# Magnetic charges in Supergravity 

Bilyana L. Tomova ${ }^{a}$<br>${ }^{a}$ DAMTP, Centre for Mathematical Sciences, University of Cambridge, Wilberforce Road, Cambridge, CB3 0WA, U.K.<br>E-mail: bt363@cam.ac.uk

ABSTRACT: In this paper we study the dual charges of $\mathcal{N}=1$ supergravity in asymptotically flat space-time. The action considered is the usual supergravity action with a topological contribution. This is the Nieh-Yan term and the magnetic term of the free Rarita-Schwinger field. Through methods of the covariant phase space formalism we construct the charges conjugate to supersymmetry, diffeomorphism and Lorentz transformations. The additional term in the action will lead to new, non-vanishing contributions to these charges. The magnetic diffeomorphism charges are equivalent to the ones previously found for gravity, while the dual supersymmetric charges are new and do not appear for the free RaritaSchwinger field. The dual Lorentz charges serve to regularize the previous two. We find that the asymptotic symmetry group for supergravity can only include globally well-defined super-rotations.

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## 1 Introduction

The usual Maxwell equations that govern the behavior of the electric and magnetic fields, seemingly forbid the existence of magnetic monopoles. This conclusion was first challenged by Dirac. His insight was that by choosing a special couple of gauge potentials, that are not everywhere continuous, and removing the origin of the coordinates, one can construct a magnetic field with a monopole.

Later, in the study of Yang-Mills theory it was shown that one can add an additional gauge and Lorentz invariant term to the action, with a coupling constant $\theta$ (hence the name theta term). In special cases where the coupling constant varies with the background, one can have magnetic field, seemingly caused by monopole. This happens, for example, when one has electromagnetic radiation propagating inside of a topological insulator $(\theta=\pi)$ and outside ( $\theta=0$ ).

There are a lot of parallels between Yang-Mills theory and the Einstein theory of gravity. The study of dual charges is no exception. In the fifties and sixties, a very peculiar solution to the vacuum Einstein equations was discovered - the Taub-NUT metric [7], [8]. This solution is characterized by the NUT parameter, which can be viewed as a gravitomagnetic monopole [16] - the dual charge of the gravitational mass. In recent years it has been shown [18], that the presence of the NUT charge requires that the subleading (with respect to the radial coordinate) components of the angular metric (written in the Bondi gauge [1]), is not continuous on the sphere. What is more, this part of the metric transforms as a gauge field under the action of the asymptotic symmetry group [9]. If one chooses this field to be the gauge potential of the Dirac monopole, one obtains the the Taub-NUT metric [9].

On the other hand, the study of asymptotic symmetries of asymptotically flat spacetimes lead to the discovery of an infinite dimensional group of asymptotic charges [13]. They represent different modes of the mass and angular momentum flux. This begs the question whether an infinite number of gravitational magnetic charges exist. Indeed these charges were found [9], [10]. Subsequently, their existence was better understood in the Hamiltonian formalism of gravity [11]. Adding an additional term to the Einstein-Palatini action, one can derive these charges via the covariant phase space formalism. This additional term is the Holst term. It is similar to the theta term in Yang-Mills theory and does not change the equations of motion. Indeed one can derive the magnetic Yang-Mills charges in a similar fashion [29]. If one couples gravity to fermions, the Holst term needs to be replaced by the Nieh-Yan term [11].

In this paper we extend these efforts to supergravity. The topological term in the action will have two contributions - the Nieh-Yan term and the magnetic term from the free Rarita-Schwinger field [15]. We choose the metric to be asymptotically flat and we write it in Bondi coordinates. The usual asymptotic symmetry group on this metric is the BMS. Here it would be enlarged to in two ways. Firstly, because of supersymmerty it will be extended to the super-BMS group [17]. Furthermore, it will have magnetic charges as well as the usual super-BMS charges.

## 2 Symplectic structure

In order to study the normal and dual Hamiltonian charges for supergravity we first need to set the tools of this analysis. Therefore, in this section, we briefly show how to construct the phase space for a gauge theory. The phase space for a field is an infinite-dimensional symplectic manifold, where every point is a specific field configuration. These configurations are constrained by the equations of motion and the boundary conditions. The conserved charges of the theory are functions on phase space. Mathematically they are are expressed as functionals of the fields. Their commutator relationship and infinitesimal variations are defined via the symplectic form. The steps for defining this symplectic form are straightforward [2], [12].

First we vary the Lagrangian,

$$
\begin{equation*}
\delta L(\phi, \delta \phi) \approx d \Theta(\phi, \delta \phi) \tag{2.1}
\end{equation*}
$$

Here and throughout this paper the sign $\approx$ means equality on shell. The boundary term $\Theta$ is the pre-symplectic potential. The pre-symplectic density and form are defined respectively as,

$$
\begin{align*}
& w\left(\phi, \delta_{1} \phi, \delta_{2} \phi\right)=\delta_{1} \Theta\left(\phi, \delta_{2} \phi\right)-\delta_{2} \Theta\left(\phi, \delta_{1} \phi\right)  \tag{2.2}\\
& \tilde{\Omega}\left(\phi, \delta_{1} \phi, \delta_{2} \phi\right)=\int_{\Sigma} w\left(\phi, \delta_{1} \phi, \delta_{2} \phi\right) \tag{2.3}
\end{align*}
$$

where $\Sigma$ is a Cauchy surface in the space-time manifold. The pre-symplectic form will be in general degenerate on the phase space. Its degenerate directions correspond to transformations, that are not physical, but are rather redundant descriptions of the same field configuration. If we quotient the phase space by the degenerate directions, we get the presymplectic form $\Omega$. The infinitesimal variation of a Hamiltonian charge, conjugate to a field variation $\delta_{\chi} \phi$ is

$$
\begin{equation*}
\not H_{\chi}[\phi]=\int_{\Sigma} w\left(\phi, \delta_{\chi} \phi, \delta \phi\right) \tag{2.4}
\end{equation*}
$$

For gauge transformations this can be re-written as a boundary term [12]. This variation is not in general an exact one form on phase space. This can be due to two to reasonsthe transformation is not canonical, or there is a charge flux through the boundary of the space-time region. In the second case, one can impose the condition that the variation of the field on the boundary is zero and thus make the Hamiltonian charge integrable [2].

The commutator of two integrable charges is defined as [12],

$$
\begin{align*}
& {\left[H_{\chi}, H_{\xi}\right][\phi]=\delta_{\xi} H_{\chi}[\phi]=\int_{\Sigma} w\left(\phi, \delta_{\chi} \phi, \delta_{\xi} \phi\right)=H_{[\xi, \chi]}+K_{\xi, \chi}[\bar{\phi}]}  \tag{2.5}\\
& K_{\xi, \chi}=\int_{\Sigma} w\left(\bar{\phi}, \delta_{\chi} \bar{\phi}, \delta_{\xi} \bar{\phi}\right)+N_{[\xi, \chi]}[\bar{\phi}] \tag{2.6}
\end{align*}
$$

Where $K$ is the central extension and $\bar{\phi}$ is a reference field configuration, which is the starting point of integration of the charges on phase space. The term $N$ is a reference charge.

## 3 The action

The action of gravity with magnetic term, coupled to fermions has been derived in [11]. It is the usual Einstein-Palatini action plus the Nieh-Yan term. The magnetic charges of the free massless Rarita-Schwinger field were studied in [15]. Here, we will combine these efforts and study the magnetic charges of the $\mathcal{N}=1$ supergravity theory.

We work in the 1.5 order formalism [14]. Initially, we consider the that the frame field, spin connection and gravitino field are all independent variables. By varying the action with respect to the spin connection, we can determine its expression in terms of the two other fields. The presence of spinor field, gives rise to a non-vanishing torsion. Once the expression for the spin connection is determined, we put it back in the action and proceed as if the only independent variables are the frame and gravitino field. For more details of this formalism see [14].

The usual action for $\mathcal{N}=1$ supergravity in the Einstein-Palatini formalism is the following,

$$
\begin{equation*}
S_{S G}=\frac{1}{2 \kappa^{2}} \int_{\mathcal{M}} \varepsilon_{a b c d} e^{a} \wedge e^{b} \wedge R^{c d}-\bar{\psi} \wedge \gamma_{5} \gamma \wedge \nabla \psi \tag{3.1}
\end{equation*}
$$

Here and throughout this paper, Greek letters $\mu, \nu$ are space-time indices, and Latin letters $a, b$ are tangent space indices.

The topological term, that will give raise to magnetic charges is,

$$
\begin{equation*}
S_{* S G}=-\frac{i \lambda}{2 \kappa^{2}} \int_{\mathcal{M}} e_{a} \wedge e_{b} \wedge R^{a b}-T^{a} \wedge T_{a}-d(\bar{\psi} \wedge \gamma \wedge \psi) \tag{3.2}
\end{equation*}
$$

where $T^{a}$ is the torsion, defined as $d e^{a}+e^{b} \wedge \omega_{b}^{a}=T^{a}$. The first part is the topological term for gravity coupled to fermions [11]. It is equall to $e_{a} \wedge e_{b} \wedge R^{a b}-T^{a} \wedge T_{a}=d\left(e_{a} \wedge T^{a}\right)$. Re-writing it like this however, hides away important terms, that are technically zero, but whose contribution to the pre-symplectic potential is not. The action is invariant under supersymmetry, diffeomorphism and local Lorentz transformations [14]. These transformations act respectively on the frame and gravitino field in the following way,

$$
\begin{align*}
& \delta_{\epsilon} e_{\mu}^{a}=\frac{1}{2} \bar{\epsilon} \gamma^{a} \psi_{\mu} \quad \delta_{\epsilon} \psi_{\mu}=\nabla_{\mu} \epsilon  \tag{3.3}\\
& \delta_{\xi} e_{\mu}^{a}=\mathcal{L}_{\xi} e_{\mu}^{a}=\xi^{\nu} \partial_{\nu} e_{\mu}^{a}+e_{\nu}^{a} \partial_{\mu} \xi^{\nu} \quad \delta_{\xi} \psi_{\mu}=\mathcal{L}_{\xi} \psi_{\mu}=\xi^{\nu} \nabla_{\nu} \psi_{\mu}+\psi_{\nu} \nabla_{\mu} \xi^{\nu}-\frac{1}{4} \nabla_{\rho} \xi_{\nu} \gamma^{\rho} \gamma^{\nu} \psi_{\mu}  \tag{3.4}\\
& \delta_{\Lambda} e^{a}=\Lambda^{a}{ }_{b} e^{b} \quad \delta_{\Lambda} \psi=-\frac{1}{4} \Lambda_{a b} \gamma^{a b} \psi \tag{3.5}
\end{align*}
$$

where $\epsilon$ is the gauge spinor, $\xi$ is the vector field, generating diffeomorphism, and $\Lambda$ generates a local Lorentz transformation.

The variation of the action is,

$$
\begin{align*}
\delta S= & \frac{1}{2 \kappa^{2}} \int_{\mathcal{M}}\left(2 \varepsilon_{a b c d} e^{b} \wedge R^{c d}-\bar{\psi} \wedge \gamma_{5} \gamma_{a} \nabla \psi\right) \wedge \delta e^{a}  \tag{3.6}\\
& +\left(2 \varepsilon_{a b f d} d e^{a} \wedge e^{b}+2 \varepsilon_{a b c d} e^{a} \wedge e^{b} \wedge \omega^{c}{ }_{f}-\frac{1}{4} \bar{\psi} \wedge \gamma_{5} \gamma \gamma_{f d} \wedge \psi\right) \wedge \delta \omega^{f d}  \tag{3.7}\\
& -\left(\nabla \bar{\psi} \gamma_{5} \wedge \gamma\right) \wedge \delta \psi  \tag{3.8}\\
& +d\left(e_{a b c d} e^{a} \wedge e^{b} \wedge \delta \omega^{c d}-\bar{\psi} \gamma_{5} \wedge \gamma \wedge \delta \psi\right)  \tag{3.9}\\
& -i \lambda\left(e_{a} \wedge e_{b} \wedge \delta \omega^{a b}-2 \delta e^{a} \wedge T_{a}-\delta(\bar{\psi} \wedge \gamma \wedge \psi)\right) \tag{3.10}
\end{align*}
$$

In the first line we expressed the variation of the action with respect to the frame field - this gives the Einstein equation. The second line is variation with respect to the spin connection and it gives the equation for the torsion and contorsion tensor,

$$
\begin{align*}
& \omega_{\mu a b}=\omega_{\mu a b}(e)+K_{\mu a b}(\psi) \quad K_{\mu[a b]}(\psi)=\frac{1}{4}\left(\bar{\psi}_{b} \gamma_{a} \psi_{\mu}-\bar{\psi}_{a} \gamma_{b} \psi_{\mu}-\bar{\psi}_{a} \gamma_{\mu} \psi_{b}\right)  \tag{3.11}\\
& T_{[\mu \nu] b}(\psi)=\frac{1}{2} \bar{\psi}_{\mu} \gamma_{b} \psi_{\nu} \tag{3.12}
\end{align*}
$$

The third line is the equation of motion for the gravitino field. The two final lines are pre-symplectic potential for $\mathcal{N}=1$ supergravity with magnetic term,

$$
\begin{align*}
\Theta= & \frac{1}{2 \kappa^{2}}\left(\varepsilon_{a b c d} e^{a} \wedge e^{b} \wedge \delta \omega^{c d}-\bar{\psi} \wedge \gamma_{5} e^{a} \gamma_{a} \wedge \delta \psi\right)  \tag{3.13}\\
& -\frac{i \lambda}{\kappa^{2}} \underbrace{\left(e_{a} \wedge e_{b} \wedge \delta \omega^{a b}-2 \delta e^{a} \wedge T_{a}-\delta(\bar{\psi} \wedge \gamma \wedge \psi)\right)}_{\Theta^{*}} \tag{3.14}
\end{align*}
$$

Before we proceed with studying the charges we note something useful about the magnetic piece in the symplectic density.

$$
\begin{align*}
& \Theta^{*}(\delta)=\delta\left(e_{a} \wedge e_{b} \wedge \omega^{a b}\right)-2 \delta e^{a} \wedge d e_{a}-\delta(\bar{\psi} \wedge \gamma \wedge \psi)  \tag{3.15}\\
& \Omega^{*}\left(\delta_{1}, \delta_{2}\right)=2 d\left(\delta_{1} e^{a} \wedge \delta_{2} e_{a}\right)=d \delta_{1}\left(e^{a} \wedge \delta_{2} e_{a}\right)-d \delta_{2}\left(e^{a} \wedge \delta_{1} e_{a}\right) \tag{3.16}
\end{align*}
$$

Because $\delta$ is the exterior derivative on phase space.

### 3.1 Asymptotic conditions

The equations of motion, that the fields $e^{a}$ and $\psi$ need to satisfy are already given in section 3. In this section, we introduce the boundary conditions of the fields. Firstly, we demand that the metric is asymptotically flat. In the Bondi gauge [1], [4], [24] an asymptotically flat metric takes the following form,

$$
\begin{align*}
& d s^{2}=-e^{2 \beta} f^{2} d u^{2}-2 e^{2 \beta} d u d r+r^{2} h_{A B}\left(d x^{A}-U^{A} d u\right)\left(d x^{B}-U^{B} d u\right)  \tag{3.17}\\
& h_{A B}=\gamma_{A B}+\frac{C_{A B}}{r}+\frac{d_{A B}}{r^{2}}+O\left(r^{-3}\right) \quad f^{2}\left(u, r, x^{A}\right)=1-\frac{2 M\left(u, x^{A}\right)}{r}+O\left(r^{-1}\right)  \tag{3.18}\\
& e^{2 \beta}=1+O\left(r^{-2}\right) \quad g_{u A}=\frac{1}{2} D_{B} C^{A B}+\frac{1}{r}\left(\frac{1}{4} C_{B C} D_{A} C^{C B}+N_{A}\right) \quad d_{A B}=\frac{1}{4} \gamma_{A B} C_{C D} C^{C D} \tag{3.19}
\end{align*}
$$

Here $D_{A}$ denotes the covariant derivative with resepect to the metric of the unit twosphere $\gamma_{A B}, M\left(u, x^{A}\right)$ is the Bondi mass, that can vary with time and $N_{A}$ is the angular momentum. The subleading tensor $C_{A B}$ characterizes a gravitational wave and is related to the time derivative of the mass. Residual gauge freedom can be used to make it traceless and to set the determinant $\operatorname{det}\left(g_{A B}\right)$ to be $r^{2} \operatorname{det}\left(\gamma_{A B}\right)$.

Since we work in the Einstein-Palatini formalism, the metric is not the gravitational field of interest. Rather we need the frame fields $e_{\mu}^{a}$, which satisfy $e_{\mu}^{a} e_{\nu}^{b} \eta_{a b}=g_{\mu \nu}$, where $\eta_{a b}$ is the Minkowski metric. The frame fields are defined up to a local Lorentz transformation. Our choice for them is,

$$
\begin{align*}
& e^{0}=\frac{e^{\beta}}{f} d r+e^{\beta} f d u \quad e^{1}=\frac{e^{\beta}}{f} d r  \tag{3.20}\\
& e^{i}=r E_{A}^{i}\left(d x^{A}-U^{A} d u\right) \quad E_{A}^{i} E_{B}^{j} \delta_{i j}=h_{A B} \quad i, j \in\{2,3\} \tag{3.21}
\end{align*}
$$

Any variation should preserve the leading terms of these fields.
The gauge condition for the gravitino field is,

$$
\begin{equation*}
\gamma^{\mu} \psi_{\mu}=0 \tag{3.22}
\end{equation*}
$$

This simplifies the equations of motion for $\psi$. The boundary conditions on $\psi$, will be dictated by the fact that supersymmetric transformations must preserve the gauge and asymptotic behavior of the metric,

$$
\begin{equation*}
\delta_{\epsilon} g_{r r}=\bar{\epsilon} \gamma_{r} \psi_{r}=0 \quad \delta_{\epsilon} g_{u r}=\frac{1}{2} \bar{\epsilon} \gamma_{(u} \psi_{r)}=O\left(r^{-2}\right) \quad \delta_{\epsilon} g_{r A}=\frac{1}{2} \bar{\epsilon} \gamma_{(r} \psi_{A)}=0 \tag{3.23}
\end{equation*}
$$

To leading order this and the fermionic gauge condition are satisfied when,

$$
\begin{array}{lll}
\epsilon \sim O(1) & \psi_{A} \sim O(1) \quad \psi_{u} \sim O\left(r^{-1}\right) \quad \psi_{r} \sim O\left(r^{-2}\right) \\
\gamma^{u} \psi_{r}=0 & \gamma^{u(0)} \psi_{A}^{0}=0 \quad \gamma^{u(0)} \psi_{u}^{(-1)}=0 \quad \gamma^{A(-1)} \psi_{A}=0 \tag{3.25}
\end{array}
$$

Notice that, because the matrix $\gamma^{u}$ is nilpotent the equations of the second line do not imply that $\psi_{r}, \psi_{A}^{(0)}$ or $\psi_{u}^{(-1)}$ are 0 . The justification for these boundary conditions are the following. Firstly, $\epsilon \sim O(1)$ comes from the fact that the commutator of the supersymmetric
transformations of the metric, should be a diffeomorphism. This is better explained in the next section. Next, because $\gamma_{A B}$ is fixed $\psi_{A}$ should be $O(1)$. It cannot be more sub leading, because the subleading tensors of $h_{A B}$ depend on $C_{A B}$. The trace-free condition requires that $\gamma^{A(-1)} \psi_{A}^{(0)}=0$. Combining this with $\delta_{\epsilon} g_{r A}$, we can reach the conclusion that $\psi_{r}^{(-1)}=0$. Last but not least, remembering that the supersymmetric variation, should also respect the linearized equations of motion for the metric, we must also have to the following constraint on $\psi_{A}^{(-1)}$,

$$
\begin{equation*}
\gamma_{(A}^{(0)} \psi_{B)}^{(0)}+\gamma_{(A}^{(1)} \psi_{B)}^{(-1)}=\frac{1}{2} \gamma_{A B} C^{D C} \gamma_{(D}^{(1)} \psi_{C)}^{0} \Rightarrow \gamma^{A(-1)} \psi_{A}^{(-1)}=0 \tag{3.26}
\end{equation*}
$$

Finally it is important to discuss what variations are allowed on the gravitino field. Super-translations preserve the leading order components of the field, while super-rotations do not. From super-symmetric transformation we require that $\delta_{\epsilon} \psi_{A}=O\left(r^{-1}\right)$ and $\delta_{\epsilon} \psi_{u}=$ $O\left(r^{-2}\right)$. This is discussed in detail in section 4.3. Finally, for integrability of the supersymmetric charges we require that any other non-specific variation $\delta \psi_{A}$ is subleading. The fact that super-rotations violate this condition will result in a contribution to the central charge.

## 4 Charges

### 4.1 Diffeomorphism charges

The diffeomorphisms that respect the boundary and gauge conditions of the fields form the BMS group. The BMS group is generated by vector fields, that take the following form [1], [4],

$$
\begin{align*}
& \xi^{u}=f\left(x^{A}\right)+\frac{u}{2} D_{B} f^{B}\left(x^{A}\right) \quad \xi^{A}=f^{A}-\partial_{B} \xi^{u} \int \frac{1}{r^{2}} e^{2 \beta} h^{A B} d r  \tag{4.1}\\
& \xi^{r}=-r \partial_{u} \xi^{u}+\frac{1}{2} \nabla^{r} \xi^{u}-\partial_{B} \xi^{u} \int U^{B} d r \quad \partial_{u} \xi^{u}=D_{A} f^{A} \tag{4.2}
\end{align*}
$$

These generators can be divided in two subcategories - super-translations and superrotations. The super-translation generators depend only on an arbitrary function on the sphere $f\left(x^{A}\right)$. When this function is taken to be the lowest order spherical harmonics, the transformation reduces to ordinary translation. Similarly, the super-rotation generators depend only on a vector field on the two sphere $f^{B}\left(x^{A}\right)$ and includes the Lorentz subgroup.

In order to calculate the diffeomorphism charge, we use the fact that the variation of any quantity is just its Lie derivative, and we can use Cartan's magic formula,

$$
\begin{gather*}
\Omega\left(\delta, \delta_{\xi}\right) \approx \delta \Theta\left(\delta_{\xi}\right)-d \iota_{\xi} \Theta(\delta)-\iota_{\xi} \delta L=\delta\left(\Theta\left(\delta_{\xi}\right)-\iota_{\xi} L\right)-d \iota_{\xi} \Theta(\delta)  \tag{4.3}\\
\not H_{\xi}=\int_{\Sigma} \delta\left(\Theta\left(\delta_{\xi}\right)-\iota_{\xi} L\right)-\int_{\partial \Sigma} \iota_{\xi} \Theta(\delta) \tag{4.4}
\end{gather*}
$$

We see that the Hamiltonian charge, naturally splits into integrable and non-integrable part. The latter quantifies the flux of the given charge through the Cauchy surface, which here is chosen to be future null-infinity. At the past boundary of $\mathcal{I}^{+}$this flux is set to be zero, by choosing appropriate $u$ fall-off conditions for gravitational radiation.

The calculations for the Hamiltonian charge density are in appendix D. The formula for the charge is,

$$
\begin{align*}
\not H_{\xi}+i \lambda \not{ }^{\prime} \tilde{H}=\frac{1}{2 \kappa^{2}} \int_{\partial \Sigma} & \delta\left(\varepsilon_{a b c c e} e^{a} \wedge e^{b} \iota_{\xi} \omega^{c d}-\bar{\psi} \wedge \gamma_{5} \gamma \iota_{\xi} \psi\right)  \tag{4.5}\\
& -\iota_{\xi}\left(\varepsilon_{a b c d} e^{a} \wedge e^{b} \wedge \delta \omega^{c d}-\bar{\psi} \wedge \gamma_{5} \gamma \wedge \delta \psi\right)+i \lambda\left(\mathcal{L}_{\xi} e^{a} \wedge \delta e_{a}\right) \tag{4.6}
\end{align*}
$$

First we note, that due to the boundary conditions on the fermionic field 3.24, the part of the charge, that depends on $\psi$, is finite and it vanishes for super-translations. It is integrable for both super-translations and super-rotations. The integrability crucially relies on $\gamma^{A(-1)} \psi_{A}^{0}=\gamma^{u(0)} \psi_{A}^{(0)}=0$. We see that in the context of supergravity, the finiteness of the global Lorentz generators of the gravitino field is a direct consequence of the asymptotic flatness of the metric. One does not need to make additional assumptions for the asymptotic behavior of the field, as is the case for the free Rarita-Schwinger field [25].

The rest of the above expression is the usual diffeomorphism charge of pure gravity. The charge has normal and magnetic part. More specifically, comparing with the results from [4], [11] and [23], the super-translation and super-rotation charges are respectively,

$$
\begin{align*}
H_{S T}+i \lambda \tilde{H}_{S T} & =\frac{1}{2 \kappa^{2}} \int_{\mathcal{I}_{-}^{+}} \sqrt{\gamma} f M+i \lambda \int_{\mathcal{I}_{-}^{+}}-C_{B[D} D_{A]} D^{B} \xi^{u}  \tag{4.7}\\
H_{S R}+i \lambda \tilde{H}_{S R} & =\frac{1}{2 \kappa^{2}} \int_{\mathcal{I}_{-}^{+}} \sqrt{\gamma}-\frac{1}{4} D_{A} f^{A} C^{2}+f^{A} N_{A}+\bar{\psi}_{[A} \gamma_{5} \gamma_{B]} \psi_{C} f^{C}  \tag{4.8}\\
& +i \lambda \int_{\mathcal{I}_{-}^{+}}-C_{B[D} D_{A]} D^{B} \xi^{u}+\frac{1}{4} f^{B} C_{C[A} D_{|B|} C_{D]}^{C}-\frac{1}{4} f_{[A} D_{D]} C^{2} \tag{4.9}
\end{align*}
$$

where $C^{2}=C_{A B} C^{A B}$.

### 4.2 Lorentz charges

In addition to diffeomorphisms, we also need to consider local Lorentz transformations. They do not affect the metric and therefore one expects that they are unphysical. The frame field transforms under diffeomorphisms and local Lorentz transformations as,

$$
\begin{equation*}
\delta e_{\mu}^{a}=\xi^{\nu} \partial_{\nu} e_{\mu}^{a}+e_{\nu}^{a} \partial_{\mu} \xi^{\nu}+\Lambda^{a}{ }_{b} e_{\mu}^{b} \tag{4.10}
\end{equation*}
$$

If we take into consideration only the diffeomorphisms, the gauge and boundary conditions are not satisfied. We need local Lorentz transformations to regularize this. Their expressions up to leading order are,

$$
\begin{align*}
& \Lambda_{i}^{0}=-e_{\mu}^{0} e_{i}^{A} \partial_{A} \xi^{\mu} \quad \Lambda_{i}^{1}=-e_{r}^{1} e_{i}^{A} \partial_{A} \xi^{r}  \tag{4.11}\\
& \Lambda_{0}^{1}=-\frac{1}{f^{2}} \partial_{r} \xi^{r} \quad \Lambda_{j}^{i}=-e_{j}^{A} \mathcal{L}_{f_{B}} e_{A}^{i}+\delta_{j}^{i} D_{A} f^{A} \tag{4.12}
\end{align*}
$$

Notice that in the third line, $\delta_{i}^{j} D_{A} f^{A}$ is the symmetric part of the term $-e_{j}^{A} \mathcal{L}_{f}{ }^{B} e_{A}^{i}$, making $\Lambda_{i j}$ asymmetric as it should be. Furthemore, we observe that these transformations are needed to preserve the boundary conditions, only if the BMS transformation in question is a super-rotation. Intuitively, this can be understood as the fact that super-rotations are a generalization of Lorentz transformations. Therefore, part of their effect on the frame field needs to be undone by a local Lorentz transformation.

We now proceed to study the charges related to these transformations. We know from [11] that $\delta_{\Lambda}\left(\varepsilon_{a b c d} e^{a} \wedge e^{b} \wedge \delta \omega^{c d}\right)=0$. We also have,

$$
\begin{equation*}
\varepsilon_{a b c d} e^{a} \wedge e^{b} \wedge \delta_{\Lambda} \omega^{c d}=d \mathcal{Q}_{\Lambda}+2 \varepsilon_{a b c d} T^{a} \wedge e^{b} \Lambda^{c d}=d \mathcal{Q}_{\Lambda}+2 \psi \gamma_{5} \gamma_{b c d} \wedge \psi \wedge e^{b} \Lambda^{c d} \tag{4.13}
\end{equation*}
$$

We know that the hamiltonian charge from $\mathcal{Q}$ in this setup is zero [11]. From the fermionic part we have:

$$
\begin{align*}
& \delta_{\Lambda} \psi=-\frac{1}{4} \Lambda_{a b}(x) \gamma^{a b} \psi \quad \delta_{\Lambda} e^{b}=\Lambda(x)^{b}{ }_{a} e^{a}  \tag{4.14}\\
& \bar{\psi} \wedge \gamma_{5} \gamma \wedge \delta_{\Lambda} \psi=\bar{\psi} \wedge \gamma_{5} \gamma \gamma_{c d} \wedge \psi \Lambda^{c d} \tag{4.15}
\end{align*}
$$

Which simplifies the second term in 4.13. Furthermore, thanks to the identity A. 5 (which works in the same way if one replaces $\omega_{a b}$ by $\Lambda_{a b}$ ), we can see that $\delta_{\Lambda}\left(\bar{\psi} \wedge \gamma_{5} \gamma \wedge \delta \psi\right)=$ 0 . We conclude that the normal Lorentz charge vanishes.

On the other hand, the magnetic term from the Lorentz charges does not vanish. This seems bizarre at a first glance. However, this is actually useful as it serves to regularize the diffeomorphism magnetic charge.

$$
\begin{equation*}
\left(\mathcal{L}_{\xi} e^{a}+\Lambda^{a}{ }_{b} e^{b}\right) \wedge \delta e_{a} \sim O(1) \tag{4.16}
\end{equation*}
$$

### 4.3 SUSY charges

The charges conjugate to supersymmetry transformations are more complicated. We will study their expression in this section. First we work out the transformations of the fields and the spin connection.

$$
\begin{align*}
\delta_{\epsilon} e_{\mu}^{a}= & \frac{1}{2} \bar{\epsilon} \gamma^{a} \psi_{\mu} \quad \delta_{\epsilon} \psi_{\mu}=\nabla_{\mu} \epsilon  \tag{4.17}\\
\delta_{\epsilon} \omega_{\mu a b}(e)= & \frac{1}{2} \partial_{\mu}\left(\bar{\epsilon} \gamma_{[a} \psi_{b]}\right)+e_{a}^{\rho} \partial_{\mu}\left(\bar{\epsilon} \gamma_{b} \psi_{\rho}\right)+\frac{1}{2} \bar{\epsilon} \gamma_{[a} \psi^{[\rho} e_{b]}^{\sigma]} \partial_{\sigma} g_{\rho \mu}+\frac{1}{2} e_{[a}^{\rho} e_{b]}^{\sigma} \partial_{\rho}\left(\bar{\epsilon} \gamma_{(\sigma} \psi_{\mu)}\right)  \tag{4.18}\\
4 \delta_{\epsilon} K_{\mu a b}= & \nabla_{b} \bar{\epsilon} \gamma_{a} \psi_{\mu}+\frac{1}{2} \bar{\epsilon} \gamma_{b} \psi^{\nu} \bar{\psi}_{\nu} \gamma_{a} \psi_{\mu}+\bar{\psi}_{b} \gamma_{a} \nabla_{\mu} \epsilon  \tag{4.19}\\
& -\frac{1}{2} \bar{\psi}_{a} \gamma^{c} \psi_{b} \bar{\epsilon} \gamma_{c} \psi_{\mu}+\nabla_{a} \bar{\epsilon} \gamma_{\mu} \psi_{b}+\frac{1}{2} \bar{\epsilon} \gamma_{a} \psi^{\nu} \bar{\psi}_{\nu} \gamma_{\mu} \psi_{b}-(a \leftrightarrow b)
\end{align*}
$$

Where $\epsilon$ is the gauge spinor. These transformations need to preserve the boundary and gauge conditions of the frame and gravitino field. This leads to various constraints. We first look at the gauge spinor,

$$
\begin{equation*}
\left[\delta_{1}, \delta_{2}\right] g_{\mu \nu}=\nabla_{(\mu} \xi_{\nu)} \quad \xi_{\nu}=\frac{1}{2} \bar{\epsilon}_{1} \gamma_{\nu} \epsilon_{2} \tag{4.20}
\end{equation*}
$$

This looks like the transformation of the metric under diffeomorphism. We already know what conditions should be satisfied by the vector field $\xi^{\mu}$ - it should generate a member of the BMS group. In particular $\xi^{u}$ should not be a function of $r$. This is achieved if the following property is satisfied by the gauge spinor $\epsilon$,

$$
\begin{equation*}
\frac{e^{\beta}}{f} \bar{\epsilon}_{1}\left(\gamma_{0}+\gamma_{1}\right) \epsilon_{2} \neq g(r) \Rightarrow \epsilon_{i}=\sqrt{\frac{f}{e^{\beta}}} f_{i}(\theta, \phi)+\gamma_{r}^{(0)} \rho_{i} \tag{4.21}
\end{equation*}
$$

From the above we see that $\epsilon^{-1}=-\frac{M}{r} \epsilon^{0}+\frac{\gamma^{u(0)}}{r} \rho^{(-1)}$, (because $\left.\gamma^{u(0)} \times \gamma^{u(0)}=0\right)$. From the gauge condition 3.22 at first and second order in $r$ the equations of motion for the gauge spinor are,

$$
\begin{align*}
O(1) & : \gamma^{u 0} \partial_{u} \epsilon^{0}=0  \tag{4.22}\\
O\left(r^{-1}\right) & : \gamma^{A(-1)} \partial_{A} \epsilon^{0}+\frac{1}{2} \cot \theta \gamma^{2} \epsilon^{0}+\gamma^{1} \epsilon^{0}+\gamma^{u 0} \partial_{u} M \epsilon^{0}+\gamma^{u 0} \partial_{u} \epsilon^{-1}=0  \tag{4.23}\\
& \gamma^{A(-1)} \partial_{A} \epsilon^{0}+\frac{1}{2} \cot \theta \gamma^{2} \epsilon^{0}+\gamma^{1} \epsilon^{0}=0 \tag{4.24}
\end{align*}
$$

This is the equation for covariantly constant spinor and it agrees with the result from [17]. It's expression is given in the appendix. This tells us that the function $\xi^{u}=\frac{1}{2} \bar{\epsilon}_{1} \gamma^{u} \epsilon_{2}$ is some linear combination of the lowest order spherical harmonics. Thus we recognize that the vector field $\xi^{\mu}$ from 4.20 generates only ordinary translations. The same conclusion was reached in [26]. Notice that in order to obtain higher order spin $1 / 2$ spherical harmonics, [3], and consequently more interesting super-translations, we would need a different constant multiplying $\gamma^{1}$. This would mean setting $U^{0}=\lambda \neq 1$. However Einstein equations tells us that $U^{0}=\frac{1}{2} R[\gamma]$. Knowing what the Ricci tensor is for the metric on the sphere, we can infer that this re-scaling would also re-scale the coefficients of $\gamma_{A B}$, so as to leave the above equation invariant. Therefore the equations of motion and the asymptotic flatness impose, that the commutator of two supersymmetric transformations can only be an ordinary translation. It would be interesting to see whether more interesting results can be obtained for other metrics. Finally, a quick calculation can show that by choosing $\rho^{(-1))}=-\gamma^{0} M \epsilon^{0}$, we can set $\delta_{\epsilon} \psi_{u} \sim O\left(r^{-2}\right)$, which is going to be useful.

With these conditions in mind, we now proceed to the calculations of the fermionic charges, arising from supersymmetry. The calculations for the part of the charge, arising from the gravitino term in the action, are almost identical to the calculation for gauge charges of the free, massless Rarita-Schwinger field [15],

$$
\begin{align*}
& \delta \Theta\left(\delta_{\epsilon}\right)-\delta_{\epsilon} \Theta(\delta) \approx \delta\left(\Theta\left(\delta_{\epsilon}\right)-I_{\epsilon} L\right)-d I_{\epsilon} \Theta(\delta)  \tag{4.25}\\
& \delta\left(\Theta\left(\delta_{\epsilon}\right)-I_{\epsilon} L\right) \approx \delta\left(\Theta\left(\delta_{\epsilon}\right)\right)=\delta\left(\bar{\psi} \wedge \gamma_{5} \gamma \wedge \nabla \epsilon\right)  \tag{4.26}\\
& \approx d\left(\delta\left(\bar{\psi} \wedge \gamma_{5} \gamma \epsilon\right)+\bar{\psi} \wedge \gamma_{5} T^{a} \gamma_{a} \epsilon\right)  \tag{4.27}\\
& I_{\epsilon} \Theta(\delta)=\bar{\epsilon} \gamma_{5} \gamma \wedge \delta \psi \tag{4.28}
\end{align*}
$$

where the operator $I_{\epsilon}$ is defined as $I_{\epsilon}=\epsilon \cdot \frac{\delta}{\delta \psi}$. In the expression for $\delta_{\epsilon} \Theta$ I have ignored the term $\bar{\psi} \wedge \gamma_{5} \delta_{\epsilon} \gamma \wedge \psi$ because this is a four fermion term that vanishes thanks to the Fierz identity A.6. Simiralry, $\bar{\psi} \wedge \gamma_{5} T^{a} \gamma_{a} \epsilon$ vanishes thanks to the cyclic identity for spinors A.8.

We now turn to the gravitational part of the symplectic form.

$$
\begin{equation*}
\Omega\left(\delta, \delta_{\epsilon}\right)=\varepsilon_{a b c d} \int_{\Sigma} \delta\left(e^{a} \wedge e^{b} \wedge \delta_{\epsilon} \omega^{c d}\right)-\delta_{\epsilon}\left(e^{a} \wedge e^{b} \wedge \delta \omega^{c d}\right) \tag{4.29}
\end{equation*}
$$

First we note two things $-\int_{\Sigma} \varepsilon_{a b c d} e^{a} \wedge e^{b} \wedge B^{c d}=\int_{\Sigma} B_{\mu}^{\mu \nu} d \Sigma_{\nu}$, for any one-form with two antisymmetrized tangent space indices. Also, the connection form $\omega$ has two parts $\omega=\omega(e)+K(\psi)$. From the expression of the supersymmetric variation of the contorsion tensor $K_{\mu a b} 4.19$ and the gauge condition 3.22, we can conclude that $\varepsilon_{a b c d} e^{a} \wedge e^{b} \wedge \delta_{\epsilon} K^{c d}=0$. Any variation $\delta$ should preserve this gauge condition. Furthermore, most of the terms of $\delta_{\epsilon}\left(\varepsilon_{a b c d} e^{a} \wedge e^{b} \wedge \delta K^{c d}\right)$ again vanish by the gauge condition. The remaining ones are of the form $n_{\nu} \delta_{\epsilon} \bar{\psi}^{\nu} \gamma^{\mu} \delta \psi_{\mu}$. This will be sub-leading because we have set $\nabla_{u} \epsilon \sim O\left(r^{-2}\right)$. We are left with the usual symplectic form of four-dimensional pure gravity. Its expression is,

$$
\begin{align*}
\Omega\left(\delta_{1}, \delta_{2}\right) & =\frac{1}{2 \kappa^{2}} \int_{\mathcal{I}^{+}} \sqrt{\gamma} d u d \theta d \phi \varepsilon^{A B} \varepsilon_{C D} \delta_{1}\left(C_{A}^{D}\right) \delta_{2}\left(\partial_{u} C_{B}^{C}\right)-(1 \leftrightarrow 2)  \tag{4.30}\\
\Omega\left(\delta, \delta_{\epsilon}\right) & =\frac{1}{2 \kappa^{2}} \int_{\mathcal{I}^{+}} \sqrt{\gamma} d u d \theta d \phi \varepsilon^{A B} \varepsilon_{C D} \bar{\epsilon} \gamma^{D} \psi_{A} \delta\left(\partial_{u} C_{B}^{C}\right)-\left(\delta \leftrightarrow \delta_{\epsilon}\right)  \tag{4.31}\\
& =-\frac{1}{2 \kappa^{2}} \int_{\mathcal{I}^{+}} d u d \theta d \phi \partial_{u}\left(\bar{\epsilon} \gamma_{5} \gamma_{D} \psi_{A} \delta C_{B}^{D} \varepsilon^{A B}\right) \tag{4.32}
\end{align*}
$$

where we have used the fact that $\gamma_{5}=\gamma_{0} \gamma_{1} \gamma_{2} \gamma_{3}$ and $\gamma_{0} \gamma_{1} \psi_{A}^{(0)}=-\psi_{A}^{(0)}$, which is a consequence of $\gamma^{u(0)} \psi_{A}^{(0)}=0$. Furthermore, writing this expression as a total derivative is possible because the leading order terms of the spinors and frame field do not depend on time. At subleading order this will no longer be true.

The overall the charge is,

$$
\begin{equation*}
\delta H_{\epsilon}+i \lambda \nless \tilde{H}_{\epsilon}=-\frac{1}{2 \kappa^{2}} \int_{\partial \Sigma} \delta\left(\bar{\epsilon} \gamma_{5} \gamma \wedge \psi\right)+\bar{\epsilon} \gamma_{5} \delta \gamma \wedge \psi-\bar{\epsilon} \gamma_{5} \gamma \wedge \delta \psi-i \lambda \bar{\epsilon} \delta \gamma \wedge \psi \tag{4.33}
\end{equation*}
$$

We see that unlike the Rarita-Schwinger case, there is contribution to the magnetic charge. Furthermore because the boundary metric is fixed and because of 3.24 both charges are finite. We can make both charges integrable by requiring that any non-specific variation (that is not BMS or susy) of $\psi_{A}^{0}$ vanishes to leading order. In the case where the variation is
supersymmetry or super-translation this still holds. When the variation is a super-rotation one may naively think that the charge is not-integrable. However, this is not the case. A simple way to understand why this is false is by noting that $\delta_{\xi} H_{\epsilon}=-\delta_{\epsilon} H_{\xi}$. However, we already know that $H_{\xi}$ is integrable up to some flux term [18]. Therefore, the obstruction that one gets can only be interpreted as a central charge. This is studied in detail in the last section.

It is worth noting that the integrable part of the normal charge is the same as the gauge charge of the Rarita-Schwinger field [15] up to an overall numerical constant. In terms of the metric and gravitino components, the integrable charges are,

$$
\begin{equation*}
H_{\epsilon}+i \lambda \tilde{H}_{\epsilon}=\frac{1}{2 \kappa^{2}} \int_{\partial \Sigma} 2 \bar{\epsilon}^{(0)} \gamma_{5} \gamma_{D}^{(1)} \psi_{[A}^{(0)} C_{B]}^{D}-i \lambda \bar{\epsilon}^{(0)} \gamma_{D}^{(1)} \psi_{[A}^{(0)} C_{B]}^{D} \tag{4.34}
\end{equation*}
$$

We see that both charges depend on the tensor $\tilde{C}_{A B}=C^{C}{ }_{(B} \varepsilon_{A) C}$, which also enters in the expression for the dual diffeomorphism charges.

## 5 Algebra of integrable charges

Before studying the algebra of the charges, we need to specify the algebra of the generators $\xi$ and $\epsilon$. The algebra for the BMS generators is well-know [12],

$$
\begin{align*}
& \xi_{f, f^{A}}=\left\{\xi_{f_{1} \cdot f_{1}^{A}}, \xi_{f_{2}, f_{2}^{A}}\right\}  \tag{5.1}\\
& f=f_{1}^{A} D_{A} f_{2}-\frac{1}{2} f_{1} D_{A} f_{2}^{A}-(1 \leftrightarrow 2) \quad f^{A}=f_{1}^{B} D_{B} f_{2}^{A}-(1 \leftrightarrow 2) \tag{5.2}
\end{align*}
$$

Furthermore, we already know that the anti-commutator of two supersymmetric parameters is a translation, characterized by the function $\bar{\epsilon}_{1} \gamma^{u(0)} \epsilon_{2}$. The problem now is to work the bracket of $\epsilon$ with $\xi$. We define the bracket between a BMS vector and the gauge spinor to be,

$$
\begin{equation*}
\left[\epsilon, \xi_{R}\right]=\frac{1}{2} \mathcal{L}_{\xi} \epsilon=f^{A} \partial_{A} \epsilon-\frac{1}{4} D_{A} f^{A}\left(\mathbb{1}+\gamma_{0} \gamma_{1}\right) \epsilon+O\left(r^{-1}\right) \tag{5.3}
\end{equation*}
$$

We see that to leading order, this vanishes for a super-translations. Asymptotically the generators of super-translation and supersymmetric transformations commute, as they should. Using the fact that $\gamma^{u(0)}\left(\mathbb{1}+\gamma_{0} \gamma_{1}\right)=2 \gamma^{u(0)}$, one can verify that with this bracket the following Jacobi identity is satisfied.

$$
\begin{equation*}
\left\{\epsilon_{1},\left[\epsilon_{2}, \xi_{S R}\right]\right\}-\left\{\epsilon_{2},\left[\epsilon_{1}, \xi_{S R}\right]\right\}=\left[\xi_{R},\left\{\epsilon_{1}, \epsilon_{2}\right\}\right] \tag{5.4}
\end{equation*}
$$

This result agrees with that in [28], where the same bracket was established in the super-symmetric extension of BMS.

We will now discuss an interesting result from this bracket. The fact that commutator of gauge spinors can only give an ordinary translation implies that only Lorentz transformations are allowed in the super BMS group. In super gravity we cannot extend the Lorentz group to include arbitrary conformal transformations of the celestial sphere. In order to understand why this is the case, using the notation of [27], we write the vector field on the sphere $f^{A}$, the spherical harmonics $f_{m . n}$ and the covariantly constant spinors $\epsilon$ as,

$$
\begin{align*}
& l_{n}=f_{n}^{A}=-z^{n+1} \partial_{z} \quad \bar{l}_{n}=\bar{f}_{n}^{A}=-\bar{z}^{n+1} \partial_{\bar{z}} \quad f_{m, n}=\frac{1}{1+z \bar{z}} z^{n} \bar{z}^{m}  \tag{5.5}\\
& \epsilon_{+}^{T}=e^{i \frac{\phi}{2}}\left(a e^{\frac{i \theta}{2}}, a e^{-\frac{i \theta}{2}}, b e^{\frac{i \theta}{2}}, b e^{-\frac{i \theta}{2}}\right)  \tag{5.6}\\
& \epsilon_{-}^{T}=e^{i \frac{-\phi}{2}}\left(c e^{\frac{i \theta}{2}},-c e^{-\frac{i \theta}{2}}, d e^{\frac{i \theta}{2}},-d e^{-\frac{i \theta}{2}}\right)  \tag{5.7}\\
& f_{0,0}=\bar{\epsilon}_{+} \gamma^{u(0)} \epsilon_{+}+\bar{\epsilon}_{-} \gamma^{u(0)} \epsilon_{-} \quad f_{1,1}=\bar{\epsilon}_{+} \gamma^{u(0)} \epsilon_{+}-\bar{\epsilon}_{-} \gamma^{u(0)} \epsilon_{-}  \tag{5.8}\\
& f_{1,0}=\bar{\epsilon}_{-} \gamma^{u(0)} \epsilon_{+} \quad f_{0,1}=\bar{\epsilon}_{+} \gamma^{u(0)} \epsilon_{-} \tag{5.9}
\end{align*}
$$

Where $z=e^{i \phi} \cot \frac{\theta}{2}$ is the standard stereographic projection, complex coordinate. the We can re-write the BMS algebra in the following way [27],

$$
\begin{align*}
& {\left[l_{n}, l_{m}\right]=(n-m) l_{n+m} \quad\left[\bar{l}_{n}, \bar{l}_{m}\right]=(n-m) \bar{l}_{n+m} \quad\left[\bar{l}_{n}, l_{m}\right]=0}  \tag{5.10}\\
& {\left[l_{k}, f_{n . m}\right]=\left(\frac{k+1}{2}-m\right) f_{m+k, n} \quad\left[\bar{l}_{k}, f_{n . m}\right]=\left(\frac{k+1}{2}-n\right) f_{m+k, n}} \tag{5.11}
\end{align*}
$$

From these equations and the Jacobi identity (5.4) we see that whenever $n>1$ the commutators of $\left[l_{n}, \epsilon\right]$ and $\left[\bar{l}_{n}, \epsilon\right]$ are equal to spinors, whose components must depend on higher order spin $1 / 2$ spherical harmonics. This is not allowed by the equations of motion for the gauge spinor. Therefore the allowed asymptotic symmetry group is the super-Poincaré group, plus the infinite dimensional super-translations.

### 5.1 Normal charges

In this section we study the algebra of the normal Hamiltonian charges, conjugate to diffeomorphic and supersymmetric transformations. They are a representation of an extension of the BMS group - the super-BMS group [17]. It contains a copy of the super-Poincaré group. The commutators of the BMS group are already well understood [12]. Curiously the presence of the gravitino field does not modify the central charge. Here, we will look only at the commutators, involving supersymmetric charges.

We first look at the commutator of two supersymmetric charges. This is the same as the commutator for gauge charges of the free Rarita-Schwinger field [22].

$$
\begin{equation*}
\left[H_{\epsilon 1}, H_{\epsilon 2}\right]=\frac{1}{\kappa^{2}} \int_{\partial \Sigma} \nabla_{[\mu} \bar{\epsilon}_{1} \gamma_{\sigma]} \gamma_{5} \epsilon_{2}-\left(\epsilon_{1} \leftrightarrow \epsilon_{2}\right) \tag{5.12}
\end{equation*}
$$

In [22] it is described in detail how the above expression is a BMS charge, generated by $\xi^{\mu}=\bar{\epsilon}_{1} \gamma^{\mu} \epsilon_{2}$. This can only be a super-translation, so the commutator is equal to,

$$
\begin{equation*}
\left[H_{\epsilon 1}, H_{\epsilon 2}\right]=\frac{1}{2 \kappa^{2}} \int_{\partial \Sigma} \sqrt{\gamma} \bar{\epsilon}_{1}^{0} \gamma^{u(0)} \epsilon_{2}^{0} M-\left(\epsilon_{1} \leftrightarrow \epsilon_{2}\right) \tag{5.13}
\end{equation*}
$$

Notice that there is no central charge. What used to be the central charge for this commutator in the free Rarita-Schwinger theory is now the super-translation charge [15], [22]. As already explained, $\bar{\epsilon}_{1}^{0} \gamma^{u(0)} \epsilon_{2}^{0}$ is just a lowest order spherical harmonic, and therefore the charge above is a global translation. Again, we note that other supertranslations can not be obtained through the commutator of fermionic charges.

Now we proceed to study the commutator of a BMS and a supersymmetric charge. Looking at the expressions from subection 4.3 and section 2 we can identify the commutator and the central charge as,

$$
\begin{equation*}
\left[H_{\epsilon}, H_{\xi}\right]=\frac{1}{\kappa^{2}} \int_{\partial \Sigma}\left(\mathcal{L}_{\xi} \bar{\epsilon}\right) \gamma_{5} \gamma \wedge \psi+\bar{\epsilon}^{(0)} \gamma_{5} \gamma_{D}^{(1)} \psi_{[A}^{(0)} \mathcal{L}_{\xi} C_{B]}^{D}=H_{[\epsilon, \xi]}+K_{\epsilon, \xi} \tag{5.14}
\end{equation*}
$$

Notice that we obtain the same result if replace $\delta$, by $\delta_{\xi}$ in 4.33 and use the act that $\int_{\mathcal{I}_{-}^{+}} \delta_{\xi}\left(\bar{\epsilon} \gamma_{5} \gamma \wedge \psi\right)=0$.

For super-translations $\mathcal{L}_{\xi} \epsilon=O\left(r^{-1}\right)$, so $H_{[\epsilon, \xi]}=0$ in agreement with the Poincarré algebra. The central charge is,

$$
\begin{equation*}
K_{\epsilon, \xi_{S T}}=\frac{i \lambda}{\kappa^{2}} \int_{\partial \Sigma} \bar{\epsilon} \gamma_{5} \gamma^{B(-1)} \psi_{[A}^{(0)} D_{D]} D_{B} f \tag{5.15}
\end{equation*}
$$

It is worth nothing that for $l=0,1$ the spherical harmonics satisfy $D_{A} D_{B} f=\gamma_{A B} f$ and the integral will vanish. Curiously, for higher order spherical harmonics it will not necessarily vanish. This is also due to the fact that, as shown in appendix $\mathrm{C}, \psi_{A}$ can depend on arbitrarily high spherical modes. This can be restricted in order to make the central charge vanish. This expression also have a term involving $\partial_{u} C_{A B}$. This is the nonintegrable part and it vanishes at the past boundary of $\mathcal{I}^{+}$. The integrable central charge for the super-rotations is,
$K_{\epsilon, \xi_{S R}}=\frac{i \lambda}{\kappa^{2}} \int_{\partial \Sigma} D_{C} f^{C} \bar{\epsilon} \gamma_{5} \gamma^{B} \psi_{[A} C_{D] B}+\bar{\epsilon} \gamma_{5} \gamma^{B(-1)} \psi_{[A}^{(0)} D_{|C|} f_{D]} C_{B}^{C}+\bar{\epsilon} \gamma_{5} \gamma^{B(-1)} \psi_{[A}^{(0)} C_{D]}^{C} D_{[B} f_{C]}$

We notice strong a parallel between this central charge and the one for super-rotations, given in equation 7.11 of [11].

### 5.2 Dual charges

The algebra of the dual charges is simpler to study. The commutator of two dual diffeomorphsim charges is already studied in [11]. It was established that the magnetic BMS charges satisfy the same algebra as the normal ones, but with slightly different central extension. Here we will look at commutators, involving dual fermioinic charges,

$$
\begin{equation*}
\left\{\tilde{H}_{\epsilon 1}, \tilde{H}_{\epsilon 2}\right\}=\frac{i \lambda}{2 \kappa^{2}} \int_{\partial \Sigma} \nabla_{[\mu} \bar{\epsilon}_{1} \gamma_{\sigma]} \epsilon_{2}-\left(\epsilon_{1} \leftrightarrow \epsilon_{2}\right)=0 \tag{5.17}
\end{equation*}
$$

This seems to defer from the usual super-Poincaré algebra. However it is worth nothing, that the dual global translation charge also vanishes [10]. Furthermore, the above expression will not vanish if $C_{A B}$ is not a continuous function on the sphere. This is precisely the condition we need for the existence of BMS magnetic charge as well. We now proceed to compute the commutator of supersymmetric and diffeomorphism charge.

$$
\begin{align*}
& \left\{\tilde{H}_{\epsilon}, \tilde{H}_{\xi}\right\}=\frac{i \lambda}{4 \kappa^{2}} \int_{\partial \Sigma} \delta_{\epsilon}\left(e^{a} \wedge \mathcal{L}_{\xi} e_{a}\right)-\mathcal{L}_{\xi}\left(e^{a} \wedge \delta_{\epsilon} e_{a}\right)=  \tag{5.18}\\
& \frac{i \lambda}{2 \kappa^{2}} \int_{\partial \Sigma} e^{a} \wedge\left(\mathcal{L}_{\xi} \epsilon\right) \gamma_{a} \psi+\delta_{\epsilon} e^{a} \wedge \mathcal{L}_{\xi} e_{a}=\tilde{H}_{[\epsilon, \xi]}+\tilde{K}_{\epsilon, \xi} \tag{5.19}
\end{align*}
$$

The first term is the magnetic supersymmetric charge, defined from a gauge spinor $\mathcal{L}_{\xi} \epsilon$. The second one is the central charge. Its expression is almost identical to the one for the normal charges,

$$
\begin{align*}
& \tilde{K}_{\epsilon, \xi_{S T}}=\frac{i \lambda}{2 \kappa^{2}} \int_{\partial \Sigma} \bar{\epsilon} \gamma^{B(-1)} \psi_{[A}^{(0)} D_{D]} D_{B} f  \tag{5.20}\\
& \tilde{K}_{\epsilon, \xi_{S R}}=\frac{i \lambda}{2 \kappa^{2}} \int_{\partial \Sigma} D_{C} f^{C} \bar{\epsilon} \gamma^{B} \psi_{[A} C_{D] B}+\bar{\epsilon} \gamma^{B(-1)} \psi_{[A}^{(0)} D_{|C|} f_{D]} C_{B}^{C}+\bar{\epsilon} \gamma^{B(-1)} \psi_{[A}^{(0)} C_{D]}^{C} D_{[B} f_{C]} \tag{5.21}
\end{align*}
$$

## 6 Conclusion

In this paper, a new type of charges was discovered for $\mathcal{N}=1$ supergravity. They are dual to the usual diffeomorphisms and supersymmetric charges. The diffeomorphism dual charges are the same as the ones studied before [10], [11]. The supersymmetric magnetic charges are new and do not appear for the free Rarita-Schwinger field [15]. We saw that their integrability and finiteness crucially depend on the conditions on the gravitino field, imposed by the asymptotically flat metric. It would be interesting to study the dual charges for more complicated supergravity theories, or for more exotic metrics. The magnetic charges in supergravity can have potential interesting applications for black holes in supergravity like the extreme Reissner-Nordström black hole.

## A Conventions and useful identities

The choice of $\gamma$ matrices is,

$$
\gamma^{0}=-i\left(\begin{array}{ll}
0 & \mathbb{1}  \tag{A.1}\\
\mathbb{1} & 0
\end{array}\right) \quad \gamma^{j}=-i\left(\begin{array}{cc}
0 & \sigma^{j} \\
-\sigma^{j} & 0
\end{array}\right) \quad \gamma^{5}=-i\left(\begin{array}{cc}
\mathbb{1} & 0 \\
0 & -\mathbb{1}
\end{array}\right) \quad C=\left(\begin{array}{cc}
0 & \mathbb{1} \\
-\mathbb{1} & 0
\end{array}\right)
$$

The covariant and Lie derivative of a spinor are defined as,

$$
\begin{align*}
\nabla_{\mu} \psi_{\nu} & =\partial_{\mu} \psi_{\mu}-\Gamma_{\mu \nu}^{\lambda} \psi_{\lambda}+\frac{1}{4} \omega_{\mu a b} \gamma^{a b} \psi_{\nu}  \tag{A.2}\\
\mathcal{L}_{\xi} \psi_{\nu} & =\xi^{\mu} \nabla_{\mu} \psi_{\nu}+\psi_{\mu} \nabla_{\nu} \xi^{\mu}-\frac{1}{4} \nabla_{\mu} \xi_{\rho} \gamma^{\mu} \gamma^{\rho} \psi_{\nu}  \tag{A.3}\\
& =\xi^{\mu} \partial_{\mu} \psi_{\nu}+\psi_{\mu} \partial_{\nu} \xi^{\mu}-\frac{1}{4} e_{a[\mu} \mathcal{L}_{\xi} e_{\nu]}^{a} \gamma^{\mu \nu} \psi_{\nu} \tag{A.4}
\end{align*}
$$

Up to leading order the last term is simplified by local a Lorentz transformation.
Simple, but useful commutation property,

$$
\begin{equation*}
-\lambda_{a b} \gamma^{a} \gamma^{b} \gamma_{5} e_{c \nu} \gamma^{c}+\lambda_{a b} \gamma_{5} e_{c \nu} \gamma^{c} \gamma^{a} \gamma^{b}+4 \lambda_{a b} e_{\nu}^{a} \gamma^{b}=0 \tag{A.5}
\end{equation*}
$$

for any $\lambda_{a b}$ that is antisymmetric in $a$ and $b$.
Fierz identity in 4 dimensions [20],

$$
\begin{align*}
\left(\bar{\lambda}_{1} \lambda_{2}\right) \lambda_{3 a}= & -\frac{1}{4}\left(\bar{\lambda}_{1} \lambda_{3}\right) \lambda_{2 a}-\frac{1}{4}\left(\bar{\lambda}_{1} \gamma_{5} \lambda_{3}\right) \gamma_{5} \lambda_{2}-\frac{1}{4}\left(\bar{\lambda}_{1} \gamma_{\mu} \lambda_{3}\right)\left(\gamma^{\mu} \lambda_{2}\right)_{a}  \tag{A.6}\\
& +\frac{1}{4}\left(\bar{\lambda}_{1} \gamma_{\mu} \gamma_{5} \lambda_{3}\right)\left(\gamma^{\mu} \gamma_{5} \lambda_{2}\right)_{a}+\frac{1}{8}\left(\bar{\lambda}_{1} \gamma_{\mu \nu} \lambda_{3}\right)\left(\gamma^{\mu \nu} \lambda_{2}\right)_{a} \tag{A.7}
\end{align*}
$$

Cyclic identity [14],

$$
\begin{equation*}
\bar{\lambda}_{[1 \mid} \gamma_{a} \lambda_{|2|} \bar{\lambda}_{\mid 3]} \gamma^{a} \tag{A.8}
\end{equation*}
$$

Where $\lambda_{i}$ are arbitrary spinors and the letter $a$ is a spinor index.

## B Covariantly constant spinor

The differential equation, satisfied by a covariantly constant spinor $\chi$, is,

$$
\begin{equation*}
\gamma^{A} \partial_{A} \chi+\gamma^{1} \chi+\frac{1}{2} \cot \theta \gamma^{2} \chi=0 \tag{B.1}
\end{equation*}
$$

With the particular choice we have made for the gamma matrices, the equations for the first and second components are coupled, and are the same as the equations for the third and forth. In particular we have,

$$
\begin{align*}
i \partial_{\theta} \chi_{2}-\frac{1}{\sin \theta} \partial_{\phi} \chi_{1}+\frac{i}{2} \cot \theta \chi_{2} & =\chi_{2}  \tag{B.2}\\
i \partial_{\theta} \chi_{1}-\frac{1}{\sin \theta} \partial_{\phi} \chi_{2}+\frac{i}{2} \cot \theta \chi_{1} & =-\chi_{1} \tag{B.3}
\end{align*}
$$

We assume separation of variables and impose, $\chi_{1}=e^{-i \frac{\theta}{2}} f(\phi)$ and $\chi_{2}=e^{i \frac{\theta}{2}} h(\phi)$,

$$
\begin{gather*}
e^{-i \frac{\theta}{2}} \partial_{\phi} f-\frac{i}{2} e^{-i \frac{\theta}{2}} h=0 \quad e^{i \frac{\theta}{2}} \partial_{\phi} h-\frac{i}{2} e^{i \frac{\theta}{2}} f=0  \tag{B.4}\\
\Rightarrow \partial_{\phi} h=\frac{i}{2} f \quad \partial_{\phi} f=\frac{i}{2} h \tag{B.5}
\end{gather*}
$$

Setting $h=a e^{i \frac{\phi}{2}}+b e^{-i \frac{\phi}{2}}$ and $f=c e^{i \frac{\phi}{2}}+d e^{-i \frac{\phi}{2}}$ we have $a=c$ and $b=-d$.

## C Equations of motion for $\psi$

In order to study the the charges and their properties, we need first to solve (to some extent) the equations of motion.Since the metric is already fixed, in this section we need to study only the gravitino field. The equations for motion for it, subject to the gauge condition 3.22, are $\gamma^{\mu} \nabla_{\mu} \psi_{\nu}=0$. Taking into consideration 3.24, at different orders and for the different components of the gravitino, the exact equations are,

- $\psi_{A}$ :

$$
\begin{align*}
O(1) & : \gamma^{u(0)} \partial_{u} \psi_{A}=0  \tag{C.1}\\
O\left(r^{-1}\right) & : \gamma^{B} \partial_{B} \psi_{A}+\frac{1}{2 r} \cot \theta \gamma^{2} \psi_{A}-\gamma^{C} \Gamma_{C A}^{B} \psi_{B}+h_{A B} \gamma^{B} \psi_{u}+\gamma^{u(0)} \partial_{u} \psi_{A}^{-1}=0 \tag{C.2}
\end{align*}
$$

- $\psi_{u}$ :

$$
\begin{align*}
& O\left(r^{-1}\right): \gamma^{u} \partial_{u} \psi_{u}=0  \tag{C.3}\\
& O\left(r^{-2}\right): r \gamma^{A} \partial_{A} \psi_{u}+\frac{1}{2} \cot \theta \gamma^{2} \psi_{u}+\gamma^{u} \partial_{u} \psi_{u}^{-2}-r \gamma^{A} \partial_{u} C_{A}^{B} \psi_{B}=0 \tag{C.4}
\end{align*}
$$

- $\psi_{r}$ :

$$
\begin{align*}
O\left(r^{-3}\right): & \gamma^{A(-1)} \partial_{A} \psi_{r}^{(-2)}+\frac{1}{2} \cot \theta \gamma^{2} \psi_{r}^{(-2)}-\gamma^{1} \psi_{r}-\gamma^{A(-1)} \psi_{A}^{(-1)}  \tag{C.5}\\
& +\gamma^{C(-1)} \frac{1}{2} C_{C}^{A} \psi_{A}^{(0)}-\gamma^{A(-2)} \psi_{A}^{(0)}=0 \tag{C.6}
\end{align*}
$$

We note that $K_{\mu \nu \rho} \gamma^{\mu} \gamma^{\nu} \gamma^{\rho} \sim O\left(r^{-2}\right)$ and that's why it does not enter these equations. Using the gauge condition 3.22 at second order the last equation can be re-written as,

$$
\begin{equation*}
\gamma^{A(-1)} \partial_{A} \psi_{r}^{(-2)}+\frac{1}{2} \cot \theta \gamma^{2} \psi_{r}^{(-2)}+\gamma^{C(-1)} \frac{1}{2} C_{C}^{A} \psi_{A}^{(0)}+\gamma^{u(0)} \psi_{u}^{(-2)}=0 \tag{C.7}
\end{equation*}
$$

We would need to components of the gravitino field to study the commutators of the charges. This is an integral evaluated at $u \rightarrow-\infty$. We would like the field to be finite in this limit, so we assume it can be decomposed as $\psi_{\mu}=\varphi_{\mu}\left(x^{A}\right)+\phi_{\mu}\left(u, x^{A}\right)$ with $\lim _{u \rightarrow-\infty} \phi_{\mu}=0$. Let's look at the equation for $\varphi_{u}^{(-1)}$.

$$
\begin{equation*}
\gamma^{A} \partial_{A} \varphi_{u}+\frac{1}{2} \cot \theta \gamma^{2} \varphi_{u}^{(-1)}=0 \tag{C.8}
\end{equation*}
$$

Remembering that $\gamma^{u(0} \varphi_{u}^{(-1)}=0$, this equation is solved by the spinor $\sqrt{\sin \theta} \varphi_{u}^{(-1)}=$ $\left(a z^{-m}, a z^{-m},-b \bar{z}^{-m}, b \bar{z}^{-m}\right)^{T}$, where $z$ is the stereographic coordinate $z=e^{i \phi} \tan \frac{\theta}{2}$, and $a$ and $b$ are arbitrary constants. Taking into consideration that $\gamma^{A(-1)} \psi_{A}^{(0)}$, the equation for $\varphi_{\phi}^{(0)}$ is,

$$
\begin{equation*}
\gamma^{A} \partial_{A} \varphi_{\phi}+\frac{1}{2} \cot \theta \gamma^{2} \varphi_{\phi}^{(-1)}+\gamma_{\phi} \varphi_{u}^{(-1)}=0 \tag{C.9}
\end{equation*}
$$

If we write $\varphi_{\phi}=(\iota \lambda)$, the equations for the components of $\lambda$ are,

$$
\begin{align*}
& \partial_{\theta}\left(\lambda_{1}+\lambda_{2}\right)-\frac{i}{\sin \theta} \partial_{\phi}\left(\lambda_{1}+\lambda_{2}\right)+\frac{1}{2} \cot \theta\left(\lambda_{1}+\lambda_{2}\right)=0  \tag{C.10}\\
& \partial_{\theta}\left(\lambda_{1}-\lambda_{2}\right)+\frac{i}{\sin \theta} \partial_{\phi}\left(\lambda_{1}-\lambda_{2}\right)+\frac{1}{2} \cot \theta\left(\lambda_{1}-\lambda_{2}\right)=\frac{2 b \sin \theta}{\sqrt{\sin \theta}} e^{i m \phi} \cot ^{m} \frac{\theta}{2} \tag{C.11}
\end{align*}
$$

We remember that because $\gamma^{u(0)} \varphi_{A}^{(0)}=0, \lambda_{1}+\lambda_{2}=0$. This means that $\lambda_{1}=-\lambda_{2}=$ $\frac{1}{2}\left(\lambda_{1}-\lambda_{2}\right)=-b \frac{\cos \theta}{\sqrt{\sin \theta}} z^{m}$. Similralrly, $\iota_{1}=\iota_{2}=\frac{1}{2}\left(\iota_{1}+\iota_{2}\right) \frac{\cos \theta}{\sqrt{\sin \theta}} \bar{z}^{m}$. From the expression of $\psi_{\phi}$ one can easily work out $\psi_{\theta}$. Curiously they take a form, very similar to the generators of the super-rotations.

$$
\begin{align*}
\psi_{m}^{\phi} & =\frac{\cos \theta}{\sqrt{\sin \theta}}\left(a z^{-m}, a z^{-m},-b \bar{z}^{-m}, b \bar{z}^{-m}\right)  \tag{C.12}\\
\psi_{m}^{\theta} & =i \sin \theta \frac{\cos \theta}{\sqrt{\sin \theta}}\left(a z^{-m}, a z^{-m}, b \bar{z}^{-m},-b \bar{z}^{-m}\right) \tag{C.13}
\end{align*}
$$

Because the algebra of super-rotations has already been established, one can easily work the Lie derivative of this siponor, along a generator of the BMS group (after one corrects with Local Lorentz transformations,

$$
\begin{equation*}
\mathcal{L}_{l_{n}+\bar{l}_{n}} \psi_{m}^{A}=\left(n-m-\frac{1+\cos ^{2} \theta}{2 \cos \theta}\right) \psi_{n+m}^{A} \tag{C.14}
\end{equation*}
$$

It may seem odd that the spinor change under the BMS group, even though the components of the metric, on which its equation of motion depend do not. However, this linear variation can be compensated by $\delta_{\xi} \psi_{u}$ and $\delta_{\xi} \psi_{A}^{(-1)}$ in the linearized equation of moiton.

From C.7, we see that the leading order $u$ - independent part of of $\psi_{r}$ depends on $\psi_{A}$ and the tensor $C_{A B}$. For Minkowksi, $\psi_{r}$ will have the same solution as $\psi_{u}$. Lastly, $m$ should be a half-integer because $\psi$ is a spinor.

## D Diffeomorphism variation of the pre-symplectic potential

In this appendix we provide in detail the calculations necessary for the deriving the expression for the diffeomorphism charge.

$$
\begin{align*}
& 2 \kappa^{2}\left(\Theta\left(\delta_{\xi}\right)-\iota_{\xi} L\right)=  \tag{D.1}\\
& \varepsilon_{a b c d} e^{a} \wedge e^{b} \wedge \mathcal{L}_{\xi} \omega^{c d}-\bar{\psi} \wedge \gamma_{5} e^{a} \gamma_{a} \wedge \mathcal{L}_{\xi} \psi  \tag{D.2}\\
& -\iota_{\xi}\left(\varepsilon_{a b c d} e^{a} \wedge e^{b} \wedge R^{c d}-\bar{\psi} \wedge \gamma_{5} \gamma \wedge \nabla \psi\right)  \tag{D.3}\\
& \approx \varepsilon_{a b c d} e^{a} \wedge e^{b} \wedge \mathcal{L}_{\xi} \omega^{c d}-\bar{\psi} \wedge \gamma_{5} e^{a} \gamma_{a} \wedge \mathcal{L}_{\xi} \psi  \tag{D.4}\\
& \varepsilon_{a b c d} e^{a} \wedge e^{b} \wedge \mathcal{L}_{\xi} \omega^{c d}=\varepsilon_{a b c d} e^{a} \wedge e^{b} \wedge\left(d \iota_{\xi} \omega^{c d}+\iota_{\xi} d \omega^{c d}\right)  \tag{D.5}\\
& \approx d\left(\varepsilon_{a b c d} e^{a} \wedge e^{b} \iota_{\xi} \omega^{c d}\right)-2 \varepsilon_{a b c d}\left(T^{a} \wedge e^{b} \iota_{\xi} \omega^{c d}\right)+2 \varepsilon_{a b c d} \xi^{a} e^{b} \wedge R^{c d} \tag{D.6}
\end{align*}
$$

We obtain in the end three terms. One is a total derivative and the other two will be simplified.

$$
\begin{align*}
& \bar{\psi} \wedge \gamma_{5} e^{a} \gamma_{a} \wedge \mathcal{L}_{\xi} \psi=\bar{\psi} \gamma_{5} e^{a} \gamma_{a}\left(\iota_{\xi} d \psi+d \iota_{\xi} \psi\right)  \tag{D.7}\\
& \approx \bar{\psi} \gamma_{5} e^{a} \gamma_{a} \iota_{\xi} d \psi+d\left(\bar{\psi} \gamma_{5} e^{a} \gamma_{a} \iota_{\xi} \psi\right)-\frac{1}{4} \bar{\psi} \gamma_{5} \wedge \omega^{c d} \gamma_{c d} \wedge e^{a} \gamma_{a} \iota_{\xi} \psi+\bar{\psi} \gamma_{5} e_{b} \wedge \omega^{a b} \gamma_{a}{ }_{\xi} \psi  \tag{D.8}\\
& -\bar{\psi} \gamma_{5} T^{a} \gamma_{a} \iota_{\xi} \psi  \tag{D.9}\\
& \approx \bar{\psi} \gamma_{5} e^{a} \gamma_{a} \iota_{\xi} d \psi+d\left(\bar{\psi} \gamma_{5} e^{a} \gamma_{a} \iota_{\xi} \psi\right)+\frac{1}{4} \bar{\psi} \gamma_{5} \wedge \gamma \wedge \gamma_{c d} \omega^{c d}{ }_{l} \psi-\bar{\psi} \gamma_{5} T^{a} \gamma_{a}{ }_{\xi} \psi \tag{D.10}
\end{align*}
$$

Notice that we have omitted the quantity $\bar{\psi} \wedge \gamma_{5} e^{a} \gamma_{a} \wedge \psi\left(\omega_{a b} \gamma^{a b}-\nabla_{a} \xi_{b} \gamma^{a} \gamma^{b}\right)$ in the Lie derivative of the gravitino, because it is compensated by the Lorentz transformation. As we already saw, the charge from this transformation is 0 .

$$
\begin{align*}
& 2 \varepsilon_{a b c d} \xi^{a} e^{b} \wedge R^{c d}=\bar{\psi} \wedge \gamma_{5} \iota_{\xi} \gamma \nabla \psi \approx \bar{\psi} \wedge \gamma_{5} \gamma \iota_{\xi} \nabla \psi  \tag{D.12}\\
& 2 \varepsilon_{a b c d}\left(T^{a} \wedge e^{b} \iota_{\xi} \omega^{c d}\right)=\frac{1}{4} \bar{\psi} \wedge \gamma_{5} \gamma \gamma_{c d} \iota_{\xi} \omega^{c d} \wedge \psi  \tag{D.13}\\
& \bar{\psi} \wedge \gamma_{5} \gamma \iota_{\xi} \nabla \psi-\frac{1}{4} \bar{\psi} \wedge \gamma_{5} \gamma \gamma_{c d} \iota_{\xi} \omega^{c d} \wedge \psi-\frac{1}{4} \bar{\psi} \gamma_{5} \wedge \gamma \wedge \gamma_{c d} \omega^{c d} \iota_{\xi} \psi=\bar{\psi} \gamma_{5} e^{a} \gamma_{a} \iota_{\xi} d \psi \tag{D.14}
\end{align*}
$$

We are left with $\bar{\psi} \gamma_{5} T^{a} \gamma_{a} \iota_{\xi} \psi$ that disappears thanks to the cyclic identity A.8.

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## References

[1] R. K. Sachs, Gravitational waves in general relativity. 8. Waves in asymptotically flat space-times, Proc. Roy. Soc. Lond. A 270 (1962), 103-126 doi:10.1098/rspa.1962.0206
[2] R. M. Wald and A. Zoupas A General Definition of "Conserved Quantities" in General Relativity and Other Theories of Gravity, 10.1103/PhysRevD.61.084027, arXiv:gr-qc/9911095
[3] J. N. Goldberg, A. J. MacFarlane, E. T. Newman, F. Rohrlich and E. C. G. Sudarshan, Spin s spherical harmonics and edth, J. Math. Phys. 8 (1967), 2155 doi:10.1063/1.1705135
[4] S. W. Hawking, M. J. Perry and A. Strominger, Superrotation Charge and Supertranslation Hair on Black Holes, JHEP 05 (2017), 161 doi:10.1007/JHEP05(2017)161 [arXiv:1611.09175 [hep-th]]
[5] H. Bondi, M. G. J. van der Burg and A. W. K. Metzner, Gravitational waves in general relativity. 7. Waves from axisymmetric isolated systems, Proc. Roy. Soc. Lond. A 269 (1962), 21-52 doi:10.1098/rspa.1962.0161
[6] A. Strominger and A. Zhiboedov, Gravitational Memory, BMS Supertranslations and Soft Theorems, JHEP 01 (2016), 086 doi:10.1007/JHEP01(2016)086 [arXiv:1411.5745 [hep-th]].
[7] E. Newman, L. Tamburino and T. Unti, Empty space generalization of the Schwarzschild metric, J. Math. Phys. 4 (1963), 915 doi:10.1063/1.1704018
[8] A. H. Taub, Empty space-times admitting a three parameter group of motion, Annals Math. 53 (1951), 472-490 doi:10.2307/1969567
[9] H. Godazgar, M. Godazgar and C. N. Pope, Taub-NUT from the Dirac monopole, Phys. Lett. B 798 (2019), 134938 doi:10.1016/j.physletb.2019.134938 [arXiv:1908.05962 [hep-th]].
[10] H. Godazgar, M. Godazgar and C. N. Pope, New dual gravitational charge, Phys. Rev. D 99 (2019) no.2, 024013 doi:10.1103/PhysRevD.99.024013
[11] H. Godazgar, M. Godazgar and M. J. Perry, Hamiltonian derivation of dual gravitational charges, JHEP 09 (2020), 084 doi:10.1007/JHEP09(2020)084 [arXiv:2007.07144 [hep-th]]
[12] G. Compère, Advanced Lectures on General Relativity, Lect. Notes Phys. 952 (2019), 150 doi:10.1007/978-3-030-04260-8
[13] A. Strominger Lectures on the Infrared Structure of Gravity and Gauge Theory arXiv:1703.05448
[14] D. Z. Freedman and A. Van Proeyen Supergravity, 978-1-139-36806-3, 978-0-521-19401-3, Cambridge Univ. Press, 2012
[15] B. L. Tomova Asymptotic symmetries of the Rarita-Schwinger field, [arXiv:2104.14904 [hep-th]]
[16] M. Nouri-Zonoz and D. Lynden-Bell, Gravomagnetic lensing by NUT space, Mon. Not. Roy. Astron. Soc. 292 (1997), 714-722 [arXiv:gr-qc/9812094 [gr-qc]]
[17] M. A. Awada, G. W. Gibbons and W. T. Shaw Conformal Supergravity, Twistors, and the Super-BMS Group
[18] H. Godazgar, M. Godazgar and C. N. Pope, Dual gravitational charges and soft theorems, JHEP 10 (2019), 123 doi:10.1007/JHEP10(2019)123 [arXiv:1908.01164 [hep-th]]
[19] U. Kol and M. Porrati, Properties of Dual Supertranslation Charges in Asymptotically Flat Spacetimes, Phys. Rev. D 100 (2019) no.4, 046019 doi:10.1103/PhysRevD.100.046019 [arXiv:1907.00990 [hep-th]]
[20] N. Lambert, Introduction to Supersymmetry Lectures at the 2011 Balkan Summer School
[21] G. Barnich and C. Troessaert, BMS charge algebra, JHEP 12 (2011), 105 doi:10.1007/JHEP12(2011)105 [arXiv:1106.0213 [hep-th]]
[22] S. G. Avery and B. U. W. Schwab, Residual Local Supersymmetry and the Soft Gravitino,, Phys. Rev. Lett. 116 (2016) no.17, 171601 doi:10.1103/PhysRevLett.116.171601 [arXiv:1512.02657 [hep-th]]
[23] H. Godazgar, M. Godazgar and C. N. Pope, Tower of subleading dual BMS charges, JHEP 03 (2019), 057 doi:10.1007/JHEP03(2019)057 [arXiv:1812.06935 [hep-th]]
[24] S. W. Hawking, M. J. Perry and A. Strominger, Superrotation Charge and Supertranslation Hair on Black Holes, JHEP 05 (2017), 161 doi:10.1007/JHEP05(2017)161 [arXiv:1611.09175 [hep-th]]
[25] O. Fuentealba, M. Henneaux, S. Majumdar, J. Matulich and T. Neogi, Asymptotic structure of the Rarita-Schwinger theory in four spacetime dimensions at spatial infinity, JHEP 02 (2021), 031 doi:10.1007/JHEP02(2021)031 [arXiv:2011.04669 [hep-th]]
[26] M. Henneaux, J. Matulich and T. Neogi, Asymptotic realization of the super-BMS algebra at spatial infinity, Phys. Rev. D 101 (2020) no.12, 126016 doi:10.1103/PhysRevD.101.126016 [arXiv:2004.07299 [hep-th]].
[27] G. Barnich and C. Troessaert, "Aspects of the BMS/CFT correspondence", JHEP 05 (2010), 062 doi:10.1007/JHEP05(2010)062 [arXiv:1001.1541 [hep-th]]
[28] K. Prabhu, A novel supersymmetric extension of BMS symmetries at null infinity,, [arXiv:2112.07186 [gr-qc]]
[29] J. D. Cohen, New Infinities of Soft Charges, [arXiv:2112.09776 [hep-th]]

