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Magic "Squares" Indeed

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NOTES

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Magic "Squares" Indeed!

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1 INTRODUCTION. Behold the remarkable property of the magic square:

$$\begin{bmatrix} 6 & 1 & 8 \\ 7 & 5 & 3 \\ 2 & 9 & 4 \end{bmatrix}$$

$$618^2 + 753^2 + 294^2 = 816^2 + 357^2 + 492^2 \quad \text{(rows)}$$

$$672^2 + 159^2 + 834^2 = 276^2 + 951^2 + 438^2 \quad \text{(columns)}$$

$$654^2 + 132^2 + 879^2 = 456^2 + 231^2 + 978^2 \quad \text{(diagonals)}$$

$$639^2 + 174^2 + 852^2 = 936^2 + 471^2 + 258^2 \quad \text{(counter-diagonals)}$$

$$654^2 + 798^2 + 213^2 = 456^2 + 897^2 + 312^2 \quad \text{(diagonals)}$$

$$693^2 + 714^2 + 258^2 = 396^2 + 417^2 + 852^2 \quad \text{(counter-diagonals)}.$$

This property was discovered by Dr. Irving Joshua Matrix [**3**], first published in [**5**] and more recently in [**1**]. We prove that this property holds for *every* 3-by-3 magic square, where the rows, columns, diagonals, and counter-diagonals can be read as 3-digit numbers in *any* base. We also describe n-by-n matrices that satisfy this condition, among them all circulant matrices and all symmetrical magic squares. For example, the 5-by-5 magic square in (1) also satisfies the square-palindromic property for every base.

$$\begin{bmatrix} 17 & 24 & 1 & 8 & 15 \\ 23 & 5 & 7 & 14 & 16 \\ 4 & 6 & 13 & 20 & 22 \\ 10 & 12 & 19 & 21 & 3 \\ 11 & 18 & 25 & 2 & 9 \end{bmatrix} \tag{1}$$

We must be careful when we read these numbers. The base 10 number represented by the first row of (1) is $17 \cdot 10^4 + 24 \cdot 10^3 + 1 \cdot 10^2 + 8 \cdot 10 + 15 = 194195$. The base 10 number based on the first row's reversal is 158357.

2 SUFFICIENT CONDITIONS. We say that a real matrix is *square-palindromic* if, for every base b, the sum of the squares of its rows, columns, and four sets of diagonals (as in the previous examples) are unchanged when the numbers are read "backwards" in base b. We can express this condition using matrix notation. Let M be an n-by-n matrix. Then the n numbers (in base b) represented by the rows of M are the entries of the vector $M\mathbf{b}$, where $\mathbf{b} = (b^{n-1}, b^{n-2}, \ldots, b, 1)^T$, and T denotes the transpose operation. The sum of the squares of these numbers is

$$(M\mathbf{b})^T(M\mathbf{b}) = \mathbf{b}^T(M^TM)\mathbf{b}.$$

Next, the n numbers represented by the rows when read "backwards" are the entries of $MR\mathbf{b}$ where the n-by-n *reversal matrix* $R = [r_{ij}]$ has $r_{ij} = 1$ if $i + j = n + 1$, and $r_{ij} = 0$ otherwise. Note that $R^T = R^{-1} = R$. The sum of the squares of these numbers is

$$(MR\mathbf{b})^T(MR\mathbf{b}) = \mathbf{b}^T(R(M^TM)R)\mathbf{b}.$$

Hence a sufficient condition for the *rows* of M to satisfy the square-palindromic property is simply $R(M^TM)R = M^TM$. Matrices A that satisfy $RAR = A$ are called *centro-symmetric* [**6**]: $a_{ij} = a_{n+1-i,n+1-j}$. Matrices A that satisfy $RAR = A^T$ are called *persymmetric* [**4**]: $a_{ij} = a_{n+1-j,\, n+1-i}$. It is easy to see that symmetric matrices that are centro-symmetric must also be persymmetric. Since M^TM is necessarily symmetric, our sufficient condition says that M^TM is centro-symmetric, or equivalently, that

$$M^TM \text{ is persymmetric.}$$

The square-palindromic condition for the *columns* of M is the square-palindromic condition for the rows of M^T. Hence it suffices to require that

$$MM^T \text{ is persymmetric.}$$

For the first set of *diagonals*, we create a matrix $\tilde{M}$ with the property that each column of $\tilde{M}$ represents a diagonal starting from the first row of M. To do this, we introduce two other special square matrices. Let $P_k = [p_{ij}]$ denote the n-by-n *projection matrix* whose only non-zero entry is $p_{kk} = 1$. Notice that $P^T = P$, and P_kM preserves the kth row of M but turns all other rows to zeros. Let $S = [s_{ij}]$ denote the n-by-n *shift operator* where $s_{ij} = 1$ if $i - j \equiv 1 \pmod n$, $s_{ij} = 0$ otherwise.

The following properties of S are easily verified: $S^n = I_n$, $S^{-1} = S^T = RSR$, and MS^k shifts the columns of M over "k steps to the left". Now define

$$\tilde{M} = \sum_{i=1}^{n} P_iMS^{i-1}.$$

Hence the i-th diagonal of M, starting from the first row becomes the i-th column of $\tilde{M}$. By the column condition, these diagonals satisfy the square-palindromic property if the (i, j) entry of $\tilde{M}\tilde{M}^T$ equals its $(n + 1 - j, n + 1 - i)$ entry.

We have

$$\tilde{M}\tilde{M}^T = \sum_{i=1}^{n} P_iMS^{i-1}\left(\sum_{j=1}^{n} P_jMS^{j-1}\right)^T = \sum_{i=1}^{n}\sum_{j=1}^{n} P_iMS^{i-j}M^TP_j.$$

It follows that $\tilde{M}\tilde{M}^T$ has the same (i, j) entry as $MS^{i-j}M^T$, and the same $(n + 1 - j, n + 1 - i)$ entry as well; if $MS^{i-j}M^T$ is persymmetric, then these entries are equal. Consequently, these diagonals obey the square-palindromic property if

$$MS^kM^T \text{ is persymmetric for } k = 1, \dots, n. \tag{2}$$

Conveniently, (2) also ensures that the counter-diagonals starting from the first row satisfy the square-palindromic property. This can be seen by mimicking the preceding explanation with $\tilde{M} = \sum_{i=1}^{n} P_iMS^{-(i-1)}$, whereby $\tilde{M}\tilde{M}^T$ has the same (i, j) and $(n + 1 - j, n + 1 - i)$ entry as $MS^{j-i}M^T$. For the other diagonal and counterdiagonal, we obtain similar results [7], which we summarize in the following theorem:

Theorem 1. *A square matrix M has the square-palindromic property if the following matrices are all persymmetric*:

1. $M^T M$,
2. MM^T,
3. $MS^k M^T$, *for* $k = 1, \ldots, n$, *and*
4. $M^T S^k M$, *for* $k = 1, \ldots, n$.

3. SQUARE-PALINDROMIC MATRICES. Next we explore classes of matrices that are square-palindromic. We say that a square matrix A is *centro-skew-symmetric* if $RAR = -A$, that is, $a_{ij} + a_{n+1-i,n+1-j} = 0$.

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 8 & 7 & 6 & 5 \\ 4 & 3 & 2 & 1 \end{bmatrix} \qquad \begin{bmatrix} a & b & c \\ d & 0 & -d \\ -c & -b & -a \end{bmatrix}$$

Centro-Symmetric Centro-Skew-Symmetric

Theorem 2. *Every centro-symmetric or centro-skew-symmetric matrix is square-palindromic.*

Proof: If M is centro-symmetric or centro-skew-symmetric, then the relations $RM = \pm MR$ and $R(S^k)R = S^{-k}$ ensure that M satisfies the conditions of Theorem 1. ■

The theorem is not at all surprising since the collection of rows, columns and diagonals of M read the same backwards and forwards. The next class of matrices, however, satisfies the conditions in a non-obvious way.

We say that A is *circulant* if every entry of each "diagonal" is the same, i.e., $a_{ij} = a_{k\ell}$ if $i - j \equiv k - \ell$ mod n or simply $SAS^{-1} = A$. We say that A is (-1)-*circulant* if $SAS = A$.

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \\ 3 & 4 & 1 & 2 \\ 2 & 3 & 4 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 1 \\ 3 & 4 & 5 & 1 & 2 \\ 4 & 5 & 1 & 2 & 3 \\ 5 & 1 & 2 & 3 & 4 \end{bmatrix}$$

Circulant (-1)-Circulant

Notice that the circulant and (-1)-circulant property is preserved under transposing. It is easy to show that the product of two circulant matrices or two (-1)-circulant matrices is circulant, while the product of a circulant and (-1)-circulant matrix is (-1)-circulant. Note that S is circulant, R is (-1)-circulant, and that all circulant matrices are persymmetric since a_{ij} and $a_{n+1-j,\, n+1-i}$ lie on the same diagonal. Consequently, if M is circulant or (-1)-circulant, the matrices $M^T M$, MM^T, $MS^k M^T$, and $M^T S^k M$ are all circulant, and thus persymmetric. From Theorem 1, it follows that

Theorem 3. *Every circulant or (-1)-circulant matrix is square-palindromic.*

Notice that four of the six square-palindromic identities are not obvious, but two of the diagonal sums are completely trivial!

4. MAGIC AND SEMIMAGIC SQUARES.

A *semi-magic square* with magic constant c is a square matrix A in which every row and column adds to c. Using matrix notation, this says that $AJ = cJ = JA$, where J is the matrix of all ones. If the main diagonal and main counter-diagonal also add to c, then the matrix is called a *magic square*. Circulant and (-1)-circulant matrices are always semi-magic, but are not necessarily magic.

A magic square A is *symmetrical* [2] if the sum of each pair of two entries that are opposite with respect to the center is $2c/n$, that is $a_{ij} + a_{n+1-i,n+1-j} = 2c/n$. Notice that a semimagic square with this property is magic.

Like the example below, magic and semi-magic squares do not necessarily satisfy the square-palindromic property.

$$\begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

Semi-Magic but not square-palindromic

However,

Theorem 4. *Every symmetrical magic square is square-palindromic.*

Proof: The trick is to notice that if M is a symmetrical magic square with magic constant c, then $M = M_0 + cJ/n$, where M_0 is a symmetrical magic square with magic constant 0. But this implies that M_0 is centro-skew-symmetric. Therefore M_0 is square-palindromic and satisfies the conditions of Theorem 1. Thus, since $M_0^T M_0$ and J are persymmetric, it follows that $M^T M = (M_0 + cJ/n)^T (M_0 + cJ/n) = M_0^T M_0 + c^2 J/n$ is also persymmetric. Hence M satisfies condition 1 of Theorem 1. To verify condition 3 (the other cases are similar), notice that

$$MS^k M^T = \left(M_0 + \frac{c}{n}J\right) S^k \left(M_0 + \frac{c}{n}J\right)^T = M_0 S^k M_0^T + \frac{c^2}{n}J$$

is persymmetric for $k = 1, \ldots, n$, since M_0 satisfies condition 3 of Theorem 1. ■

Although not all magic squares are square-palindromic, it is easy to see that all 3-by-3 magic squares are symmetrical. Consequently, we have

Theorem 5. *All* 3-*by*-3 *magic squares are square-palindromic.*

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An Elementary Proof of Binet's Formula for the Gamma Function

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The present note presents an elementary proof of the following important result of J. P. M. Binet [**3**, p. 249].

Theorem 1. *For* $x > 0$ *we have*

$$\Gamma(x+1) = \left(\frac{x}{\mathrm{e}}\right)^x \sqrt{2\pi x} \cdot \mathrm{e}^{\theta(x)} \tag{1}$$

where

$$\theta(x) = \int_0^\infty \left(\frac{1}{\mathrm{e}^t - 1} - \frac{1}{t} + \frac{1}{2}\right) \mathrm{e}^{-xt} \frac{1}{t}\, dt.$$

Here Γ *denotes the gamma function defined by*

$$\Gamma(x) = \int_0^\infty t^{x-1} \mathrm{e}^{-t}\, dt.$$

Since $\lim_{x \to \infty} \theta(x) = 0$, *from* (1) *we immediately obtain Stirling's formula*

$$n! = \Gamma(n+1) \sim \left(\frac{n}{\mathrm{e}}\right)^n \sqrt{2\pi n}\,.$$

Binet's formula can also be used to prove a more precise version of Stirling's asymptotic expansion

$$\log \frac{n!}{(n/\mathrm{e})^n \sqrt{2\pi n}} = \sum_{j=1}^{\infty} \frac{B_{2j}}{2j(2j-1)n^{2j-1}} = \frac{1}{12n} - \frac{1}{360n^3} + \frac{1}{1260n^5} - \cdots,$$

where the B_{2j}*'s denote the Bernoulli numbers defined by*

$$\frac{1}{\mathrm{e}^t - 1} - \frac{1}{t} + \frac{1}{2} = \sum_{j=1}^{\infty} \frac{B_{2j}}{(2j)!} t^{2j-1}.$$

For, by problem 154 *in Part I, Chapter* 4 *of* [**2**], *the inequalities*

$$\sum_{j=1}^{2N} \frac{B_{2j}}{(2j)!} t^{2j-1} < \frac{1}{\mathrm{e}^t - 1} - \frac{1}{t} + \frac{1}{2} < \sum_{j=1}^{2N+1} \frac{B_{2j}}{(2j)!} t^{2j-1}$$