PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 80, Number 3, November 1980

MAD FAMILIES AND ULTRAFILTERS

MARTIN WEESE[1]

ABSTRACT. For each almost disjoint family X let $F(X) = \{a \subseteq \omega$: card $\{s \in X$: $s \setminus a$ is finite$\} = 2^\omega\}$, $I(X) = \{a \subseteq \omega$: card $\{s \in X$: card $(s \cap a) = \omega\} = 2^\omega\}$. Assuming $P(2^\omega)$ we show that for each nonprincipal ultrafilter p there exist a maximal almost disjoint family X and an almost disjoint family Y with $F(X) = I(Y) = p$.

1. Introduction. We refer the reader to [2] for unexplained notions. Let A be a set; $\mathcal{P}(A)$ denotes the power set of A and card A denotes the cardinality of A. Fin denotes the set of finite subsets of ω. For $a, b \in \mathcal{P}(A)$ we write $a \subseteq_* b$ if $a \setminus b$ is finite and we write $a =_* b$ if $a \subseteq_* b$ and $b \subseteq_* a$.

Let $X \subseteq \mathcal{P}(\omega) \setminus \text{Fin}$. X has the fip (finite intersection property) if for any finite subset S of X, $\bigcap S$ is infinite. X is almost disjoint if (i) for $a, b \in X$ with $a \neq b$, $a \cap b \in \text{Fin}$ and (ii) for any finite subset S of X, $\omega \setminus \bigcup S$ is infinite. X is called mad family if it is a maximal almost disjoint family and X is called ad family if it is an almost disjoint family.

Let $P(2^\omega)$ be the following proposition (considered by Rothberger [5]):

If $F \subseteq \mathcal{P}(\omega)$ has the fip and card $F < 2^\omega$ *then there is $d \in \mathcal{P}(\omega) \setminus$*Fin *with $a \subseteq_* b$ for each $b \in F$.*

The proposition $P(2^\omega)$ is weaker than Martin's axiom (see [4]).

For X an ad family we set

$$F(X) = \left\{a \subseteq \omega \colon \text{card}\left\{s \in X \colon s \subseteq_* a\right\} = 2^\omega\right\};$$
$$I(X) = \{a \subseteq \omega \colon \text{card}\{s \in X \colon \text{card}(s \cap a) = \omega\} = 2^\omega\}.$$

Then for each ad family X, $F(X) \subseteq I(X)$; for X a mad family, $I(X) = \{a \subseteq \omega$: for each finite subset S of X, $\text{card}(a \setminus \bigcup S) = \omega\}$. We show:

THEOREM 1. *Assume $P(2^\omega)$. Then for any nonprincipal ultrafilter p on ω there exists a mad family X with $F(X) = p$.*

THEOREM 2. *Assume $P(2^\omega)$. Then for any nonprincipal ultrafilter p on ω there exists an ad family X with $I(X) = p$.*

Received by the editors August 11, 1978 and, in revised form, July 16, 1979.

AMS (MOS) subject classifications (1970). Primary 02K05; Secondary 06A40, 54A25, 54D35.

Key words and phrases. Almost disjoint family, Stone-Čech compactification, 2^ω-point, superatomic Boolean algebra.

[1]The author wishes to thank the referee and J. E. Baumgartner for many valuable remarks.

2. Proof of Theorems 1 and 2. Let p be any nonprincipal ultrafilter on ω, let $\{a_i: i < 2^\omega\}$ be an enumeration of p such that for each $b \in p$ we have $\operatorname{card}\{i < 2^\omega: b = a_i\} = 2^\omega$ and let $\{b_i: i < 2^\omega\}$ be an enumeration of $\{b \subseteq \omega: b \notin p, \operatorname{card} b = \omega\}$. Let $A_k = \{a_i: i < k\}$, $B_k = \{b_i: i < k\}$. We construct increasing sequences $\{X_i: i < 2^\omega\}$, $\{Y_i: i < 2^\omega\}$ of almost disjoint sets such that for each $i < 2^\omega$:

(i) $\operatorname{card} X_i < 2^\omega$ and $\operatorname{card} Y_i < 2^\omega$;
(ii) $(X_i \cup Y_i) \cap p = \varnothing$;
(iii) $X_i \cap Y_i = \varnothing$;
(iv) there is $c \in X_{i+1} \setminus X_i$ with $c \subseteq a_i$;
(v) there is $d \in Y_{i+1}$ with $\operatorname{card}(d \cap b_i) = \omega$;
(vi) for $i < k < 2^\omega$, if $c \in X_k \setminus X_i$, then $\operatorname{card}(c \cap b_i) < \omega$;
(vii) for $i < k < 2^\omega$, if $d \in Y_k \setminus Y_i$, then $\operatorname{card}(d \cap b_i) = \omega$.

Let $X = \bigcup \{X_i: i < 2^\omega\}$, $Y = \bigcup \{Y_i: i < 2^\omega\}$. Then X is an ad family and (v) implies that $X \cup Y$ is a mad family. (iv) implies that for each $a \in p$, $a \in F(X)$ and $a \in F(X \cup Y)$. (vi) implies that for each $a \subseteq \omega$ with $a \notin p$, $a \notin I(X)$. (vii) implies that for each $a \subseteq \omega$ with $a \notin p$, $a \notin F(X \cup Y)$. Thus $I(X) = F(X \cup Y) = p$.

Now we describe the construction of the X_i and Y_i. We set $X_0 = Y_0 = \varnothing$. Assume $i < 2^\omega$ and for each $k < i$, X_k and Y_k are constructed. For i a limit ordinal we set $X_i = \bigcup \{X_k: k < i\}$, $Y_i = \bigcup \{Y_k: k < i\}$.

Now let i be a successor ordinal, $i = k + 1$. Let $S = A_i \cup \{\omega \setminus b: b \in B_i\} \cup \{\omega \setminus x: x \in X_k\}$. Then S has the fip and $\operatorname{card} S < 2^\omega$. $P(2^\omega)$ implies that there is $a \subseteq \omega$ with $a \setminus s \in \mathrm{Fin}$ for each $s \in S$. Let $a^* \subseteq a \cap a_i$ be such that $a^* \notin p$ and $\operatorname{card} a^* = \omega$. Then we set $X_i = X_k \cup \{a^*\}$. Assume there is $s \in X_i \cup Y_k$ with $\operatorname{card}(s \cap b_i) = \omega$. Then we set $Y_i = Y_k$. Assume now that no such s exists. Let $T = A_i \cup \{\omega \setminus b: b \in B_i\} \cup \{\omega \setminus x: x \in X_i\}$. Then T has the fip and $\operatorname{card} T < 2^\omega$. $P(2^\omega)$ implies that there is $c \subseteq \omega$ with $c \setminus s \in \mathrm{Fin}$ for each $s \in T$. Let $c^* \subseteq c \cap a_i$ be such that $c^* \notin p$ and $\operatorname{card} c^* = \omega$. Then we set $Y_i = Y_k \cup \{c^* \cup b_i\}$. It is now easy to see that (i)–(vii) are satisfied.

3. Topological consequences. Let N be the discrete countable space and let βN be the Stone-Čech compactification of N. Then $\beta N \setminus N$ can be represented by the set of all nonprincipal ultrafilters over ω and the topology generated by the following basis $\mathfrak{A}$: For each $a \subseteq \omega$ let $\hat{a} = \{p \in \beta N \setminus N: a \in p\}$ and $\mathfrak{A} = \{\hat{a}: a \subseteq \omega\}$. Then $\hat{a} \supseteq \hat{b}$ iff $b \subseteq_* a$. Then Theorems 1 and 2 can be reformulated as follows:

THEOREM 1$'$. *Assume $P(2^\omega)$. Then for each $p \in \beta N \setminus N$ there is a dense system $\mathfrak{U}_p$* of open sets such that for each $a \subseteq \omega$, $a \in p$ iff $\operatorname{card}\{U \in \mathfrak{U}_p: U \subseteq \hat{a}\} = 2^\omega$.

THEOREM 2$'$. *Assume $P(2^\omega)$. Then for each $p \in \beta N \setminus N$ there is a system $\mathfrak{U}_p$ of open sets such that for each $a \subseteq \omega$, $a \in p$ iff* $\operatorname{card}\{U \in \mathfrak{U}_p: U \cap \hat{a} \neq \varnothing\} = 2^\omega$.

$p \in \beta N \setminus N$ is a 2^ω-point if there is a family $\{U_i: i < 2^\omega\}$ of pairwise disjoint open sets with $p \in (\mathrm{cl}_{\beta N} U_i) \setminus N$. We can use Theorem 1 to derive the following theorem of Hindman [3] (Hindman used CH but there is little difficulty adapting his proof to $P(2^\omega)$):

THEOREM 3. *Assume $P(2^\omega)$. Then each $p \in \beta N \setminus N$ is a 2^ω-point.*

PROOF. Let $X = \{c_i : i < 2^\omega\}$ be a mad family with $F(X) = p$. For each $i < 2^\omega$ choose an ad family $\{d_{ik} : k < 2^\omega\}$ with $d_{ik} \subseteq c_i$ for each $k < 2^\omega$. For $k < 2^\omega$ let

$$U_k = \bigcup \{\hat{d}_{ik} : i < 2^\omega\}.$$

Then the U_k are pairwise disjoint open sets and p is in the closure of each U_k.

REMARK. Balcar and Vojtáš [1] proved Theorem 3 without any set-theoretical assumption. It is also unknown whether Theorem 1 holds without any set-theoretical assumption.

4. Applications to superatomic Boolean algebras. Let $\mathfrak{A}$ be a Boolean algebra. $a \in |\mathfrak{A}|$ is an atom if $a \neq 0$ and for each $b \in |\mathfrak{A}|$, $a \cap b = a$ or $a \cap b = 0$. $\mathfrak{A}$ is atomic if for each $b \in |\mathfrak{A}|$ there is an atom a with $a \leqslant b$. $\mathfrak{A}$ is superatomic if each homomorphic image of $\mathfrak{A}$ is atomic. $\underline{2}$ denotes the two-element Boolean algebra, $\mathrm{Pow}(\omega)$ denotes the power set Boolean algebra over ω. For $A \subseteq \mathrm{Pow}(\omega)$ let $\mathrm{Pow}(\omega)[A]$ denote the subalgebra of $\mathrm{Pow}(\omega)$ generated by $A \cup \omega$. For each Boolean algebra $\mathfrak{A}$, $\mathfrak{A}^{(1)}$ denotes $\mathfrak{A}$ factorized by the ideal generated by the atoms and for each $k \in \omega$ we set $\mathfrak{A}^{(k+1)} = (\mathfrak{A}^{(k)})^{(1)}$. If X is a mad family then $\mathrm{Pow}(\omega)[X]$ is a superatomic Boolean algebra whose set of atoms is ω and $(\mathrm{Pow}(\omega)[X])^{(2)} \cong \underline{2}$.

THEOREM 4. *Assume $P(2^\omega)$. Then there are 2^{2^ω} nonisomorphic superatomic Boolean algebras $\mathfrak{A}$ whose set of atoms is ω and with $\mathfrak{A}^{(2)} \cong \underline{2}$.*

PROOF. Let $\mathfrak{X}$ be the class of all mad families X such that $F(X)$ is a nonprincipal ultrafilter. Let $X, Y \in \mathfrak{X}$. X and Y ar called equivalent if there are $a \in X$, $b \in Y$ and a one-one function f from a onto b such that for each $s \in X$ with $s \subseteq_* a$ there is $t \in Y$ with $f[s] =_* t$. That means, X and Y are equivalent iff $F(X)$ and $F(Y)$ are equivalent with respect to the Rudin-Keisler order of ultrafilters. Now there are 2^{2^ω} nonprincipal ultrafilters on ω and each equivalence class with respect to the Rudin-Keisler order contains 2^ω ultrafilters. Let $\mathfrak{Y} \subseteq \mathfrak{X}$ be such that card $\mathfrak{Y} = 2^{2^\omega}$ and the elements of $\mathfrak{Y}$ are pairwise nonequivalent. Let

$$\mathfrak{K} = \{\mathrm{Pow}(\omega)[X] : X \in \mathfrak{Y}\}.$$

Then $\mathfrak{K}$ is the desired class of superatomic Boolean algebras.

ADDED IN PROOF. As I was informed by Baumgartner, it is impossible to prove Theorem 1 without any set-theoretical assumption.

REFERENCES

1. B. Balcar and P. Vojtás, *Almost disjoint refinement of families of subsets of N* (preprint).

2. W. Comfort and S. Negrepontis, *The theory of ultrafilters*, Springer-Verlag, Berlin and New York, 1974.

3. N. Hindman, *On the existence of τ-points in $\beta N \setminus N$*, Proc. Amer. Math. Soc. **21** (1969), 277–280.

4. K. Kunen and F. D. Tall, *Between Martin's axiom and Souslin's hypothesis*, Fund. Math. **102** (1979), 173–181.

5. F. Rothberger, *On some problems of Hausdorff and of Sierpiński*, Fund. Math. **35** (1948), 29–46.

HUMBOLDT-UNIVERSITÄT ZU BERLIN, UNTER DEN LINDEN 6, 108 BERLIN, GERMAN DEMOCRATIC REPUBLIC