dV \\ &= - \int_{\Omega_N} \left[\frac{\partial \mathbf{H}}{\partial t} \cdot \nabla u + \mathbf{H} \cdot \frac{\partial}{\partial t} (\nabla u) \right] dV - \int_{\Gamma} (\mathbf{H} \cdot \nabla u) v_n dS. \end{aligned}$$

Taking the divergence of equation (12a) yields

$$\frac{\partial u}{\partial t} = 0.$$

Hence, using (12a) and (15) we have

$$\begin{aligned} \frac{d}{dt} \int_{\Omega_N} u^2 dV &= \int_{\Omega_N} ((\operatorname{curl})^2 \mathbf{H} \cdot \nabla u) dV + \int_{\Gamma} (\operatorname{curl} \mathbf{H} \wedge \mathbf{n}) \cdot \nabla u dS \\ &= \int_{\Omega_N} \operatorname{div}(\operatorname{curl} \mathbf{H} \wedge \nabla u) dV + \int_{\Gamma} (\operatorname{curl} \mathbf{H} \wedge \mathbf{n}) \cdot \nabla u dS \\ &= \int_{\Gamma} (\operatorname{curl} \mathbf{H} \wedge \nabla u) \cdot \mathbf{n} dS - \int_{\Gamma} (\operatorname{curl} \mathbf{H} \wedge \nabla u) \cdot \mathbf{n} dS \\ &= 0. \end{aligned}$$

Hence

$$\int_{\Omega_N} u^2 dV = 0,$$

since initially $\operatorname{div} \mathbf{H}_0 = 0$ everywhere. Hence

$$\operatorname{div} \mathbf{H} = 0 \quad \text{in } \Omega_N,$$

as required.

We prove also that $\operatorname{div} \mathbf{H}$ is zero when (12a) is replaced by (12b) and $v_n \geq 0$, which is the case of interest. As before, we have

$$\int_{\Omega_N} u^2 dV = - \int_{\Omega_N} \mathbf{H} \cdot \nabla u dV.$$

Differentiating this equation with respect to t we have

$$\begin{aligned}
 \frac{d}{dt} \int_{\Omega_N} u^2 dV &= - \frac{d}{dt} \int_{\Omega_N} \mathbf{H} \cdot \nabla u dV \\
 &= - \int_{\Omega_N} \left[\frac{\partial \mathbf{H}}{\partial t} \cdot \nabla u + \mathbf{H} \cdot \frac{\partial}{\partial t} (\nabla u) \right] dV - \int_{\Gamma} (\mathbf{H} \cdot \nabla u) v_n dS \\
 &= - \int_{\Omega_N} \left[(\Delta \mathbf{H} \cdot \nabla u) + \mathbf{H} \cdot \frac{\partial}{\partial t} (\nabla u) \right] dV \\
 &\quad - \int_{\Gamma} (\operatorname{curl} \mathbf{H} \wedge \mathbf{n}) \cdot \nabla u dS,
 \end{aligned}$$

by (12b) and (15). Now

$$\Delta \mathbf{H} = \nabla u - (\operatorname{curl})^2 \mathbf{H}.$$

Also

$$\begin{aligned}
 \int_{\Omega_N} \operatorname{div} \left(\frac{\partial u}{\partial t} \mathbf{H} \right) dV &= \int_{\Omega_N} u \frac{\partial u}{\partial t} + \mathbf{H} \cdot \nabla \left(\frac{\partial u}{\partial t} \right) dV \\
 &= \int_{\Gamma} \frac{\partial u}{\partial t} \mathbf{H} \cdot \mathbf{n} dS \\
 &= 0,
 \end{aligned}$$

since $\mathbf{H} \cdot \mathbf{n} = 0$ on Γ . Hence

$$\begin{aligned}
 \frac{d}{dt} \int_{\Omega_N} u^2 dV &= \int_{\Omega_N} \left((\operatorname{curl})^2 \mathbf{H} \cdot \nabla u - |\nabla u|^2 + u \frac{\partial u}{\partial t} \right) dV \\
 &\quad - \int_{\Gamma} (\operatorname{curl} \mathbf{H} \wedge \mathbf{n}) \cdot \nabla u dS \\
 &= \int_{\Omega_N} \frac{1}{2} \frac{\partial u^2}{\partial t} - |\nabla u|^2 dV,
 \end{aligned}$$

since

$$\int_{\Omega_N} (\operatorname{curl})^2 \mathbf{H} \cdot \nabla u dV - \int_{\Gamma} (\operatorname{curl} \mathbf{H} \wedge \mathbf{n}) \cdot \nabla u dS = 0,$$

as before. We also have that

$$\frac{d}{dt} \int_{\Omega_N} u^2 dV = \int_{\Omega_N} \frac{\partial u^2}{\partial t} dV - \int_{\Gamma} u^2 v_n dS.$$

Hence

$$\frac{1}{2} \int_{\Omega_N} \frac{\partial u^2}{\partial t} dV = - \int_{\Omega_N} |\nabla u|^2 dV - \int_{\Gamma} u^2 v_n dS.$$

Hence, for $v_n \geq 0$,

$$\int_{\Omega_N} \frac{\partial u^2}{\partial t} dV \leq 0.$$

Hence

$$\frac{d}{dt} \int_{\Omega_N} u^2 dV \leq 0.$$

However, $\int_{\Omega_N} u^2 dV \geq 0$, and $\int_{\Omega_N} u^2 dV = 0$ initially. Hence

$$\int_{\Omega_N} u^2 dV = 0,$$

and, therefore, $\operatorname{div} \mathbf{H} = 0$, as required.

Appendix 2. The Ginzburg–Landau free energy functional. The expanded and truncated Ginzburg–Landau free energy density is, in dimensional variables⁷,

$$\mathcal{E} = \alpha(T)|\Psi|^2 + \frac{\beta(T)}{2}|\Psi|^4 + \frac{\mu}{2}|\mathbf{H}|^2 + \frac{1}{4m}|-i\hbar\nabla\Psi - 2e\mathbf{A}\Psi|^2$$

where m is the electron mass, e the electronic charge, μ the permeability, and \hbar Planck's constant. Consider first a uniform superconductor in the absence of a magnetic field, so that

$$\mathcal{E} = \alpha(T)|\Psi|^2 + \beta\frac{(T)}{2}|\Psi|^4.$$

In equilibrium $\partial\mathcal{E}/\partial|\Psi|^2 = 0$, $\partial^2\mathcal{E}/\partial(|\Psi|^2)^2 > 0$, with $|\Psi|^2 = 0$ for $T \geq T_c$, $|\Psi|^2 > 0$ for $T < T_c$. It follows that $\alpha(T_c) = 0$, $\beta(T_c) > 0$, $\alpha(T) < 0$ for $T < T_c$. For temperatures in the vicinity of T_c the coefficients α, β may be expanded in powers of $\theta = (T - T_c)/T_c$ and only the first nonzero terms retained. Then

$$\alpha(T) = a \left(\frac{T - T_c}{T_c} \right), \quad \beta(T) = b, \quad a, b > 0.$$

⁷In Gaussian units the free energy density is $\alpha(T)|\Psi|^2 + \frac{\beta(T)}{2}|\Psi|^4 + \frac{|\mathbf{H}|^2}{8\pi} + \frac{1}{4m}|-i\hbar\nabla\Psi - \frac{2e}{c}\mathbf{A}\Psi|^2$.

Thus in equilibrium for $T \leq T_c$,

$$|\Psi|^2 = -\frac{\alpha}{\beta} = \frac{a}{b} \left(\frac{T_c - T}{T_c} \right).$$

We nondimensionalize by setting

$$\begin{aligned} \mathbf{H} &= H_e \tilde{\mathbf{H}}, & \mathbf{A} &= H_e \ell \mu \tilde{\mathbf{A}}, \\ T &= T_c + T_c \theta, & \Psi &= \sqrt{\frac{a}{b}} \tilde{\Psi}, \\ \mathbf{x} &= \ell \tilde{\mathbf{x}}, & \mathcal{E} &= \frac{a^2}{b} \mathcal{E}. \end{aligned}$$

Then

$$\mathcal{E} = \theta |\Psi|^2 + \frac{|\tilde{\Psi}|^4}{2} + \tilde{\gamma}^2 |\tilde{\mathbf{H}}|^2 + | -i\tilde{\xi} \nabla \tilde{\Psi} - \frac{\tilde{\gamma}}{\tilde{\lambda}} \tilde{\Psi} \tilde{\mathbf{A}}|^2,$$

where

$$\tilde{\gamma} = \frac{H_e}{a} \sqrt{\frac{\mu b}{2}}, \quad \tilde{\xi} = \frac{\hbar}{2\ell \sqrt{ma}}, \quad \tilde{\lambda} = \frac{1}{e\ell} \sqrt{\frac{mb}{2a\mu}},$$

and θ is included as a variable; $\tilde{\gamma}, \tilde{\lambda}, \tilde{\xi}$ are independent of temperature. Under isothermal conditions we may rescale

$$\begin{aligned} \tilde{\Psi} &= |\theta|^{1/2} \Psi', & \mathcal{E} &= \theta^2 \mathcal{E}', \\ \tilde{\mathbf{H}} &= \mathbf{H}', & \tilde{\mathbf{A}} &= \mathbf{A}', & \tilde{\mathbf{x}} &= \mathbf{x}', \end{aligned}$$

to give

$$\mathcal{E}' = -|\Psi'|^2 + \frac{|\Psi'|^4}{2} + \gamma^2 |\mathbf{H}'|^2 + | -i\xi \nabla \Psi' - \frac{\gamma}{\lambda} \Psi' \mathbf{A}'|^2,$$

where

$$\xi = \frac{\tilde{\xi}}{|\theta|^{1/2}}, \quad \lambda = \frac{\tilde{\lambda}}{|\theta|^{1/2}}, \quad \gamma = \frac{\tilde{\gamma}}{|\theta|},$$

so that the temperature appears in the equations only through the parameters ξ, λ, γ . In [18] it is noted that $H_c = |\alpha|/\sqrt{\mu\beta} = a|\theta|/\sqrt{\mu b}$, and hence $\gamma = H_e/\sqrt{2}H_c$. The nonlinear dependence of H_c on T shown in Fig. 1 may be obtained by retaining further powers of θ .

Appendix 3. Time-dependent “Ginzburg–Landau” equations. Gor’kov and Éliashberg [21] average the microscopic BCS theory and arrive at the following macroscopic equations:⁸

$$\frac{\partial \Psi}{\partial t} + 2ei\Psi\phi + \frac{\tau_S}{3} \left[-\pi^2(T_c^2 - T^2) + \frac{|\Psi|^2}{2} \right] \Psi - D(\nabla - 2ie\mathbf{A})^2\Psi = 0,$$

$$\begin{aligned} \mathbf{j} &= \frac{1}{\mu} \text{curl}^2 \mathbf{A} \\ &= -\sigma \left(\frac{\partial \mathbf{A}}{\partial t} + \nabla \phi \right) - 2\sigma\tau_S \left(\mathbf{A}|\Psi|^2 - \frac{i}{4e}(\Psi^*\nabla\Psi - \Psi\nabla\Psi^*) \right), \end{aligned}$$

where τ_S is a microscopic parameter, D is the diffusion coefficient, σ is the conductivity of the normal electrons, e is the electronic charge, μ is the permeability, T is the temperature, and T_c is the critical temperature. Under isothermal conditions we nondimensionalize by setting

$$\mathbf{x} = \ell \mathbf{x}', \quad \Psi = \pi \sqrt{2(T_c^2 - T^2)} \Psi',$$

$$\mathbf{H} = H_e \mathbf{H}', \quad \mathbf{A} = \mu \ell H_e \mathbf{A}',$$

$$t = \mu \sigma \ell^2 t', \quad \phi = \frac{H_e}{\sigma} \phi',$$

to give (dropping the primes)

$$\alpha \xi^2 \frac{\partial \Psi}{\partial t} + \frac{\alpha \xi \gamma}{\lambda} i \Psi \phi + \Psi |\Psi|^2 - \Psi - \left(\xi \nabla - i \frac{\gamma}{\lambda} \mathbf{A} \right)^2 \Psi = 0,$$

$$-\lambda^2 \text{curl}^2 \mathbf{A} = \lambda^2 \frac{\partial \mathbf{A}}{\partial t} + \lambda^2 \nabla \phi + \frac{\xi \lambda}{2\gamma} i (\Psi^* \nabla \Psi - \Psi \nabla \Psi^*) + |\Psi|^2 \mathbf{A},$$

where

$$\alpha = \frac{1}{\mu \sigma D}, \quad \xi = \frac{1}{\ell \pi} \sqrt{\frac{3D}{\tau_S(T_c^2 - T^2)}},$$

⁸In Gaussian units these equations are

$$\begin{aligned} \frac{\partial \Psi}{\partial t} + 2ei\Psi\phi + \frac{\tau_S}{3} \left[-\pi^2(T_c^2 - T^2) + \frac{|\Psi|^2}{2} \right] \Psi - D \left(\nabla - \frac{2ie}{c} \mathbf{A} \right)^2 \Psi &= 0 \\ -\frac{c}{4\pi} (\text{curl})^2 \mathbf{A} &= \sigma \left(\frac{1}{c} \frac{\partial \mathbf{A}}{\partial t} + \nabla \phi \right) + \frac{2\sigma\tau_S}{c} \left(|\Psi|^2 \mathbf{A} - \frac{ic}{4e} (\Psi^* \nabla \Psi - \Psi \nabla \Psi^*) \right). \end{aligned}$$

$$\lambda = \frac{1}{2\pi\ell\sqrt{\sigma\tau_S\mu(T_c^2 - T^2)}}, \quad \gamma = \frac{eH_e}{\pi^2\tau_S(T_c^2 - T^2)}\sqrt{\frac{3D\mu}{\sigma}}.$$

These expressions give the values of λ, ξ, γ in terms of microscopic parameters and hence determine the coefficients a, b in the Ginzburg–Landau free energy in terms of these parameters.

Writing

$$\mathbf{Q} = \mathbf{A} - \frac{\lambda\xi}{\gamma}\nabla\chi,$$

$$\Phi = \phi + \frac{\lambda\xi}{\gamma}\frac{\partial\chi}{\partial t}$$

leads to equations (26)–(28). Retaining temperature as a variable we nondimensionalize by setting

$$\begin{aligned} \mathbf{x} &= \ell\mathbf{x}', & \Psi &= 2\pi T_c\Psi', \\ \mathbf{H} &= H_e\mathbf{H}', & \mathbf{A} &= \mu\ell H_e\mathbf{A}', \\ t &= \mu\sigma\ell^2 t', & \phi &= \frac{H_e}{\sigma}\phi', \\ T &= T_c + \theta T_c \end{aligned}$$

to give

$$\alpha\tilde{\xi}^2\frac{\partial\Psi}{\partial t} + \frac{\alpha\tilde{\xi}\tilde{\gamma}}{\tilde{\lambda}}\Psi\phi i + \Psi|\Psi|^2 + \theta\left(1 + \frac{\theta}{2}\right)\Psi - \left(\tilde{\xi}\nabla - i\frac{\tilde{\gamma}}{\tilde{\lambda}}\mathbf{A}\right)^2\Psi = 0,$$

$$-\tilde{\lambda}^2\text{curl}^2\mathbf{A} = \tilde{\lambda}^2\frac{\partial\mathbf{A}}{\partial t} + \tilde{\lambda}^2\nabla\phi + \frac{\tilde{\xi}\tilde{\lambda}}{2\tilde{\gamma}}i(\Psi^*\nabla\Psi - \Psi\nabla\Psi^*) + |\Psi|^2\mathbf{A},$$

where

$$\alpha = \frac{1}{\mu\sigma D}, \quad \tilde{\xi} = \frac{1}{T_c\ell\pi}\sqrt{\frac{3D}{2\tau_S}},$$

$$\tilde{\lambda} = 1/2\pi\ell T_c\sqrt{2\sigma\tau_S\mu}, \quad \tilde{\gamma} = \frac{eH_e}{2\pi^2\tau_S T_c^2}\sqrt{\frac{3D\mu}{\sigma}}$$

are independent of temperature. Neglecting the quadratic term in θ leads to (46)–(48).

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