

Henning Struchtrup

Macroscopic Transport Equations for Rarefied Gas Flows

Approximation Methods in Kinetic Theory

With 35 Figures

Contents

1	Introduction	1
1.1	Contents and scope	1
1.2	What is an ideal gas?	6
1.3	Length scales	7
1.4	Pressure and energy	9
1.5	Estimates for viscosity and heat conductivity	10
1.6	Knudsen number and microscale effects	11
	Problems	12
2	Basic quantities and definitions	15
2.1	Phase space and phase density	15
2.2	Moments of the phase density	16
2.2.1	Number, mass density, and velocity	16
2.2.2	Energy, pressure, and temperature	17
2.2.3	General moments	19
2.2.4	The relevance of moments	21
2.3	Equilibrium and the Maxwellian distribution	21
2.4	Boundary conditions for the phase density	22
	Problems	24
3	The Boltzmann equation and its properties	27
3.1	The Boltzmann equation	27
3.1.1	Space-time evolution of f	27
3.1.2	Binary collisions	29
3.1.3	Potential and differential cross section	31
3.1.4	The Stosszahlansatz	32
3.1.5	The Boltzmann equation	33
3.2	Equilibrium	34
3.3	Equation of transfer and conservation laws	35
3.3.1	Equation of transfer	35
3.3.2	Conservation laws	36
3.3.3	Moment equations	38

3.4	Entropy and the H-theorem	38
3.4.1	H-theorem	38
3.4.2	$H = k \ln W$ —entropy and disorder	39
3.4.3	Gibbs paradox	40
3.4.4	Equilibrium and maximum disorder	41
3.4.5	Irreversibility	42
3.5	Collision frequency	43
3.6	Kinetic Models	45
3.6.1	The BGK model	46
3.6.2	The ES-BGK model	47
3.7	The Direct Simulation Monte Carlo method	48
	Problems	49
4	The Chapman-Enskog method	53
4.1	Basic principles	53
4.1.1	The closure problem	53
4.1.2	Dimensionless and scaled Boltzmann and ES-BGK equations	55
4.1.3	The Chapman-Enskog expansion	56
4.2	The CE expansion for the ES-BGK model	57
4.2.1	General arguments	57
4.2.2	Zeroth order: Euler equations	58
4.2.3	First order: Navier-Stokes-Fourier equations	59
4.2.4	Second order: Burnett equations	60
4.3	CE expansion for power potentials	62
4.3.1	First order: Navier-Stokes-Fourier laws	62
4.3.2	Second order: Burnett equations	67
4.3.3	Third order: Super-Burnett equations	68
4.3.4	Augmented Burnett equations	69
4.3.5	BGK-Burnett equations	70
4.4	The second law	70
4.4.1	The second law to first order	70
4.4.2	The H-theorem in higher order expansions	71
	Problems	73
5	Moment equations	75
5.1	Moment equations as infinite coupled system	75
5.2	Equations for trace-free central moments	77
5.2.1	Generic moment equation	77
5.2.2	Conservation laws	78
5.2.3	Scalar moments	79
5.2.4	Vector moments	79
5.2.5	Rank-2 tensor moments	80
5.2.6	General equation	80

5.3	Production terms	80
5.3.1	BGK model	81
5.3.2	ES-BGK model	81
5.3.3	Maxwell molecules	82
5.3.4	Linear production terms	84
5.3.5	Eigenfunctions	85
6	Grad's moment method	87
6.1	Closed moment systems by Grad's method	87
6.1.1	General outline	87
6.1.2	Grad's 13 moment equations	89
6.1.3	Grad's 26 moment equations	91
6.1.4	Extended moment sets	92
6.2	Remarks on Grad-type equations	93
6.2.1	Large moment numbers	93
6.2.2	Moment systems and its competitors	95
6.2.3	Mathematical properties	95
6.3	CE expansion of Grad's moment equations	96
6.3.1	Grad and CE phase density for Maxwell molecules and BGK model	96
6.3.2	13 moments	97
6.3.3	26 moments	99
6.3.4	Maxwellian iteration	99
6.4	Reinecke-Kremer-Grad method	101
6.5	Moments of the collision term	102
6.6	Entropy maximization/Extended Thermodynamics	104
6.6.1	Brief outline of entropy maximization	104
6.6.2	Properties and problems	105
6.6.3	Linearization and Grad method	106
	Problems	107
7	Regularization of Grad equations	109
7.1	Grad distributions as pseudo-equilibrium manifolds	109
7.2	Basic equations	111
7.3	Expansion around $f_{ 13}$	112
7.4	Euler and Navier-Stokes-Fourier equations	114
7.5	Linearized equations	115
7.6	Discussion	116
7.7	Jin-Slemrod regularization	118
7.7.1	Basic equations	118
7.7.2	Comparison with R13 equations	119

8	Order of magnitude approach	123
8.1	Introduction	123
8.2	The order of magnitude of moments	126
8.2.1	Zeroth and first order expansion	126
8.2.2	Second order	127
8.2.3	Minimal number of moments of order $\mathcal{O}(\varepsilon)$	127
8.3	The transport equations with 2^{nd} order accuracy	129
8.3.1	The conservation laws and the definition of $\lambda - th$ order accuracy	129
8.3.2	Zeroth order accuracy: Euler equations	129
8.3.3	Collision moments for vectors and tensors	130
8.3.4	Equations for pressure deviator and heat flux	130
8.3.5	First order accuracy: Navier-Stokes-Fourier equations	131
8.3.6	Second order accuracy: 13 moment theory	133
8.3.7	Burnett equations	135
8.4	Third order accuracy: R13 equations	136
8.5	Discussion	138
8.5.1	Higher order accuracy	138
8.5.2	Comparison with Chapman-Enskog method	139
8.5.3	Comparison with the Grad method	140
8.5.4	Comparison with the original derivation of the R13 equations	141
8.5.5	Comparison with consistently ordered extended thermodynamics	142
8.5.6	Comparison with Jin-Slemrod equations	143
9	Macroscopic transport equations for rarefied gas flows	145
9.1	Relations between the equations	145
9.2	3-D non-linear equations	146
9.2.1	Conservation laws	146
9.2.2	Chapman-Enskog expansion	147
9.2.3	Moment equations for Maxwell molecules	150
9.2.4	Moment equations for general molecule types	152
9.3	One-dimensional equations	153
9.3.1	Conservation laws	154
9.3.2	Chapman-Enskog expansion	154
9.3.3	Moment equations for Maxwell molecules	155
9.3.4	Moment equations for general molecule types	157
9.4	Linear dimensionless equations	157
9.4.1	Conservation laws	158
9.4.2	Chapman-Enskog expansion	158
9.4.3	Moment equations for Maxwell molecules	159
9.4.4	Moment equations for general molecule types	160
	Problems	160

10	Stability and dispersion	161
10.1	Linear stability	161
10.1.1	Plane harmonic waves	161
10.1.2	Linear one-dimensional equations	163
10.1.3	Euler equations, speed of sound	163
10.1.4	Linear stability in time	164
10.1.5	Linear stability in space	166
10.1.6	Discussion	166
10.2	Dispersion and Damping	168
10.3	Stability analysis for the ES-BGK Burnett equations	170
	Problems	172
11	Shock structures	175
11.1	The 1-D shock structure problem	175
11.1.1	Basic definitions	175
11.1.2	Conservation laws and Rankine-Hugoniot relations	176
11.1.3	Entropy production over the shock	177
11.1.4	Shock thickness and asymmetry	178
11.1.5	Shocks and Knudsen number	179
11.1.6	Solution method	179
11.2	Comparison with DSMC results	181
11.2.1	Failure of classical methods	182
11.2.2	Maxwell molecules	183
11.2.3	Hard Spheres	185
11.2.4	Jin-Slemrod equations	186
11.3	Solution behavior	188
11.3.1	Transition from Grad's 13 moment equations	188
11.3.2	Temperature overshoot	189
11.3.3	Shock thickness	190
11.3.4	Shock asymmetry	192
11.3.5	Positivity of the phase density	192
11.4	Concluding remarks	194
12	Boundary value problems	197
12.1	Boundary conditions for moments	197
12.1.1	Basic considerations	197
12.1.2	Tangential momentum	200
12.1.3	Energy flux	200
12.1.4	Maxwell-Smoluchowski boundary conditions	201
12.1.5	Knudsen layer correction	202
12.2	Plane Couette flow	203
12.2.1	Couette geometry and conservation laws	203
12.2.2	Navier-Stokes-Fourier equations	204
12.2.3	Grad 13 equations	206
12.2.4	R13 equations	207

12.3	Bulk equations	208
12.4	Linear Knudsen boundary layers	210
12.4.1	Scaling and Knudsen layers	210
12.4.2	Navier-Stokes-Fourier and Grad-13 equations	211
12.4.3	Burnett equations	211
12.4.4	Super-Burnett and augmented Burnett equations	212
12.4.5	Regularized 13 moment equations	214
12.4.6	The heat flux parallel to the flow	214
12.4.7	26 and more moments	216
12.5	Superpositions	217
12.6	A boundary condition for normal stress	225
12.7	Further discussion	227
	Problems	228
A	Appendix	229
A.1	Tensor index notation	229
A.2	Symmetric and trace-free tensors	231
A.2.1	Symmetry	231
A.2.2	Trace-free tensors	231
A.2.3	Spherical harmonics	233
A.3	Integrals of Gaussians and Maxwellians	234
A.3.1	Gaussian integrals	234
A.3.2	Integrals of the Maxwellian	235
A.3.3	Half-space moments of the Maxwellian	236
A.3.4	Integrals of the anisotropic Gaussian	238
A.4	The integrals (5.21)	239
A.5	Lagrange multipliers	240
A.6	Equations for the computation of generalized 13 moment equations	241
A.6.1	Moment equations for w_i^α	241
A.6.2	Moment equations for w_{ij}^α	242
A.6.3	Coefficients in (8.18, 8.19)	244
	References	247
	Index	255