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MAGNETIC BRAKING BY A STELLAR WIND—I

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Summary

The stellar wind theory is formulated for a rotating magnetic star, surrounded by an isothermal corona, and with the magnetic and rotation axes parallel. A critical surface C is defined by the Alfvénic points on each field-line, at which the wind-speed just equals the local Alfvén speed. The limiting field-line, with its Alfvénic point on the equator, separates the wind zone (which includes the polar regions) from the dead zone, in which outward expansion is prevented by the magnetic pressure. Within C, the roughly dipolar field is strong enough to force the flow to follow the field, and to keep the gas approximately co-rotating with the star. Beyond C, the gas flow drags the field to follow the flow, and each element approximately conserves its angular momentum. If the coronal temperature is too low for a thermal wind, the magnetically-controlled centrifugal forces drive a centrifugal wind, so that there is again a surface C limiting the zone of co-rotation.

The rate of mass-loss, $-\dot{M}$ and angular momentum loss, $-\dot{J}$ are found as functions of the coronal base density ρ_s, the star's rotation Ω_s, and three non-dimensional numbers: l, the ratio of gravitational to thermal energy density, ζ, the ratio of magnetic to thermal energy density, and κ, the ratio of centrifugal force to gravity, all three computed at the coronal base r_s. For $\zeta \gg 1$, $-\dot{J}$ varies slowly with ζ, but $-\dot{M}$ decreases sharply with increasing ζ, so illustrating Schatzman's suggestion that angular momentum loss per unit mass-loss may be greatly increased by magnetic control of the outflowing gas. When the corona is hot, $-\dot{J}/\Omega_s r_s^2$ is almost independent of κ—the wind is essentially thermal; but at lower temperatures it is high only when $\kappa \simeq 1$—the wind is essentially centrifugal.

The theory is applied to stars in thermal equilibrium, and to pre-main sequence stars contracting along the Hayashi and post-Hayashi tracks. In a close binary system of proto-stars, coupling between spin and orbital motion may be sufficiently strong to ensure that spin angular momentum lost is restored at the expense of the orbital angular momentum, so that the system remains a close binary in spite of the contraction of the individual stars.

1. *Introduction.* It is well-known that the angular momentum of the interstellar gas clouds is *prima facie* a formidable obstacle for the condensation of new stars. The difficulty can be resolved if some physical process transfers angular momentum from a condensation to the surrounding gas, and it is possible that the galactic magnetic field may play just this role, and without inevitably replacing the ' angular momentum problem ' by the ' magnetic flux problem ' (e.g. Mestel 1965). However, it seems likely that at each stage of the break-up of a rotating magnetic cloud, the centrifugal forces of spin are always close to gravity. This is because the magnetic energy of a gravitationally-bound condensation can never exceed the gravitational energy, so that the time of travel of Alfvén waves through the condensation is at least equal to the free-fall time, and will usually be considerably longer. We therefore expect that in the pre-opaque phases, when thermal pressure is weak, the magnetic torques will not be strong enough to equalize the angular velocities of a condensa-

tion and the surrounding medium, but rather that steady removal of angular momentum will allow condensations to contract, but in approximate centrifugal balance.

By contrast, one of the most striking features of the normal main-sequence stars is the great difference in the angular momentum per unit mass between the early-type and late-type stars, with a sharp drop at type F. We are forced to conclude that after the formation of ' proto-stars '—opaque bodies, with strong thermal pressure fields maintaining hydrostatic equilibrium—there is a new process of braking that acts preferentially on the stars of low mass, and with an efficiency that is sensitive to mass-variations near type F.

A promising idea was introduced by Schatzman (1962), who pointed out that if gas emitted by a star is magnetically constrained to co-rotate with the star out to distances large compared with the stellar radius, then a comparatively small amount of mass-loss would yield a proportionately much greater loss of angular momentum, because of the effective increase of the moment of inertia of the out-flowing gas. Schatzman suggested that this form of magnetic braking would operate with greater or less efficiency according to the strength of sub-photospheric convection, which is closely correlated with the mass of main-sequence stars. The convection is responsible for enhanced surface activity, with consequent corpuscular emission; also, the magnetic field is supposed generated by a dynamo process, with the rotation and the convection the two essential types of motion (Cowling, in Lüst 1965), as in Parker's dynamo model (1955) and the later somewhat different model of Babcock (1961).

In this paper we attempt to link up this general picture with the theory of a quasi-steady stellar wind, as developed by Parker (1963), but with important modifications caused by the presence of strong centrifugal and magnetic fields. Again, it is sub-photospheric convection that is probably an essential feature. Waves generated in the convection zone are dissipated above the surface, so supplying the heat responsible for the formation of the corona; however, we shall see that the centrifugal forces on the co-rotating gas can lead to an outward motion—a ' centrifugal wind '—in cases where the coronal temperatures are too low for the Parker thermal wind to flow. As in Parker's models, the wind velocity accelerates outwards from very low values below the corona to supersonic values far from the star. Provided that the magnetic field has an energy density at least comparable with the thermal energy at the coronal base, there is then no difficulty in ensuring that the wind speed remains less than the Alfvén speed until the gas has travelled a substantial distance from the star. This condition, essential for this form of magnetic braking to be operative, was first made explicit by Cowling (Lüst 1965); we shall see below how the ' Alfvénic ' points appear critically in the mathematical theory.

By analogy with the solar case, we can expect a dynamo process to operate in the atmosphere of a rotating star with a convective envelope, generating either a steady or a fluctuating surface magnetic field. The axis of the external dipole component will be more or less parallel to the rotation axis, and its magnitude will depend on the angular velocity and perhaps also on the strength of the convection. However, it can be argued (e.g. Mestel 1967) that the most plausible explanation of the fields of the strongly magnetic stars is that they are large-scale ' fossil ' fields (Cowling 1945), with both strength and overall direction determined largely by conditions in the primeval gas clouds in which the stars formed. In this paper we shall assume the field to be symmetric about the rotation axis, so that it could be either primeval or dynamo-driven, but in subsequent papers we shall present approximate studies of the problem with the dipole at an arbitrary angle to the angular momentum vector. A general field of energy high enough to withstand tangling by the Hayashi turbulence will inevitably interfere somewhat with convective energy transport, but is unlikely to suppress it (cf. the recent phenomenological studies of sunspot structure by Chitre (1963) and Deinzer (1965): we may still expect generation of a corona, though the waves travelling up will be hydromagnetic rather than sound or gravity waves.

The most difficult part of the complicated interaction between the magnetic, thermal, gravitational and centrifugal fields is the determination of the detailed structure of the coronal magnetic field. A crucial parameter for an accurate estimate of both the mass and angular momentum flux is the position of the limiting field-line, separating the wind-zone from the dead zone, consisting of gas prevented from expanding by the pressure of the closed magnetic field loops. This field-line is sensitive to variations in the strength of the magnetic dipole, the coronal temperature and the stellar rotation; the qualitative and quantitative conclusions of this paper are therefore in part significantly different from those of the earlier treatments (Mestel 1966, 1967).

Since Schatzman's original paper and Hayashi's pioneering paper (1961), much more work has been done on pre-main sequence contraction. It is now agreed that stars of moderate or small mass are fully convective during their descent of the Hayashi track, and that they retain extensive outer convective zones during the subsequent Kelvin–Helmholtz-type approach to the main sequence. We may therefore plausibly expect generation of an expanding corona and magnetic braking during pre-main sequence contraction. Observations of T Tauri stars by Kuhi (1964, 1966) have confirmed that mass-loss can be very pronounced in these phases. The integrations of Iben (1965) are particularly valuable in showing how the time spent in the Hayashi and post-Hayashi phases increases sharply with decreasing mass. The variation in main-sequence rotation may therefore turn out to originate essentially in the contraction phase rather than on the main sequence itself.

At the low-mass end of the main sequence, the W Ursae Majoris stars are exceptions to the rule of low rotation. At the upper end, the typical stars of abnormally low rotation are the Ap and Am stars. Again, many of these are binary members—according to Abt (1961), all the Am stars, and a high proportion of the Ap stars (1966). This suggests that in both cases the abnormal rotation has resulted from an interchange of angular momentum between the spin and the orbital motion, leading to synchronization of the two periods: in the W Ursae Majoris case, yielding an abnormally rapid rotation, while slowing up the Ap and Am stars. Continuing this line of thought, we are led to conjecture that the same process of magnetic braking is responsible for the formation of the short-period and contact binary systems: the abnormal spin may be just an indirect consequence (through spin–orbit coupling) of the mutual approach of the two components as they lose angular momentum.

2. *The wind from a rotating magnetic star*. We begin by summarizing the arguments for a spherically-symmetric, thermally-driven wind from a slowly rotating, weakly magnetic star surrounded by a hot corona (Parker 1963). The alternative of a *static* corona requires that the heat fed in at the coronal base r_s be all lost either

by radiation, conduction or convection, but without any laminar mass motions. In the solar corona radiative losses (mainly by free–free emission) are comparatively small, and a static model must involve a nearly constant energy flux L, carried outwards by thermal conduction from r_s to the radius r_u at which the conductive temperature gradient becomes convectively unstable, after which the gradient is nearly adiabatic out to infinity. With the thermal conductivity $\sigma = \sigma_0 T^{5/2}$, the temperature for $r_s \leqslant r \leqslant r_u$ is

$$T^{7/2} - T_u^{7/2} = \frac{7L}{8\pi\sigma_0}\left(\frac{1}{r} - \frac{1}{r_u}\right), \tag{1}$$

with the pressure given by

$$\frac{d}{dr}\log p = -\frac{GM\mu}{\mathscr{R}T_u}\cdot\frac{1}{r^2(7r_u/2r - 5/2)^{2/7}} \tag{2}$$

in standard notation. For $r > r_u$ the adiabatic relation $p \propto \rho^{5/3}$ yields

$$\frac{5}{2}\frac{p_u}{\rho_u}\left(\frac{\rho}{\rho_u}\right)^{2/3} = \frac{GM}{r_u}\left(\frac{r_u}{r}\right), \tag{3}$$

where we have used the condition $p \to 0$ at infinity (the pressure of the interstellar medium being small enough for this to be an adequate approximation). At $r = r_u$, equation (3) reduces to

$$\frac{5}{2}\frac{p_u}{\rho_u} = \frac{GM}{r_u}, \tag{4}$$

so that equation (3) is equivalent to

$$\frac{T}{T_u} = \left(\frac{\rho}{\rho_u}\right)^{2/3} = \frac{r_u}{r}. \tag{5}$$

The adiabatic and conductive gradients must be identical at $r = r_u$, where

$$L = 4\pi\sigma_0 r_u T_u^{7/2} = \frac{4\pi\sigma_0\left(\dfrac{2\mu GM}{5\mathscr{R}}\right)^{7/2}}{r_u^{5/2}}, \tag{6}$$

and equation (1) simplifies to

$$\left(\frac{T}{T_u}\right)^{7/2} = \frac{7}{2}\frac{r_u}{r} - \frac{5}{2}. \tag{7}$$

The radius r_u at the onset of convection is fixed in terms of the temperature T_s at the coronal base by equations (4) and (7):

$$T_s = \left(\frac{2\mu GM}{5\mathscr{R}r_s}\right)\left(\frac{r_s}{r_u}\right)\left(\frac{7}{2}\frac{r_u}{r_s} - \frac{5}{2}\right)^{2/7}. \tag{8}$$

Since $(\frac{7}{2}X - \frac{5}{2})^{2/7}/X$ is a monotonic decreasing function of X when $X > 1$, a necessary and sufficient condition for such a static model is

$$T_s < \bar{T}_s \equiv \frac{2GM\mu}{5\mathscr{R}r_s}. \tag{9}$$

For the solar case, $\bar{T}_s \simeq 3{\cdot}9 \times 10^6$ °K. But with too great an input of heat at the coronal base forcing T_s above $\bar{T}_s$, not even a completely convective corona can remain in hydrostatic equilibrium, and there results a *thermally-driven* stellar wind.*

The argument is modified somewhat if conduction is not the only source of heat for the upper corona—e.g. if the waves from the sub-photospheric convection zone penetrate a considerable distance beyond r_s before being damped, yielding a corona with a weaker temperature gradient. In the limit of a strictly isothermal corona at temperature T_s, the hydrostatic condition yields a pressure which decreases exponentially to the value $p(r_s)\exp(-l)$ at infinity, where the parameter $l \equiv \dfrac{\mu GM}{\mathscr{R} T_s r_s}$ measures the ratio of gravitational energy to thermal energy at the coronal base. Even with l taken as high as 20—corresponding in the solar case to a value for T_s appropriate to the outer corona rather than the base—$p(\infty)$ is still well above the thermal and magnetic pressures of the interstellar medium: in the absence of a 'wall' the gas again settles into a steady state of expansion with velocities increasing outwards. A further large increase in l would allow the corona to achieve equilibrium with the interstellar gas. It is in fact found convenient below to put the background pressure strictly to zero, so that an isothermal corona of any temperature always generates a wind, but the expansion of a 'cool' corona yields much smaller dynamical pressures, and correspondingly negligible loss of mass and angular momentum.

We now wish to study the changes in the theory when the star has both a rotation and a strong dipolar magnetic field. A complete treatment must include an estimate of the temperature field, taking into account the effect of the magnetic field on the propagation and damping of waves, and as the cause of anisotropy in the thermal conductivity. However, as we are at the moment concerned to bring out the effect of the magnetic and centrifugal fields on the dynamics of the wind, we bypass the formidable thermal problem by assuming the corona isothermal, but allowing its temperature (or equivalently the parameter l) to vary through an appropriate range of values.

If the surface magnetic field $\mathbf{H}_p$ is sufficiently weak, the stellar wind theory is not much affected: the high conductivity merely ensures that the field-lines are drawn out by the flow so as to be nearly radial. The kinetic energy density $\frac{1}{2}\rho v_p^2$ associated with a wind speed v_p is a measure of the outward driving forces (the thermal pressure and the centrifugal force). An obvious rough criterion for no magnetic interference with the wind is $\frac{1}{2}\rho v_p^2 \gg H_p^2/8\pi$, or $v_p \gg H_p/\sqrt{4\pi\rho}$—the wind speed much greater than the Alfvén speed at the point considered. But the same criterion also ensures that the field is *in*effective at transporting angular momentum—the Alfvén waves that try to keep the gas co-rotating with the star are unable to catch up with the wind. (This vital point will emerge from the mathematical discussion in Section 3.) A star with a sufficiently *strong* primeval field can easily exert the magnetic torques required to maintain co-rotation, but equally it is now not legitimate to ignore the effect of the poloidal magnetic forces on the stellar wind: instead of the magnetic field being distorted by the effectively unimpeded stellar wind, it is the flow which will tend to be channelled so as to follow the field-lines. Thus consider a star with a dipole-type poloidal field

$$H_r = \frac{\bar{H}\cos\theta}{(r/r_s)^3}, \qquad H_\theta = \frac{1}{2}\frac{\bar{H}\sin\theta}{(r/r_s)^3}. \tag{10}$$

* The author acknowledges the help of Professor P. A. Sweet in clarifying the physics of this problem.

The field-line emerging from the coronal base r_s at the point with colatitude θ_s has the equation

$$\frac{\varpi^2}{r^3} \equiv \frac{\sin^2\theta}{r} = \frac{\sin^2\theta_s}{r_s}, \tag{11}$$

where $\varpi \equiv r\sin\theta$ is the axial distance at the point (r, θ). The total field strength at (r, θ) is

$$\mathbf{H}^2(r,\theta) = \frac{\bar{H}^2}{(r/r_s)^6}(1-\tfrac{3}{4}\sin^2\theta) = \frac{\bar{H}^2}{(r/r_s)^6}\left(1-\frac{3}{4}\frac{r}{r_s}\sin^2\theta_s\right). \tag{12}$$

We suppose that the field at the coronal base has an energy density greater than the thermal energy density, so that an initially subsonic flow will follow the field-lines. Gas starting at sufficiently low latitudes will reach the equator at points not too far from the star, where the magnetic energy density is still larger than the thermal. Even if there were no hot gas outside the region defined by the loop ABA' in Fig. 1, exerting an inward pressure, the gas within ABA' would reach equilibrium: a very slight denting of the field-lines would generate the discontinuity in the magnetic pressure $\mathbf{H}^2/8\pi$ that would balance the discontinuity in thermal pressure. But gas expanding along field lines such as EC cannot reach such a state of hydrostatic equilibrium. Before it has expanded far enough to reach the equator, it will find that its pressure exceeds the magnetic pressure, so that it will cease to flow along prescribed, nearly dipole field-lines: instead it will expand more-or-less radially, dragging the field with it.

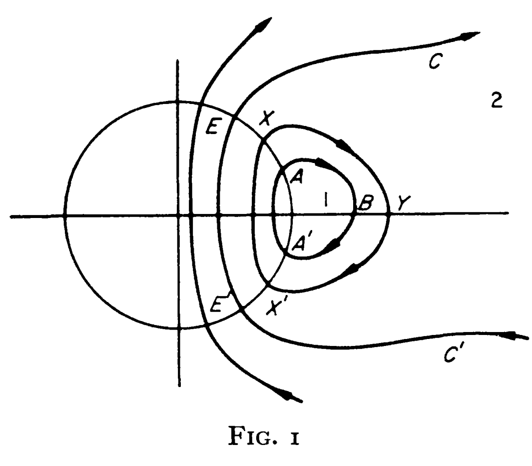

FIG. 1

The picture we arrive at finally is as in Fig. 1. There is a dead zone (1) in which the closed, approximately dipolar field-loops hold in the gas and keep it rotating with the star's angular velocity Ω_s. The density field ρ along each field-line is given by the component of hydrostatic support along the field: assuming isothermality with sound speed a,

$$-\frac{a^2}{\rho}\frac{\partial\rho}{\partial s}+\frac{\partial}{\partial s}\left(\frac{GM}{r}+\tfrac{1}{2}\Omega_s^2\varpi^2\right) = 0, \tag{13}$$

or

$$\log\left(\frac{\rho}{\rho_s}\right) = l\left(\frac{r_s}{r}-1\right)+\tfrac{1}{2}\kappa l\sin^2\theta_s\left(\left(\frac{r}{r_s}\right)^3-1\right), \tag{14}$$

where $\kappa \equiv \Omega_s^2 r_s^3/GM$ is the ratio at the coronal base r_s of centrifugal force to the opposing component of gravity, and we have used the ϖ–r relation (11) for a dipole field. In zone (2), outside a limiting field-line XYX', a modified stellar wind flows. As long as isothermality and zero viscosity are good approximations, the density, velocity and rotation fields, and the toroidal magnetic field generated by the wind are again determined by equations relating to individual field lines (see Sections 3 and 4). The difficulty is to determine accurately the shape of the field-lines. At points in either zone near enough to the star, the field remains approximately dipolar; for with the magnetic energy density (by hypothesis) large compared with the non-magnetic, if $|\nabla \times \mathbf{H}_p|$ were not much smaller than the 'order-of magnitude' estimate H_p/r the magnetic force density would be much greater than the trans-field components of the non-magnetic forces (including the inertial force on the wind in zone (2)), and one component of the equation of motion would not be satisfied. Far from the star an unconstrained stellar wind will flow, with the lines of the poloidal field drawn out to be straight. Beyond Y, the sharp change in direction of the field-lines near the equator requires an equatorial current sheet.

It is in the transitional region that the equations to the magnetic and dynamical fields are mutually coupled. As a first approximation we take each (poloidal) field-line to be dipolar (given by equation (11)) out to a critical point C, and radial beyond it. The critical point is determined by the condition that the energy density of the wind just equals that of the dipolar field.* (This fairly obvious choice has the advantage of being identical with the critical point in the equations that fix the angular velocity and toroidal magnetic fields—see Section 3). On the limiting field-line XYX', the critical point coincides with the equatorial point Y. In region 1 the whole field is dipolar. The distribution over each sphere of the (radial) magnetic flux far from the star is not determined by the equations to this region, but by continuity with the field nearer to the star.

An improved model would have a smooth transition from dipole-type to radial field-lines. The density field on the windward side of the limiting field-line will differ from that in the dead zone, implying a corresponding large gradient in magnetic pressure. This in turn requires some local drift of gas across the field (driven by a difference in the thermal plus magnetic pressures on either side of XYX'), so as to offset the Ohmic diffusion that becomes large in regions of large field gradient. The number of field-lines in the wind-zone is an important parameter for estimating the total rate of angular momentum transport (cf. Section 5). As noted, we are assuming that both the wind zone and the dead zone have the same temperature, as given by the parameter l. If the wind is essentially driven by centrifugal forces, this is probably a tolerable approximation. But if the wind is thermally-driven—i.e. if it is the transport of heat energy by the wind that enables thermal balance to be reached—then the suppression of part of the wind by the magnetic field removes this sink of energy, while the magnetic field also sharply reduces the transverse thermal conductivity. The continuing input of heat must increase the temperature in the dead zone; the steady state reached may be such

* A model similar in some respects has been independently developed by Pneuman (1966a, b); however, near the equator *all* of his field lines are imagined distorted from the curl-free state so as to become part of the radial field beyond C. The model of Weber & Davis (1967) assumes a radial poloidal field all the way to the Sun. Both these models are consistent with the transverse component of the poloidal equation of motion only if there is a very high equatorial density all the way to the solar surface, able to balance the strong pinching forces exerted by the magnetic field.

that the thermal conduction down the field-lines *into* the star balances the input from wave damping. The number of field-lines in the dead zone will certainly be reduced, and the efficiency of angular momentum transport correspondingly increased: the numbers given in Section 5, computed assuming a uniform coronal temperature, are therefore likely to be lower limits.

It will be confirmed in Section 3 that within the critical point C the field does maintain approximate co-rotation, while beyond C the gas moves effectively free of torque. If the star has a moderate or high rotation, the centrifugal forces are important for the dynamics of the wind (cf. Sections 4 and 5). In fact, the parameter l can be so large that the purely thermal wind would be negligible, yet a strong magnetic field enforcing co-rotation out to large distances will drive a ' centrifugal wind ', with its energy coming not from the heat input at the coronal base, but from the centrifugal energy of the star, transmitted to the coronal gas by the magnetic torques. The theory also embraces cases in which the heat input to the corona is balanced by radiative losses—e.g. from an excess density of heavy ions—so that there is no energy available to drive a *thermal* wind; again a centrifugal wind will in general be driven by a rotating magnetic star.

This result provides a natural resolution of a long-standing difficulty—how to keep the zone of co-rotation surrounding a magnetic star of finite extent. Lüst & Schlüter (1954) pointed out that their force-free fields, with toroidal flux linking poloidal loops, break up into separated zones, so that all the field-lines emanating from the star are kept within a finite sphere, of radius that can be adjusted by altering the constant of proportionality between current density and field strength. This resolution of the co-rotation difficulty is rather arbitrary: no explanation is given as to how the field acquired its topology. More recently, Ferraro & Bhatia (1967) have postulated that in the absence of a wind the stellar field is kept of finite extent by the pressure of an external field, separated from the stellar field by a nul-sphere carrying surface currents. This system again appears rather *ad hoc*, and likely to be unstable. The difficulties disappear if we accept that the centrifugal forces themselves will drive a wind, with again the zone of co-rotation limited by the set of Alfvénic points. Provided the field is strong enough, the gas velocity will exceed the velocity of escape when it reaches the Alfvénic points, so that it will flow to infinity with nearly uniform velocity. If the field is somewhat weaker, the gas will flow through the Alfvénic points to collect on the equator, subsequently to move in approximately Keplerian orbits, with the magnetic field slowly feeding in angular momentum and so causing a slow outward drift (Hazlehurst 1966).

3. *The theory of magnetic braking*. We now study the dynamics of the wind region. One component of the equation of motion has been used in Section 2 to fix the approximate field structure. The component along the field lines fixes the wind velocity; this will be studied in Section 4. The total velocity field will have a poloidal component $\mathbf{v}_p$ (the wind velocity) and a toroidal component $\mathbf{v}_t = \varpi\Omega\mathbf{t}$, where $\mathbf{t}$ is the unit toroidal vector, and Ω is in general non-uniform. The magnetic field will now have besides its poloidal component $\mathbf{H}_p$ a toroidal component $H_t\mathbf{t}$, generated by the non-uniform rotation that is itself a consequence of the stellar wind. In a steady state, the velocity, magnetic and density fields must be consistent with the hydromagnetic, continuity and torque equations.

In a steady-state, the electric field $\mathbf{E}$ is curl-free, and the hydromagnetic equation

$$\nabla\times(\mathbf{v}\times\mathbf{H}) = -c\nabla\times\mathbf{E} = 0 \tag{15}$$

has general integrals (Chandrasekhar 1956; Mestel 1961)

$$\mathbf{v}_p = \kappa \mathbf{H}_p, \tag{16}$$

$$\Omega - \frac{\kappa H_t}{\varpi} = \alpha, \tag{17}$$

or jointly

$$\mathbf{v} = \kappa \mathbf{H} + \varpi \alpha \mathbf{t}, \tag{18}$$

where κ is a scalar function of position and α is a constant on each field-stream line. From the combined form (18) we see that the general motion consistent with equation (15) is an arbitrary flow parallel to the field, superposed on an arbitrary uniform rotation of each field line—a simple generalization of Ferraro's law (1937). The continuity equation yields

$$\frac{\rho v_p}{H_p} = \rho\kappa = \eta = \text{constant on each field-line.}^{\star} \tag{19}$$

The toroidal equation of motion asserts that in a steady state the magnetic torque gives to each unit volume of gas the same amount of angular momentum that the streaming gas removes:

$$\varpi \left(\frac{(\nabla \times \mathbf{H}) \times \mathbf{H}}{4\pi}\right)_t = \nabla . (\rho \mathbf{v} \Omega \varpi^2). \tag{20}$$

This has a general integral

$$-\frac{\varpi H_t}{4\pi} + \rho\kappa\Omega\varpi^2 = -\frac{\beta}{4\pi} = \text{constant on each field-line.} \tag{21}$$

If we consider an infinitesimal flux-tube of (variable) area of cross-section A, we have that the transport of angular momentum along the tube by the gas flow is $\rho v_p \Omega \varpi^2 A = (\rho\kappa\Omega\varpi^2) H_p A$, or $\rho\kappa\Omega\varpi^2$ per unit flux-tube. This is not in general constant, but a piling-up of angular momentum is prevented by the magnetic stresses, which transport angular momentum $-(\varpi H_t/4\pi)(H_p A)$ per second along the flux tube (Lüst & Schlüter 1955). We can therefore immediately interpret the constant of integration $-\beta/4\pi$ as the total rate of transport of angular momentum along a unit flux-tube. At different points of the same flux-tube either the material transport or the magnetic transport may dominate. When the material transport is dominant, the steady-state integral (21) is effectively the constancy of the angular momentum along the stream lines; when the magnetic transport dominates, equation (21) reduces to the torque-free condition $\varpi H_t = \beta$ (Lüst & Schlüter 1954).

The integrals (17), (19) and (21) combine to yield

$$\Omega = \frac{\alpha + \dfrac{\eta\beta}{\rho\varpi^2}}{1 - \dfrac{4\pi\eta^2}{\rho}}, \qquad H_t = \frac{\dfrac{\beta}{\varpi} + 4\pi\eta\alpha\varpi}{1 - \dfrac{4\pi\eta^2}{\rho}}. \tag{22}$$

$\star$ Ferraro & Bhatia (1967) assert that the step from $\mathbf{H}_p . \nabla(\rho\kappa) = 0$ (constancy of $\rho\kappa$ along a poloidal field-line) to $\mathbf{H} . \nabla(\rho\kappa) = 0$ (constancy along a total field-line) cannot be made unless $\mathbf{v}$ is parallel to $\mathbf{H}$ (i.e. $\alpha = 0$). In fact, since we are assuming axial symmetry, $\nabla(\rho\kappa)$ has no component in the toroidal direction, $\mathbf{H}_t . \nabla(\rho\kappa) = 0$, and $\mathbf{H} . \nabla(\rho\kappa) = \mathbf{H}_p . \nabla(\rho\kappa)$. The vanishing or non-vanishing of α is quite irrelevant.

The denominator is $1 - v_p^2/v_A^2$, where v_A is the Alfvén speed $H_p/\sqrt{4\pi\rho}$ at the point considered. A complete solution demands knowledge of η, α and β. The flow constant η for each field-streamline is given by the modified stellar wind theory (see Section 4). For the moment we take it as a prescribed parameter, except for the condition that the field strength H_p is high enough near the star for $4\pi\eta^2/\rho$ to be less than unity (sub-Alfvénic flow at the coronal base). However, since in any realistic solution the density must decrease outwards, there must be a point C at which

$$\rho_c = 4\pi\eta^2. \tag{23}$$

The expressions for both Ω and H_t become singular at this point unless the numerators vanish simultaneously with the denominator, i.e. unless

$$\beta = -4\pi\eta\alpha\varpi_c^2 = -\rho_c\varpi_c^2\alpha/\eta. \tag{24}$$

In this way, the parameter β is determined as a sort of eigen-value for each field-streamline. The angular velocity then becomes

$$\frac{\Omega}{\alpha} = \frac{\left(1 - \dfrac{\rho_c\varpi_c^2}{\rho\varpi^2}\right)}{\left(1 - \dfrac{\rho_c}{\rho}\right)}, \tag{25}$$

with ρ_c and ϖ_c given by equation (23) and (11), and α fixed by the prescribed stellar rotation at the coronal base (given by suffix s):

$$\frac{\Omega_s}{\alpha} = \frac{\left(1 - \dfrac{\rho_c\varpi_c^2}{\rho_s\varpi_s^2}\right)}{\left(1 - \dfrac{\rho_c}{\rho_s}\right)}. \tag{26}$$

The critical point C is identical with the change-over from dipolar to radial field structure in the approximation of Section 2. The toroidal magnetic field is given by

$$\varpi H_t = -4\pi\eta\alpha\varpi_c^2 \frac{\left(1 - \dfrac{\varpi^2}{\omega_c^2}\right)}{\left(1 - \dfrac{\rho_c}{\rho}\right)}. \tag{27}$$

Values of Ω and H_t at C can be found from l'Hôpital's rule.

Comparing parts of the same field-line in the northern and southern hemispheres respectively, we see from equations (16) and (27) that in our equatorially-symmetric system H_t at corresponding points is equal in magnitude but opposite in sign, so that $-\beta H_p$ has the same sign in both hemispheres. Within the star, the steady-state approximation is not valid—there is a magnetic torque density $\mathbf{H}.\nabla(\varpi H_t/4\pi)$; this ensures that in moving, e.g. from E to E' in Fig. 1, ϖH_t changes to $-(\varpi H_t)$ (Mestel 1959). Thus in both hemispheres the flow of angular momentum is in the same direction. The type of solution we want must yield outward transport: this is ensured by a positive value for

$$-\frac{\beta}{4\pi} H_p = \eta\alpha\varpi_c^2 H_p = \rho v_p \alpha\varpi_c^2, \tag{28}$$

equivalent to α greater than 0, since ρv_p is essentially positive. From equation (26), we see that the sign of α is determined by the behaviour of $\rho\varpi^2$ and ρ along the field-streamlines; in particular, with ρ decreasing outwards, α is positive if and only if $\rho\varpi^2$ decreases outwards. If $\rho_s\varpi_s^2 \gg \rho_c\varpi_c^2$, then $\alpha \simeq \Omega_s$, and for $\varpi \ll \varpi_c$ equations (25) and (27) reduce to

$$\Omega \simeq \alpha \tag{29}$$

and

$$\varpi H_t \simeq -4\pi\eta\alpha\varpi_c^2; \tag{30}$$

the magnetic field is so strong as to be nearly torque-free, and the torque it does exert is sufficient to keep the rotation nearly uniform, in spite of the transport of angular momentum by the wind. Thus near enough to the star Ω is effectively determined by the hydromagnetic equation (17), and H_t by the dynamical equation (21). Well beyond the point C, equations (25) and (27) reduce to

$$\Omega\varpi^2 \simeq \Omega_s\varpi_c^2 \tag{31}$$

and

$$\frac{H_t}{\Omega_s} \simeq -\rho\varpi/\eta. \tag{32}$$

The wind speed so much exceeds the Alfvén speed that each element of gas effectively conserves its angular momentum; far from the star it is Ω that is determined by the dynamical equation (21). The toroidal field (32) is determined by the hydromagnetic equation (17) with the angular velocity field (31) substituted, but the torque it exerts is small compared with the Coriolis torque on the streaming gas.

Thus solutions with both ρ and $\rho\varpi^2$ decreasing outwards (to be called 'normal solutions') have all the required properties—proper asymptotic behaviour, and an outward transport of angular momentum. The continuity equation (19) applied to flow along a dipolar field (10) yields

$$\frac{\rho v_p \left(\dfrac{r}{r_s}\right)^3}{\left(1-\dfrac{3}{4}\dfrac{r}{r_s}\sin^2\theta_s\right)^{1/2}} = \frac{\rho v_p \left(\dfrac{\varpi}{r_s}\right)^2}{\sin^2\theta_s\left(1-\dfrac{3}{4}\dfrac{r}{r_s}\sin^2\theta_s\right)^{1/2}} = \frac{\rho_s v_s}{(1-\frac{3}{4}\sin^2\theta_s)^{1/2}}, \tag{33}$$

where v_s is the wind speed at the coronal base. *Thus provided v_p increases outwards* (as in solutions similar to Parker's critical solution—see Section 4 below), then $\rho\varpi^2$ will decrease outwards along both the dipolar part of the field, and also along the approximately radial part, for which the continuity equation reduces to $\rho v_p\varpi^2 =$ constant.

If, however, the wind speed were to decrease outwards sufficiently fast for $\rho\varpi^2$ as given by equation (33) to increase outwards, then with ρ still assumed to decrease outwards, α and hence $(-\beta/4\pi)H_p$ would be negative, corresponding to an *inward* transport of angular momentum from infinity into the star. Near to the star the field H_t is still approximately torque-free and so given by equation (30), but since α is negative, the sign is now positive in the northern hemisphere and negative in the southern. The approximation to Ω is now

$$\frac{\Omega}{\Omega_s} \simeq \frac{\rho_s\varpi_s^2}{\rho\varpi^2}; \tag{34}$$

the relevant approximation to the hydromagnetic equation (17) is now not $\Omega \simeq \alpha$ but $\Omega \simeq \kappa H_t/\varpi$. Far from the star

$$\frac{\Omega}{\Omega_s} \simeq \left(\frac{\rho_s \varpi_s^2}{\rho_c \varpi_c^2}\right)\left(\frac{\rho}{\rho_c}\right), \tag{35}$$

and

$$\frac{H_t}{\Omega_s} \simeq -\frac{\alpha \varpi \rho}{\Omega_s \eta} = \left(\frac{\rho_s \varpi_s^2}{\rho_c \varpi_c^2}\right)\left(\frac{\varpi \rho}{\eta}\right). \tag{36}$$

These result from $\kappa H_t/\varpi \simeq -\alpha$ being again the relevant approximation to equation (17), but with the dynamical equation (21) reducing to

$$\Omega \varpi^2 \simeq \varpi H_t/4\pi\eta \tag{37}$$

—far from the star the inward transport of angular momentum by the magnetic field is only slightly greater than the outward transport by the wind. Such a solution of the equations is clearly of no relevance to our problem. But we are forced to conclude that if the velocity were to decrease outwards—a 'ballistic' velocity field, as used by Kuhi (1964) to interpret his observations of mass-loss from T Tauri stars—then magnetic braking could not be treated as a steady-state problem, but explicit account would have to be taken of the changes with time of the Ω-field.

Another solution with *outward* angular momentum transport would result if not only $\rho\varpi^2$ but ρ increased outwards. The asymptotic behaviour of Ω and H_t for $\varpi \ll \varpi_c$ would be similar to that in the normal solution for $\varpi \gg \varpi_c$, and *vice versa*; matter would flow out effectively conserving its angular momentum up to ϖ_c, but would be magnetically constrained to rotate uniformly beyond ϖ_c. However, as the postulated positive ρ-gradient is wholly unrealistic, we can again ignore this mathematical curiosity, and confine attention from now on to the normal solution.

The total rate $-\dot{J}$ of angular momentum transport is given by integrating expression (28) over all the field-lines up to the limiting one (labelled $\bar{\theta}_s$): e.g. over the sphere r_s,

$$-\dot{J} = 2\int_0^{\bar{\theta}_s} \left(\frac{H_r}{H_p}\right)_s \left(-\frac{\beta}{4\pi} H_p\right)_s 2\pi r_s^2 \sin\theta_s d\theta_s \tag{38}$$

$$= 4\pi\Omega_s r_s^4 \int_0^{\bar{\theta}_s} \bar{H} \cos\theta_s \eta \left(\frac{\alpha}{\Omega_s}\right)\left(\frac{\varpi_c}{r_s}\right)^2 \sin\theta_s d\theta_s \tag{39}$$

$$= 4\pi\Omega_s r_s^4 \int_0^{\bar{\theta}_s} \rho_s v_s \left(\frac{\alpha}{\Omega_s}\right)\left(\frac{\varpi_c}{r_s}\right)^2 \frac{\sin\theta_s \cos\theta_s}{(1-\frac{3}{4}\sin^2\theta_s)^{1/2}} d\theta_s. \tag{40}$$

If magnetic stiffening were ignorable, the same degree of mass-loss would yield the angular momentum loss

$$4\pi\Omega_s r_s^4 \int_0^{\bar{\theta}_s} \rho_s v_s \frac{\sin^3\theta_s \cos\theta_s d\theta_s}{(1-\frac{3}{4}\sin^2\theta_s)^{1/2}}. \tag{41}$$

Remembering that if a critical point is not too close to the star, $\alpha/\Omega_s \simeq 1$, we see how the magnetic field effectively extends the surface of the star to that defined by the points C. However, the magnetic field also limits the amount of mass loss by keeping $\bar{\theta}_s$ below $\pi/2$; this effect is important, and will show up in the numerical integrations.

From the definition (23) of the critical point, the continuity equation (33) and the dipole field-lines (11),

$$\frac{\varpi_c^2}{r_s^2} = x_c^3 \sin^2 \theta_s = \frac{2 \sin^2 \theta_s (1 - \frac{3}{4} x_c \sin^2 \theta_s)^{1/2} (1 - \frac{3}{4} \sin^2 \theta_s)^{1/2}}{\left(\dfrac{v_s}{a}\right)\left(\dfrac{v_c}{a}\right)} \zeta, \tag{42}$$

where

$$\zeta = \frac{\bar{H}^2}{8\pi\rho_s a^2} \tag{43}$$

and a is again the coronal sound-speed. Anticipating that v_c/a will usually turn out to be of order unity, and v_s/a either slightly or greatly less than unity, we see how a large ratio of magnetic to thermal energy at the coronal base yields a correspondingly large ratio ϖ_c/r_s for the field-line considered.

A new form of the integral (40) results from substituting (42) and (43); this is

$$-\dot{J} = \frac{(\bar{H} r_s^2)^2}{a} \Omega_s I, \tag{44}$$

where

$$I = \frac{\sqrt{2}}{4} \int_0^{\sin^2 \bar{\theta}_s} \left(\frac{\alpha}{\Omega_s}\right)\left(\frac{S}{U_c}\right) (1 - \tfrac{3}{4} x_c S)^{1/2} dS, \tag{45}$$

U_c is the non-dimensional velocity $v_c/\sqrt{2}a$, and the variable of integration $S \equiv \sin^2 \theta_s$ is used. It will be noted that, the density ρ_s at the coronal base does not enter this form *explicitly*; for although a higher mass-loss along a *given field-line* corresponds to a higher transport of angular momentum at the surface, the higher density yields a smaller region of approximate co-rotation (the point C nearer the star), and the two effects cancel. However, the most useful form for $-\dot{J}$ is given by substituting from equation (43) for $\bar{H}^2$ in terms of ζ:

$$-\dot{J} = 8\pi \rho_s a \Omega_s r_s^4 (\zeta I). \tag{46}$$

The associated rate of mass-loss is

$$-\frac{dM}{dt} = 2 \int_0^{\bar{\theta}_s} \left(\frac{H_r}{H_p}\right)_s \rho_s v_s 2\pi r_s^2 \sin \theta_s d\theta_s \tag{47}$$

$$= 2\sqrt{2}\pi a \rho_s r_s^2 K,$$

with

$$K = \int_0^{\bar{S}} \frac{U_s dS}{(1 - \frac{3}{4} S)^{1/2}}, \quad U_s = \frac{v_s}{\sqrt{2}a}. \tag{48}$$

Both the integrals I and K depend on ζ, principally through the upper limit $\bar{S} = \sin^2 \bar{\theta}_s$. The ratio

$$\frac{dJ/dt}{(dM/dt)\Omega_s r_s^2} = \frac{2^{3/2}(\xi I)}{K} = \Gamma, \tag{49}$$

say, measures the effect of the magnetic field in increasing the angular momentum carried away by a given amount of mass-loss.

For comparison, consider the angular momentum carried by a spherically symmetric wind, free of magnetic constraint. This can be written variously as

$$\int_0^\pi (2\pi r_s^2 \sin\theta_s d\theta_s)\rho_s v_s(\Omega_s r_s^2 \sin^2\theta_s) = \frac{8\pi}{3}\rho_s v_s r_s^2(\Omega_s r_s^2)$$

$$= (8\pi\rho_s a r_s^4\Omega_s)\left(\frac{1}{3}\frac{v_s}{a}\right) = -\tfrac{2}{3}(\Omega_s r_s^2)\frac{dM}{dt}. \quad (50)$$

The factor Γ is now $2/3$, and the quantity ζI in (46) is replaced by $(v_s/3a)$; v_s/a will be fixed by the spherically symmetric stellar wind theory (cf. Section 4). The essence of the Schatzman process is the increase of Γ, by the action of the magnetic field, over the value $2/3$, or the somewhat greater value appropriate to a rotationally distorted star.

4. *The wind velocity field.* To complete the theory, we now study the component of the equation of motion parallel to the field, in the form of the generalization of Bernoulli's equation for an isothermal rotating system (Freeman & Mestel 1966):

$$\tfrac{1}{2}v_p^2 + \tfrac{1}{2}\Omega^2\varpi^2 + a^2\log\rho - \frac{GM}{r} - \alpha\Omega\varpi^2 = \text{constant on streamlines,} \quad (51)$$

where a is the isothermal sound speed. This is the energy integral in a non-rotating frame, and is derived by taking the scalar product of the vector equation of motion with the total velocity $(\mathbf{v}_p + \mathbf{v}_t)$. The new terms are $\frac{1}{2}\Omega^2\varpi^2$, the kinetic energy of rotation, and $\alpha\Omega\varpi^2$, representing the work done on the streaming gas by the magnetic torque (the velocity and magnetic vectors are parallel only when $\alpha = 0$ —cf. equation (18)). The equation can also be derived from Jacobi's quasi-energy integral for the frame rotating with angular velocity α, in which the velocity $\mathbf{v}_p + (\mathbf{v}_t - \varpi\alpha\mathbf{t})$ is parallel to $\mathbf{H}$. Written in the form

$$\nabla.\left[\rho\mathbf{v}\left(\tfrac{1}{2}v_p^2 + \tfrac{1}{2}\Omega^2\varpi^2 + a^2\log\rho - \frac{GM}{r}\right)\right]$$

$$= \nabla.[\rho\mathbf{v}(\alpha\Omega\varpi^2)] = \nabla.(\rho\kappa\alpha\Omega\varpi^2\mathbf{H}_p)$$

$$= \nabla.\left(\alpha\frac{\varpi H_t}{4\pi}\mathbf{H}_p\right) \qquad \text{by equation (21)}$$

$$= -\nabla.\left[-\frac{1}{4\pi}(\mathbf{v}\times\mathbf{H})\times\mathbf{H}\right] \qquad \text{by (18)}$$

$$= -\nabla.\left[\frac{c}{4\pi}(\mathbf{E}\times\mathbf{H})\right], \quad (52)$$

the equation is recognizable as the relevant form of Poynting's theorem.

Since $\Omega \simeq \alpha$ near the star, and decreases further out, we see that the new terms assist the outward motion—the work done by the magnetic field in increasing the angular momentum of the streaming gas more than compensates for the increase in the kinetic energy of rotation. Thus as already noted, when these terms are important they imply that the energy driving the stellar wind comes not only from the heat supplied at the coronal base, but also from the centrifugal energy of the star. Further, the angular momentum loss rate (46) will now depend on the rotation of the star not only explicitly, but also through the integral I, in that x_c, U_c and (most important of all) the upper limit $\bar{\theta}_s$ will also depend on the instantaneous value of Ω_s.

Following Parker (1963), we introduce non-dimensional quantities as follows:

$$v_p^2 = 2a^2U^2, \quad \frac{\rho}{\rho_s} = z, \quad \frac{r}{r_s} = x,$$

$$\frac{\Omega}{\alpha} = \omega, \quad \frac{\alpha^2 r_s^2}{GM} = \kappa, \quad l = \frac{GM}{r_s a^2}, \quad \kappa l = \frac{\alpha^2 r_s^2}{a^2}. \tag{53}$$

The continuity equation (33) in the dipolar field region can be written

$$\frac{Uzx^3}{(1-\frac{3}{4}x\sin^2\theta_s)^{1/2}} = \frac{U_s}{(1-\frac{3}{4}\sin^2\theta_s)^{1/2}}. \tag{54}$$

The Bernoulli integral (51) becomes

$$U^2+\log z-\frac{l}{x}-\kappa l\sin^2\theta_s x^3\omega(1-\tfrac{1}{2}\omega) = \text{constant}. \tag{55}$$

Elimination of z yields

$$(U^2-\log U)-\left(\frac{l}{x}+3\log x-\tfrac{1}{2}\log(1-\tfrac{3}{4}x\sin^2\theta_s)+\kappa l\sin^2\theta_s x^3\omega(1-\tfrac{1}{2}\omega)\right)$$
$$= U_s^2-\log U_s-(l-\tfrac{1}{2}\log(1-\tfrac{3}{4}\sin^2\theta_s)+\kappa l\sin^2\theta_s\omega_s(1-\tfrac{1}{2}\omega_s)) \equiv A. \tag{56}$$

Beyond C, with the field-lines roughly radial, the continuity equation becomes

$$Uzx^2 = \text{constant}; \tag{57}$$

the Bernoulli integral (51) becomes

$$U^2+\log z-\frac{l}{x}-\kappa l x_c\sin^2\theta_s x^2\omega(1-\tfrac{1}{2}\omega) = \text{constant}, \tag{58}$$

and equation (56) is replaced by

$$(U^2-\log U)-\left(\frac{l}{x}+2\log x+\kappa l x_c\sin^2\theta_s x^2\omega(1-\tfrac{1}{2}\omega)\right) = \text{constant}. \tag{59}$$

Since in a normal solution $\omega x^2 \simeq$ constant beyond C, the rotation terms in equation (59) become effectively constant for $x \gg x_c$. The constants in equations (58) and (59) are fixed by continuity with the equations (54) and (56).

The solutions of equations (56) and (59) are analogous to those of the similar equations (without the terms in ω) discussed by Parker (1963). We again select the 'critical' solutions, with velocities which increase outwards from subsonic values near the star, become sonic at a particular point X_A on each field-line, and are supersonic far from the star and have no discontinuities in their gradients. We have already noted that solutions of this type allow a quasi-steady treatment of the braking problem. For moderate or strong magnetic fields the critical point C turns out to be further out than X_A, so that equation (56) is the relevant one. Further, when determining the constant A in equation (56) it is a good approximation to put $\omega = 1$, so that $\omega(1-\frac{1}{2}\omega) = \frac{1}{2}$. (Even at the Alfvénic point C, the computed values of $\omega(1-\frac{1}{2}\omega)$ are usually near 0·375, and $\omega \to 1$ for $x \ll x_c$). The second bracket in equation (56) is now an explicit function of x only, with a minimum at the point given by

$$\frac{3x_a}{l}+\tfrac{3}{2}\kappa S x_a^4+\frac{3Sx_a^2}{8l(1-\frac{3}{4}x_a S)}-1 = 0, \tag{60}$$

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where again $S \equiv \sin^2 \theta_s$. The function $(U^2 - \log U)$ has a minimum when $U^2 = \frac{1}{2}$, corresponding to sonic flow. A smooth transition from subsonic to supersonic flow occurs if $U^2 = \frac{1}{2}$ at the 'nozzle-point' given by equation (60). A discontinuous jump by means of a shock would violate the condition of entropy increase. This gravitationally determined 'nozzle-point' appeared in studies of gravitational accretion of finite-temperature gas (Bondi 1952; Mestel 1954; Schatzman 1955), before the advent of solar-wind theory.

The constant of integration A is now fixed for each field-line at

$$A = \tfrac{1}{2}(1+\log 2) - \left(\frac{l}{x_a} + 3\log x_a - \tfrac{1}{2}\log\left(1-\tfrac{3}{4}x_a S\right) + \tfrac{1}{2}\kappa l S x_a^3\right), \qquad (61)$$

and the non-dimensional velocity U_s at the coronal base satisfies

$$U_s^2 - \log U_s = A + l\left(1+\frac{\kappa S}{2}\right) - \tfrac{1}{2}\log\left(1-\tfrac{3}{4}S\right). \qquad (62)$$

If x_a from equation (60) is greater than unity, the coronal base is in the subsonic region, and the value of U_s less than $1/\sqrt{2}$ must be taken. For κ and S large enough and l small enough, x_a will fall below unity, and $U_s > 1/\sqrt{2}$. If the 'coronal base' is regarded just as a convenient normalization level within the isothermal region, there is no objection to values of x_a less than unity. However, if we regard $x = 1$ as the level at which the heat input from sub-photospheric waves reaches its maximum, then for $x < 1$ the temperature gradient will not be zero or weakly negative but positive and possibly large, and strictly a Bernoulli equation with a different equation of state should be used; however, this is one of the refinements postponed for later treatment.

At the critical point C, the velocity U_c is given by

$$U_c^2 - \log U_c = \frac{l}{x_c} + 3\log x_c - \tfrac{1}{2}\log\left(1-\tfrac{3}{4}x_c S\right) + x_c^3 \kappa l S \omega_c\left(1-\tfrac{1}{2}\omega_c\right) + A. \qquad (63)$$

The condition (42) can now be written

$$\frac{U_c U_s x_c^3}{\left(1-\tfrac{3}{4}x_c S\right)^{1/2}\left(1-\tfrac{3}{4}S\right)^{1/2}} = \zeta. \qquad (64)$$

From equations (25), (54) and (11),

$$\omega = \frac{\left(1 - \dfrac{U}{U_c}\dfrac{\left(1-\frac{3}{4}x_c S\right)^{1/2}}{\left(1-\frac{3}{4}x S\right)^{1/2}}\right)}{\left(1 - \dfrac{U}{U_c}\dfrac{x^3}{x_c^3}\dfrac{\left(1-\frac{3}{4}x_c S\right)^{1/2}}{\left(1-\frac{3}{4}x S\right)^{1/2}}\right)}, \qquad (65)$$

and

$$\omega(x_c) \equiv \omega_c = \frac{\left(\left(\dfrac{xU'}{U}\right)_c + \dfrac{3x_c S}{8\left(1-\frac{3}{4}x_c S\right)}\right)}{\left(3 + \left(\dfrac{xU'}{U}\right)_c + \dfrac{3x_c S}{8\left(1-\frac{3}{4}x_c S\right)}\right)}. \qquad (66)$$

If U were independent of ω, then U_c and x_c would be given by the simultaneous solution of equations (63) and (64). As it is, U_c from equation (63) depends by equation (66) on U' at x_c. Differentiation of equation (63) would similarly lead to a dependence on $U''(x_c)$, and so on: strictly the solution for U at all points on the field-line depends in a highly non-linear way on the boundary condition at x_c, which is itself not a fixed point but determined simultaneously with U_c and ω_c. To take some account of the variation of ω, we approximate by putting ω near x_c equal to a *constant* value ω_c. The derivative U' at x_c is therefore given by

$$\left(\frac{xU'}{U}\right)_c (2U_c^2 - 1) = -\frac{l}{x_c} + 3 + \frac{3Sx_c}{8(1-\frac{3}{4}x_cS)} + 3\kappa lSx_c^3\omega_c(1-\tfrac{1}{2}\omega_c) \qquad (67)$$

—the term $\kappa lSx_c^4\omega'(x_c)(1-\omega_c)$ has been dropped. In this way, the determination of x_c, U_c and ω_c is reduced to iteration on the algebraic equations (63), (64), (66) and (67). In most cases, ω_c is found to be between 0·4 and 0·6. The use of $\omega = 1$ in equation (60) for x_a does not lead to any great errors. The final expression for the integral I in equation (46) is

$$I = \frac{\sqrt{2}}{4}\int_0^{\bar{S}} \frac{S}{U_c}(1-\tfrac{3}{4}x_cS)^{1/2}\left\{\frac{U_c(1-\frac{3}{4}S)^{1/2} - U_s(1-\frac{3}{4}x_cS)^{1/2}/x_c^3}{U_c(1-\frac{3}{4}S)^{1/2} - U_s(1-\frac{3}{4}x_cS)^{1/2}}\right\} dS. \qquad (68)$$

As in Parker's work, the selection of the critical-type solution is forced on us by the boundary conditions. Near the star the flow must be subsonic—for by continuity, as the density approaches a high photospheric value, the velocity drops, and the equation of motion reduces effectively to the equation (13) of hydrostatic support along the field-lines. A flow which starts with velocities too low for the sound speed to be reached at x_a will remain subsonic, with the velocity decreasing outwards from the maximum attained at x_a; z would be given by equation (55) with U^2 ultimately negligible. If l is not too large, then even if $\kappa = 0$ this is sufficient to dispose of these cases, for as noted in Section 2, the pressure far from the star will again be too great for the interstellar pressure to balance. We shall, however, also be interested in cases with l possibly as large as 25; the pressure far from the star in a non-rotating corona in hydrostatic equilibrium would by equation (14) be $\simeq \rho_s e^{-25}a^2$, which may very well be less than the interstellar pressure. However, with κ not too small, the centrifugal term in equation (55) forces up the density far from the star. Since $4\pi\eta^2/\rho$ would decrease outwards, the Alfvénic point C would never be reached in such solutions, ω would stay near unity, and ρ would be given closely by equation (14). Again such a state of equilibrium would require a wall at infinity; in reality the gas would expand outwards to reach the critical solution, the only one which satisfies the boundary condition near and far from the star.

In our computations we shall in fact include cases with κ so small and l so large that it might be more realistic to assume hydrostatic equilibrium maintained by a background pressure rather than a stellar wind. However, these cases are included to show how the loss of angular momentum decreases sharply with decreasing κ when l is large; the difference between the loss rate computed and the strictly zero rate in hydrostatic equilibrium is of little significance.

5. *Numerical results.* The problem as formulated depends on the three parameters:

$$l = \frac{GM}{r_s a^2} = \text{the ratio of gravitational energy to thermal,}$$

$$\zeta = \frac{\bar{H}^2}{8\pi\rho_s a^2} = \text{the ratio of polar magnetic energy to thermal,}$$

$$\kappa = \frac{\Omega_s^2 r_s^3}{GM} = \text{the ratio of centrifugal force to gravity,}$$

all three referring to the coronal base r_S. Solutions were found for $3 \leqslant l \leqslant 25$, $0 \leqslant \kappa \leqslant 5$, $2 \leqslant \zeta \leqslant 200$. Table I lists the principal integral properties of the solutions, the last three columns being proportional respectively to the angular momentum-loss rate, the mass-loss rate, and the angular momentum loss per unit mass-loss.

Table II contains some details of eight representative solutions. The last column defines the limiting field-line, and the others give (in non-dimensional form) the positions of the sonic and Alfvénic points, and the velocities at the coronal base and the Alfvénic points, on respectively the axial and the limiting field-lines.

Several qualitative features emerge from the computations.

(i) With l and κ fixed, the product ζI is remarkably *in*sensitive to changes in ζ, so that by equation (46), if ρ_S also stays constant, then once ζ greatly exceeds unity, further increase in the magnetic field strength changes the angular momentum transport by at most a factor 2 or 3. The reason for this unexpected behaviour is that although an increase in ζ causes the Alfvénic point on a given field-streamline to move out, it also cuts down the number of field-lines in the wind zone, and the two effects nearly cancel. As already emphasized, this result depends on the assumption that the whole corona has a common temperature. Excess heating of the dead zone will move field-lines into the wind zone, so that the tabulated values of ζI are probably lower limits.

(ii) As ζ decreases through unity, the Alfvénic points move below the coronal base, so that magnetic stiffening disappears. If centrifugal forces are ignorable, the flow will be spherically-symmetric. Provided the chromosphere and the corona below r_S are forced to co-rotate with the star (either by magnetic torques or turbulent friction), then from equation (50) the angular momentum transport is given by equation (46) with ζI replaced by $v_S/3a$ (tabulated in Table III), and with Ω_S again the angular velocity of the *star*. In a ' hot ' corona—e.g. with $l = 3$—the effect of magnetic stiffening above r_S is shown by (ζI) being greater than $v_S/3a$ by

TABLE I

l	κ	ζ	ζI	K	Γ
3	5	200	1·5	0·91	4·5
		100	1·5	1·3	3·3
		50	1·7	1·95	2·5
3	2	200	1·1	0·49	6·3
		100	1·1	0·67	4·4
		50	1·1	0·92	3·3
		20	1·3	1·6	2·3
		10	1·05	1·7	1·7
3	1	200	1·1	0·34	9·1
		100	1·0	0·45	6·5
		50	1·0	0·60	4·75
		20	1·1	0·92	3·1
		10	1·4	1·4	2·75
3	0·1	200	1·0	0·21	14·0
		100	0·91	0·27	9·5
		50	0·81	0·35	6·65
		20	0·75	0·50	4·2
		10	0·75	0·68	3·1
		5	0·84	1·1	2·2

TABLE I (*continued*)

l	κ	ζ	ζI	K	Γ
3	0	200	1·0	0·19	15
		100	0·89	0·25	10
		50	0·79	0·32	7·1
		20	0·70	0·45	4·45
		10	0·69	0·61	3·2
		5	0·75	0·9	2·4
		2	0·49	0·91	1·5
5	5	200	1·5	1	4
		100	1·6	1·6	2·9
		50	1·9	2·6	2·1
		20	1·1	2·1	1·4
5	2	200	0·91	0·42	6·2
		100	0·91	0·59	4·3
		50	0·95	0·87	3·1
		20	1·2	1·6	2·1
		10	0·84	1·5	1·5
5	1	200	0·74	0·22	9·5
		100	0·71	0·30	6·7
		50	0·70	0·42	4·7
		20	0·76	0·68	3·1
		10	0·94	1·05	2·5
5	0·1	200	0·51	0·077	19
		100	0·45	0·098	13
		50	0·4	0·13	8·9
		20	0·39	0·18	5·6
		10	0·34	0·24	4·0
5	0	200	0·49	0·058	24
		100	0·41	0·073	16
		50	0·35	0·093	10
		20	0·30	0·13	6·5
		10	0·275	0·17	4·55
7·5	5	200	1·6	1·3	3·5
		100	1·8	2·0	2·5
		50	2·0	3·0	1·9
		20	0·86	1·95	1·2
7·5	2	200	0·81	0·38	6·0
		100	0·84	0·58	4·1
		50	0·91	0·9	2·9
		20	1·2	1·75	1·9
		10	0·7	1·4	1·4
7·5	1	200	0·53	0·14	10·5
		100	0·525	0·21	7·1
		50	0·54	0·31	4·9
		20	0·60	0·55	3·1
		10	0·75	0·88	2·4
7·5	0·1	200	0·21	0·019	32
		100	0·19	0·024	22
		50	0·17	0·032	15
		20	0·15	0·048	8·9
		10	0·14	0·067	6·1

TABLE I (*continued*)

l	κ	ζ	ζI	K	Γ
7·5	0	200	0·18	0·0094	53
		100	0·15	0·012	35
		50	0·12	0·015	23
		20	0·11	0·0205	14
		10	0·088	0·0265	9·4
10	5	200	1·8	1·5	3·1
		100	2·1	2·6	2·3
		50	1·8	3·0	1·7
		20	0·74	1·9	1·1
10	2	200	0·76	0·375	5·4
		100	0·81	0·6	3·9
		50	0·92	0·96	2·7
		20	1·1	1·8	1·8
		10	0·66	1·4	1·3
10	1	200	0·4	0·097	12
		100	0·42	0·16	7·6
		50	0·45	0·25	5·0
		20	0·52	0·48	3·1
		10	0·66	0·8	2·3
10	0·1	200	0·075	$3·5 \times 10^{-3}$	61
		100	0·070	$4·8 \times 10^{-3}$	41
		50	0·065	$6·65 \times 10^{-3}$	27·5
		20	0·060	$1·1 \times 10^{-2}$	16
		10	0·058	$1·55 \times 10^{-2}$	10·5
10	0	200	0·055	$1·1 \times 10^{-3}$	147
		100	0·046	$1·3 \times 10^{-3}$	97
		50	0·038	$1·7 \times 10^{-3}$	61
		20	0·03	$2·3 \times 10^{-3}$	37
		10	0·025	$2·9 \times 10^{-3}$	25
12·5	1	200	0·31	0·069	13
		150	0·33	0·087	11
		100	0·35	0·12	8
		50	0·39	0·21	5·1
12·5	0·5	200	0·11	0·013	31
		150	0·12	0·013	25
		100	0·12	0·019	19
		50	0·14	0·035	11
12·5	0·25	200	0·047	2×10^{-3}	66
		150	0·047	$2·4 \times 10^{-3}$	55
		100	0·048	$3·3 \times 10^{-3}$	41·5
		50	0·051	$5·6 \times 10^{-3}$	26
12·5	0·1	200	0·024	$5·2 \times 10^{-4}$	130
		150	0·023	$5·9 \times 10^{-4}$	110
		100	0·022	$7·4 \times 10^{-4}$	86
		50	0·021	$1·1 \times 10^{-3}$	57

TABLE I (*continued*)

l	κ	ζ	ζI	K	Γ
15	5	200	2·1	2·2	2·7
		100	2·7	3·8	2·0
		50	1·5	2·9	1·5
15	2	200	0·73	0·39	5·3
		100	0·82	0·67	3·5
		50	0·98	1·15	2·4
		20	1.0	1·7	1·6
15	1	200	0·25	0·049	14
		150	0·26	0·065	11·5
		100	0·29	0·0975	8·4
		50	0·34	0·187	5·2
		20	0·43	0·414	3·0
15	0·5	200	0·054	$3{\cdot}2\times10^{-3}$	48
		150	0·059	$4{\cdot}4\times10^{-3}$	38
		100	0·067	$7{\cdot}05\times10^{-3}$	27
		50	0·083	$1{\cdot}6\times10^{-2}$	15
15	0·25	200	0·016	$3{\cdot}3\times10^{-4}$	134
		150	0·016	$4{\cdot}3\times10^{-4}$	108
		100	0·017	$5{\cdot}9\times10^{-4}$	81
		50	0·019	$1{\cdot}15\times10^{-3}$	47
15	0·1	200	$6{\cdot}45\times10^{-3}$	$6{\cdot}2\times10^{-5}$	294
		150	$6{\cdot}3\times10^{-3}$	$7{\cdot}2\times10^{-5}$	248
		100	$5{\cdot}9\times10^{-3}$	$8{\cdot}5\times10^{-5}$	197
		50	6×10^{-3}	$1{\cdot}3\times10^{-4}$	128
20	1	200	0·145	0·022	18
		150	0·17	0·035	14
		100	0·21	0·063	9·3
		50	0·28	0·15	5·3
20	0·5	200	$6{\cdot}2\times10^{-3}$	$9{\cdot}6\times10^{-5}$	183
		150	$7{\cdot}4\times10^{-3}$	$1{\cdot}5\times10^{-4}$	136
		100	0·01	$3{\cdot}4\times10^{-4}$	84
		50	0·019	$1{\cdot}5\times10^{-3}$	35
20	0·1	200	$2{\cdot}6\times10^{-4}$	$4{\cdot}6\times10^{-7}$	$1{\cdot}6\times10^{3}$
		150	$2{\cdot}4\times10^{-4}$	$4{\cdot}8\times10^{-7}$	$1{\cdot}4\times10^{3}$
		100	3×10^{-4}	$7{\cdot}6\times10^{-7}$	$1{\cdot}1\times10^{3}$
		50	$3{\cdot}3\times10^{-4}$	$1{\cdot}2\times10^{-6}$	780
25	1	200	$6{\cdot}6\times10^{-2}$	$6{\cdot}9\times10^{-3}$	27
		150	$9{\cdot}8\times10^{-2}$	$1{\cdot}6\times10^{-2}$	17
		100	0·15	4×10^{-2}	10
		50	0·23	0·125	5·3
25	0·5	200	$1{\cdot}9\times10^{-4}$	$4{\cdot}3\times10^{-7}$	$1{\cdot}3\times10^{3}$
		150	$2{\cdot}7\times10^{-4}$	8×10^{-7}	$9{\cdot}1\times10^{2}$
		100	$3{\cdot}3\times10^{-4}$	$1{\cdot}3\times10^{-6}$	720
		50	$6{\cdot}8\times10^{-4}$	$6{\cdot}5\times10^{-6}$	290
25	0·25	200	$5{\cdot}5\times10^{-5}$	$5{\cdot}3\times10^{-8}$	$2{\cdot}9\times10^{3}$
		150	$5{\cdot}1\times10^{-5}$	$5{\cdot}5\times10^{-8}$	$2{\cdot}7\times10^{3}$

TABLE II

ζ	$(x_a)_0$	$(x_a)_{\bar{S}}$	$(x_c)_0$	$(x_c)_{\bar{S}}$	$(u_s)_0$	$(u_s)_{\bar{S}}$	$(u_c)_0$	$(u_c)_{\bar{S}}$	$\bar{S} = \sin^2 \bar{\theta}_s$
				$l = 3$	$\kappa = 1$				
200	1	0·78	5·2	2·85	0·71	1·05	2·0	3·5	0·35
100	1	0·75	4·2	2·3	0·71	1·1	1·9	3·0	0·43
50	1	0·73	3·45	1·85	0·71	1·2	1·7	2·5	0·54
20	1	0·69	2·65	1·4	0·71	1·3	1·5	1·9	0·72
				$l = 3$	$\kappa = 0$				
200	1	0·96	5·2	4·0	0·71	0·74	2·0	2·0	0·255
100	1	0·95	4·2	3·2	0·71	0·75	1·9	1·85	0·32
50	1	0·94	3·45	2·5	0·71	0·77	1·7	1·725	0·40
20	1	0·91	2·65	1·85	0·71	0·80	1·5	1·47	0·53
10	1	0·88	2·2	1·5	0·71	0·84	1·4	1·3	0·68
				$l = 10$	$\kappa = 1$				
200	3·3	1·1	19	3·1	0·0145	0·52	2·1	5·8	0·33
100	3·3	1·0	15	2·4	0·0145	0·65	1·9	4·5	0·41
50	3·3	0·97	12·5	1·9	0·0145	0·8	1·8	3·4	0·52
20	3·3	0·9	9·6	1·4	0·0145	1·05	1·6	2·25	0·7
10	3·3	0·85	7·8	1·1	0·0145	1·3	1·4	1·5	0·87
				$l = 10$	$\kappa = 0·1$				
200	3·3	2·1	19	7·9	0·0145	0·038	2·1	5·1	0·13
100	3·3	2·0	15	6·4	0·0145	0·043	1·9	4·2	0·16
50	3·3	1·9	12·5	5·15	0·0145	0·05	1·8	3·4	0·195
20	3·3	1·7	9·55	3·85	0·0145	0·062	1·6	2·55	0·26
10	3·3	1·7	7·85	3·1	0·0145	0·074	1·4	2·0	0·32
				$l = 10$	$\kappa = 0$				
200	3·3	3·2	19	14·5	0·0145	0·015	2·1	2·065	0·07
100	3·3	3·2	15	11·5	0·0145	0·016	1·9	1·9	0·087
50	3·3	3·15	12·5	9·6	0·0145	0·016	1·8	1·8	0·107
20	3·3	3·10	9·55	7·15	0·0145	0·017	1·6	1·6	0·145
10	3·3	3·0	7·85	5·7	0·0145	0·0175	1·43	1·41	0·175
				$l = 12·5$	$\kappa = 0·1$				
200	4·2	1·1	33	3·1	0·0023	0·44	2·3	6·5	0·32
150	4·2	1·1	31	2·8	0·0023	0·49	2·25	5·8	0·355
100	4·2	1·0	27	2·4	0·0023	0·58	2·2	4·9	0·41
50	4·2	0·99	22	1·9	0·0023	0·75	2·0	3·7	0·52
				$l = 12·5$	$\kappa = 0·5$				
200	4·2	1·4	33	4·9	0·0023	0·10	2·3	7·45	0·205
150	4·2	1·4	31	4·4	0·0023	0·12	2·25	6·6	0·230
100	4·2	1·3	27	3·8	0·0023	0·15	2·2	5·6	0·265
50	4·2	1·3	22	2·9	0·0023	0·21	2·0	4·2	0·34
				$l = 12·5$	$\kappa = 0·1$				
200	4·2	2·4	33	11·6	0·0023	0·0088	2·3	7·6	0·09
150	4·2	2·3	31	10·0	0·0023	0·0093	2·25	7·2	0·098
100	4·2	2·3	27	9·0	0·0023	0·011	2·2	6·3	0·112
50	4·2	2·15	22	7·2	0·0023	0·0125	2·0	5·1	0·14

significant (though not very large) ratios: varying from about 17 for the extreme case $l = 3$, $\kappa = 5$, $\zeta = 50$; to near 10 for $\zeta \gg 1$, and $0 \cdot 1 \leqslant \kappa < 1$, and for $\kappa = 0$, $\zeta = 200$; and down to $\simeq 5$ for the case $\kappa = 0$, $\zeta = 2$. But if the corona is a little cooler—$l = 10$—then $v_S/3a$ is well below ζI for all values of κ and ζ: by a factor 15 for $\kappa = 0$, $\zeta = 10$; by 32 for $\kappa = 0$, $\zeta = 200$; and by 240–400 for $\kappa = 1$ and different values of ζ. Table III shows how $v_S/3a$ becomes negligibly small as l increases further by no more than a factor 1·5. And if the angular velocity field of the gas below the corona is in fact unconstrained by either magnetic or frictional forces, so that the gas reaches the coronal base (by spherical flow) with each element conserving its angular momentum, then Ω_S is less than the stellar rotation by the factor $(R/r_S)^2$ when R is the photospheric radius—a further reduction by $\frac{1}{2}$ if $r_S \simeq 1 \cdot 4R$, as in the solar case.

TABLE III

l	3	5	7·5	10	12·5	15
$\frac{1}{3}\frac{v_S}{a}$	0·096	0·064	$1 \cdot 2 \times 10^{-2}$	$1 \cdot 7 \times 10^{-3}$	$2 \cdot 2 \times 10^{-4}$	$2 \cdot 6 \times 10^{-5}$

(iii) With l and κ fixed, the ratio Γ *increases* with increasing ζ. This is a consequence of the insensitivity of ζI to increase in ζ, and of the steady decrease in $\bar{S}$. Thus if ρ_S and a^2 can be taken as constants as ζ varies (though this is by on means obvious), then increase of ζ through values well above unity enables roughly the *same amount* of angular momentum-loss to be achieved by means of a correspondingly *smaller* amount of mass-loss.

(iv) In a hot corona ($l = 3$ or less), the product ζI does not change much, for fixed ζ, as κ decreases from 5, through unity, to zero. Although the number of field-lines in the wind zone decreases somewhat with decreasing κ (cf. Table II), the reduced centrifugal force enables the gas to reach greater distances before its velocity becomes Alfvénic: the decrease in the upper limit $\bar{S}$ is nearly off-set by the increase of ϖ_c in equation (40), equivalent to a decrease of U_c in equation (45). The ratio Γ does increase noticeably with decrease in κ—again because the slowly-varying angular momentum-loss is achieved by a smaller wind zone. However, when l is somewhat larger—the coronal temperature lower by no more than a factor 4 or 5—ζI is highly sensitive to decrease in κ. Thus when $\kappa = 5$, ζI for $l = 15$ is comparable with or greater than ζI for $l = 3$; assuming ρ_S is unchanged, but allowing for the decrease in the sound speed a, the angular momentum-loss rate is smaller by at most a factor 2 than that for $l = 3$. For $\kappa = 1$, $l = 15$, ζI is lower by a factor 3 or 4, and $-\dot{J}$ by 7–11. When $\kappa = 0 \cdot 1$—Ω_S lower by only a factor $\simeq 3$—ζI is lower still by factors of 40 or 50. Further reduction in κ makes little difference, for when $\kappa < 0 \cdot 1$, the overall effect of the centrifugal force is small compared with the isothermal pressure corresponding to $l = 15$. A further increase of l to 20 shows the dependence on κ still more strikingly (though the numerical integrations are now not very accurate): ζI decreases from 0·15 to 3×10^{-4} as κ decreases from 1 to 0·1. Thus in a cool but rapidly rotating corona, it is the magnetically-controlled centrifugal forces that keep a strong wind blowing, with a total angular momentum transport that though somewhat less than the rate for $l = 3$, is not negligible. But a modest reduction in the centrifugal force at the coronal base cuts down the wind force and yields a much reduced angular momentum transport. These results illustrate strikingly the concept of a ' centrifugal wind ', discussed in Section 2.

6. *Applications to single stars.* A complete theory of the formation and expansion of a stellar corona will predict the whole density–temperature–velocity field, so that both the rate of mass-loss $-dM/dt$ and angular momentum loss $-dJ/dt$ will be known independently. Thus assuming the isothermal model is realistic, we require the theory to fix both the temperature parameter l and the density ρ_s at the coronal base r_s, including any dependence on latitude, in terms of the structure of the star, its magnetic field and its rotation. As remarked in Section 2, we bypass this problem by taking l as a free parameter. We also follow Schatzman (1962) and eliminate ρ_s by expressing $-\dot{J}$ in terms of $-\dot{M}$, so bringing out the effect of the the magnetic field in increasing the specific angular momentum-loss, and replacing ρ_s by $-\dot{M}$; we can then check whether the amount of mass-loss required by theory is observationally plausible.

For a single star of mass M, photospheric radius R, and moment of inertia kMR^2, the angular momentum equation is

$$\frac{d}{dt}(kM\Omega R^2) = \Gamma\alpha^2\Omega R^2 \frac{dM}{dt}, \tag{69}$$

where the suffix s has been dropped, and the radius r_s is written as αR, with α greater than unity. (If instead we write the moment of inertia of the star in terms of r_s, k would be replaced by $k(R/r_s)^2 = k/\alpha^2$, and we arrive at the same equation (69)). The parameter $\Gamma\alpha^2$ will clearly be unaltered (for given l, κ and ζ) whether r_s is taken as the coronal base, or any other level within the isothermal corona. Observation suggests that the base is at a radius $r_s \simeq 10^{11}$ cm, with $\alpha = 1{\cdot}4$ (Parker 1963); however, for some illustrative purposes it is convenient to adopt the somewhat larger value $\alpha = 2$, with r_s in the solar case located in the 'middle corona' (van de Hulst 1953).

Equation (69) integrates to

$$\Omega R^2 \propto M^{(\alpha^2\Gamma/k-1)}. \tag{70}$$

(Even though it is only the least massive main sequence stars and stars on the Hayashi track that are fully convective, we assume that coupling between a radiative core and a convective envelope is sufficient for uniform Ω to be an adequate approximation.) The power of M is not proportional to Γ because of the factor M appearing in the left-hand side of equation (64), but is nearly so if $\alpha^2\Gamma/k \gg 1$. For a star in thermal equilibrium (e.g. on the main sequence) R is constant, and equation (70) is a straight equation for the angular velocity Ω, or equivalently for the angular momentum J, yielding for the latter an e-folding time

$$\tau_J = \frac{\tau_M k}{\Gamma\alpha^2}, \tag{71}$$

where τ_M is the e-folding time for mass-loss.

It is tempting to apply the present theory to the Sun. This amounts to assuming that the interplanetary magnetic field is the extension by the solar wind of the dipolar field emanating from the poles, estimated to have a photospheric strength of $\simeq 1$ gauss. At our conventional base $r_s = 2R_\odot$ the density in the polar regions is $\simeq 5/3 \times 10^{-19}$ g at sunspot minimum (Parker 1963, p. 17). With the coronal gas fully ionized hydrogen at 2×10^6 °K, $l = 3$ and $\zeta \simeq 11{\cdot}5$. Since the solar rotation is so low, we can use the solution with $\kappa = 0$. From the Tables for $l = 3$, $\kappa = 0$ we find that $\zeta \simeq 11{\cdot}5$ corresponds to $\Gamma \simeq 3{\cdot}4$ and $\bar{S} \simeq 0{\cdot}66$, and that the Alfvénic points are no further out than $x_c \simeq 2{\cdot}3$. The rotational velocity at r_E (= one astronomical unit = $1{\cdot}5 \times 10^{13}$ cm) is then at the most $\Omega_s r_c{}^2/r_E \simeq 0{\cdot}2$ km/s, much below the estimates of 4–9 km/s deduced from observation (Brandt 1967). The predicted density and velocity at r_E are $\simeq 5 m_M/\text{cm}^3$ and 6×10^7 cm/s, not too different from the observed values, and yielding an e-folding time for mass-loss of $\simeq 5 \times 10^{13}$ years. An approximate estimate of k is found by adopting a stellar density law $\rho = \rho_c \exp(-r^2/\lambda^2 R^2)$; normalization by $\rho_c/\bar{\rho} \equiv 4\pi R^3 \rho_c/3M = 100$ fixes $\lambda \simeq 0{\cdot}196$ and $k \simeq 3{\cdot}85 \times 10^{-2}$. Thus we arrive at an e-folding time for angular momentum loss of $\simeq 1{\cdot}4 \times 10^{11}$ years, an order of magnitude longer than the estimates of Weber & Davis (1967) and of Brandt (1967). Further, the present model predicts an interplanetary poloidal field H_p at r_E of $(1/8)(1/2{\cdot}3)^3(2{\cdot}3/100)^2 \simeq 5{\cdot}5 \times 10^{-6}$ gauss. From equations (32) and (19) the associated toroidal field is $\simeq (r_E \Omega_s/v_p) H_p \simeq 0{\cdot}7 H_p$ (as compared with the observational ‘ fire-hose angle ’ $\tan^{-1}|H_\phi/H_p| \simeq 45°$). Thus the total predicted field at the Earth is well below the observed strength of 5×10^{-5} (the value assumed by the above authors). An increase of the photospheric polar-field to two gauss still yields Alfvénic points that are comparatively near the Sun, leading to a long braking time and a rotational velocity and magnetic field-strength near the Earth that disagree with observation. The conclusion is that the lines of the interplanetary field near the Earth are not part of the Sun's polar field, but emerge from regions of the solar surface that have somewhat stronger fields than the polar regions. And in fact satellite observations have revealed a striking correlation between both magnitude and sense of the interplanetary field at the Earth, and the local photospheric field at low rather than high latitudes (Wilcox 1966). The observations also seem to confirm the general picture in which some emerging field-lines are strong enough to resist dragging out by the wind, returning instead to the photosphere, though the relationship between the ‘ limiting field-line ’ in these local fields and the Alfvénic points still need to be elucidated. As regards the magnetic braking problem in general, the solar evidence shows not only that stars without strong primeval fields can acquire a large-scale dynamo-built component which contributes to braking, but that a much larger contribution may come from local fields, which presumably also arise as part of the dynamo process, but which may not make an observable contribution to the integrated Zeeman effect.

The next example is of a star contracting towards the main sequence, either down the Hayashi track, with the bulk of the star convective, or on the subsequent Kelvin–Helmholtz-type path, but with the outer layers still strongly convective and so still generating a corona. For a fully convective star, built on the 3/2 polytropic model, $k = 0{\cdot}134$ (Roxburgh 1966). As an illustration, suppose the star to contract by a factor 50, and ask that enough angular momentum be lost for the ratio of centrifugal force to surface gravity to remain constant. Thus

$$\Omega^2 R^3 \propto GM \tag{72}$$

and equations (70) and (72) jointly yield

$$R \propto M^{(2\alpha^2\Gamma/k-3)}, \tag{73}$$

so fixing the amount of mass-loss that will preserve the relation (72) as the star contracts.

If the star had no corona and only a weak magnetic field, any mass emitted would carry with it just the angular momentum of the photospheric levels: $\alpha = 1$,

$\Gamma = 2/3$ (cf. equation (50)), and with $k = 0{\cdot}134$, $\Omega R^2 \propto M^4$. This comparatively low exponent implies that a very large fraction of the mass must be lost if Ω is to be kept within bounds as the star contracts, and in fact (73) becomes $R \propto M^7$, requiring a 40 per cent mass loss as R shrinks by a factor 50.

Now let the fully convective star have a corona and a magnetic field just strong enough to keep the gas beneath the coronal base r_s co-rotating with the star, but unable to influence the stellar wind above r_s, which therefore flows approximately spherically symmetrically from r_s. Again $\Gamma = 2/3$, $k = 0{\cdot}134$, and if $\alpha = 2$, equations (70) and (73) become $\Omega R^2 \propto M^{19}$ and $R \propto M^{37}$ respectively. This is already a substantial gain over the previous case—now only ten per cent loss is required as R shrinks by a factor 50.

Now let the magnetic field be strong enough to control the wind far above the coronal base. Inspection of the tables in Section 5 shows that for a wide parameter range $\Gamma \gg 1$, so that as R shrinks by a factor 50, equation (73) requires a fractional mass-loss

$$\frac{\Delta M}{M} \simeq \frac{3{\cdot}9}{\left(\dfrac{2\alpha^2\Gamma}{k} - 3\right)}. \tag{74}$$

Again taking $\alpha = 2$, $k = 0{\cdot}134$, we have that a one per cent mass loss suffices for Γ no greater than $6{\cdot}5$, and correspondingly smaller percentages with Γ larger. In cases of interest, $2\alpha^2\Gamma/k \gg 3$ so that $\Delta M\Gamma \simeq$ constant—the mass required for the prescribed condition (72) varies inversely with the specific angular-momentum-loss parameter Γ. But we have seen that for $\zeta \gg 1$, the actual rate of angular momentum loss is only weakly dependent on ζ ($\zeta I \simeq$ constant for κ and l fixed); a change in ζ does in fact lead to approximately inverse variation in Γ and $-dM/dt$. Hence we conclude that if e.g. the condition (72) is satisfied during the Hayashi descent by a star with one particular surface field, it will also be approximately satisfied if the field is imagined altered, subject only to the conditions $\zeta \gg 1$, and κ and l fixed.

7. *Application to a binary system.* The fragmentation of a gas cloud into a cluster of proto-stars is likely to yield some sub-systems consisting of a pair of proto-stars that are much more closely bound to each other than to the rest of the cluster. If the mutual distance d between the two components is comparable with the sum of their radii, we would describe the pair as forming a ' close binary system ' of proto-stars. However, the individual stars will contract further by factors of 20–50 as they move along the Hayashi and Henyey-type curves towards the main sequence; they will therefore not form a main sequence close binary system unless some process removes enough orbital angular momentum for d to decrease by a similar factor.

A strong, magnetically-controlled stellar wind blowing from a component of a binary system will tend to bring the angular velocity of spin Ω_s below the orbital angular velocity $\bar{\Omega}$. But if there is strong coupling between the spin and orbital motions, tending to synchronize the two angular velocities, the effect will be off-set by a feeding-in of spin angular momentum at the expense of the orbital angular momentum, implying a steady reduction in the mutual distance d. We have noted in the Introduction that contact binaries of W Ursae Majoris type rotate rapidly, in spite of their being late-type stars. Rather than having to find a reason why these stars have not suffered the same braking as single stars of the same type, it seems preferable to argue that magnetic braking has occurred, but that the stars have been re-supplied with angular momentum: their rapid rotations are then just an inevitable consequence of the process by which they have remained components of a ' close ' binary system, in spite of their contraction to the main sequence.*

For simplicity, consider two equal stars of mass M, moving in circular orbits, so that

$$\bar{\Omega}^2 d^3 = 2GM, \tag{75}$$

and the total orbital angular momentum is

$$2M\bar{\Omega}\left(\frac{d}{2}\right)^2 = \left(\frac{GM^3 d}{2}\right)^{1/2}. \tag{76}$$

Although the wind problem will clearly be more complicated in a close binary system, we shall continue using the formula (69) for the angular momentum loss-rate from each star. Provided close synchronization is maintained, we may write $\Omega_s \simeq \bar{\Omega}$ and the equation to the variation of d is

$$\begin{aligned}\frac{d}{dt}\left[\frac{GM^3 d}{2}\right]^{1/2} &= 2\Gamma\alpha^2\Omega_s R^2 \dot{M} \\ &= 2\Gamma\alpha^2(2GM)^{1/2}\left(\frac{R}{d}\right)^2 d^{1/2}\dot{M}.\end{aligned} \tag{77}$$

(The contribution of the spin angular momentum is small, because of the central condensation that yields $k = 0{\cdot}134$ or less, and so is neglected in equation (77).) If the system does remain a contact binary, $R/d \simeq \frac{1}{2}$, and equation (77) reduces to

$$d \propto M^\lambda \tag{78}$$

$$\lambda = 2\Gamma\alpha^2 - 3. \tag{79}$$

The index λ is less than the corresponding one in equation (73), because of the need to dispose of much more angular momentum in this case. Thus if d is to contract by a factor 50, parameters $\Gamma = 10$, $\alpha = 2$ require a mass-loss of five per cent from each star; $\Gamma = 5$, of about ten per cent.

The computation assumes that the time for synchronization of Ω_s and $\bar{\Omega}$ is always short compared with the time for the radius of each star to decrease substantially. The classical process of synchronization is tidal friction (Darwin 1908). Under the mutual gravitational field each star is distorted from a spherical into a prolate shape. If Ω_s is equal to $\bar{\Omega}$, then as seen from a frame rotating with angular velocity $\bar{\Omega}$, a steady state clearly exists with each prolate star pointing towards the other, and with no internal motions. Provided all dissipative processes can be ignored, there again exists a steady state symmetrical about the instantaneous line of centres even when $\Omega_s \neq \bar{\Omega}$, but it is now maintained by internal motions that compensate for the difference between $\bar{\Omega}$ and Ω_s—each element of gas performs an oscillation of period $\pi/|\bar{\Omega} - \Omega_s|$. When dissipation is included, the orbital and internal motions become out of phase, and the density field ceases to be symmetric about the line of centres. The consequent mutual gravitational torque acts so as to equalize $\bar{\Omega}$ and Ω_s, interchanging angular momentum of spin and orbital motion. In the Earth–Moon system, tidal friction is slowly bringing the ' day ' and the ' month ' into equality, with the Moon moving slowly away as it gains orbital angular momentum from the spin angular momentum of the Earth. In our problem,

* A similar scheme has been proposed independently by Su-Shu Huang (1966).

the flow of angular momentum is reversed, with the mutual distance d being reduced as $\bar{\Omega}$ and Ω_s are equalized.

The essence of the process is the destruction of the energy of the tidal motions within each star. A simple first-order estimate of the dissipation rate is found by first constructing the tidal velocity field within each star as if there were no dissipative forces, and then computing the frictional drag on this motion. In a radiative zone, ordinary radiative or molecular viscosity is negligibly small. The departures from adiabaticity due to finite radiative conductivity are more important, as in the theory of stellar stability. However, in a star that is fully or largely convective, the turbulence provides a much more powerful dissipative mechanism, with an effective coefficient of viscosity.

$$\mu_t = \rho v_t \lambda, \tag{80}$$

where λ is the mixing length, and v_t the mean turbulent velocity, given in terms of the luminosity L, radius R and mass M by

$$v_t^3 \simeq \left(\frac{2LR}{M}\right)\left(\frac{\lambda}{R}\right). \tag{81}$$

It is found (Mestel 1966) that the time for a spin Ω_s much less than $\bar{\Omega}$ to be increased to $\simeq 0{\cdot}9\,\bar{\Omega}$ is about $(d/R)^6$ years. (This strong dependence on d/R is characteristic of the tidal friction process, because the tidal velocities are $\propto (R/d)^3$). The time spent in the Hayashi phase is about 10^6 years (Iben 1965). Thus whereas contact binaries will stay synchronized, this gravitational coupling will be too slow if $d/R \gg 10$: once the components have succeeded in contracting substantially without a comparable reduction in d, then subsequent magnetic braking will slow up the spin of the individual stars, without causing any further mutual approach. Equally, if magnetic braking becomes small, increase of spin due to subsequent contraction of individual stars will persist, as the spin–orbit coupling is too weak to transfer angular momentum to the orbital motion.

An alternative and possibly more important cause of synchronization is a magnetic field linking the two stars. Such a system could arise naturally in a gas cloud with the magnetic and angular momentum vectors nearly perpendicular (Mestel 1965). Magnetic braking of the individual stars can again take place, (though it is not so easy to analyse as in the axisymmetric case—cf. Mestel 1968); but now there is immediate magnetic coupling between orbital and spin motion. The same turbulent friction will act so as to damp out the resulting hydromagnetic oscillations, so that as long as the magnetic linkage persists, a rapid enough angular momentum loss can again cause a secular decrease in the distance between the two stars.

8. *Comments*. The illustrations given show that the principal aim of the Schatzman-type process is attained: with a degree of mass-loss well within the range deduced from observation (Kuhi 1964, 1966) of pre-main sequence (T Tauri) stars, the corresponding angular momentum-loss can be greatly increased by magnetic control. But it must again be emphasized that the equations (69) and (77) include only part of what is demanded of a complete theory. The results tabulated show what happens (as functions of l, κ and ζ) if a given amount of mass-loss is assumed, whereas the ultimate theory will determine ρ_s and so also the mass-loss rate as part of the density-temperature structure of the expanding corona.

However, we can check that the magnetic fields demanded by the applications of Sections 6 and 7 are not unreasonable. If a star loses a fraction $\Delta M/M$ of its mass in a time τ, then the coronal base density ρ_s is given from equation (47) by

$$\rho_s = \left(\frac{\Delta M}{M}\Big/\tau\right)(M/2^{3/2}\pi a r_s^2 K). \tag{82}$$

The field $\bar{H}$ is then given by $\bar{H}^2 = 8\pi\rho_s a^2\zeta$, with the condition that $\zeta \gg 1$. Thus

$$\bar{H}^2 r_s^4 = \left(\frac{\Delta M}{M}\Big/\tau\right)\left(\frac{8GM^3 r_s^3}{l}\right)^{1/2}\left(\frac{\zeta}{K}\right) \tag{83}$$

$$= 3{\cdot}8\times 10^{48}\left(\frac{\Delta M}{M}\Big/\tau_y\right)\left(\frac{\bar{M}^3\bar{r}_s^3}{l}\right)^{1/2}\left(\frac{\zeta}{K}\right), \tag{84}$$

where τ_y is a time expressed in multiples of 10^6 years, this being a time characteristic of the Hayashi track for stars of solar order (Iben 1965). The sound speed a has been expressed in terms of l, and $\bar{M} = M/M_\odot$, $\bar{r}_s = r_s/R_\odot$, $R_\odot$ being the solar photospheric radius. Now suppose the field is primeval, being quasi-uniform and of strength $\tilde{H}$ out to a radius r_e, and effectively dipolar beyond r_e; then its energy is $\simeq \frac{1}{3}\tilde{H}^2 r_e^3$, and by the virial theorem we may certainly write

$$\frac{1}{3}\frac{\tilde{H}^2 r_e^4}{r_e} = \eta\frac{GM^2}{R}, \tag{85}$$

where η must be well below unity (e.g. Mestel 1965). Thus at the radius r_s

$$\bar{H}^2 r_s^4 = \tilde{H}^2\left(\frac{r_e}{r_s}\right)^6 r_s^4 = (\tilde{H}^2 r_e^4)\left(\frac{r_e}{r_s}\right)^2$$

$$= 3\eta\frac{GM^2}{\alpha^2}\left(\frac{r_e}{R}\right)^3 = \frac{8\times 10^{59}}{\alpha^2}\eta\bar{M}^2\left(\frac{r_e}{R}\right)^3. \tag{86}$$

If the central condensation of the field is similar to that of the slowest decaying mode discussed by Cowling (1945), $R/r_e \simeq 4$. However, if the field is strong, the flux must be much more centrally condensed, otherwise the magnetic force density will not be less than the gravitational at each individual point of the star (Mestel 1967): it is then more reasonable to use a ratio $R/r_e \simeq 10$ or more.

We now verify that the flux $\bar{H} r_s^2$ required by equation (84) easily satisfies equation (86) with $\eta \ll 1$. Thus take $\alpha = 2$, $\Gamma = 10$, and suppose $\Delta M/M = 0{\cdot}05$ during the Hayashi phase (cf. the discussion of (78) in Section 7). From the Tables, we see that the parameter sets $l = 3$, $\kappa = 0$, $\zeta = 100$, and $l = 3$, $\kappa = 1$, $\zeta = 200$ both yield $\Gamma = 10$ with $K = 0{\cdot}25$ and $0{\cdot}34$ respectively. With $\bar{r}_s = 100$—an extreme value—and with $\tau_y = 1$, $\bar{H}^2 r_s^4$ must be $\simeq 4{\cdot}4\times 10^{53}\bar{M}^{3/2}$ and $6{\cdot}6\times 10^{53}\bar{M}^{3/2}$ respectively. If the corona is very cool, with $l = 25$, but $\kappa = 1$, so that an efficient centrifugal wind is blowing, then if $\zeta = 100$, $\Gamma = 10$ and $K = 0{\cdot}04$, and (84) yields $\bar{H}^2 r_s^4 \simeq 10^{54}\bar{M}^{3/2}$. From (86) with $\alpha = 2$ and $r_e/R = 0{\cdot}1$, $\eta \leqslant 5\times 10^{-4}/\bar{M}^{1/2}$ for all three cases.

As a second example, again suggested by the binary problem, take $\Gamma = 5$ and $\Delta M/M = 0{\cdot}1$. Possible parameter sets include $l = 3$, $\kappa = 1$, $\zeta = 20$, with $K = 0{\cdot}92$; $l = 3$, $\kappa = 0$, $\zeta = 10$, with $K = 0{\cdot}69$; and $l = 25$, $\kappa = 1$, $\zeta = 50$, with $K = 0{\cdot}125$. Again with $\bar{r}_s = 100$, and $\tau_y = 1$, the values of $\bar{H}^2 r_s^4$ from (84) are

$4{\cdot}8\times10^{52}\overline{M}^{3/2}$, $3{\cdot}2\times10^{52}\overline{M}^{3/2}$ and $3\times10^{53}\overline{M}^{3/2}$, and with $\alpha = 2$, (86) yields respectively $10^4\eta\overline{M}^{1/2} = 2{\cdot}4$, $1{\cdot}6$, 15.

Tentative cosmogonical arguments (Mestel 1965) suggest that the energy of a primeval magnetic field may be a much larger fraction of the gravitational energy than the numbers found above. In fact it may be questioned whether a general field of energy less than that of the Hayashi turbulence will survive the tangling during this phase. Opinions on this are divided (e.g. Spitzer 1957; Cowling, in Lüst 1965): but it seems fairly safe to invert the proposition, and argue that only fields of energy at least comparable with that of the Hayashi turbulence will certainly survive. From (81) the mean-squared turbulent velocity is given by

$$v_t^2 \simeq 4\times10^6\left(\frac{\bar{L}\bar{R}}{\overline{M}}\right)^{2/3}, \tag{87}$$

where the mixing length λ is taken as $R/10$ and barred quantities are in solar units. The gravitational energy per unit mass $\simeq GM/R \simeq 2\times10^{15}\overline{M}/\bar{R}$. In the Hayashi phase $\bar{L} = A\bar{R}^2$ approximately, with A somewhat less than unity. Thus if $\bar{R}$ is as much as 40—at the top of the Hayashi track—$v_t^2/(GM/R) \simeq 2\times10^{-4}$; whereas at the lower end, the ratio is $\simeq 2\times10^{-9}$.* If we assume that the star has a field certainly strong enough to withstand the Hayashi turbulence at all epochs, then $\eta \simeq 10^{-4}$ at least. However, as already noted, the main effect of any increase in ζ will be not to change substantially the angular momentum loss rate—cf. the weak ζ-dependence of ζI in (46), for fixed κ and l—but mainly to *increase* Γ, *through a reduction in* $-dM/dt$. It may even be conjectured that the retention by a protostar of a very high primeval magnetic flux may be cosmogonically important not for its increasing the angular momentum loss-rate—we have seen how this flattens to an asymptotic value for $\zeta \gg 1$—but by its *cutting down the mass-loss to a level that is insignificant for stellar evolution.*

However, one should not deduce that a star which enters the Hayashi phase with a ' weak ' primeval field has a subsequent history that is qualitatively different. We may expect the strong turbulence always to generate an expanding corona. Further, even if the primeval flux is permanently washed out of the star, there is every reason to expect a temporary surface field to be generated by a dynamo process, as is presumed to operate in the Sun, driven by the joint effects of convection and rotation (Babcock 1961). Such a field is likely to be weaker than the surface manifestation of a strong internal primeval field, but provided $\zeta \gg 1$, the difference will again show itself more in the amount of mass lost rather than in any change in the rate of angular momentum loss. It is also possible that sometimes a dynamo-built field may yield $\zeta \simeq 1$, so that $-dJ/dt$ is somewhere between the roughly asymptotic value reached when $\zeta \gg 1$, and the much lower value when magnetic stiffening is absent. There is evidence (Deutsch 1965) that some normal AO stars, showing none of the spectral peculiarities characteristic of the observably magnetic stars, have abnormally low rotations as compared with most AO stars. Deutsch interprets them as stars that have evolved into the giant region, lost mass and angular momentum, and then returned to the main sequence. This again suggests that during the giant phase, when the stars acquire extensive sub-photospheric convec-

* Primeval stellar magnetic fields may be classified loosely (Mestel 1967) as (i) ' strong ', with η less than unity, but not by more than one or two orders of magnitude; (ii) ' moderate ', with $\eta \ll 1$, but with the field energy sufficient to withstand the Hayashi turbulence; and (iii) ' weak ', if the field energy is much below the maximum Hayashi turbulent energy.

tion zones, a strong stellar wind blows, and a surface dynamo generates a temporary surface field which again increases the efficiency of angular momentum loss.*

In Sections 6 and 7 we applied the same magnetic braking process to both the single and binary star problems. For the same value of Γ, the binary problem demands a much larger mass-loss, simply because far more angular momentum has to be lost. However, if $-dJ/dt$—measured by $-\Gamma dM/dt$—has essentially the same value whether the star is single or a member of a binary system, we are faced with a dilemma. If the pre-main sequence angular momentum loss rate is just sufficient to account (at least semi-quantitatively) for the angular momenta that stars of different mass possess as they reach the main sequence, then the loss is inadequate for the close binary application. Conversely, if $-dJ/dt$ is sufficient for the binary problem, then comparison of (70) and (78) shows that the angular momentum-loss from individual stars would be far too great: for example, if we took $\gamma = 5$, $\alpha = 2$, and $\Delta M/M = 0{\cdot}1$—a set of values satisfying the requirements of (78) and (79)—then (70) yields a decrease in ΩR^2 by $\simeq 10^{-7}$. If $\Omega^2 R^3/GM$ is of order unity at the beginning of the Hayashi track, then on the main sequence this ratio would have decreased to $\simeq 10^{-12}$, as compared with $\simeq 10^{-5}$ as actually observed in the Sun.

To avoid this difficulty, we need to find a physical reason why $-dJ/dt$ is about an order of magnitude greater for a star in a close binary system than for a single star. There is no obvious reason why the magnetic field-strength should be greater for a binary member, and in any case we have seen that in our models with isothermal coronas, when $\zeta \gg 1$, a further increase in ζ leaves $-dJ/dt$ roughly unaltered. It is also difficult to find reasons why $\rho_s a$ should be larger in a binary system. However, we have seen that at least for cool coronas, ζI depends quite sharply on the rotation of the star, as measured by κ. Inspection of the Tables shows e.g. that for $l = 10$, ζI (or equivalently ΓK) decreases by factors 5–11 as κ is decreased from 1 to $0{\cdot}1$, and by factors $7{\cdot}5$–26 between $\kappa = 1$ and $\kappa = 0$. Thus the theory is flexible enough not to exclude simultaneous application to both the single and binary star cases. Provided l is not too small, the angular momentum loss decreases with decreasing κ, so that the braking of a single star is slowed up as the ratio of centrifugal force to gravity decreases. But in a close binary system centrifugal force is necessarily close to gravity, so that the braking process should work near to its maximum efficiency. The rate of angular momentum loss $-dJ/dt$ can therefore be markedly different in two cases, so that there is no contradiction in requiring $-\Gamma(dM/dt)\tau$ to vary by an order of magnitude, even though τ is for both cases just the Hayashi descent time.

It is hoped that the ultimate complete theory will be able to account for the

* Roxburgh (1966) has revived the classical scheme for the formation of close binaries by rotational fission. His model depends critically on the assumptions (i) that the ratio of centrifugal force to gravity at the *photospheric* equator stays close to unity as the parent star descends the Hayashi track; and (ii) that the parent star has no primeval magnetic field, so that subsequent contraction of the radiative core occurs with conservation of angular momentum. If a weak primeval field is washed out of the star then (ii) may be satisfied. However, the remarks in the text on dynamo generation of a surface field and on coronal expansion suggests that condition (i) is at the least far from obvious. If the angular momentum loss did no more than keep the parameter κ below unity, then with $r_s/R \simeq 2$, centrifugal force at the photosphere would be $\simeq$ one eighth of gravity. Following subsequent contraction of the radiative core, the ratio of centrifugal force to gravity deep in the star will still not reach the critical values demanded for rotational fission to be possible.

variation of angular momentum along the main-sequence, in particular, the sharp drop in spin near type *F*. Particularly significant are the differences in the times spent by stars of different mass in the Hayashi phase, and in the immediate post-Hayashi phase in which the stars retain extensive outer convection zones, and the amounts of mass lost in these phases. One would like information on the distribution with stellar mass of rotation in young clusters, to see whether a substantial part of the difference in spin arises through magnetic braking subsequent to arrival at the main sequence.

The slow rotations of the observably magnetic A stars as compared with the normal A stars must be fitted into the theory. Stars that are partners in short-period binary systems—according to Abt (1961, 1966), all the Am stars and a fair proportion of the Ap stars—we would expect to tend to rotate in synchronism with the orbital period; if the scheme of Section 7 is applicable, their rotation may be just a necessary consequence of their formation process. It has been urged by Renson (1963; Ledoux & Renson 1966) that the magnetic binary is the most probable model for the variable Ap stars, though as yet no observer has claimed that all Ap stars are binary members. If, then, *single*, slowly-rotating, strongly-magnetic Ap stars exist, theory must compare them with normal, rapidly-rotating A stars. Excess magnetic braking is presumably the answer, but the details of the process must discriminate between the quasi-permanent (presumably primeval) field of an Ap star, and the dynamo-built surface field that we expect all stars to acquire while they have strong convection zones. Perhaps the excess angular momentum is lost during later stages, when the convection is too weak to generate an effective surface field. The upward transport of energy by hydromagnetic waves may enable an Ap star to retain a hot corona longer than a normal A star.

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