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MAGNETIC BRAKING BY A STELLAR WIND—III

THE OBLIQUE ROTATOR WITH A QUASI-RADIAL FIELD

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SUMMARY

It is known that if a rotating magnetic star emits a stellar wind during part of its lifetime, the consequent magnetic torque not only brakes the star, but in general causes the instantaneous axis of rotation to precess through the star, in particular altering the angle χ between the rotation and magnetic axes. This suggests that the same magnetic coupling may be responsible for both the abnormally slow rotations of the magnetic variables and for the large values of χ required if the oblique rotator model is to account for field reversal.

In this paper, the magnitude and sign of the precessional torque component are computed for the special case with the magnetic field outside the star the sum of a basic, split monopole field, with radial field-lines and an equatorial current-sheet, a perturbation field, of order ϵ, due to a small angle-dependent flux distribution over the stellar surface, and further perturbations due to the stellar rotation α, of order α and $\epsilon\alpha$ respectively. The field of order α yields a braking torque, but has too much symmetry to yield a precessional component, which results only from the part of order $\alpha\epsilon$. If the surface flux distribution is symmetric about the magnetic axis, the precessional torque vanishes when χ is either 0 or $\pi/2$; while if the flux has a non-axisymmetric part, and in addition is not completely antisymmetric in the magnetic equator, then the precessional torque vanishes for two orthogonal values of χ, with $|\chi-\pi/2| \lessgtr \pi/4$ respectively. The details of the photospheric flux distribution also determine whether χ approaches the larger or the smaller value. When the flux distribution is axisymmetric, then for the most important cases computed the magnetic and rotation axes tend to align when the perturbation flux is more concentrated to the magnetic poles, and to become orthogonal when there is less flux at the poles than at the equator. The flux distribution actually present is presumably determined by internal stellar hydromagnetics.

With this assumed external field structure, it is found that the ratio of the precessional and braking torques is usually too small for a sizeable change in χ to be associated with a reasonable degree of braking. However, if the field outside the star has the type of structure discussed in Paper I of this series, we may expect this torque ratio to be big enough for the process to be significant.

1. INTRODUCTION

In Paper II of this series (Mestel 1968b) the theory of the magnetically-controlled stellar wind, developed e.g. in Paper I (Mestel 1968a) for axisymmetric geometry, was partially extended to the essentially non-axisymmetric oblique rotator model. It was shown that in general the magnetic field—as distorted by the wind and the consequent non-uniform rotation—lacks the degree of symmetry that would ensure that the integrated magnetic torque on the star is parallel to the instantaneous axis of rotation: the magnetic tensions have non-cancelling moments about lines perpendicular to the rotation axis. There results a slow motion of the rotation axis through the star, and a corresponding rotation of the magnetic axis in space. If the magnetic flux emerging from the star is symmetric about the magnetic axis, then the precessional torque vanishes when the angle χ between the rotation and magnetic axes is either zero or $\pi/2$. It was anticipated that one of these two states is unstable, in the sense that a small departure yields a magnetic torque that steadily changes χ until the other (stable) state is reached. With a more general flux distribution the angle χ is likely to settle at a value between 0 and $\pi/2$.

The interest in the result lies in the possibility that if the abnormally slow rotations of the magnetic stars are to be explained by excess magnetic braking, then the same process may simultaneously account for the observed magnetic field reversals, by spontaneously leading to a large angle of inclination between magnetic and rotation axes, whatever the initial value of χ. We shall confirm that the details of the flux distribution over the stellar surface are crucial in deciding whether the angle χ approaches a large or small value. It is therefore clear that the wind theory cannot by itself solve the problem: all it can do is indicate which flux distributions will yield large angles, as required by observation. The reason for any particular flux distribution must lie in the internal hydromagnetics of the star, and is outside the scope of this paper.

Some progress was made in Paper II towards computing the torque on the star; in particular, we derived an integral transform (equation (67)—a special case of that found in Section 2 below) which enables approximate estimates to be made (Selley 1970 in preparation). However, there remained an undetermined pressure integral over the Alfvénic surface, and although plausible arguments can be given why this should be smaller than the other terms, it is desirable to construct rigorously at least one detailed model, which can then be a guide to non-rigorous treatment of the more realistic but mathematically less tractable cases.

The difficulties of the non-axisymmetric theory arise principally from there no longer being an integral describing the steady transport of one component of angular momentum along field-streamlines. Instead, all three components of angular momentum are interchanged between field-lines by magnetic and thermal pressures, so that the angular momentum integral—equation (42) of Paper II—is replaced by a partial differential equation. However, a comparatively simple solution by a perturbation procedure is possible if we assume (1) the star's rotation α is low, so that the centrifugal forces and the magnetic forces quadratic in α may be dropped in the first approximation; (2) the wind is powerful enough to draw out the field-lines into a quasi-radial structure; and (3) the magnetic flux emerging from the star has a weak dependence on angle, except for a discontinuous change in sign at the magnetic equator. Of these, (1) and (3) are legitimate restrictions, undertaken for mathematical convenience and limiting the quantitative but not the qualitative reliability of the conclusions. There is more question about (2), as it has been argued (e.g. in Paper I) that near a strongly magnetic star the field will tend to be curl-free, with some of the field-lines closed, and with the corona divided into a 'dead zone' and a 'wind zone'. Other workers (e.g. Weber & Davis 1967) have in fact adopted a quasi-radial field model, with the oppositely-directed field-lines at the cut kept apart by a high thermal pressure. We feel that the detailed structure of the magnetic and velocity fields outside a strongly magnetic star—and near the active magnetic regions on the solar surface—remains a challenge to theorists. Limited support for the general contention that a strong field will not be pulled out by the wind comes from observations of coronal structure made during a recent solar eclipse (Schatten 1969). Our adoption of a quasi-radial field is therefore from mathematical expediency rather than conviction; and in fact the quantitative discussion in Section 6 suggests strongly that a self-consistent theory, capable of explaining both the low rotations and the high obliquities of the magnetic stars, can be constructed only if the field is more nearly curl-free rather than quasi-radial.

Current ideas on pulsars involve models of rotating magnetized neutron stars (Gold 1968), but with the external magnetic energy at least comparable with the rest energy, so that familiar approximations of non-relativistic hydromagnetics break down even in regions near enough to the star for the velocity of corotation to be well below c (Goldreich & Julian 1969). However, the same general symmetry considerations as in Paper II must yield a precessional as well as a braking torque acting on models with the rotation and magnetic axes distinct (Pacini 1968; Ostriker & Gunn 1969; Davis 1969 private communication). It would be of great interest if radiation from pulsars could be shown to exhibit a secular change consistent with a steady change in the obliquity.

2. THE GENERALIZED TORQUE INTEGRAL

In a frame rotating with angular velocity $\alpha\mathbf{k}$, the equation to the steady flow of inviscid gas is

$$\frac{\partial}{\partial x_k}(\rho V_i V_k) + 2\alpha\epsilon_{ijk}k_j\rho V_k = -\frac{\partial p}{\partial x_i} + \rho\frac{\partial \phi}{\partial x_i} - \frac{\partial T_{ik}}{\partial x_k} + \rho\epsilon_{ijk}(\alpha k_j)\,\epsilon_{klm}x_l(\alpha k_m), \tag{1}$$

where ϵ_{ijk} is the alternating tensor, V_i the velocity in this frame, ϕ the gravitational potential, ρ the density, p the pressure, and T_{ik} the magnetic part of the Maxwell stress tensor:

$$T_{ik} = \frac{\mathbf{H}^2}{8\pi}\delta_{ik} - \frac{H_i H_k}{4\pi}. \tag{2}$$

The first term in equation (1) incorporates the continuity equation

$$\frac{\partial}{\partial x_k}(\rho V_k) = 0; \tag{3}$$

the second is the Coriolis force $2\alpha\mathbf{k}\times\rho\mathbf{V}$; and the last is the centrifugal force $\rho\{\alpha\mathbf{k}\times(\mathbf{r}\times\alpha\mathbf{k})\} = \rho\alpha^2\{\mathbf{r}-\mathbf{k}(\mathbf{r}.\mathbf{k})\}$. The velocity $\mathbf{V}$ also satisfies the equation

$$\nabla\times(\mathbf{V}\times\mathbf{H}) = 0 \tag{4}$$

in the magnetohydrodynamic approximation. In an axisymmetric system the analogous equation in the inertial frame has the general solution (Chandrasekhar 1956; Mestel 1961)

$$\mathbf{v} = \kappa\mathbf{H} + \varpi\alpha\mathbf{t}, \tag{5}$$

where $\mathbf{v}$ is the velocity in the inertial frame, $\mathbf{t}$ is the unit vector in the direction of the rotational velocity, ϖ the perpendicular distance from the axis, κ a scalar and α the angular velocity of the particular field-line considered. If α is constant over the whole field, then the solution (5) implies that in the frame rotating with angular velocity $\alpha\mathbf{k}$,

$$\mathbf{V} = \kappa\mathbf{H} \tag{6}$$

along all field-lines. In the oblique rotator problem we can write down steady-state equations only if we use the rotating frame; and although it is not now immediately obvious that (6) is the only relevant solution of equation (4), we shall in fact adopt the solution (6), so that the continuity equation (3) implies

$$\rho\kappa = \eta = \text{constant on a field-line.} \tag{7}$$

The Reynolds stress tensor $\rho V_i V_k$ then becomes

$$\rho V_i V_k = 4\pi\rho\kappa^2(H_i H_k/4\pi) = (4\pi\eta^2/\rho)(H_i H_k/4\pi) \tag{8}$$

and so is proportional to the tension terms in T_{ik}. As ρ will decrease outwards in any plausible solution, there will exist an Alfvénic surface S_A consisting of the points where the velocity in the rotating frame equals the Alfvén speed defined by the *total* field $\mathbf{H}$:

$$1 = \mathbf{V}^2/(\mathbf{H}^2/4\pi\rho) = 4\pi\rho\kappa^2 = 4\pi\eta^2/\rho. \tag{9}$$

By equation (8), the Reynolds and magnetic tension tensors coincide on S_A.

Consider now a volume τ, fixed in the rotating frame, surrounded by a surface S with outward normal n_l. We form the quantity

$$\int_S F_{il} n_l \, dS \equiv \int_S \epsilon_{ijk} x_j \{T_{kl} + p\delta_{kl} + \rho V_l [V_k + \alpha(\mathbf{k}\times\mathbf{r})_k]\} n_l \, dS. \tag{10}$$

With the origin of coordinates fixed on the rotation axis—e.g. at the mass-centre of the star—this is the outflow across S of the i-component of angular momentum about the origin, due respectively to the magnetic stresses, the thermal pressure, and the macroscopic gas flow. Since the first three terms in the brackets are symmetric tensors, they can be written

$$\int_\tau d\tau \, \epsilon_{ijk} x_j \frac{\partial}{\partial x_l}(T_{kl} + p\delta_{kl} + \rho V_k V_l)$$
$$= \int_\tau d\tau \, \epsilon_{ijk} x_j \left\{\rho \frac{\partial \phi}{\partial x_k} + \rho\alpha^2(\mathbf{k}\times(\mathbf{r}\times\mathbf{k}))_k - 2\alpha\epsilon_{kpl} k_p \rho V_l\right\} \tag{11}$$

by the equation of motion (1). The centrifugal term in expression (11) reduces at once to

$$-\alpha\left[\mathbf{k}\times\int_\tau \{\mathbf{r}\times\rho(\alpha\mathbf{k}\times\mathbf{r})\} \, d\tau\right]_i. \tag{12}$$

The Coriolis term becomes

$$-2\alpha\int_\tau d\tau \, \epsilon_{kij}\epsilon_{kpl} x_j k_p \rho V_l = -2\alpha\int_\tau d\tau(\rho V_l)(k_i x_l - k_j x_j \delta_{il}), \tag{13}$$

and combines with the fourth term in expression (10)

$$\alpha\int_\tau d\tau \rho V_l \frac{\partial}{\partial x_l}(\epsilon_{ijk} x_j \epsilon_{kpq} k_p x_q) = \alpha\int_\tau d\tau \rho V_l (2k_i x_l - x_i k_l - k_j x_j \delta_{il}) \tag{14}$$

to yield

$$\alpha \int_\tau d\tau \rho V_l (k_j x_j \delta_{il} - x_i k_l) = -\left\{\alpha \mathbf{k} \times \int_\tau (\mathbf{r} \times \rho \mathbf{V})\, d\tau\right\}_i. \tag{15}$$

Thus the surface integral (10) can be written

$$\int_S F_{il} n_l \, dS = \int_\tau \{\mathbf{r} \times (\rho \nabla \phi)\}_i \, d\tau - (\alpha \mathbf{k} \times \mathbf{h})_i, \tag{16}$$

where

$$\mathbf{h} \equiv \int_\tau \mathbf{r} \times \rho(\mathbf{V} + \alpha \mathbf{k} \times \mathbf{r})\, d\tau \tag{17}$$

is the angular momentum of the gas flowing instantaneously through τ. The first term on the right of equation (16) is the integrated moment of the gravitational force density; it will vanish if we ignore distortions of the star from sphericity. The last term implies that in order to rotate the ‘ flywheel ’ of angular momentum $\mathbf{h}$ at the rate $\alpha\mathbf{k}$, the matter within τ must be given angular momentum at the rate $\alpha\mathbf{k} \times \mathbf{h}$.

We now specialize by taking τ to be the volume between the Alfvénic surface S_A (defined by equation (9)) and an inner surface S_1 surrounding the star at the base of the stellar corona (also fixed in the rotating frame). The total torque $\mathbf{L}$ exerted across S_i is given by

$$\begin{aligned} -L_i &= \int_{S_1} F_{il} \tilde{n}_l \, dS \\ &= (\alpha \mathbf{k} \times \mathbf{h})_i + \int_{S_A} \left(p + \frac{\mathbf{H}^2}{8\pi}\right)(\mathbf{r} \times \mathbf{n})_i \, dS + \int_{S_A} \rho(\mathbf{V}.\mathbf{n})\{\mathbf{r} \times (\alpha \mathbf{k} \times \mathbf{r})\}_i \, dS, \end{aligned} \tag{18}$$

where $\tilde{n}_l \equiv -n_l$ is the normal drawn *into* volume τ, i.e. *out* of the surface S_1. Since the wind speed decreases exponentially below the sonic point (Parker 1963), the only important motion within S_1 is the stellar rotation. The torque $\mathbf{L}$ in general brakes the star, and also causes the instantaneous axis of rotation to move through the star, in particular altering the angle χ between the magnetic and rotation axes (Paper II, Section 4).

If $\chi = 0$ and the magnetic field structure is symmetric about the rotation axis $\mathbf{k}$, then the first two terms on the right of equation (18) vanish identically, and the last has a non-zero component only in the direction $\mathbf{k}$. The total torque in the axisymmetric case is seen to be equivalent to that given by assuming the gas to corotate with the star out to the Alfvénic surface S_A. Further, in this axisymmetric case there is a natural geometrical decomposition of $\mathbf{H}$ and $\mathbf{V}$ into poloidal and toroidal parts which are mutually orthogonal, so that the definition (9) of the (axisymmetric) Alfvénic surface S_A is equivalent to the more familiar definition with $\mathbf{H}$ replaced by $\mathbf{H}_p$, and $\mathbf{V}$ by $\mathbf{V}_p \equiv \mathbf{v}_p$, the poloidal component in the inertial frame (cf. equation (43) of Paper II). The transformed expression for the torque on the star does not involve explicitly either the poloidal or toroidal components of the field, but merely the flow across S_A. The poloidal field structure is of course implicit, in that the velocity field $\mathbf{v}_p$ and $\mathbf{H}_p$ are not only parallel but must satisfy jointly the component of the equation of motion perpendicular to their common field-streamlines; but at least for slowly rotating stars with hot coronas, the toroidal field $\mathbf{H}_t$ generated by the non-uniform rotation set up by the wind will not greatly affect $\mathbf{v}_p$ and $\mathbf{H}_p$. *The total torque on a slowly rotating* *star can be computed from the transform* (18) *in terms of the flow and magnetic fields in the absence of rotation.*

There are easily-defined analogues of poloidal and toroidal vectors for the oblique rotator problem, called $\bar{\mathbf{V}}$ and $\mathbf{V}'$, $\bar{\mathbf{H}}$ and $\mathbf{H}'$ in Paper II, but they are not in general orthogonal, and a non-approximate treatment must use the definition (9) for S_A. In Paper II the problem was simplified by assuming that ρ and κ were invariant under the reflection $\mathfrak{R}$ in the plane defined by the two axes. Even so, the equation (67) of Paper II that yields the two principal components of torque—essentially an analogue of the present equation (18)—required knowledge of $\mathbf{H}'$ in order to compute the moment of $(\bar{\mathbf{H}}.\mathbf{H}')/4\pi$ over the Alfvénic surface. There seems therefore little virtue in prior assumptions about ρ and κ; in order to get trustworthy results for the magnitude and sign of the precessional torque on the star, we must find a self-consistent method of approximation.

The general problem is formidable. However, we can make progress by means of a double perturbation procedure. We consider first a non-rotating star, with a magnetic field that has the form of a split monopole: the field-lines are strictly radial and the field-strength independent of angle, but the sign of the field changes at the magnetic equator. The star is supposed to have a corona, taken as isothermal for simplicity, and hot enough to expand steadily to infinity along radial stream-lines, as described by Parker's critical solution (1963). With the pole-strength of the field fixed, the zero-order Alfvénic surface S_A is a *sphere* of known radius r_c.

We then give the star a rotation α about an arbitrary axis $\mathbf{k}$ inclined at an angle χ to the magnetic axis (i.e. the axis perpendicular to the magnetic equator). The perturbed fields $\mathbf{H}'$ etc. are found in Section 3 to the *first order* in the rotation. Since the centrifugal forces are quadratic in α, the wind remains purely thermal, and we shall see that the Alfvénic surface is still a sphere. The stresses exerted by the perturbed magnetic field brake the star, but exert no precessional torque. As long as the zero-order field is radial and independent of angle (and we shall see in Section 5 that these two conditions are not independent), then the distorted field exerts stresses that take no cognizance of the magnetic axis*.

The magnetic flux emerging from a non-rotating star is then supposed to have a small angular dependence of order ϵ, and the mutually self-consistent magnetic-velocity fields are computed (Section 5). If the star is then given a slow rotation α, the corrections of order $\alpha\epsilon$ to the first-order rotational distortion $\mathbf{H}'$ can be found by a generalization of the method of Section 3. The system now has low enough symmetry for the net torque exerted by $\mathbf{H}'$ to have a precessional component of order $\alpha\epsilon$, in addition to the braking component of order α. However, if all we want is the *torque on the star* to order $\alpha\epsilon$, we do not in fact require the perturbed field $\mathbf{H}'$ to this order. Analogously to the result quoted for the axi-symmetric case, *the transform* (18) *yields the torque to order* $\alpha\epsilon$ *in terms of the magnetic and wind fields of order* ϵ.

3. THE ROTATING STAR WITH A SPLIT MONOPOLE FIELD

We set up two sets of rectangular Cartesian axes $O(X, Y, Z)$, $O(x, y, z)$, where OZ is the rotation axis, $OX = Ox$, Oxz is the magnetic equator and the angle $y\widehat{O}Z$ between the rotation and magnetic axes is called χ. We introduce

* This remark was made by Professor P. A. Sturrock before the analysis of this Section was done.

spherical polar coordinates (r, θ, λ) based on Oz:

$$\begin{aligned} x &= -r \sin\theta \sin\lambda, & X &= x, \\ y &= r \sin\theta \cos\lambda, & Y &= y \sin\chi - z\cos\chi, \\ z &= r\cos\theta, & Z &= y\cos\chi + z\sin\chi. \end{aligned} \tag{19}$$

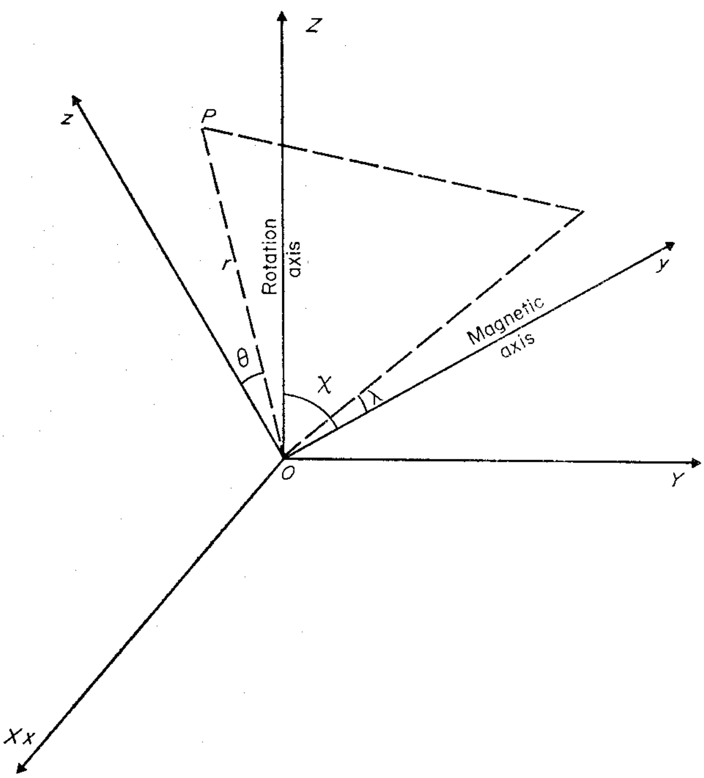

FIG. 1.

The unit vectors **i**, **j**, **k**, directed along OX, OY, OZ can be written

$$\begin{aligned} \mathbf{i} &= -\sin\theta\sin\lambda\,\hat{\mathbf{r}} - \cos\theta\sin\lambda\,\hat{\boldsymbol{\theta}} - \cos\lambda\,\hat{\boldsymbol{\lambda}}, \\ \mathbf{j} &= (\sin\theta\cos\lambda\sin\chi - \cos\theta\cos\chi)\,\hat{\mathbf{r}} + (\cos\theta\cos\lambda\sin\chi + \sin\theta\cos\chi)\,\hat{\boldsymbol{\theta}} \\ &\qquad - (\sin\lambda\sin\chi)\,\hat{\boldsymbol{\lambda}}, \\ \mathbf{k} &= (\sin\theta\cos\lambda\cos\chi + \cos\theta\sin\chi)\,\hat{\mathbf{r}} \\ &\qquad + (\cos\theta\cos\lambda\cos\chi - \sin\theta\sin\chi)\,\hat{\boldsymbol{\theta}} - (\sin\lambda\cos\chi)\,\hat{\boldsymbol{\lambda}}, \end{aligned} \tag{20}$$

where $\hat{\mathbf{r}}$, $\hat{\boldsymbol{\theta}}$, $\hat{\boldsymbol{\lambda}}$ are unit vectors in the three curvilinear directions.

The zero-order (split monopole) field has (r, θ, λ) components conveniently written as

$$\begin{aligned} \bar{\mathbf{H}}_0 &\equiv \bar{H}_0(1, 0, 0) \\ &= \bar{H}_c \left(\frac{r_c}{r}\right)^2 \operatorname{sign}\left(\frac{\pi}{2} - |\lambda|\right)(1, 0, 0), \end{aligned} \tag{21}$$

where $\bar{H}_c$ is the magnitude of the field at the zero-order Alfvénic sphere r_c. The zero-order wind field has radial velocity $\bar{V}_0$ and density $\bar{\rho}_0$ satisfying the familiar equations

$$\begin{aligned} \bar{\rho}_0 \bar{V}_0 r^2 &= \text{constant}, && \text{(continuity)} \\ \bar{V}_0 &= \bar{\kappa}_0 \bar{H}_0, && \text{(hydromagnetic)} \end{aligned} \tag{22}$$

yielding jointly

$$\bar{\rho}_0\bar{\kappa}_0 \equiv \bar{\eta}_0 = \text{constant}\left[\text{sign}\left(\frac{\pi}{2}-|\lambda|\right)\right]; \tag{23}$$

and the Bernoulli equation

$$a^2 \log \bar{\rho}_0 - \frac{GM}{r} + \frac{1}{2}\bar{V}_0{}^2 = \text{constant}. \tag{24}$$

We select Parker's critical solution (1963), with this last constant fixed so that $\bar{V}_0$ attains the isothermal sound speed a at the point $r_a = GM/2a^2$.

There is a current sheet across the cut $|\lambda| = \pi/2$, and the associated magnetic pinching forces require an excess thermal pressure at the cut to prevent the oppositely-directed field-lines from diffusing into each other. If we ignore the outflow of gas along the cut, we may make the model strictly self-consistent by postulating the excess density at the cut (proportional to $1/r^4$ in an isothermal corona) that balances the pressure of the radial magnetic field. But if the wind flows in the equatorial zone it will establish its own density field; and in fact if the flow were everywhere strictly radial, there would be no excess pressure at the cut. Thus even if the field emerging from the stellar surface has no angular dependence (except for the sign change at the cut), the magnetic and velocity fields would adjust themselves into a steady state in which $\bar{\mathbf{H}}_0$ and $\bar{\mathbf{V}}_0$ have small transverse components, and the radial components are not strictly independent of angle. There appears to be no adequate treatment of this problem in the literature, and we shall in fact ignore these zero-order departures from sphericity. But there is a lacuna in the theory at this point; a full analysis would almost certainly predict a small but non-vanishing precessional torque.

When the star is given a rotation α, there result perturbations $\mathbf{H}_0'$, $\mathbf{V}_0'$, ρ_0', p_0', κ_0'. The terms of order α in the equation of motion (1) yield

$$-\nabla p_0' - \rho_0' \frac{GM}{r^2} + (\nabla \times \mathbf{H}_0') \times \bar{\mathbf{H}}_0/4\pi$$
$$= \rho_0' \nabla(\tfrac{1}{2}\bar{V}_0{}^2) + \bar{\rho}_0\{\nabla(\bar{\mathbf{V}}_0 . \mathbf{V}_0') - \bar{\mathbf{V}}_0 \times (\nabla \times \mathbf{V}_0')\} + 2\bar{\rho}_0 \alpha \mathbf{k} \times \bar{\mathbf{V}}_0. \tag{25}$$

The terms involving $\nabla \times \bar{\mathbf{H}}_0$ and $\nabla \times \bar{\mathbf{V}}_0$ vanish. The notation $\bar{\mathbf{H}}_0$, $\mathbf{H}_0'$ etc., is used so as to agree with that of Paper II: it is easily verified that the various fields have the appropriate properties under the reflection $\mathfrak{R}$ in OYZ (cf. the relations (20) of Paper II).

For brevity, we shall drop the suffix zero in the rest of this Section, restoring it later. Then the (r, θ, λ) components of equation (25) are respectively

$$-\frac{\partial p'}{\partial r} - \rho' \frac{GM}{r^2} = \rho' \frac{\partial}{\partial r}(\tfrac{1}{2}\bar{V}^2) + \bar{\rho}\frac{\partial}{\partial r}(\bar{V}V_r'), \tag{26}$$

$$-\frac{1}{r}\frac{\partial p'}{\partial \theta} - \frac{1}{4\pi r}\frac{\partial}{\partial \theta}(\bar{H}H_r') + \frac{1}{4\pi r^3}\frac{\partial}{\partial r}(r^3 \bar{H} H_\theta')$$
$$= \frac{\bar{\rho}\bar{V}}{r}\frac{\partial}{\partial r}(rV_\theta') - 2\alpha\bar{\rho}\bar{V}\cos\chi\sin\lambda, \tag{27}$$

$$-\frac{1}{r\sin\theta}\frac{\partial p'}{\partial \lambda} - \frac{1}{4\pi r\sin\theta}\frac{\partial}{\partial \lambda}(\bar{H}H_r') + \frac{1}{4\pi r^3}\frac{\partial}{\partial r}(r^3\bar{H}H_\lambda')$$
$$= \frac{\bar{\rho}\bar{V}}{r}\frac{\partial}{\partial r}(rV_\lambda') + 2\alpha\bar{\rho}\bar{V}(\sin\theta\sin\chi - \cos\theta\cos\lambda\cos\chi). \tag{28}$$

The hydromagnetic integral (6) yields

$$V_r' = \bar{\kappa} H_r' + \kappa' \bar{H}, \qquad V_\theta' = \bar{\kappa} H_\theta', \qquad V_\lambda' = \bar{\kappa} H_\lambda', \tag{29}$$

and the continuity equation (3)

$$\bar{H} \frac{\partial}{\partial r} (\bar{\rho}\kappa' + \rho'\bar{\kappa}) = 0, \tag{30}$$

the term $\mathbf{H}' . \nabla(\bar{\rho}\bar{\kappa})$ vanishing by (23). Finally $\mathbf{H}'$ must satisfy the divergence condition

$$\frac{\partial}{\partial r}(r^2 H_r') + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(r \sin\theta H_\theta') + \frac{1}{\sin\theta}\frac{\partial}{\partial\lambda}(rH_\lambda') = 0. \tag{31}$$

Equations (27) and (28) can be re-written with the help of equations (22) and (29):

$$-\frac{1}{r}\frac{\partial}{\partial\theta}\left(p' + \frac{\bar{H}H_r'}{4\pi}\right) + \frac{1}{4\pi r^3}\frac{\partial}{\partial r}\left[r^3 \bar{H} H_\theta'\left(1 - \frac{4\pi\bar{\eta}^2}{\bar{\rho}}\right)\right] = -2\alpha\bar{\rho}\bar{V}\cos\chi\sin\lambda, \tag{32}$$

$$-\frac{1}{r\sin\theta}\frac{\partial}{\partial\lambda}\left(p' + \frac{\bar{H}H_r'}{4\pi}\right) + \frac{1}{4\pi r^3}\frac{\partial}{\partial r}\left[r^3 \bar{H} H_\lambda'\left(1 - \frac{4\pi\bar{\eta}^2}{\bar{\rho}}\right)\right] = 2\alpha\bar{\rho}\bar{V}(\sin\theta\sin\chi - \cos\theta\cos\lambda\cos\chi). \tag{33}$$

(a) The magnetic and rotation axes parallel

When $\chi = 0$, Oy coincides with OZ, and Oz with $O(-Y)$. The solution of equations (26)–(33) must reduce to the axisymmetric solution of Paper I for the special case of a split monopole poloidal field and a purely thermal wind. The perturbation field is now azimuthal about the axis OZ, so that $H_r' = 0$. Also, p', ρ', κ' all vanish: the only effect the rotation has on the radial velocity and so on the density–pressure field in the axisymmetric problem is through terms of the *second* order in α, such as the centrifugal force and the quadratic magnetic force term $(\nabla \times \mathbf{H}') \times \mathbf{H}'/4\pi$. Thus equations (26) and (30) are satisfied identically and equations (32) and (33) yield

$$\left.\begin{matrix}(H_\theta')_0 \\ (H_\lambda')_0\end{matrix}\right\} = -4\pi\alpha\frac{(\bar{\rho}\bar{V}r^2)}{(\bar{H}r^2)}\frac{(r^2 - r_c^2)}{\left(1 - \frac{4\pi\bar{\eta}^2}{\bar{\rho}}\right)}\frac{1}{r}\begin{cases}\sin\lambda, \\ \cos\theta\cos\lambda,\end{cases} \tag{34}$$

where we have used the condition that there be no singularity at the Alfvénic sphere to fix the constants of integration (cf. Paper I). The divergence condition (31) is readily seen to be satisfied. If we introduce new spherical polar coordinates based on OZ, such that

$$\begin{aligned}(X, Y, Z) \underset{(\chi=0)}{\equiv} & \; r(-\sin\theta\sin\lambda, \; -\cos\theta, \; \sin\theta\cos\lambda) \\ \equiv & \; r(\sin\Theta\cos\Lambda, \; \sin\Theta\sin\Lambda, \; \cos\Theta),\end{aligned} \tag{35}$$

it is easy to verify that the field (34) has (r, Θ, Λ) components (again dropping the suffix zero)

$$H_r' = H_\Theta' = 0,$$

$$H_\Lambda' = -4\pi\alpha \frac{(\bar{\rho}\bar{V}r^2)}{(\bar{H}r^2)} \frac{\sin\Theta}{r} \frac{(r_c^2-r^2)}{\left(1-\dfrac{4\pi\bar{\eta}^2}{\bar{\rho}}\right)} = -\frac{4\pi\alpha\bar{\eta}\sin\Theta}{r} \frac{(r_c^2-r^2)}{\left(1-\dfrac{4\pi\bar{\eta}^2}{\bar{\rho}}\right)} \tag{36}$$

—just the solution found in Paper I, written in its 'natural' coordinates.

Because of the change in sign of $\bar{\mathbf{H}}_0$ across the cut, H_Λ' also changes sign. However, the magnetic pressure to order α is $(\bar{\mathbf{H}}^2+2\bar{\mathbf{H}}.\mathbf{H}')/8\pi = \bar{\mathbf{H}}^2/8\pi$, since $\bar{\mathbf{H}}$ and $\mathbf{H}'$ are orthogonal; and as $p' = 0$, there is no unacceptable discontinuity in the total pressure at the cut. The tangential components of the magnetic stresses at the magnetic equator are zero on either side of the cut, and so automatically satisfy the continuity condition.

(b) The magnetic and rotation axes perpendicular

Now consider the case with $\chi = \pi/2$. The θ-equation (32) becomes homogeneous, and the λ-equation reduces to

$$-\frac{1}{r\sin\theta}\frac{\partial}{\partial\lambda}\left(p'+\frac{\bar{H}H_r'}{4\pi}\right)+\frac{1}{4\pi r^3}\frac{\partial}{\partial r}\left[r^3\bar{H}H_\lambda'\left(1-\frac{4\pi\bar{\eta}^2}{\bar{\rho}}\right)\right] = 2\alpha\bar{\rho}\bar{V}\sin\theta. \tag{37}$$

The forcing term on the right is independent of λ—its Fourier expansion in $\cos(n\lambda)$ consists of only the term '$c_0/2$' in standard notation. Also, the perturbed total pressure $(p'+\bar{H}H_r'/4\pi)$ must again be continuous across the cut $|\lambda| = \pi/2$, so that the Fourier series of its λ-gradient is just the λ-gradient of its Fourier series. We arrive at the solution

$$H_\lambda' = -4\pi\alpha \frac{(\bar{\rho}\bar{V}r^2)}{(\bar{H}r^2)} \frac{(r_c^2-r^2)}{\left(1-\dfrac{4\pi\bar{\eta}^2}{\bar{\rho}}\right)} \frac{\sin\theta}{r}, \tag{38}$$

with

$$H_r' = H_\theta' = 0,$$

$$p', \rho', \kappa' = 0. \tag{39}$$

Thus the perturbed pressure is in fact zero everywhere, and not just at the cut. The tangential magnetic stress at the cut is radial and equal to $\bar{H}H_\lambda'/4\pi$; its continuity at the cut (assured by equation (38)) is in fact a consequence of the general parity relations (13) of Paper II. The field is again a twist about the rotation axis, but with the significant difference that H_λ' changes sign with $\bar{H}_r$ at the cut. Thus although it is readily seen that the divergence condition (31) is satisfied (since H_λ' is independent of λ), there is a discontinuity at the cut. This does not, however, require a fictitious layer of magnetic poles on $|\lambda| = \pi/2$, but merely implies a first-order shift in the position of the cut. Let the coordinates of a point on the cut, originally at $(r, \theta, \pm\pi/2)$, be now $(r, \theta, \pm\tilde{\lambda})$ with

$$\pm\frac{\pi}{2} = \tilde{\lambda}+\alpha\mu(r, \theta, \tilde{\lambda}) \tag{40}$$

and $\mu(r, \theta, \tilde{\lambda})$ a function to be determined. The coordinates $(dr, d\theta, d\tilde{\lambda})$ of an arbitrary displacement on one sheet of the cut are related by

$$d\tilde{\lambda} = -\left(\alpha \frac{\partial \mu}{\partial r} dr + \alpha \frac{\partial \mu}{\partial \theta} d\theta\right) \Big/ \left(1 + \alpha \frac{\partial \mu}{\partial \tilde{\lambda}}\right)$$
$$\simeq -\alpha \frac{\partial \mu}{\partial r} dr - \alpha \frac{\partial \mu}{\partial \theta} d\theta \tag{41}$$

to the first order in α. This displacement coincides with a line of the distorted magnetic field

$$\mathbf{H} = \{\bar{H}(r),\ 0,\ H_\lambda'(r, \theta)\}$$

if

$$\frac{dr}{\bar{H}} = \frac{r d\theta}{0} = \frac{r \sin\theta\, d\tilde{\lambda}}{H_\lambda'}. \tag{42}$$

Thus the shape of the distorted cut is given by

$$H_\lambda' + \alpha \bar{H} r \sin\theta (\partial\mu/\partial r) = 0, \tag{43}$$

yielding

$$\alpha\mu = 4\pi \frac{(\bar{\rho}\bar{V}r^2)}{(\bar{H}r^2)^2} \alpha r_c^2 \int_0^r \frac{\left(1 - \frac{r^2}{2}\right)}{\left(1 - \frac{\bar{\rho}_c}{\bar{\rho}}\right)} dr, \tag{44}$$

where the choice of zero for the lower limit ensures that the cut coincides with the plane OXZ in the limit $r \to 0$, when $|H_\lambda'/\bar{H}| \to 0$. An upper limit to the integral in expression (44) is given by assuming the wind speed is constant all the way from the star to the Alfvénic surface, so that $\bar{\rho} r^2 = \bar{\rho}_c r_c^2$. (In a thermally-driven wind, the speed increases slowly after passing through the sonic point, so the approximation is good in the supersonic region.) Thus

$$\frac{\alpha\mu}{\pi/2} < \frac{2}{\pi} \frac{4\pi(\bar{\rho}_c \bar{V}_c r_c^2)}{\bar{H}_c^2 r_c^4} \alpha r_c^2 r = \frac{2}{\pi}\left(\frac{\alpha \bar{r}_c}{\bar{V}_c}\right)\left(\frac{r}{r_c}\right), \tag{45}$$

and this is necessarily small compared with unity in an approximation where the centrifugal force terms in the Bernoulli equation are assumed small. But equally if one tries to build a model with a centrifugally-driven wind, and with a split monopole for the zero-order field, one is likely to find large distortions to the field, at least near the Alfvénic surface.

(*c*) *The magnetic and rotation axes arbitrarily inclined*

We now return to the problem with χ neither 0 nor $\pi/2$, so that the (θ, λ) equations are (32) and (33), with the polar coordinates referred to the tilted axis Oz in Fig. 1. It is at once clear that the solution for $\mathbf{H}'$ is now

$$\mathbf{H}' = \cos\chi\, (\mathbf{H}')_{\chi=0} + \sin\chi\, (\mathbf{H}')_{\chi=\pi/2}$$
$$= -\frac{4\pi(\bar{\rho}\bar{V}r^2)\alpha}{\bar{H}r^2} \frac{(r_c^2 - r^2)}{\left(1 - \frac{4\pi\bar{\eta}^2}{\bar{\rho}}\right)} \frac{1}{r} \left\{ \begin{array}{c} -\sin\lambda \cos\chi\, \hat{\boldsymbol{\theta}} \\ + \\ (\sin\theta \sin\chi - \cos\theta \cos\lambda \cos\chi)\, \hat{\boldsymbol{\lambda}} \end{array} \right\}; \tag{46}$$

for this satisfies both the equations (26–31) and the conditions at the cut. *Such a decomposition is always possible, whatever the structure of the basic field* $\bar{\mathbf{H}}$ *as* *long as we work just to the first order in* α; for then the angle χ appears explicitly only in the forcing terms on the right of equations (32) and (33), and not in the velocity $\bar{V}$ and the mass-flow parameter $\bar{\eta}$, as it would if the centrifugal forces were important in the Bernoulli equation (cf. equation (47) of Paper II). With the basic field a split monopole, the perturbation field (46) is recognizable as a superposition of a twist about Oy due to a rotation $\alpha \cos \chi$, and a twist about Oz due to a rotation $\alpha \sin \chi$.

By successive scalar multiplication of the field (46) with **i, j, k**, given by equations (20), we reduce $\mathbf{H}'$ to Cartesian components along $O(X, Y, Z)$:

$$\mathbf{H}' = -4\pi \frac{(\bar{\rho}\bar{V}r^2)}{(\bar{H}r^2)} \alpha \frac{\left(1-\dfrac{r^2}{r_c^2}\right)}{\left(1-\dfrac{\bar{\rho}_c}{\bar{\rho}}\right)} \frac{r_c^2}{r^2} \{-Y, X, O\}. \tag{47}$$

The total perturbation field is now seen to consist of a twist about the rotation axis. The outflow of angular momentum from the star is easily computed from the magnetic stress tensor integral taken over a sphere near to the star, where $r \ll r_c$ and $\bar{\rho}_c \ll \bar{\rho}$ (cf. equation (8) of Paper II):

$$\frac{1}{4\pi}\int -|\bar{H}| \operatorname{sign}\left(\frac{\pi}{2}-|\lambda|\right)\left(-\frac{4\pi(\bar{\rho}\bar{V}r^2)\alpha}{\bar{H}\operatorname{sign}\left(\dfrac{\pi}{2}-|\lambda|\right)}\frac{r_c^2}{r^2}\right)(X, Y, Z)\times(-Y, X, O)\,r^2\,d\Omega$$

$$= \alpha(\bar{\rho}_c\bar{V}_c r_c^2)\int(-XZ, -ZY, (X^2+Y^2))\,d\Omega. \tag{48}$$

It will be seen that the (X, Y) components in the integrand are anti-symmetric in Z, so that only the Z-component survives, to yield

$$\bar{\rho}_c\bar{V}_c\alpha r_c^4\int_0^\pi \sin^2\theta \sin\theta\, 2\pi\, d\theta = \frac{8\pi}{3}\bar{\rho}_c\bar{V}_c\alpha r_c^4. \tag{49}$$

The same calculation can be performed for each of the fields $\cos\chi\,(\mathbf{H}')_{\chi=0}$ and $\sin\chi\,(\mathbf{H}')_{\chi=\pi/2}$ separately. It is easily shown that they yield (Y, Z) components

$$\frac{8\pi}{3}\alpha r_c^2(\bar{\rho}\bar{V}r^2)\begin{Bmatrix} \sin\chi\cos\chi, & \cos^2\chi \\ -\sin\chi\cos\chi, & \sin^2\chi \end{Bmatrix} \tag{50}$$

respectively.

In fact, these results are more easily derived from the general expression (18). The Alfvénic surface defined by equation (9) remains a sphere, since ρ' and κ' are zero. The integral (17) for **h** vanishes by symmetry, so there remains just the flow integral across the sphere S_A:

$$\alpha\mathbf{k}\int \bar{\rho}_c\bar{V}_c r_c^2 \sin\theta\, d\theta\, d\lambda (r_c^2 \sin^2\theta) = \frac{8\pi}{3}\alpha r_c^2(\bar{\rho}\bar{V}r^2)\,\mathbf{k}. \tag{51}$$

This solution for the split monopole in fact hardly differs from the axisymmetric problem. The perturbation in the magnetic pressure $\bar{\mathbf{H}}.\mathbf{H}'/4\pi$ vanishes, and the magnetic stresses carry the Z-component of angular momentum without any mutual interchange between the zero-order field-lines. But although the coordinates (19) turn out to be redundant for describing the distorted split monopole, they are useful for computing the rotational distortion $\mathbf{H}'$ for more general zero-order fields.

4. THE PERTURBED TORQUE-INTEGRAL

The discussion of the preceding Section has again illustrated how the net torque on the star can be computed by use of the theorem (18), without a previous computation of the perturbed magnetic field. However, the results contained nothing very new: the differences from the strictly axisymmetric case with $\chi = 0$ were slight, the perturbed field being due to a simple twist about the rotation axis, with an associated distortion to the cut. The main use of this solution is that we can now apply the theorem (18) to a star with a basic field that departs slightly from the split monopole. This field is written

$$\bar{\mathbf{H}} = \bar{\mathbf{H}}_0 + \epsilon \bar{\mathbf{H}}_1, \tag{52}$$

where $\bar{\mathbf{H}}_1$ depends in general on θ and λ as well as r. There will be associated velocity, density and pressure fields

$$\begin{aligned} \mathbf{V} &= \bar{\mathbf{V}}_0 + \epsilon \bar{\mathbf{V}}_1, \\ \bar{p} &= \bar{p}_0 + \epsilon \bar{p}_1, \\ \bar{\rho} &= \bar{\rho}_0 + \epsilon \bar{\rho}_1, \end{aligned} \tag{53}$$

which must be found from the equations to the thermal wind theory. When the star is given a rotation α, the fields suffer a further perturbation: we therefore write

$$\begin{aligned} \mathbf{H} &= \bar{\mathbf{H}} + \mathbf{H}' = (\bar{\mathbf{H}}_0 + \epsilon \bar{\mathbf{H}}_1) + (\mathbf{H}_0' + \epsilon \mathbf{H}_1'), \\ \mathbf{V} &= \bar{\mathbf{V}} + \mathbf{V}' = (\bar{\mathbf{V}}_0 + \epsilon \bar{\mathbf{V}}_1) + (\mathbf{V}_0' + \epsilon \mathbf{V}_1'), \\ p &= \bar{p} + p' = (\bar{p}_0 + \epsilon \bar{p}_1) + (\epsilon p_1'), \\ \rho &= \bar{\rho} + \rho' = (\bar{\rho}_0 + \epsilon \bar{\rho}_1) + (\epsilon \rho_1'), \\ \kappa &= \bar{\kappa} + \kappa' = (\bar{\kappa}_0 + \epsilon \bar{\kappa}_1) + (\epsilon \kappa_1'). \end{aligned} \tag{54}$$

The suffix zero always refers to the solution of Section 3, with the split monopole basic field, and the suffix unity to the consequences of the perturbation of order ϵ; the bar and prime again are attached respectively to terms of zero order and first order in the rotation α. We note that the terms p_0', ρ_0' and κ_0' are zero.

The straightforward procedure for computation of the torque on the star begins by choosing a mutually self-consistent set of fields $\bar{\mathbf{H}}_1$, $\bar{\mathbf{V}}_1$, etc. Equations analogous to (26)–(31) will then fix $\mathbf{H}_1'$, $\mathbf{V}_1'$, etc., and the torque can then be found again by integrating the moment of the Maxwell stress tensor over a sphere deep in the stellar corona. Besides the familiar Z-torque, with a dominant term of order α, there should now be a Y-component, of order $\alpha\epsilon$. However, we now see how the theorem (18) enables us to compute the torque on the star to order $\alpha\epsilon$ in terms of $\bar{\mathbf{H}}_1$ etc., but without requiring prior computation of $\mathbf{H}_1'$. Both the flow integral across S_A and the flywheel term are already explicitly of order α, so that only barred quantities need be substituted. In the pressure integral term, $(\mathbf{r} \times \mathbf{n})$ is of order ϵ, since in an isothermal corona the Alfvénic surface is non-spherical (to order α) only because of the angular dependence of the field. The integrals of order ϵ may therefore be computed over the zero-order (spherical) surface S_A. The term of order ϵ vanishes, since

$$\left(\bar{p}_0+\frac{\bar{H}_0^2}{8\pi}\right)\int_{S_A}\epsilon_{ijk}x_j n_k\,dS \quad\propto\quad \int \delta_{jk}\epsilon_{ijk}\,d\tau = 0; \tag{55}$$

and the term of order $\alpha\epsilon$ vanishes since p_0' and $\bar{\mathbf{H}}_0.\mathbf{H}_0'$ are both zero (cf. equation (47)). Thus the first surviving terms are of order $\alpha\epsilon^2$, and so are dropped. (If $(p_0'+\bar{\mathbf{H}}_0.\mathbf{H}_0'/4\pi)$ were not zero, the pressure integral would yield a contribution of order $\alpha\epsilon$, but computation of $\mathbf{H}_1'$ etc, would still be unnecessary for estimates of this order.)

The flywheel term $\alpha\mathbf{k}\times\mathbf{h}$ has a j-component of order $\alpha\epsilon$:

$$\mathbf{j}.\alpha\mathbf{k}\times\int_{S_A-S_1}\mathbf{r}\times(\bar{\rho}_0\bar{\mathbf{V}}_1)\epsilon\,d\tau = \alpha\epsilon\int\mathbf{i}.(\mathbf{r}\times\bar{\rho}_0\bar{\mathbf{V}}_1)\,d\tau, \tag{56}$$

and a similar i-component

$$-\alpha\epsilon\int\mathbf{j}.(\mathbf{r}\times\bar{\rho}_0\bar{\mathbf{V}}_1)\,d\tau. \tag{57}$$

Both these terms are shown below to vanish identically.

The flow integral term has a dominant Y-component

$$\mathbf{j}.\int_{S_A}\bar{\rho}\bar{V}_l n_l\alpha(\mathbf{k}r^2-\mathbf{r}(\mathbf{k}.\mathbf{r}))\,dS = -\alpha\int_{S_A}(\bar{\rho}\bar{V}_l n_l r^2)(\hat{\mathbf{r}}.\mathbf{j})(\hat{\mathbf{r}}.\mathbf{k})\,dS, \tag{58}$$

where the integral is over the perturbed Alfvénic surface, and $\hat{\mathbf{r}}$ is the unit radial vector. The X-component is identical, except for $(\hat{\mathbf{r}}.\mathbf{i})$ replacing $(\hat{\mathbf{r}}.\mathbf{j})$. The unit normal n_l can be written

$$\mathbf{n} = \hat{\mathbf{r}}+\epsilon\mathbf{t}_1 \tag{59}$$

with

$$1 = \mathbf{n}^2 = \hat{\mathbf{r}}^2+2\epsilon(\hat{\mathbf{r}}.\mathbf{t}_1)+O(\epsilon^2)$$

so that

$$\hat{\mathbf{r}}.\mathbf{t}_1 = 0 \tag{60}$$

up to order ϵ. The surface element dS and the area of the sphere r cut off by the cone of solid angle $d\Omega$ are related by

$$r^2\,d\Omega = \hat{\mathbf{r}}.(\mathbf{n}\,dS) = dS(1+O(\epsilon^2)). \tag{61}$$

The integral (58) becomes

$$-\alpha\int(\hat{\mathbf{r}}.\mathbf{j})(\hat{\mathbf{r}}.\mathbf{k})(r_c+\epsilon\bar{r}_1)^2\{(\hat{\mathbf{r}}+\epsilon\mathbf{t}_1).(\bar{\rho}\bar{\mathbf{V}}r^2)_{r_c+\epsilon\bar{r}_1}\}\,d\Omega, \tag{62}$$

where $r_c+\epsilon\bar{r}_1$ is the radius (to order ϵ) of the perturbed Alfvénic surface in a given direction.

The bracketed quantity expands into

$$\begin{aligned}(r_c+\epsilon\bar{r}_1)^2\{\bar{\rho}_0(r_c+\epsilon\bar{r}_1)+\epsilon\bar{\rho}_1(r_c)\}\{\bar{\mathbf{V}}_0(r_c+\epsilon\bar{r}_1)+\epsilon\bar{\mathbf{V}}_1\}.(\hat{\mathbf{r}}+\epsilon\mathbf{t}_1)\\ = (\bar{\rho}_0\bar{V}_0r^2)_{(r_c+\epsilon\bar{r}_1)}+\epsilon\{\bar{\rho}_1(r_c)\bar{V}_0(r_c)r_c^2+\bar{\rho}_0(r_c)\bar{V}_{1r}(r_c)r_c^2\}\\ = (\bar{\rho}_0\bar{V}_0r^2)_{r_c}\left\{1+\epsilon\left(\frac{\bar{\rho}_1}{\bar{\rho}_0}+\frac{\bar{V}_{1r}}{\bar{V}_0}\right)+O(\epsilon^2)\right\},\end{aligned} \tag{63}$$

where the term in $\mathbf{\bar{V}}_0.\mathbf{t}_1$ vanishes by condition (60), and the zero-order continuity condition (22) has been used. Thus the integral (58) becomes

$$-\alpha\epsilon(\bar{\rho}_c\bar{V}_c r_c^4)\int\left(\frac{2\bar{r}_1}{r_c}+\frac{\bar{\rho}_1}{\bar{\rho}_0}+\frac{\bar{V}_{1r}}{\bar{V}_0}\right)(\hat{\mathbf{r}}.\mathbf{j})(\hat{\mathbf{r}}.\mathbf{k})\,d\Omega, \tag{64}$$

since the term independent of ϵ vanishes.

By definition (9), $\bar{r}_1$ is given by

$$4\pi\{\bar{\rho}_0(r_c+\epsilon\bar{r}_1)+\epsilon\bar{\rho}_1(r_c)\}\{\mathbf{\bar{V}}_0(r_c+\epsilon\bar{r}_1)+\epsilon\mathbf{\bar{V}}_1(r_c)\}^2 = \{\mathbf{\bar{H}}_0(r_c+\epsilon\bar{r}_1)+\epsilon\mathbf{\bar{H}}_1(r_c)\}^2, \tag{65}$$

or

$$\left\{\left(\frac{4\pi\bar{\rho}_0\mathbf{\bar{V}}_0^2}{\bar{H}_0^2}\right)_{r_c}+\epsilon r_1\frac{d}{dr}\left(\frac{4\pi\bar{\rho}_0\bar{V}_0^2}{\bar{H}_0^2}\right)\right\}\left\{1+\epsilon\left(\frac{2\bar{V}_{1r}}{\bar{V}_0}+\frac{\bar{\rho}_1}{\bar{\rho}_0}\right)\right\} = \left(1+\frac{2\epsilon\bar{H}_{1r}}{\bar{H}_0}\right). \tag{66}$$

Since r_c is the zero-order Alfvénic radius, this reduces with the help of equation (22) to

$$\left(\frac{\bar{r}_1}{r_c}\right)\left\{2+\frac{r\bar{V}_0'}{\bar{V}_0}\right\} = \left\{\frac{2\bar{H}_{1r}}{\bar{H}_0}-\frac{\bar{\rho}_1}{\bar{\rho}_0}-\frac{2\bar{V}_{1r}}{\bar{V}_0}\right\}, \tag{67}$$

where all quantities are now computed at r_c.

5. COMPUTATION OF THE FIELD TO ORDER ϵ

To compute $\bar{r}_1/r_c$ from equation (67) and the flow integral (64) we need the fields $\bar{\rho}_1$, $\bar{V}_{1r}$ etc., at the zero-order Alfvénic surface. The crucial difference from the model of Section 3 is that the magnetic flux emerging from the star has a prescribed angular dependence of order ϵ, so that it is no longer adequate to assume radial field-streamlines and a spherical Alfvénic surface. Instead we must find the mutually compatible fields $\bar{\rho}_1$, $\mathbf{\bar{V}}_1$, $\mathbf{\bar{H}}_1$ from the (isothermal) wind theory, subject to the condition of a prescribed flux distribution at the stellar surface.

It is convenient in this Section to use spherical polar coordinates (r, θ, λ) based on the magnetic axis Oy in Fig. 1, rather than on Oz. Thus we shall write

$$\left.\begin{aligned} x &= r\sin\theta\sin\lambda \\ y &= r\cos\theta \\ z &= r\sin\theta\cos\lambda, \end{aligned}\right\} \tag{68}$$

it being clearly understood that the angles θ and λ differ from those used in Section 3 and indicated in Fig. 1.

The perturbed radial equation of motion becomes

$$-a^2\frac{d\bar{\rho}_1}{dr}-\frac{GM}{r^2}\bar{\rho}_1 = \bar{\rho}_1\bar{V}_0\frac{d\bar{V}_0}{dr}+\bar{\rho}_0\frac{d}{dr}(\bar{V}_0\bar{V}_{1r}), \tag{69}$$

where a is the isothermal sound speed. The comparative simplicity of the right-hand side is due to $\mathbf{\bar{V}}_0$ and $\mathbf{\bar{H}}_0$ both being radial and with zero curl. Use of the zero-order radial equation reduces equation (69) to

$$\frac{d}{dr}(\bar{V}_0\bar{V}_{1r}) = -a^2\frac{d}{dr}\left(\frac{\bar{\rho}_1}{\bar{\rho}_0}\right), \tag{70}$$

so that

$$a^2\frac{\bar{\rho}_1}{\bar{\rho}_0}+\bar{V}_0\bar{V}_{1r} = a^2\bar{C}_1(\theta,\lambda). \tag{71}$$

The perturbed hydromagnetic integral (6) yields

$$\bar{\mathbf{V}}_1 = \bar{\kappa}_0\bar{\mathbf{H}}_1+\bar{\kappa}_1\bar{\mathbf{H}}_0, \tag{72}$$

so that

$$\bar{V}_{1r} = \bar{\kappa}_0\bar{H}_{1r}+\bar{\kappa}_1\bar{H}_0 = \left(\bar{H}_{1r}+\frac{\bar{\kappa}_1}{\bar{\kappa}_0}\bar{H}_0\right)\frac{\bar{V}_0}{\bar{H}_0}, \tag{73}$$

$$\bar{V}_{1\theta} = \bar{\kappa}_0\bar{H}_{1\theta}, \qquad \bar{V}_{1\lambda} = \bar{\kappa}_0\bar{H}_{1\lambda}. \tag{74}$$

The perturbed continuity equation (3) then yields

$$\bar{\mathbf{H}}_0 . \nabla(\bar{\rho}_0\bar{\kappa}_1+\bar{\rho}_1\bar{\kappa}_0)+\bar{\mathbf{H}}_1 . \nabla(\bar{\rho}_0\bar{\kappa}_0) = 0; \tag{75}$$

and since $\bar{\rho}_0\bar{\kappa}_0$ is constant over each hemisphere, equation (75) integrates to

$$\bar{\rho}_0\bar{\kappa}_1+\bar{\rho}_1\bar{\kappa}_0 = \bar{\rho}_0\bar{\kappa}_0\bar{f}_1(\theta,\lambda). \tag{76}$$

From equations (73) and (76) $\bar{V}_{1r}$ can be rewritten

$$\bar{V}_{1r} = \frac{\bar{\eta}_0}{\bar{\rho}_0}\left\{\bar{H}_{1r}+\left(\bar{f}_1-\frac{\bar{\rho}_1}{\bar{\rho}_0}\right)\bar{H}_0\right\}, \tag{77}$$

and equation (71) reduces to

$$\begin{aligned}\left(\bar{C}_1-\frac{\bar{\rho}_1}{\bar{\rho}_0}\right) &= \left(\frac{\bar{V}_0}{a}\right)^2\left(\frac{\bar{V}_{1r}}{\bar{V}_0}\right)\\ &= \left(\frac{\bar{V}_0}{a}\right)^2\left\{\frac{\bar{H}_{1r}}{\bar{H}_0}+\left(\bar{f}_1-\frac{\bar{\rho}_1}{\bar{\rho}_0}\right)\right\},\end{aligned} \tag{78}$$

or

$$\left(\frac{\bar{\rho}_1}{\bar{\rho}_0}\right)\{\bar{M}^2-1\} = \bar{M}^2\left(\bar{f}_1+\frac{\bar{H}_{1r}}{\bar{H}_0}\right)-\bar{C}_1, \tag{79}$$

where

$$\bar{M}\equiv\frac{\bar{V}_0}{a} = \text{Mach number of zero-order flow.} \tag{80}$$

The non-dimensional radius

$$x = \frac{r}{r_c} \tag{81}$$

is used, $x = 1$ corresponding to the zero-order Alfvénic sphere $r = r_c$.

The perturbation in the density must be finite at the zero-order sonic point $x = x_a$, so that

$$\bar{f}_1+\left(\frac{\bar{H}_{1r}}{\bar{H}_0}\right)_{x_a}-\bar{C}_1 = 0. \tag{82}$$

Below the sonic point $\bar{M}$ becomes exponentially small and $\bar{\rho}_0$ exponentially large as the Bernoulli equation goes over into the condition of hydrostatic support, whereas we shall see below that $\bar{H}_{1r}/\bar{H}_0$ increases only algebraically. Thus $\bar{\rho}_1/\bar{\rho}_0\to\bar{C}_1$ as $x\to 0$; and since we require the density perturbation to become small near the star, we must choose

$$\bar{C}_1 = 0,$$

$$\bar{f}_1 = -\left(\frac{\bar{H}_{1r}}{\bar{H}_0}\right)_{x_a}. \tag{83}$$

Hence

$$\left(\frac{\bar{\rho}_1}{\bar{\rho}_0}\right)\{\bar{M}^2-1\} = \bar{M}^2\left\{\left(\frac{\bar{H}_{1r}}{\bar{H}_0}\right)-\left(\frac{\bar{H}_{1r}}{\bar{H}_0}\right)_{x_a}\right\}, \tag{84}$$

and from equation (77),

$$\left(\frac{\bar{V}_{1r}}{\bar{V}_0}\right)(\bar{M}^2-1) = -\left\{\left(\frac{\bar{H}_{1r}}{\bar{H}_0}\right)-\left(\frac{\bar{H}_{1r}}{\bar{H}_0}\right)_{x_a}\right\}, \tag{85}$$

$$\left(\frac{\bar{\rho}_1}{\bar{\rho}_0}+\frac{\bar{V}_{1r}}{\bar{V}_0}\right) = \left\{\left(\frac{\bar{H}_{1r}}{\bar{H}_0}\right)-\left(\frac{\bar{H}_{1r}}{\bar{H}_0}\right)_{x_a}\right\}; \tag{86}$$

and from equation (67)

$$\left(\frac{\bar{r}_1}{r_c}\right) = \left[\frac{\bar{M}^2\left\{\left(\frac{\bar{H}_{1r}}{\bar{H}_0}\right)+\left(\frac{\bar{H}_{1r}}{\bar{H}_0}\right)_{x_a}\right\}-2\left(\frac{\bar{H}_{1r}}{\bar{H}_0}\right)_{x_a}}{(\bar{M}^2-1)(2+x\bar{M}'/\bar{M})}\right]_{x=1} \tag{87}$$

We now expand $\bar{H}_{1r}/\bar{H}_0$ as follows

$$\frac{\bar{H}_{1r}}{\bar{H}_0} = \epsilon \sum_{n=1}^{\infty} \Phi_n(x_a)\{\phi_n(x) Y_n(\theta, \lambda)\}, \tag{88}$$

with the normalization

$$\phi_n(x_a) = 1. \tag{89}$$

Here $Y_n(\theta, \lambda)$ is a surface harmonic of order n; it satisfies the equation

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\,\frac{\partial Y_n}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2 Y_n}{\partial\lambda^2} = -n(n+1)\,Y_n \tag{90}$$

and can be written as

$$Y_n(\theta, \lambda) = a_0(n)\,P_n(\cos\theta) + \sum_{k=1}^{n}\{a_k(n)\cos k\lambda+b_k(n)\sin k\lambda\}\,P_n^k(\cos\theta) \tag{91}$$

in standard notation. From equations (83)

$$\bar{f}_1 = -\epsilon \sum_{n=1}^{\infty} \Phi_n(x_a)\,Y_n(\theta, \lambda). \tag{92}$$

Similarly, we write

$$\frac{\bar{\kappa}_1}{\bar{\kappa}_0} = \epsilon \sum_{n=1}^{\infty} \Phi_n(x_a)\{\bar{\bar{\psi}}_n(x)\,Y_n(\theta, \lambda)\}, \tag{93}$$

and

$$\frac{\bar{\rho}_1}{\bar{\rho}_0} = \epsilon \sum_{n=1}^{\infty} \Phi_n(x_a)\{\bar{\psi}_n(x)\,Y_n(\theta, \lambda)\}; \tag{94}$$

whence from equations (76) and (92)

$$\bar{\psi}_n(x) + \bar{\bar{\psi}}_n(x) = -1, \tag{95}$$

and from equation (84)

$$\bar{\psi}_n(x) = \frac{\bar{M}^2(1-\phi_n(x))}{(1-\bar{M}^2)}. \tag{96}$$

The θ- and λ-components of the equation of motion reduce respectively to

$$\frac{\partial}{\partial\theta}\bar{H}_{1r} = \frac{\partial}{\partial x}\{x(1-M_A{}^2)\,\bar{H}_{1\theta}\} - \frac{4\pi a^2}{\bar{H}_0}\frac{\partial}{\partial\theta}\bar{\rho}_1 \tag{97}$$

and

$$\frac{\partial}{\partial\lambda}\bar{H}_{1r} = \frac{\partial}{\partial x}\{x(1-M_A{}^2)\sin\theta\,\bar{H}_{1\lambda}\} - \frac{4\pi a^2}{\bar{H}_0}\frac{\partial\bar{\rho}_1}{\partial\lambda} \tag{98}$$

where

$$M_A = \bar{V}_0/(|\bar{H}_0|/\sqrt{4\pi\rho_0}) = \text{Alfvén Mach number of zero-order flow.} \tag{99}$$

The final equation of the system is the divergence condition on the perturbation field $\bar{\mathbf{H}}_1$

$$\frac{1}{x}\frac{\partial}{\partial x}(x^2\bar{H}_{1r}) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\bar{H}_{1\theta}) + \frac{1}{\sin\theta}\frac{\partial}{\partial\lambda}\bar{H}_{1\lambda} = 0. \tag{100}$$

Writing

$$\frac{\bar{H}_{1\theta}}{\bar{H}_0} = \epsilon\sum_1^\infty \Phi_n(x_a)\psi_n(x)\frac{\partial Y_n}{\partial\theta}, \qquad \frac{\bar{H}_{1\lambda}}{\bar{H}_0} = \epsilon\sum_1^\infty \Phi_n(x_a)\frac{\psi_n(x)}{\sin\theta}\frac{\partial Y_n}{\partial\lambda}, \tag{101}$$

we have from equations (90) and (100)

$$\psi_n(x) = \frac{1}{n(n+1)}\,x\frac{\partial}{\partial x}\phi_n(x), \tag{102}$$

and both equations (97) and (98) reduce to

$$\frac{d}{dx}\left[(1-M_A{}^2)\frac{d\phi_n}{dx}\right] - n(n+1)\left[\frac{\phi_n}{x^2} + \frac{\bar{M}}{\bar{M}_c}\frac{(1-\phi_n)}{(1-\bar{M}^2)}\right] = 0. \tag{103}$$

Thus with each surface harmonic of order n there is associated a normalized function $\phi_n(x)$, the solution of equation (103) that is finite at both the sonic point and the Alfvénic point. At the sonic point $\phi_n(x_a) = 1$ by convention, but $\phi_n{}'(x_a)$ is free to be chosen; at the Alfvénic point $\phi_n(1)$ is free, but its choice will fix $\phi_n{}'(1)$; this two-point boundary condition problem therefore fixes simultaneously both $\phi_n{}'(x_a)$ and $\phi_n(1)$ and so also the function $\phi_n(x)$ for all x.

As $x \to 0$, the exponential drop in $\bar{M}$ reduces equation (103) effectively to the *curl-free* condition

$$\phi_n{}'' - \frac{n(n+1)\,\phi_n}{x^2} = 0, \tag{104}$$

so that the Y_n part of the field component $\bar{H}_{1r}$ behaves like

$$Ax^{n-1} + B/x^{n+2}, \tag{105}$$

where A and B are constants (fixed by the finiteness conditions at the two singularities). The fact that the perturbation fields increase inwards more rapidly than the split monopole field $\bar{\mathbf{H}}_0$ is no difficulty, as the wind theory is valid only beyond the chromospheric-coronal interface. One should rather interpret the approximation (105) as showing that the higher order harmonics fall off more rapidly than $\bar{\mathbf{H}}_0$, at least between the coronal base and the sonic point. With the chromospheric temperature much lower than the coronal, we may expect the force-free condition (104) to remain a good approximation down to the photosphere, so that the solutions of equations (103) describe the field $\bar{\mathbf{H}}_1$ all the way from the Alfvénic surface to the photosphere.

If the only contributions to the expansions (88), (91) etc., have $(n-k)$ even, then $\bar{H}_{1r}$ is antisymmetric in the equator, like $\bar{H}_0$; $\bar{H}_{1\theta}$ is symmetric, and therefore vanishes on the equator; and $\bar{\rho}_1(\pi/2)\neq 0$ by equation (94). There is therefore no shift in the cut, but a change of order ϵ in the density within the cut required to balance the total pressure perturbation $\{a^2\bar{\rho}_0(\pi/2)+\bar{H}_0\bar{H}_{1r}(\pi/2)/4\pi\}$ just outside. By contrast, the contributions with $(n-k)$ odd yield $\bar{\rho}_1(\pi/2)$ and $\bar{H}_{1r}(\pi/2)$ both zero, but $\bar{H}_{1\theta}(\pi/2)\neq 0$ and changes sign with $\bar{H}_0$ across the cut. They require no change in the density in the cut, but there is a distortion in the cut, described by a theory similar to that of equations (40)–(45) which describe the distortion associated with the field $\mathbf{H}_0'$. The tangential components of the magnetic stresses at the cut are either zero ($(n-k)$ even) or non-zero but continuous ($(n-k)$ odd), as required for equilibrium.

We have found it convenient to normalize the radial functions in terms of values at the sonic point, whereas in the physical problem it is the flux distribution $\bar{H}_{1r}(x_s)$ at the coronal base that should be taken as prescribed. Since the harmonic Y_n, as defined in equation (91), already has an arbitrary factor, we may write

$$\frac{\bar{H}_{1r}(x_s)}{\bar{H}_0(x_s)} = \epsilon \sum_{n=1}^{\infty} Y_n(\theta, \lambda). \tag{106}$$

Comparing equations (88) and (106), we have

$$\Phi_n(x_a) = 1/\phi_n(x_s). \tag{107}$$

Thus by continuing the integration of equation (103) from the singularity nearer the star towards the coronal base, we find $\phi_n(x_s)$ and so relate the magnetic–velocity–density fields in the expanding corona to the field deep down. Since the radial function $\phi_n(x)$ does not depend on k, the prescribed proportions $a_0(n)$: $a_k(n)$: $b_k(n)$ (k running from 1 to n) are retained throughout the expanding corona.

We can now derive expressions for the X- and Y-torques on the star. In terms of the new coordinates (θ, λ) the unit vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$ along $O(X, Y, Z)$ can be written as

$$\begin{aligned} \mathbf{i} &= \sin\theta \sin\lambda\, \hat{\mathbf{r}} + \sin\lambda \cos\theta\, \hat{\boldsymbol{\theta}} + \cos\lambda\, \hat{\boldsymbol{\lambda}}, \\ \mathbf{j} &= (\cos\theta \sin\chi - \sin\theta \cos\lambda \cos\chi)\, \hat{\mathbf{r}} \\ &\quad -(\sin\theta \sin\chi + \cos\theta \cos\lambda \cos\chi)\, \hat{\boldsymbol{\theta}} + \sin\lambda \cos\chi\, \hat{\boldsymbol{\lambda}}, \\ \mathbf{k} &= (\cos\theta \cos\chi + \sin\theta \cos\lambda \sin\chi)\, \hat{\mathbf{r}} \\ &\quad +(-\sin\theta \cos\chi + \cos\lambda \cos\theta \sin\chi)\, \hat{\boldsymbol{\theta}} - \sin\lambda \sin\chi\, \hat{\boldsymbol{\lambda}}. \end{aligned} \tag{108}$$

Corresponding to the expansions (88) etc., we may write

$$\left(\frac{2\bar{r}_1}{r_c}+\frac{\bar{\rho}_1}{\bar{\rho}_0}+\frac{\bar{V}_{1r}}{\bar{V}_0}\right)_{x=1} = \epsilon \sum_{n=1}^{\infty} F_n(1)\, \Phi_n(x_a)\, Y_n(\theta, \lambda), \tag{109}$$

where

$$\epsilon\Phi_n(x_a)\,F_n(x) \equiv \left(\frac{2\bar{r}_1}{r_c}+\frac{\bar{\rho}_1}{\bar{\rho}_0}+\frac{\bar{V}_{1r}}{\bar{V}_0}\right)_n \tag{110}$$

is defined by the normalized solution $\phi_n(x)$ of equation (103), and the corresponding functions for the radial parts of $\bar{\rho}_1/\bar{\rho}_0$ etc. Thus the Y-component (58) of the flow integral consists of terms such as

$$-\alpha\epsilon\bar{\rho}_c\bar{V}_c r_c{}^4 F_n(1)\Phi_n(x_a)\iint(\cos\theta\sin\chi-\sin\theta\cos\lambda\cos\chi)$$
$$\times(\cos\theta\cos\chi+\sin\theta\cos\lambda\sin\chi)Y_n(\theta,\lambda)\sin\theta\,d\theta\,d\lambda. \tag{111}$$

By use of the orthogonality relations for trigonometric and associated Legendre functions, it is readily seen that only the P_2, $P_2{}^1\cos\lambda$ and $P_2{}^2\cos 2\lambda$ terms survive integration, so that the expression (111) reduces to

$$-\frac{4\pi}{5}\,\alpha\epsilon(\bar{\rho}_c\bar{V}_c r_c{}^4)F_2(1)\Phi_2(x_a)\{\sin 2\chi(\tfrac{1}{2}a_0(2)-a_2(2))-\cos 2\chi(a_1(2))\}. \tag{112}$$

A similar treatment shows that the X-component of the flow integral—given by equation (64) with $(\mathbf{r}.\mathbf{i})$ replacing $(\mathbf{r}.\mathbf{j})$—reduces to

$$-\frac{4\pi}{5}\,\alpha\epsilon(\bar{\rho}_c\bar{V}_c r_c{}^4)F_2(1)\Phi_2(x_a)\cos\chi\, b_1(2). \tag{113}$$

The Y-component (56) of the flywheel term reduces to

$$\epsilon\alpha\int_{S_A-S_1} r\bar{\rho}_0\bar{V}_0\left\{-\frac{\bar{H}_{1\lambda}}{\bar{H}_0}\sin\lambda\cos\theta+\frac{\bar{H}_{1\theta}}{\bar{H}_0}\cos\lambda\right\}d\tau. \tag{114}$$

The partial integral at fixed radius is proportional to

$$\iint\left(\cos\lambda\frac{\partial Y_n}{\partial\theta}-\frac{\sin\lambda\cos\theta}{\sin\theta}\frac{\partial Y_n}{\partial\lambda}\right)\sin\theta\,d\theta\,d\lambda. \tag{115}$$

Of the terms in the expansion (91) of Y_n, only those in $a_1(n)$ survive the λ-integration. A typical term is then

$$a_1(n)\iint\left(\sin\theta\frac{\partial}{\partial\theta}P_n{}^1\cos^2\lambda+\sin^2\lambda\cos\theta P_n{}^1\right)d\theta\,d\lambda$$
$$= a_1(n)\pi\int_0^\pi d\theta\frac{\partial}{\partial\theta}(\sin\theta P_n{}^1) = 0. \tag{116}$$

A similar argument shows that the X-component (57) also vanishes.

We are left with expressions (112) and (113) for the negative precessional torques, computed to order $\alpha\epsilon$. A direct computation of the field $\mathbf{H}_1{}'$ is laborious but has been carried out for the case $\bar{H}_{1r}\propto P_2(\cos\theta)$ so that $a_0(2)\neq 0$ but $a_1(2)$, $a_2(2)$, $b_1(2)$ all vanish. The Y-component of the torque, computed directly from the moment of the Maxwell stress tensor integrated over a sphere deep in the coronal base, agrees satisfactorily with the expression (112).

As noted in Paper II, the X-component of torque causes a rotation of the instantaneous axis of rotation through the star at constant angle χ. This changes the coefficients $a_k(n)$, $b_k(n)$, since the plane through the rotation axis and the frozen-in magnetic axis—defined by $\lambda = 0$, π—steadily rotates through the star, until the coefficient $b_1(2)$ vanishes. If the field $\bar{H}_{1r}$ were symmetric in the plane $OYZ-\mathfrak{R}$-symmetric, in the terminology of Paper II—then all the $b_k(n)$ coefficients would automatically vanish. In general, the effect of the X-torque is to make the Y_2 part of the field $\overline{\mathfrak{R}}$-symmetric.

We now express the Y-torque on the star—the negative of expression (112)—as a fraction of the Z-torque, given by the negative of expression (51):

$$\frac{L_Y}{L_Z} = -0 \cdot 3\epsilon \frac{F_2(1)}{\phi_2(x_s)} [\{\tfrac{1}{2}a_0(2) - a_2(2)\} \sin 2\chi - a_1(2) \cos 2\chi], \tag{117}$$

where we have substituted for $\Phi_2(x_a)$ from equation (107).

To complete the problem we need to solve equation (103) for the case $n = 2$. Once $\phi_2(x)$ is known, then $F_2(1)$ is given from the definitions (109) and (110) and equations (86), (87) etc., as

$$F_2(1) = [\phi_2(1) - 1] + \frac{[\bar{M}^2(1)\{\phi_2(1) + 1\} - 2]}{(\bar{M}^2 - 1)\left(1 + \frac{1}{2}\frac{x\bar{M}'}{\bar{M}}\right)_{x=1}}. \tag{118}$$

The crucial question is the sign of the ratio $F_2(1)/\phi_2(x_s)$ appearing in equation (117).

The solutions of equations (103) form a one-parameter family: once the ratio $x_a \equiv r_a/r_c$ is chosen, then the critical Parker solution fixes the functions $\bar{M}$ and M_A and the value $\bar{M}_c$ at the Alfvénic point $x = 1$. The zero-order ratio of magnetic to thermal energy is known at all points:

$$\frac{\bar{H}_0^2}{8\pi\rho_0 a^2} = \frac{(\bar{H}_c^2/x^4)}{8\pi\left(\frac{\bar{\rho}_0}{\bar{\rho}_c}\right)\bar{\rho}_c a^2} = \frac{4\pi\bar{\rho}_c\bar{M}_c^2 a^2}{8\pi\left(\frac{\bar{\rho}_0}{\bar{\rho}_c}\right)x^4\bar{\rho}_c a^2} = \left(\frac{\bar{\rho}_c}{\bar{\rho}_0}\right)\frac{\bar{M}_c^2}{2x^4} = \frac{\bar{M}_c\bar{M}}{2x^2}. \tag{119}$$

In particular, when we choose a radius $x_s < x_a$ which we can identify with the coronal base of density $\bar{\rho}_s$ and field-strength $\bar{H}_s$, then the ratio (119) becomes the parameter ζ introduced in Paper I. The associated parameter $l = GM/r_s a^2$ relates the coronal base and the sonic point

$$x_a = \tfrac{1}{2} l x_s. \tag{120}$$

Thus there is a single infinity of flows which correspond to the same value of x_a, and so to the same solution of equation (103) and to the same torque ratio (117).

The integrations were performed on the Manchester University ATLAS machine. Series expansions were employed near the two singularities, and the integrations continued using the Runge-Kutta technique. The most important results are presented in Table I. The suffix 2 on ϕ, F, $\bar{\psi}$ has been dropped for short. The suffix N on $[\bar{\rho}_1(1)/\bar{\rho}_0(1)]_N$ etc., implies that we are selecting the *normalized* Y_2 component in e.g., equation (94), without the factor $\epsilon\Phi_2(x_a)$. Except when $r_c/r_a \simeq 1$ the integrations show that the function ϕ approximates closely to a curl-free form some way beyond the sonic point.

6. DISCUSSION OF RESULTS

The dynamical effect of the X- and Y-torques on the star has been studied in Section 4 of Paper II. Since it is unlikely that in any realistic star the magnetic forces over the bulk are comparable with the gravitational or even the centrifugal

TABLE I

x_a	r_c/r_a	$\bar{M}_c$	$M_A(x_a)$	$\phi(1)$	$\phi(\frac{1}{2}x_a)$	$\psi(1) \equiv \left[\frac{\bar{\rho}_1(1)}{\bar{\rho}_0(1)}\right]_N$	$\left[\frac{\bar{r}_1(1)}{r_c}\right]_N$	$\left[\frac{\bar{V}_{1r}(1)}{\bar{V}_0(1)}\right]_N$	$F(1)$
1	1	1	1	1		−0·5	0·5	0·5	1
0·9524	1·05	1·049	0·9300	0·9512	13·56	−0·5370	0·4954	0·4882	0·9421
0·9091	1·1	1·095	0·8687	0·9050	8·74	−0·5713	0·4908	0·4763	0·8865
0·8696	1·15	1·140	0·8146	0·8613	7·006	−0·6031	0·4861	0·4644	0·8335
0·8333	1·2	1·182	0·7665	0·8203	6·118	−0·6324	0·4815	0·4528	0·7834
0·7692	1·3	1·261	0·6850	0·7460	5·232	−0·6843	0·4729	0·4303	0·6919
0·7143	1·4	1·334	0·6185	0·6810	4·804	−0·7284	0·4652	0·4094	0·6114
0·6667	1·5	1·401	0·5632	0·6243	4·561	−0·7658	0·4584	0·3901	0·5412
0·6250	1·6	1·463	0·5167	0·5749	4·409	−0·7975	0·4526	0·3724	0·4800
0·5882	1·7	1·521	0·4769	0·5316	4·310	−0·8247	0·4475	0·3563	0·4266
0·5556	1·8	1·576	0·4426	0·4937	4·240	−0·8479	0·4432	0·3415	0·3801
0·5263	1·9	1·627	0·4127	0·4602	4·190	−0·8678	0·4396	0·3280	0·3393
0·5	2·0	1·674	0·3864	0·4307	4·154	−0·8850	0·4365	0·3157	0·3036
0·4	2·5	1·878	0·2919	0·3244	4·068	−0·9432	0·4269	0·2676	0·1782
0·3333	3·0	2·037	0·2335	0·2603	4·040	−0·9745	0·4232	0·2348	0·1067
0·2857	3·5	2·168	0·1940	0·2184	4·030	−0·9927	0·4222	0·2111	$6·28 \times 10^{-2}$
0·25	4·0	2·279	0·1656	0·1894	4·027	−1·004	0·4225	0·1933	$3·44 \times 10^{-2}$
0·2222	4·5	2·374	0·1442	0·1683	4·024	−1·011	0·4234	0·1795	$1·506 \times 10^{-2}$
0·2	5·0	2·457	0·1276	0·1524	4·023	−1·016	0·4246	0·1683	$1·48 \times 10^{-3}$
0·1818	5·5	2·530	0·1143	0·1399	4·023	−1·019	0·4259	0·1592	$-8·35 \times 10^{-3}$
0·1667	6·0	2·597	0·1034	0·1299	4·023	−1·022	0·4272	0·1515	$-1·57 \times 10^{-2}$
0·1539	6·5	2·656	0·09439	0·1218	4·023	−1·023	0·4285	0·1450	$-2·12 \times 10^{-2}$
0·1333	7·5	2·761	0·08024	0·1093	4·023	−1·025	0·4310	0·1345	$-2·87 \times 10^{-2}$
0·1	10	2·964	0·05809	0·09020	4·023	−1·027	0·4362	0·1169	$-3·74 \times 10^{-2}$
0·08	12·5	3·114	0·04534	0·07935	4·024	−1·027	0·4402	0·1059	$-4·03 \times 10^{-2}$
0·04	25	3·545	0·02124	0·05800	4·025	−1·024	0·4509	0·0814	$-4·03 \times 10^{-2}$
0·02	50	3·933	0·01009	0·04605	4·025	−1·020	0·4590	0·06594	$-3·60 \times 10^{-2}$

forces (e.g., Wright 1969), we shall take the angular momentum vector **K** parallel to the rotation axis:

$$\mathbf{K} = C\alpha\mathbf{k}, \tag{121}$$

with

$$\frac{d}{dt}(C\alpha) = L_Z, \tag{122}$$

$$\dot{\chi} = -L_Y/C\alpha. \tag{123}$$

These equations are identical with equations (71), (81) and (87) of Paper II, except that we have allowed for the star's being in a state of contraction by keeping the moment of inertia C within the time-derivative in equation (122). We have already noted in Section 5 the precession at constant χ due to the X-component of the torque.

It is convenient to combine equations (122) and (123) into

$$\dot{\chi} = -\left(\frac{L_Y}{L_Z}\right)\frac{d}{dt}\log(C\alpha). \tag{124}$$

The torque ratio L_Y/L_Z is given by equation (117), and depends in particular on r_c/r_a through the factor $F_2(1)/\phi_2(x_s)$. Inspection of Table I shows that this factor has a maximum of $\simeq 0·132$ near $r_c/r_a = 1·3$; it stays close to 0·1 until $r_c/r_a \simeq 1·9$ and then falls off to zero near $r_c/r_a = 5$, remaining negative and small beyond.

For the effect studied to be important, we require that the decrease in angular momentum from an initial value $(C\alpha)_i$—e.g., corresponding to a point on the Hayashi track—to a final value $(C\alpha)_f$ should yield a change in χ of order unity.

The maximum that the present theory can yield is

$$|\Delta\chi| \simeq 0 \cdot 04\epsilon \log \{(C\alpha)_i/(C\alpha)_f\}. \tag{125}$$

(We here assume that $a_0(2)$ etc., are of order unity—their absolute order of magnitude is absorbed into ϵ.) Suppose that the star starts at the top of the Hayashi track with a radius $\simeq 100R_\odot$ and rotating near the centrifugal limit, and ends on the main sequence with a radius $\simeq 2R_\odot$ and with a ratio of centrifugal force to gravity of $\simeq 5 \times 10^{-4}$—typical of a magnetic star with a mass $\simeq 2M_\odot$ and a presumed rotation period of about ten days. Then $(C\alpha)_i/(C\alpha)_f \simeq 300$, yielding $|\Delta\chi| \simeq 0 \cdot 23\epsilon$.

Thus even with the extreme value $\epsilon = 1$ the present model predicts only a modest effect; and then only for a limited range of r_c/r_a; for most values of this ratio the effect will be at least a factor 10 smaller.

At first sight the results are disappointing, suggesting that the effect is usually negligible. However, we prefer to interpret them as a further argument in favour of the type of field adopted in Paper I, with the *whole* structure nearly curl-free near the star. The reason for the unsatisfactory results is essentially the smallness of the ratio $|L_Y/L_Z|$, and this is due not only to the factor ϵ appearing in L_Y, but also to the assumption that the basic field falls off like $1/r^2$, whereas our integrations show that the Y_2 part of the perturbation field falls off more rapidly than $1/r^3$. We can reverse the argument leading to equation (125) and say that in order for $|\Delta\chi|$ to be substantial, the star would need to suffer a much greater loss of angular momentum than is consistent with observation. If instead the whole field is e.g., quasi-dipolar, we simultaneously reduce the braking torque on the star, and give the *whole* field an asymmetry with respect to the axis OY, so that it contributes to the precessional torque. And in fact computations—necessarily approximate, but suggestive—done for dipolar fields do in fact yield ratios $|L_Y/L_Z|$ that are $\simeq 0 \cdot 2$ (Selley 1970).

From equation (124), the sign of $\dot{\chi}$ is the same as that of L_Y/L_Z. With $F_2(1)/\phi_2(x_s)$ positive (certainly for that limited range of ratios r_c/r_a for which $\dot{\chi}$ is non-negligible), it follows that

$$\dot{\chi} \gtrless 0 \tag{126}$$

according as

$$[\tfrac{1}{2}a_0(2) - a_2(2)] \sin 2\chi - a_1(2) \cos 2\chi \lessgtr 0. \tag{127}$$

Thus if $a_1(2) = 0$—the Y_2 part of the radial field $\bar{H}_{1r}$ strictly antisymmetric in the equator—then χ approaches 0 or $\pi/2$ according as $[\tfrac{1}{2}a_0(2) - a_2(2)] \gtrless 0$. In particular, if the star has an axisymmetric field, with its P_2 part more concentrated to the magnetic pole, χ tends to 0—the two axes tend to align; with the P_2 part more concentrated to the equator, χ will approach $\pi/2$. With $a_1(2) \neq 0$, the states with zero Y-torque have

$$\tan 2\chi = \frac{a_1(2)}{\tfrac{1}{2}a_0(2) - a_2(2)} \equiv \tan 2\chi_0, \tag{128}$$

say. If $\tan 2\chi_0 > 0$, the two equilibria χ_0, $\chi_0 + \pi/2$ lie respectively between 0 and $\pi/4$ and $\pi/2$ and $3\pi/4$; the smaller angle is stable if both $a_1(2)$ and $(\tfrac{1}{2}a_0(2) - a_2(2))$ are positive, and the larger one if both are negative. If $\tan 2\chi_0 < 0$, then the two equilibria lie between $\pi/4$ and $\pi/2$ and $3\pi/4$ and π respectively, the former case being stable if $a_1(2) > 0$ and $(\tfrac{1}{2}a_0(2) - a_2(2))$ negative, and the latter being stable if the signs are reversed. To get reversal of the integrated Zeeman shifts from the oblique rotator model we need large angles χ; on the present theory, the stable equilibria with $|\chi - \pi/2| < \pi/4$ occur when $(\frac{1}{2}a_0(2) - a_2(2))$ is negative, the sign of $a_1(2)$ determining whether χ is greater than or less than $\pi/2$.

We have already noted that the present model is unsatisfactory, both because it fails in general to give a sufficiently large torque ratio $|L_Y/L_Z|$, and more basically, because the field $\bar{\mathbf{H}}$ is more likely to approximate near the star to a curl-free rather than a split monopolar structure. However, we can expect that solution of the problem with a more realistic field $\bar{\mathbf{H}}$ will again yield a sign for the Y-torque that depends critically on the flux distribution over the stellar surface. And in fact the approximate computations (Selley 1970) done for a basic field that is dipolar near the star do predict asymptotic alignment of the rotation and magnetic axes, in agreement with the corresponding results for $a_1(2)$, $a_2(2)$ both zero and $a_0(2) > 0$ (i.e. for an axisymmetric field $\bar{\mathbf{H}}_1$ with a P_2-flux distribution that is more concentrated towards the magnetic pole).

The general conclusion drawn is that the same subphotospheric hydromagnetic processes that are presumably responsible for the distribution of flux over the stellar surface may also be responsible, through coupling with the stellar wind, for the large mutual inclination of the two axes. One requires a convincing dynamical theory of the flux distribution. It may also be possible to make semi-empirical preliminary checks, e.g., by comparing with observation the computed Zeeman shifts and cross-over effects for those flux distributions that should yield large angles of inclination according to the present theory. The next step in the stellar wind theory is clearly a generalization to more realistic external fields.

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