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Magic square and three-zero textures for Dirac neutrinos

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We show a matrix decomposition of flavor mass matrix for Dirac neutrinos M by sum as $M = M' + M^0$ where M' obeys the feature of the magic square and M^0 is three-zero texture. The favorable three-zero textures in the context of magic square are explored. It turned out that so-called the normal ordering of neutrino masses is favored over the inverted ordering in the context of magic square.

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1. Introduction

The texture of the flavor neutrino mass matrix is a long standing problem in particle physics. Many possible textures are proposed in the literature, such as, tri-bi maximal texture,¹⁻⁴ μ - τ symmetric texture,⁵⁻²⁷ and textures under discrete symmetries e.g., A_n and S_n .²⁸

One of the schemes of simple textures for neutrino mass matrix is known as texture zeros.²⁹⁻⁶⁹ There are two usages of the term “texture zeros.” To begin, if a flavor neutrino mass matrix contains zero elements, the texture of this flavor mass matrix is referred to as texture zeros. Second, even though the flavor neutrino mass matrix M has no vanishing element, if a matrix M is related to another matrix M^0 and M^0 has vanishing elements, we say that M^0 has a structure of the texture zeros for flavor neutrino mass matrix M . For example, according to the seesaw

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mechanism for generating tiny neutrino masses, the flavor neutrino mass matrix is decomposed as $M \simeq M_D M_R^{-1} M_D$, where M_D represents the Dirac mass matrix and M_R represents the mass matrix for heavy right-handed neutrinos. If the Dirac mass matrix has vanishing elements, e.g., $M_D = M^0$, the texture of the Dirac mass matrix is called texture zeros. Aside from the terminology, the texture zeros scheme is concerned with the allowed number of zero elements in the mass matrix.

In this paper, we use the term “texture zeros” as a second meaning. Instead, of the usual matrix decomposition by-product such as $M \simeq M^0 M_R^{-1} M^0$, we take a matrix decomposition by sums such as $M = M' + M^0$ where M' does not have zero elements and M^0 has zero elements. We show that if the matrix M' obeys the magic square’s feature,^{70–76} the texture of M^0 can naturally become a three-zero texture.

We note that the structure of the neutrino flavor mass matrix under symmetries with magic square has been studied previously,^{77–84} however, the constraint of the number of zero elements in the flavor neutrino mass matrix with magic square under decomposition by sum $M = M' + M^0$ is revealed for the first time in this paper.

This paper is organized as follows. In section 2, we present a brief review the of magic square and make a matrix decomposition of flavor mass matrix for Dirac neutrinos by sum. Then, we show that if the matrix M' obeys the property of the magic square, the texture of M^0 can naturally become three-zero texture. In addition, 24 three-zero textures that can be compared with observations are shown concretely. Section 3, selects the most favorable three-zero texture based on numerical calculations. A summary is provided in Section 4.

2. Magic square and three-zero textures

2.1. Magic square

A magic square of order n is a $n \times n$ square grid filled with distinct natural numbers.^{70–76} Each cell contains a number in the range $1, 2, \dots, n^2$. The total of the numbers in each row, column, and diagonal is equal. For example, a magic square of order 3 is schematically shown as

$$\begin{array}{ccc|c}
 2 & 7 & 6 & \leftarrow 15 \\
 9 & 5 & 1 & \leftarrow 15 \\
 4 & 3 & 8 & \leftarrow 15 \\
 \nearrow & \uparrow & \uparrow & \uparrow & \nwarrow \\
 15 & 15 & 15 & 15 & 15
 \end{array} \tag{1}$$

where the sum (which is called magic constant or magic sum) is 15.

It is known that the following Lucas’s matrix is a simple realization of the magic squares:^{71–75}

$$M^{\text{magic}} = \begin{pmatrix} \gamma - \beta & \gamma + \alpha + \beta & \gamma - \alpha \\ \gamma - \alpha + \beta & \gamma & \gamma + \alpha - \beta \\ \gamma + \alpha & \gamma - \alpha - \beta & \gamma + \beta \end{pmatrix}, \tag{2}$$

where, α, β and γ are complex numbers.

2.2. Decomposition and three-zero textures

With Lucas's matrix, the Dirac flavor neutrino mass matrix

$$M = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}, \quad (3)$$

can be decomposed

$$M = M^{\text{magic}} + M^0, \quad (4)$$

where

$$M^0 = \begin{pmatrix} a_0 & b_0 & c_0 \\ d_0 & e_0 & f_0 \\ g_0 & h_0 & i_0 \end{pmatrix}. \quad (5)$$

This decomposition applies to any arbitrary complex matrix, not just the Dirac flavor neutrino mass matrix.

The number of complex elements in the complex matrix M is nine. Since the number of complex elements in Lucas's form M^{magic} is three, the number of independent complex elements in M^0 is six. Three complex elements in M^0 can vanish without loss of generality. As a result if we decompose the Dirac flavor neutrino mass matrix by sum with M^{magic} and M^0 as shown in Eq.(4), the most natural form of M^0 may be three-zero textures.

2.3. Three-zero textures

We construct concrete three-zero textures M_0 for comparison with observations. The nine elements of Lucas's matrix are a linear combination of α, β, γ . The (2,2) component of Lucas's matrix is composed simply of one complex element γ .⁷¹ As a result of the experimental results, we can directly identify γ . We take $e = \gamma$, in other words $e_0 = 0$. In this case, the following 28 three-zero textures are obtained:

$$\begin{aligned} M_1^0 &= \begin{pmatrix} 0 & 0 & c_0 \\ d_0 & 0 & f_0 \\ g_0 & h_0 & i_0 \end{pmatrix}, & M_2^0 &= \begin{pmatrix} 0 & b_0 & 0 \\ d_0 & 0 & f_0 \\ g_0 & h_0 & i_0 \end{pmatrix}, & M_3^0 &= \begin{pmatrix} 0 & b_0 & c_0 \\ 0 & 0 & f_0 \\ g_0 & h_0 & i_0 \end{pmatrix}, \\ M_4^0 &= \begin{pmatrix} 0 & b_0 & c_0 \\ d_0 & 0 & 0 \\ g_0 & h_0 & i_0 \end{pmatrix}, & M_5^0 &= \begin{pmatrix} 0 & b_0 & c_0 \\ d_0 & 0 & f_0 \\ 0 & h_0 & i_0 \end{pmatrix}, & M_6^0 &= \begin{pmatrix} 0 & b_0 & c_0 \\ d_0 & 0 & f_0 \\ g_0 & 0 & i_0 \end{pmatrix}, \\ M_7^0 &= \begin{pmatrix} a_0 & 0 & 0 \\ d_0 & 0 & f_0 \\ g_0 & h_0 & i_0 \end{pmatrix}, & M_8^0 &= \begin{pmatrix} a_0 & 0 & c_0 \\ 0 & 0 & f_0 \\ g_0 & h_0 & i_0 \end{pmatrix}, & M_9^0 &= \begin{pmatrix} a_0 & 0 & c_0 \\ d_0 & 0 & 0 \\ g_0 & h_0 & i_0 \end{pmatrix}, \end{aligned}$$

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$$\begin{aligned}
M_{10}^0 &= \begin{pmatrix} a_0 & 0 & c_0 \\ d_0 & 0 & f_0 \\ 0 & h_0 & i_0 \end{pmatrix}, & M_{11}^0 &= \begin{pmatrix} a_0 & 0 & c_0 \\ d_0 & 0 & f_0 \\ g_0 & h_0 & 0 \end{pmatrix}, & M_{12}^0 &= \begin{pmatrix} a_0 & b_0 & 0 \\ 0 & 0 & f_0 \\ g_0 & h_0 & i_0 \end{pmatrix}, \\
M_{13}^0 &= \begin{pmatrix} a_0 & b_0 & 0 \\ d_0 & 0 & 0 \\ g_0 & h_0 & i_0 \end{pmatrix}, & M_{14}^0 &= \begin{pmatrix} a_0 & b_0 & 0 \\ d_0 & 0 & f_0 \\ g_0 & 0 & i_0 \end{pmatrix}, & M_{15}^0 &= \begin{pmatrix} a_0 & b_0 & 0 \\ d_0 & 0 & f_0 \\ g_0 & h_0 & 0 \end{pmatrix}, \\
M_{16}^0 &= \begin{pmatrix} a_0 & b_0 & c_0 \\ 0 & 0 & f_0 \\ 0 & h_0 & i_0 \end{pmatrix}, & M_{17}^0 &= \begin{pmatrix} a_0 & b_0 & c_0 \\ 0 & 0 & f_0 \\ g_0 & 0 & i_0 \end{pmatrix}, & M_{18}^0 &= \begin{pmatrix} a_0 & b_0 & c_0 \\ 0 & 0 & f_0 \\ g_0 & h_0 & 0 \end{pmatrix}, \\
M_{19}^0 &= \begin{pmatrix} a_0 & b_0 & c_0 \\ d_0 & 0 & 0 \\ 0 & h_0 & i_0 \end{pmatrix}, & M_{20}^0 &= \begin{pmatrix} a_0 & b_0 & c_0 \\ d_0 & 0 & 0 \\ g_0 & 0 & i_0 \end{pmatrix}, & M_{21}^0 &= \begin{pmatrix} a_0 & b_0 & c_0 \\ d_0 & 0 & 0 \\ g_0 & h_0 & 0 \end{pmatrix}, \\
M_{22}^0 &= \begin{pmatrix} a_0 & b_0 & c_0 \\ g_0 & 0 & f_0 \\ 0 & 0 & i_0 \end{pmatrix}, & M_{23}^0 &= \begin{pmatrix} a_0 & b_0 & c_0 \\ d_0 & 0 & f_0 \\ 0 & h_0 & 0 \end{pmatrix}, & M_{24}^0 &= \begin{pmatrix} a_0 & b_0 & c_0 \\ d_0 & 0 & f_0 \\ g_0 & 0 & 0 \end{pmatrix}, \tag{6}
\end{aligned}$$

and

$$\begin{aligned}
M_{25}^0 &= \begin{pmatrix} 0 & b_0 & c_0 \\ d_0 & 0 & f_0 \\ g_0 & h_0 & 0 \end{pmatrix}, & M_{26}^0 &= \begin{pmatrix} a_0 & 0 & c_0 \\ d_0 & 0 & f_0 \\ g_0 & 0 & i_0 \end{pmatrix}, & M_{27}^0 &= \begin{pmatrix} a_0 & b_0 & 0 \\ d_0 & 0 & f_0 \\ 0 & h_0 & i_0 \end{pmatrix}, \\
M_{28}^0 &= \begin{pmatrix} a_0 & b_0 & c_0 \\ 0 & 0 & 0 \\ g_0 & h_0 & i_0 \end{pmatrix}. \tag{7}
\end{aligned}$$

For 24 three-zero textures of $M_1^0, M_2^0, \dots, M_{24}^0$ in Eq.(6), the Dirac flavor neutrino mass matrix M is obtained as

$$M = M_i^{\text{magic}} + M_i^0, \quad (i = 1, 2, \dots, 24), \tag{8}$$

and we can determine the non-zero elements in M_i^0 and M_i^{magic} in the terms of the elements a, b, c, \dots, i in the Dirac flavor neutrino mass matrix as follows:

$$M_1^{\text{magic}} : \alpha = a + b - 2e, \quad \beta = -a + e, \quad \gamma = e, \tag{9}$$

$$\begin{aligned}
M_1^0 : & c_0 = a + b + c - 3e, \quad d_0 = 2a + b + d - 4e, \quad f_0 = -2a - b + 2e + f, \\
& g_0 = -a - b + e + g, \quad h_0 = b - 2e + h, \quad i_0 = a - 2e + i,
\end{aligned}$$

$$M_2^{\text{magic}} : \alpha = -c + e, \quad \beta = -a + e, \quad \gamma = e, \tag{10}$$

$$\begin{aligned}
M_2^0 : & b_0 = a + b + c - 3e, \quad d_0 = a - c + d - e, \quad f_0 = -a + c - e + f, \\
& g_0 = c - 2e + g, \quad h_0 = -a - c + e + h, \quad i_0 = a - 2e + i,
\end{aligned}$$

$$M_3^{\text{magic}} : \alpha = -a - d + 2e, \quad \beta = -a + e, \quad \gamma = e, \quad (11)$$

$$M_3^0 : b_0 = 2a + b + d - 4e, \quad c_0 = -a + c - d + e, \quad f_0 = d - 2e + f, \\ g_0 = a + d - 3e + g, \quad h_0 = -2a - d + 2e + h, \quad i_0 = a - 2e + i,$$

$$M_4^{\text{magic}} : \alpha = -a + f, \quad \beta = -a + e, \quad \gamma = e, \quad (12)$$

$$M_4^0 : b_0 = 2a + b - 2e - f, \quad c_0 = -a + c - e + f, \quad d_0 = d - 2e + f, \\ g_0 = a - e - f + g, \quad h_0 = -2a + f + h, \quad i_0 = a - 2e + i,$$

$$M_5^{\text{magic}} : \alpha = -e + g, \quad \beta = -a + e, \quad \gamma = e, \quad (13)$$

$$M_5^0 : b_0 = a + b - e - g, \quad c_0 = c - 2e + g, \quad d_0 = a + d - 3e + g, \\ f_0 = -a + e + f - g, \quad h_0 = -a - e + g + h, \quad i_0 = a - 2e + i,$$

$$M_6^{\text{magic}} : \alpha = a - h, \quad \beta = -a + e, \quad \gamma = e, \quad (14)$$

$$M_6^0 : b_0 = b - 2e + h, \quad c_0 = a + c - e - h, \quad d_0 = 2a + d - 2e - h, \\ f_0 = -2a + f + h, \quad g_0 = -a - e + g + h, \quad i_0 = a - 2e + i,$$

$$M_7^{\text{magic}} : \alpha = -c + e, \quad \beta = b + c - 2e, \quad \gamma = e, \quad (15)$$

$$M_7^0 : a_0 = a + b + c - 3e, \quad d_0 = -b - 2c + d + 2e, \quad f_0 = b + 2c - 4e + f, \\ g_0 = c - 2e + g, \quad h_0 = b - 2e + h, \quad i_0 = -b - c + e + i,$$

$$M_8^{\text{magic}} : \alpha = \frac{1}{2}(b - d), \quad \beta = \frac{1}{2}(b + d - 2e), \quad \gamma = e, \quad (16)$$

$$M_8^0 : a_0 = \frac{1}{2}(2a + b + d - 4e), \quad c_0 = \frac{1}{2}(b + 2c - d - 2e), \quad f_0 = d - 2e + f, \\ g_0 = \frac{1}{2}(-b + d - 2e + 2g), \quad h_0 = b - 2e + h, \quad i_0 = \frac{1}{2}(-b - d + 2i),$$

$$M_9^{\text{magic}} : \alpha = \frac{1}{2}(b - 2e + f), \quad \beta = \frac{1}{2}(b - f), \quad \gamma = e, \quad (17)$$

$$M_9^0 : a_0 = \frac{1}{2}(2a + b - 2e - f), \quad c_0 = \frac{1}{2}(b + 2c - 4e + f), \quad d_0 = d - 2e + f, \\ g_0 = \frac{1}{2}(-b - f + 2g), \quad h_0 = b - 2e + h, \quad i_0 = \frac{1}{2}(-b - 2e + f + 2i),$$

$$M_{10}^{\text{magic}} : \alpha = -e + g, \quad \beta = b - g, \quad \gamma = e, \quad (18)$$

$$M_{10}^0 : a_0 = a + b - e - g, \quad c_0 = c - 2e + g, \quad d_0 = -b + d - 2e + 2g, \\ f_0 = b + f - 2g, \quad h_0 = b - 2e + h, \quad i_0 = -b - e + g + i,$$

$$M_{11}^{\text{magic}} : \alpha = b - i, \quad \beta = -e + i, \quad \gamma = e, \quad (19)$$

$$M_{11}^0 : a_0 = a - 2e + i, \quad c_0 = b + c - e - i, \quad d_0 = b + d - 2i, \\ f_0 = -b - 2e + f + 2i, \quad g_0 = -b - e + g + i, \quad h_0 = b - 2e + h,$$

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$$M_{12}^{\text{magic}} : \alpha = -c + e, \quad \beta = -c + d, \quad \gamma = e, \quad (20)$$

$$M_{12}^0 : a_0 = a - c + d - e, \quad b_0 = b + 2c - d - 2e, \quad f_0 = d - 2e + f, \\ g_0 = c - 2e + g, \quad h_0 = -2c + d + h, \quad i_0 = c - d - e + i, \quad (21)$$

$$M_{13}^{\text{magic}} : \alpha = -c + e, \quad \beta = -c + 2e - f, \quad \gamma = e, \quad (22)$$

$$M_{13}^0 : a_0 = a - c + e - f, \quad b_0 = b + 2c - 4e + f, \quad d_0 = d - 2e + f, \\ g_0 = c - 2e + g, \quad h_0 = -2c + 2e - f + h, \quad i_0 = c - 3e + f + i,$$

$$M_{14}^{\text{magic}} : \alpha = -c + e, \quad \beta = c - h, \quad \gamma = e, \quad (23)$$

$$M_{14}^0 : a_0 = a + c - e - h, \quad b_0 = b - 2e + h, \quad d_0 = -2c + d + h, \\ f_0 = 2c - 2e + f - h, \quad g_0 = c - 2e + g, \quad i_0 = -c - e + h + i,$$

$$M_{15}^{\text{magic}} : \alpha = -c + e, \quad \beta = -e + i, \quad \gamma = e, \quad (24)$$

$$M_{15}^0 : a_0 = a - 2e + i, \quad b_0 = b + c - e - i, \quad d_0 = -c + d + e - i, \\ f_0 = c - 3e + f + i, \quad g_0 = c - 2e + g, \quad h_0 = -c - e + h + i,$$

$$M_{16}^{\text{magic}} : \alpha = -e + g, \quad \beta = d - 2e + g, \quad \gamma = e, \quad (25)$$

$$M_{16}^0 : a_0 = a + d - 3e + g, \quad b_0 = b - d + 2e - 2g, \quad c_0 = c - 2e + g, \\ f_0 = d - 2e + f, \quad h_0 = d - 4e + 2g + h, \quad i_0 = -d + e - g + i,$$

$$M_{17}^{\text{magic}} : \alpha = \frac{1}{2}(-d + 2e - h), \quad \beta = \frac{1}{2}(d - h), \quad \gamma = e, \quad (26)$$

$$M_{17}^0 : a_0 = \frac{1}{2}(2a + d - 2e - h), \quad b_0 = b - 2e + h, \quad c_0 = \frac{1}{2}(2c - d - h), \\ f_0 = d - 2e + f, \quad g_0 = \frac{1}{2}(d - 4e + 2g + h), \quad i_0 = \frac{1}{2}(-d - 2e + h + 2i),$$

$$M_{18}^{\text{magic}} : \alpha = -d + i, \quad \beta = -e + i, \quad \gamma = e, \quad (27)$$

$$M_{18}^0 : a_0 = a - 2e + i, \quad b_0 = b + d - 2i, \quad c_0 = c - d - e + i, \\ f_0 = d - 2e + f, \quad g_0 = d - e + g - i, \quad h_0 = -d - 2e + h + 2i,$$

$$M_{19}^{\text{magic}} : \alpha = -e + g, \quad \beta = -f + g, \quad \gamma = e, \quad (28)$$

$$M_{19}^0 : a_0 = a - e - f + g, \quad b_0 = b + f - 2g, \quad c_0 = c - 2e + g, \\ d_0 = d - 2e + f, \quad h_0 = -2e - f + 2g + h, \quad i_0 = -e + f - g + i,$$

$$M_{20}^{\text{magic}} : \alpha = \frac{1}{2}(f - h), \quad \beta = \frac{1}{2}(2e - f - h), \quad \gamma = e, \quad (29)$$

$$M_{20}^0 : a_0 = \frac{1}{2}(2a - f - h), \quad b_0 = b - 2e + h, \quad c_0 = \frac{1}{2}(2c - 2e + f - h), \\ d_0 = d - 2e + f, \quad g_0 = \frac{1}{2}(-2e - f + 2g + h), \quad i_0 = \frac{1}{2}(-4e + f + h + 2i),$$

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$$M_{21}^{\text{magic}} : \alpha = -2e + f + i, \quad \beta = -e + i, \quad \gamma = e, \quad (30)$$

$$M_{21}^0 : a_0 = a - 2e + i, \quad b_0 = b + 2e - f - 2i, \quad c_0 = c - 3e + f + i, \\ d_0 = d - 2e + f, \quad g_0 = e - f + g - i, \quad h_0 = -4e + f + h + 2i,$$

$$M_{22}^{\text{magic}} : \alpha = -e + g, \quad \beta = 2e - g - h, \quad \gamma = e, \quad (31)$$

$$M_{22}^0 : a_0 = a + e - g - h, \quad b_0 = b - 2e + h, \quad c_0 = c - 2e + g, \\ d_0 = d - 4e + 2g + h, \quad f_0 = 2e + f - 2g - h, \quad i_0 = -3e + g + h + i,$$

$$M_{23}^{\text{magic}} : \alpha = -e + g, \quad \beta = -e + i, \quad \gamma = e, \quad (32)$$

$$M_{23}^0 : a_0 = a - 2e + i, \quad b_0 = b + e - g - i, \quad c_0 = c - 2e + g, \\ d_0 = d - e + g - i, \quad f_0 = -e + f - g + i, \quad h_0 = -3e + g + h + i,$$

$$M_{24}^{\text{magic}} : \alpha = 2e - h - i, \quad \beta = -e + i, \quad \gamma = e, \quad (33)$$

$$M_{24}^0 : a_0 = a - 2e + i, \quad b_0 = b - 2e + h, \quad c_0 = c + e - h - i, \\ d_0 = d + 2e - h - 2i, \quad f_0 = -4e + f + h + 2i, \quad g_0 = -3e + g + h + i.$$

On the other hand, the rank of the remaining four matrices $M_{25}^0, M_{26}^0, M_{27}^0, M_{28}^0$ in Eq.(7) is not enough to determine non-zero elements $\{a_0, b_0, \dots, i_0\}$ in M^0 and $\{\alpha, \beta, \gamma\}$ in M^{magic} in the term of the elements $\{a, b, c, \dots, i\}$ in M . In this paper, we pick up only the 24 three-zero textures $M_1^0, M_2^0, \dots, M_{24}^0$.

2.4. Usual magic symmetry and Magic square condition

The usual magic symmetry is given by the Z_2 symmetry.⁸² In the following matrix

$$M = \begin{pmatrix} a + e & a + c + d & a - b \\ a - b + d & a & a + c + e \\ a + c & a - b + e & a + d \end{pmatrix}, \quad (34)$$

sum of each row and column is equal to $3a - b + c + d + e$. This matrix M is under the following Z_2 symmetry:

$$S_2 M S_2^T = M, \quad (35)$$

where

$$S_2 = \begin{pmatrix} 1/3 & -2/3 & -2/3 \\ -2/3 & 1/3 & -2/3 \\ -2/3 & -2/3 & 1/3 \end{pmatrix}, \quad (36)$$

and

$$S_2^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad (37)$$

which is usual magic symmetry.⁸⁰

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The matrix M in Eq.(34) has 5 parameters, and the magic square matrix under discussion in this paper can be thought of as a further diagonal and anti-diagonal sum conditions added from here. The following two conditions may be used:

- (1) $\text{Tr}M = 3a$,
- (2) anti-trace $\text{aTr}M \equiv M_{13} + M_{22} + M_{31} = 3a$.

From the first condition, we obtain $e = -d$. From the second condition, $b = c$ is required. The matrix M should be

$$M = \begin{pmatrix} a-d & a+b+d & a-b \\ a-b+d & a & a+b-d \\ a+b & a-b-d & a+d \end{pmatrix}, \quad (38)$$

which is magic square matrix in this paper.

Because the number of parameters is reduced from 5 to 3, the magic square condition in Eq.(38) is extremely severe. The exact magic square condition should be violated for Dirac neutrinos and these violation will appear in three-zero textures. In the next subsection, we focus our attention on the three-zero textures in the context of the magic square.

3. Favorable three-zero textures

3.1. Criteria

For previous section, we showed that 24 three-zero textures can be compared with observations. Which of 24 three-zero textures is the most favored?

In this paper, it is assumed that the Dirac flavor neutrino mass matrix M can be decomposed by sum containing the Lucas's matrix M^{magic} . This is the most important assumption. Thus, the case of $M = M_i^{\text{magic}}$ may be regarded as the complete favorable case.

According to Ref.⁸⁵ we use the following dimensionless parameters to quantify the strength of the violations of the magic-square feature

$$\epsilon_i^{\ell m} = \left| \frac{M_{\ell m} - (M_i^{\text{magic}})_{\ell m}}{(M_i^{\text{magic}})_{\ell m}} \right|. \quad (39)$$

In the most favorable case, we obtain

$$\epsilon_i = \frac{1}{6} \sum_{\ell, m=1,2,3} \epsilon_i^{\ell m} = 0. \quad (40)$$

where 1/6 is a normalization factor. We will estimate ϵ_i for neutrino parameters in 3σ region and pick up its minimum values. If the minimum value ϵ_i^{min} close to 0, we regard the combination $\{M_i^{\text{magic}}, M_i^0\}$ are favored with the context of the magic square.

3.2. Neutrino parameters

The flavor neutrino mass matrix for Dirac neutrinos is reconstructed as

$$M = U \text{diag.}(m_1, m_2, m_3) V^\dagger, \quad (41)$$

with m_1, m_2 and m_3 being the neutrino mass eigenstates, U being the left-handed lepton mixing matrix, and V being the transformation matrix of three right-handed neutrinos.⁸⁶ We assume that the charged leptons mass matrix is diagonal and real. In this case, U becomes the Pontecorvo-Maki-Nakagawa-Sakata mixing matrix,^{87–90} and whose elements are

$$\begin{aligned} U_{e1} &= c_{12}c_{13}, & U_{e2} &= s_{12}c_{13}, & U_{e3} &= s_{13}e^{-i\delta_{\text{CP}}}, \\ U_{\mu 1} &= -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta_{\text{CP}}}, \\ U_{\mu 2} &= c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta_{\text{CP}}}, & U_{\mu 3} &= s_{23}c_{13}, \\ U_{\tau 1} &= s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta_{\text{CP}}}, \\ U_{\tau 2} &= -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta_{\text{CP}}}, & U_{\tau 3} &= c_{23}c_{13}. \end{aligned} \quad (42)$$

We used the abbreviations $c_{ij} = \cos \theta_{ij}$ and $s_{ij} = \sin \theta_{ij}$ where θ_{ij} ($ij = 12, 23, 13$) is a mixing angle and the Dirac CP phase is denoted by δ_{CP} .

The unitary matrix V has three rotation angles ϕ_{ij} ($ij = 12, 23, 13$), six phases ω_i ($i = 1, 2, \dots, 5$) and δ_V where $\phi_{ij} \in (0, \pi/2)$ and $\omega_i, \delta_V \in (-\pi, \pi)$. The matrix V can be parametrize as^{86, 91}

$$V = \text{diag.}(e^{i\omega_1}, e^{i\omega_2}, e^{i\omega_3}) \tilde{V}(\phi_{ij}, \delta_V) \text{diag.}(e^{i\omega_4}, e^{i\omega_5}, 1), \quad (43)$$

where the elements of \tilde{V} are written in a way similar to the PMNS matrix as

$$\begin{aligned} \tilde{V}_{e1} &= \tilde{c}_{12}\tilde{c}_{13}, & \tilde{V}_{e2} &= \tilde{s}_{12}\tilde{c}_{13}, & \tilde{V}_{e3} &= \tilde{s}_{13}e^{-i\delta_V}, \\ \tilde{V}_{\mu 1} &= -\tilde{s}_{12}\tilde{c}_{23} - \tilde{c}_{12}\tilde{s}_{23}\tilde{s}_{13}e^{i\delta_V}, \\ \tilde{V}_{\mu 2} &= \tilde{c}_{12}\tilde{c}_{23} - \tilde{s}_{12}\tilde{s}_{23}\tilde{s}_{13}e^{i\delta_V}, & \tilde{V}_{\mu 3} &= \tilde{s}_{23}\tilde{c}_{13}, \\ \tilde{V}_{\tau 1} &= \tilde{s}_{12}\tilde{s}_{23} - \tilde{c}_{12}\tilde{c}_{23}\tilde{s}_{13}e^{i\delta_V}, \\ \tilde{V}_{\tau 2} &= -\tilde{c}_{12}\tilde{s}_{23} - \tilde{s}_{12}\tilde{c}_{23}\tilde{s}_{13}e^{i\delta_V}, & \tilde{V}_{\tau 3} &= \tilde{c}_{23}\tilde{c}_{13}, \end{aligned} \quad (44)$$

with the abbreviations $\tilde{c}_{ij} = \cos \phi_{ij}$ and $\tilde{s}_{ij} = \sin \phi_{ij}$.

Note that due to the unphysical phases corresponding to the rephasing of three left-handed neutrinos, the three phases ($\phi_e, \phi_\mu, \phi_\tau$) appearing at the left side of the mixing matrix U . Therefore, we rewrite the Dirac flavor neutrino mass matrix as

$$M = \text{diag.}(e^{i\phi_e}, e^{i\phi_\mu}, e^{i\phi_\tau}) U \text{diag.}(m_1, m_2, m_3) V^\dagger. \quad (45)$$

A global analysis of current data shows the following the best fit values of the squared mass differences $\Delta m_{ij}^2 = m_i^2 - m_j^2$ and the mixing angles for the so-called

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normal mass ordering (NO), $m_1 < m_2 < m_3$, of the neutrinos:⁹²

$$\begin{aligned}
\frac{\Delta m_{21}^2}{10^{-5}\text{eV}^2} &= 7.42_{-0.20}^{+0.21} \quad (6.82 \rightarrow 8.04), \\
\frac{\Delta m_{31}^2}{10^{-3}\text{eV}^2} &= 2.510_{-0.027}^{+0.027} \quad (2.430 \rightarrow 2.593), \\
\theta_{12}/^\circ &= 33.45_{-0.75}^{+0.77} \quad (31.27 \rightarrow 35.87), \\
\theta_{23}/^\circ &= 42.1_{-0.9}^{+1.1} \quad (39.7 \rightarrow 50.9), \\
\theta_{13}/^\circ &= 8.62_{-0.12}^{+0.12} \quad (8.25 \rightarrow 8.98), \\
\delta_{\text{CP}}/^\circ &= 230_{-25}^{+36} \quad (144 \rightarrow 350),
\end{aligned} \tag{46}$$

where the \pm represents the 1σ region and the parentheses denote the 3σ region. On the other hand, for the so-called inverted mass ordering (IO), $m_3 < m_1 \lesssim m_2$, we have

$$\begin{aligned}
\frac{\Delta m_{21}^2}{10^{-5}\text{eV}^2} &= 7.42_{-0.20}^{+0.21} \quad (6.82 \rightarrow 8.04), \\
\frac{\Delta m_{32}^2}{10^{-3}\text{eV}^2} &= -2.490_{-0.028}^{+0.026} \quad (-2.574 \rightarrow -2.410), \\
\theta_{12}/^\circ &= 33.45_{-0.75}^{+0.78} \quad (31.27 \rightarrow 35.87), \\
\theta_{23}/^\circ &= 49.0_{-1.3}^{+0.9} \quad (39.8 \rightarrow 51.6), \\
\theta_{13}/^\circ &= 8.61_{-0.12}^{+0.14} \quad (8.24 \rightarrow 9.02), \\
\delta_{\text{CP}}/^\circ &= 278_{-30}^{+22} \quad (194 \rightarrow 345).
\end{aligned} \tag{47}$$

In addition, we have the constraint

$$\sum m_i < 0.12 \text{ eV}, \tag{48}$$

from observation of the cosmic microwave background radiation.^{93,94}

3.3. Numerical analysis

We randomly generated 10^7 parameters sets $\{m_1, m_2, m_3, \theta_{12}, \theta_{23}, \theta_{13}, \delta_{\text{CP}}\}$ that satisfy the 3σ region of Eq.(46), and Eq.(48) for NO with $\phi_{ij} \in (0, \pi/2)$, $\omega_i, \delta_V, \phi_e, \phi_\mu, \phi_\tau \in (-\pi, \pi)$. We also randomly generated 10^7 parameter sets that satisfy the 3σ region of Eq.(47), and Eq.(48), for IO. We estimate ϵ_i^{min} using each parameter sets and the results are shown in Table.1 and Table.2. Table.1 shows ϵ_i^{min} , and $\epsilon_i^{\ell m}$ when ϵ_i^{min} is obtained. Table.2 shows ϵ_i^{min} , and neutrino parameters when ϵ_i^{min} is obtained.

First, we focus our attention on the NO case. Table.1 indicates that the M_{14}^0 ($\epsilon_{14}^{\text{min}} = 0.327$) is the most favorable three-zero texture. The magnitude of $\epsilon_{14}^{\ell m}$ for non-zero element in M_{14}^0 is estimated as

$$\begin{aligned}
\epsilon_{14}^{11} &= 0.117, & \epsilon_{14}^{12} &= 0.503, & \epsilon_{14}^{21} &= 0.228, \\
\epsilon_{14}^{23} &= 0.455, & \epsilon_{14}^{31} &= 0.0260, & \epsilon_{14}^{33} &= 0.398.
\end{aligned} \tag{49}$$

Although the (1, 2) element violates the magic-square nature over 50%, the other elements obey the magic-square nature with $\epsilon_{14}^{\ell m} < 0.46$. More small $\epsilon_{14}^{\ell m}$ is preferable in the context of the magic square, however, we conclude that the decomposition of the Dirac flavor mass matrix with $M = M_{14}^{\text{magic}} + M_{14}^0$ is reasonable and that the most favorable three-zero texture in the context of magic square is M_{14}^0 for NO.

The predicted neutrino parameters for the most favorable texture M_{14}^0 for NO

Table 1. ϵ_i^{\min} , and $\epsilon_i^{\ell m}$ when ϵ_i^{\min} is obtained.

	M_i^0	ϵ_i^{\min}	ϵ_i^{11}	ϵ_i^{12}	ϵ_i^{13}	ϵ_i^{21}	ϵ_i^{22}	ϵ_i^{23}	ϵ_i^{31}	ϵ_i^{32}	ϵ_i^{33}
NO	M_{14}^0	0.327	0.117	0.503	0.000	0.228	0.000	0.455	0.260	0.000	0.398
	M_2^0	0.337	0.000	0.522	0.000	0.244	0.000	0.527	0.260	0.143	0.326
	M_{17}^0	0.357	0.227	0.571	0.0768	0.000	0.000	0.547	0.170	0.000	0.552
	M_{11}^0	0.358	0.633	0.000	0.215	0.468	0.000	0.386	0.302	0.146	0.000
	M_6^0	0.367	0.000	0.503	0.262	0.277	0.000	0.596	0.240	0.000	0.326
	M_{12}^0	0.368	0.252	0.559	0.000	0.000	0.000	0.547	0.172	0.125	0.554
	M_{24}^0	0.370	0.633	0.145	0.128	0.570	0.000	0.415	0.328	0.000	0.000
	M_{10}^0	0.376	0.263	0.000	0.639	0.434	0.000	0.127	0.000	0.445	0.348
	M_9^0	0.381	0.556	0.000	0.000	0.657	0.000	0.0985	0.0690	0.350	0.556
	M_4^0	0.387	0.000	0.700	0.332	0.497	0.000	0.000	0.264	0.315	0.216
	M_3^0	0.389	0.000	0.443	0.406	0.000	0.000	0.406	0.420	0.452	0.208
	M_3^0	0.393	0.000	0.000	0.683	0.267	0.000	0.331	0.293	0.610	0.176
	M_5^0	0.395	0.000	0.721	0.447	0.329	0.000	0.220	0.000	0.436	0.216
	M_{22}^0	0.399	0.375	0.590	0.385	0.267	0.000	0.319	0.000	0.000	0.455
	M_{18}^0	0.402	0.146	0.209	0.761	0.000	0.000	0.354	0.400	0.543	0.000
	M_9^0	0.404	0.381	0.000	0.655	0.466	0.000	0.000	0.151	0.445	0.327
	M_{15}^0	0.416	0.766	0.290	0.000	0.251	0.000	0.425	0.696	0.0688	0.000
	M_{20}^0	0.418	0.364	0.684	0.387	0.129	0.000	0.000	0.224	0.000	0.721
	M_{23}^0	0.421	0.115	0.541	0.382	0.322	0.000	1.132	0.000	0.0371	0.000
	M_{19}^0	0.425	0.496	0.266	0.639	0.466	0.000	0.000	0.000	0.368	0.317
	M_8^0	0.434	0.745	0.000	0.430	0.000	0.000	0.340	0.435	0.146	0.506
	M_{16}^0	0.437	0.417	0.106	0.178	0.000	0.000	0.686	0.000	0.261	0.976
	M_{21}^0	0.444	0.146	0.280	0.746	0.313	0.000	0.000	0.787	0.393	0.000
	M_{13}^0	0.450	0.269	0.630	0.000	0.129	0.000	0.000	0.272	0.705	0.694
IO	M_{24}^0	0.312	0.103	0.0234	0.188	0.171	0.000	0.995	0.391	0.000	0.000
	M_{11}^0	0.325	0.103	0.000	0.207	0.200	0.000	0.996	0.396	0.0517	0.000
	M_6^0	0.330	0.000	0.0234	0.108	0.141	0.000	0.989	0.359	0.000	0.357
	M_{15}^0	0.334	0.103	0.134	0.000	0.126	0.000	0.988	0.322	0.330	0.000
	M_1^0	0.341	0.000	0.000	0.157	0.749	0.000	0.186	0.754	0.0931	0.109
	M_5^0	0.344	0.000	0.0764	0.000	0.172	0.000	0.984	0.322	0.152	0.357
	M_{18}^0	0.347	0.103	0.156	0.153	0.000	0.000	0.986	0.347	0.336	0.000
	M_{17}^0	0.354	0.0906	0.0234	0.131	0.000	0.000	0.986	0.366	0.000	0.526
	M_4^0	0.366	0.000	0.0903	0.0682	0.780	0.000	0.000	0.922	0.194	0.141
	M_3^0	0.369	0.000	0.113	0.184	0.000	0.000	0.986	0.375	0.200	0.357
	M_{14}^0	0.372	0.161	0.350	0.000	1.007	0.000	0.0447	0.233	0.000	0.434
	M_8^0	0.374	0.109	0.000	0.128	0.000	0.000	0.986	0.367	0.0517	0.601
	M_{20}^0	0.374	0.140	0.350	0.0382	1.008	0.000	0.000	0.286	0.000	0.424
	M_{13}^0	0.380	0.123	0.405	0.000	1.008	0.000	0.000	0.233	0.0986	0.414
	M_{19}^0	0.385	0.0749	0.333	0.0525	1.189	0.000	0.000	0.000	0.222	0.439
	M_9^0	0.389	0.144	0.000	0.000	0.229	0.000	0.978	0.322	0.0517	0.612
	M_{22}^0	0.391	0.213	0.145	0.167	1.004	0.000	0.363	0.000	0.000	0.452
	M_5^0	0.391	0.000	0.332	0.0525	1.183	0.000	0.0543	0.000	0.272	0.454
	M_{16}^0	0.394	0.506	0.133	0.133	0.000	0.000	0.203	0.000	1.349	0.0396
	M_{23}^0	0.396	0.0373	0.272	0.0834	0.775	0.000	0.258	0.000	0.948	0.000
	M_{21}^0	0.405	0.225	0.290	0.151	0.588	0.000	0.000	0.843	0.334	0.000
	M_{12}^0	0.413	0.143	0.0948	0.000	0.000	0.000	1.054	0.287	0.769	0.130
	M_9^0	0.413	0.178	0.000	0.519	0.774	0.000	0.000	0.281	0.607	0.121
	M_{10}^0	0.423	0.345	0.000	0.304	0.737	0.000	0.199	0.000	0.588	0.364

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are

$$\begin{aligned}
m_1 &= 0.0270\text{eV}, & m_2 &= 0.0283\text{eV}, & m_3 &= 0.0566\text{eV}, \\
\theta_{12} &= 35.3^\circ, & \theta_{23} &= 44.0^\circ, & \theta_{13} &= 8.4^\circ, & \delta_{\text{CP}} &= 179.9^\circ,
\end{aligned}
\tag{50}$$

Table 2. ϵ_i^{min} , and neutrino parameters when ϵ_i^{min} is obtained.

	M_i^0	ϵ_i^{min}	m_1/eV	m_2/eV	m_3/eV	$\theta_{12}/^\circ$	$\theta_{23}/^\circ$	$\theta_{13}/^\circ$	$\delta_{\text{CP}}/^\circ$
NO	M_{14}^0	0.327	0.0270	0.0283	0.0566	35.3	44.0	8.4	179.9
	M_2^0	0.337	0.0270	0.0283	0.0566	35.3	44.0	8.4	179.9
	M_{17}^0	0.357	0.0297	0.0310	0.0582	35.2	45.0	8.3	207.5
	M_{11}^0	0.358	0.0294	0.0307	0.0582	34.5	48.1	8.4	223.8
	M_6^0	0.367	0.0270	0.0283	0.0566	35.3	44.0	8.4	179.9
	M_{12}^0	0.368	0.0297	0.0310	0.0582	35.2	45.0	8.3	207.5
	M_{24}^0	0.370	0.0294	0.0307	0.0582	34.5	48.1	8.4	223.8
	M_{10}^0	0.376	0.0221	0.0237	0.0541	32.5	48.2	8.8	219.6
	M_7^0	0.381	0.0177	0.0196	0.0524	33.1	50.0	8.5	146.7
	M_4^0	0.387	0.0284	0.0296	0.0580	35.1	48.7	8.7	277.4
	M_3^0	0.389	0.0269	0.0282	0.0568	32.3	48.3	8.5	324.8
	M_1^0	0.393	0.0220	0.0237	0.0543	31.9	49.8	8.4	341.3
	M_5^0	0.395	0.0284	0.0296	0.0580	35.1	48.7	8.7	277.4
	M_{22}^0	0.399	0.0266	0.0279	0.0566	33.0	44.5	8.4	184.9
	M_{18}^0	0.402	0.0296	0.0309	0.0581	34.6	50.0	8.6	278.8
	M_9^0	0.404	0.0221	0.0237	0.0541	32.5	48.2	8.8	219.6
	M_{15}^0	0.416	0.0298	0.0310	0.0589	34.7	42.3	8.9	327.1
	M_{20}^0	0.418	0.0258	0.0273	0.0569	32.8	47.8	8.7	298.5
	M_{23}^0	0.421	0.0205	0.0221	0.0542	32.8	50.8	8.6	146.3
	M_{19}^0	0.425	0.0221	0.0237	0.0541	32.5	48.2	8.8	219.6
M_8^0	0.434	0.0294	0.0307	0.0582	34.5	48.1	8.4	223.8	
M_{16}^0	0.437	0.0295	0.0308	0.0588	33.6	48.9	8.7	293.1	
M_{21}^0	0.444	0.0296	0.0309	0.0581	34.6	50.0	8.6	278.8	
M_{13}^0	0.450	0.0258	0.0273	0.0569	32.8	47.8	8.7	298.5	
IO	M_{24}^0	0.312	0.0502	0.0510	0.00792	32.1	48.1	8.9	250.5
	M_{11}^0	0.325	0.0502	0.0510	0.00792	32.1	48.1	8.9	250.5
	M_6^0	0.330	0.0502	0.0510	0.00792	32.1	48.1	8.9	250.5
	M_{15}^0	0.334	0.0502	0.0510	0.00792	32.1	48.1	8.9	250.5
	M_1^0	0.341	0.0507	0.0514	0.0132	35.6	46.2	9.0	316.7
	M_2^0	0.344	0.0502	0.0510	0.00792	32.1	48.1	8.9	250.5
	M_{18}^0	0.347	0.0502	0.0510	0.00792	32.1	48.1	8.9	250.5
	M_{17}^0	0.354	0.0502	0.0510	0.00792	32.1	48.1	8.9	250.5
	M_4^0	0.366	0.0512	0.0519	0.0116	34.3	48.7	8.9	194.3
	M_3^0	0.369	0.0502	0.0510	0.00792	32.1	48.1	8.9	250.5
	M_{14}^0	0.372	0.0498	0.0505	0.00971	35.2	49.4	8.7	235.9
	M_8^0	0.374	0.0502	0.0510	0.00792	32.1	48.1	8.9	250.5
	M_{20}^0	0.374	0.0498	0.0505	0.00971	35.2	49.4	8.7	235.9
	M_{13}^0	0.380	0.0498	0.0505	0.00971	35.2	49.4	8.7	235.9
	M_{19}^0	0.385	0.0500	0.0508	0.00329	33.4	44.2	9.0	277.7
	M_7^0	0.389	0.0502	0.0510	0.00792	32.1	48.1	8.9	250.5
	M_{22}^0	0.391	0.0487	0.0496	0.00386	34.0	45.4	8.7	304.4
	M_5^0	0.391	0.0500	0.0508	0.00329	33.4	44.2	9.0	277.7
	M_{16}^0	0.394	0.0514	0.0521	0.0129	33.4	40.5	8.8	331.7
	M_{23}^0	0.396	0.0506	0.0513	0.0126	34.6	43.3	8.9	250.7
M_{21}^0	0.405	0.0513	0.0520	0.0153	34.0	44.9	8.2	271.6	
M_{12}^0	0.413	0.0513	0.0519	0.0114	34.7	47.7	8.8	340.0	
M_9^0	0.413	0.0512	0.0519	0.0146	32.0	47.8	8.8	280.6	
M_{10}^0	0.423	0.0511	0.0519	0.0154	34.8	42.8	8.8	305.5	

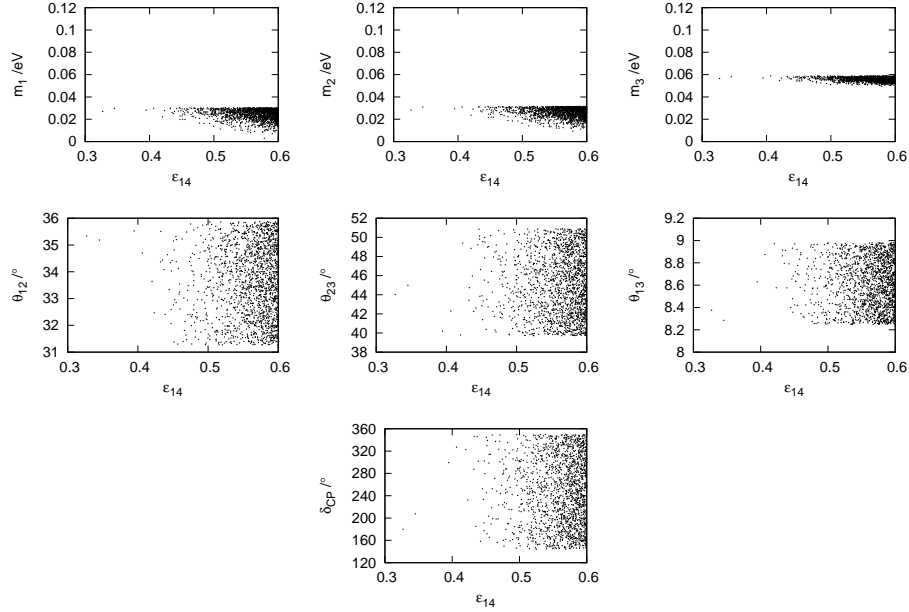


Fig. 1. Dependence of neutrino parameters $m_i, \theta_{ij}, \delta_{\text{CP}}$ on ϵ_i for M_{14}^0 in the case of NO.

as shown in Table. 2. We would like to comment on the T2K and NOvA tension.^{95,96} NOvA and T2K reported very different best fit value of δ_{CP} . In fact, for NO case, $\delta_{\text{CP}} \sim 145^\circ$ is reported from NOvA,⁹⁵ but it is reported as $\delta_{\text{CP}} \sim 250^\circ$ from T2K.⁹⁶ Unfortunately, the predicted value $\delta_{\text{CP}} \sim 179.9^\circ$ for the most favorable M_{14}^0 does not match the best fit values of δ_{CP} from either NOvA or T2K; however, the predicted value is not excluded in the 90% C.L..^{95,96}

Figure.1 shows the correlation between ϵ_{14} and neutrino parameters $m_1, m_2, m_3, \theta_{12}, \theta_{23}, \theta_{13}, \delta_{\text{CP}}$ for M_{14}^0 in the NO case. We have not plotted all 10^7 points on Figure.1 to reduce the file size of the figure. Figure.1 indicates that there is no significant correlation between ϵ_{14} and neutrino parameters. Therefore, the small ϵ_{14} is realized under a fine-tuning of the neutrino parameters. Although this is undesirable result, we believe it is meaningful to show that the Dirac flavor mass matrix has a part of the nature of magic square for a specific parameters.

Next, we see the IO case. From Table.1, the candidate of the favored three-zero textures in the context of a magic square is M_{24}^0 ($\epsilon_{24}^{\text{min}} = 0.312$) and M_{11}^0 ($\epsilon_{11}^{\text{min}} = 0.325$). Although, these ϵ_i^{min} are smaller than $\epsilon_{14}^{\text{min}} = 0.327$ for NO, $\epsilon_{24}^{23} = 0.995$ for M_{24}^0 and $\epsilon_{11}^{23} = 0.996$ for M_{11}^0 violate the magic-square nature over 99%. Thus, we conclude that NO case is favored over IO in the context of magic square.

4. Summary

The number of zero elements permitted in the texture zeros neutrino flavor mass matrix is an important consideration. In this paper, we decomposed the Dirac flavor neutrino mass matrix as $M = M' + M^0$, where M' is a matrix without zero elements and M^0 is a matrix with zero elements. We have shown that if M' obeys a magic square, the three elements of M^0 can be naturally zero. The 24 three-zero textures that can be compared to the experimental results are numerically estimated, and we chose the preferred matrices in the context of magic square. It turned out that NO case is favored over IO, in the context of magic square.

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