Magnetic coordinates for equilibria with a continuous symmetry

R. L. Dewar[a] and D. A. Monticello

Plasma Physics Laboratory, Princeton University, Princeton, New Jersey 08544

W. N.-C. Sy

Department of Theoretical Physics, Research School of Physical Sciences, The Australian National University, Canberra A.C.T. 2601, Australia

Magnetic coordinates for hydromagnetic equilibria are defined which treat toroidal and "straight" helical plasmas equivalently and yet exploit the existence of a continuous symmetry to derive relations between various geometrical and physical qualities. This allows the number of equilibrium quantities which must be known to be reduced to a minimal, or primitive, set. Practical formulas for various quantities required in hydromagnetic stability calculations (interchange, ballooning, and global) are given in terms of this primitive set.

I. INTRODUCTION

It has long been known[1,2] that a general coordinate system for toroidal plasmas can be set up which simplifies analytic and numerical work by making the magnetic field lines straight when graphed in the space of the coordinates. All that needs to be assumed about the system is that the magnetic field lines map out nested toroidal surfaces. The authors cited restrict their coordinate system further by the requirement that the Jacobian of the transformation from magnetic coordinates to Cartesian coordinates be constant on each magnetic surface. We shall call such a restricted set *Hamada coordinates*, but shall work mainly with the more general class of *straight-field-line coordinates*, which include Hamada coordinates as a special case, but allow more flexibility in the choice of grid for numerical work.[3,4]

Most detailed calculations to date have been done on systems which, at least approximately, have a continuous symmetry; either axisymmetry[5,6] or helical symmetry.[7-10] The assumption of symmetry has the advantages that the existence of magnetic surfaces is assured, that the existence of an ignorable coordinate allows the dimensionality of the system to be reduced by one, and that there are relations between the various metric-type quantities which may be used to reduce the amount of information stored in computational work.

In exploiting these symmetries, however, there is a temptation to build in so much geometric information specific to the system being treated that easy application to other systems is precluded. In particular, the PEST2[3] stability code was formulated specifically for axisymmetric systems. The ERATO[10] code for axisymmetric systems has been converted to a helical symmetry code, HERA,[11,12] by replacing the axisymmetric expressions with the appropriate helical expressions. It is the purpose of this paper to show instead that, for evaluating plasma stability, it is not necessary to treat the axisymmetric and helical cases separately, thus making future generalization to nonsymmetric geometries

easier. We construct a straight-field-line coordinate system appropriate equally to axisymmetric or helically symmetric systems. Using only the existence of an ignorable coordinate, various geometric relations are derived. We use only standard vector calculus identities rather than a tensor calculus formalism, and wherever possible use only quantities which are invariant under change of coordinates. When a symbol is defined for a noninvariant quantity, we shall distinguish it by use of script letter. We shall call the general system of coordinates *universal coordinates*.

We show how to reduce formulas required for evaluation of interchange, ballooning, and global stability to algebraic expressions involving a minimal set of scalar quantities. We call this the *primitive set* and have written a mapping code FMAP2.5 which takes the output file (EQDSK) of a new helical flux coordinate code (FEQ2.5) and tabulates the primitive quantities in a file METDSK. This file is used by a ballooning stability code BAL2.5, described in Sec. IX, and a new version of PEST2, PEST2.5. These have been used for evaluating the stability of heliac configurations,[13] but the purpose of the present paper is simply to present the basic formalism in as accessible a fashion as possible, as its simplicity and generality raises the hope that it will be found useful in a variety of applications.

In Sec. II we introduce the straight-field-line coordinates and general notation. In Sec. III we define a set of orthogonal, invariant basis vectors used for resolving vector quantities, and also define the local shear and the integrated residual shear. Up to this point, we have not used the existence of a continuous symmetry, but in Secs. IV and V we use this fact to find expressions for metric-type quantities and the plasma current density in terms of a primitive set of quantities, defined in Sec. V. In Secs. VI–VII we define and calculate fluxes and curvatures, and in Sec. VIII we describe a procedure for actually calculating equilibria and the primitive set.

In Secs. IX–X we give formulas for evaluating local and global stability. In Appendix A we describe the high-β tokamak ordering used for checking that the large-aspect-ratio limit of the various expressions is well-defined, and in Appendix B we show how to calculate the gradient of an arbitrary vector in terms of the primitive set.

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[a] Permanent address: Department of Theoretical Physics, Research School of Physical Sciences, The Australian National University, Canberra A.C.T. 2601, Australia.
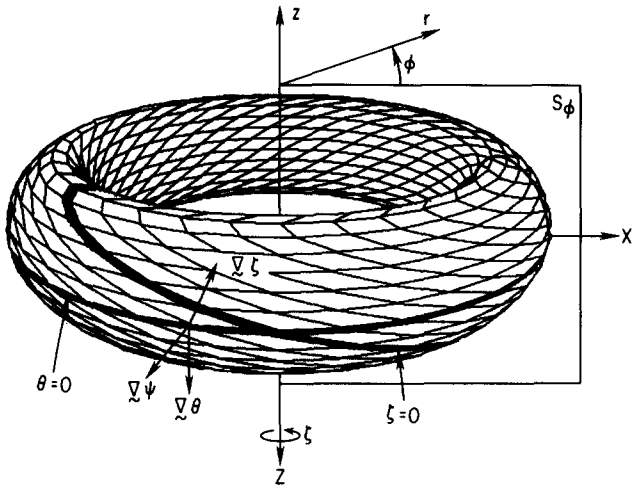
FIG. 1. Axisymmetric toroidal magnetic surface $\psi =$ const showing a θ, ζ mesh, the r, ϕ, z coordinates, the surface of section $S_\phi = \{r|\phi = \text{const}\}$, and the X, Z, Y coordinates.

II. STRAIGHT-FIELD-LINE COORDINATES

Figure 1 illustrates an axisymmetric toroidal magnetic surface, and Fig. 2 a helical surface. In both cases the equilibrium is doubly 2π-periodic, with a "poloidal" angle θ and a "toroidal" angle ζ. The mapping from magnetic coordinates to Cartesian coordinates will be discussed in more detail in Sec. VIII. Here it suffices to know only that ζ is *ignorable*, that is, that any scalar equilibrium quantity is independent of ζ. As a flux surface label we use ψ, such that $2\pi\psi$ is the helical magnetic flux (or poloidal flux in the axisymmetric toroidal case). That is, $2\pi\psi$ is the flux threading a ribbon $\theta =$ const bounded by the magnetic axis, the magnetic surface, and the surfaces $\zeta = 0$ and $\zeta = 2\pi$. We can use $\nabla \cdot \mathbf{B} = 0$ (where $\mathbf{B}$ is the magnetic field) to show that[14]

$$\psi = (2\pi)^{-2} \int_V d\tau \, \mathbf{B} \cdot \nabla\theta , \qquad (1)$$

where $V(\psi)$ is the volume enclosed by the bounding surfaces. By the assumption of the existence of magnetic surfaces we have
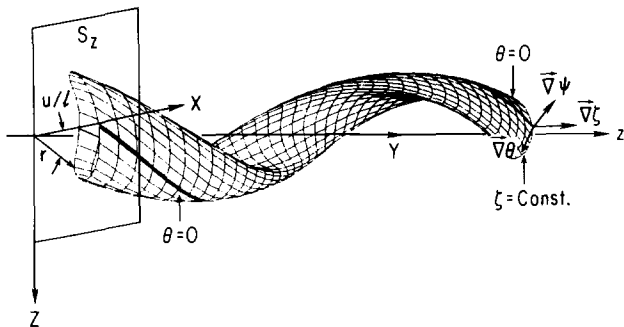
$$\mathbf{B} \cdot \nabla\psi = 0 . \qquad (2)$$

FIG. 2. Helical magnetic surface $\psi =$ const, showing a θ, ζ mesh, the r, ϕ, z coordinates the surface of section $S_z = \{r|z = \text{const}\}$ and the X, Z, Y coordinates. This is an $l = 1$ system.

The conditions that $\mathbf{B}$ be divergence-free, that $\mathbf{B}$ be orthogonal to $\nabla\psi$, and that the field lines be straight in (ψ,θ,ζ) space can be expressed as the requirement that $\mathbf{B}$ be given by

$$\mathbf{B} = \nabla\zeta \times \nabla\psi + q(\psi)\nabla\psi\times\nabla\theta . \qquad (3)$$

The constant q is the slope of the magnetic field lines in the (θ,ζ) plane and is called the "safety factor" in tokamaks. In the helical case, with magnetic surfaces having l-fold symmetry (Sec. VIII), q is related to the rotational transform[14] ι per helical period (i.e., ζ increasing by 2π) by

$$\frac{\iota}{2\pi} = \frac{1}{q} + \frac{1}{l} . \qquad (4)$$

This remains true in the tokamak, but with the term $1/l$ deleted because of the different interpretation of ψ (Sec. VIII).

From Eq. (3) we can calculate the current density

$$\mathbf{j} = \nabla\times\mathbf{B} . \qquad (5)$$

By crossing the equilibrium equation

$$\mathbf{j}\times\mathbf{B} = \nabla p , \qquad (6)$$

where $p(\psi)$ is the plasma pressure, with $\nabla\psi$ we see that

$$\mathbf{j} \cdot \nabla\psi = 0 . \qquad (7)$$

Now since $\nabla\times\nabla\mu = 0$, where μ is any scalar, we have from Eq. (5),

$$\mathbf{j} \cdot \nabla\mu = \nabla \cdot (\mathbf{B}\times\nabla\mu) . \qquad (8)$$

Using Eq. (3) in Eq. (8) with $\mu = \psi$ we see that the condition [Eq. (7)] that the current lies within a magnetic surface gives the following constraint on the ζ and θ coordinates:

$$\nabla \cdot (|\nabla\psi|^2\nabla_s\zeta - q|\nabla\psi|^2\nabla_s\theta) = 0 , \qquad (9)$$

where $\mathbf{f}_s$ is the component of any vector $\mathbf{f}$ within a magnetic surface,

$$\mathbf{f}_s \equiv (\mathbf{I} - \nabla\psi\nabla\psi/|\nabla\psi|^2) \cdot \mathbf{f} , \qquad (10)$$

$\mathbf{I}$ being the unit dyadic.

The Jacobian of the transformation from Cartesian coordinates to (ψ,θ,ζ) coordinates is denoted by $\mathcal{J}$,

$$\mathcal{J} \equiv (\nabla\psi \cdot \nabla\theta \times\nabla\zeta)^{-1} , \qquad (11)$$

and may be chosen in any convenient way apart from a ψ-dependent normalization determined by periodicity. If $\mathcal{J} = \mathcal{J}(\psi)$ we have *Hamada coordinates*. Denoting the operators which take the partial derivatives with respect to ψ, θ, and ζ by ∂_ψ, ∂_θ, and ∂_ζ, respectively, we have

$$\nabla \equiv (\nabla\psi)\partial_\psi + (\nabla\theta)\partial_\theta + (\nabla\zeta)\partial_\zeta . \qquad (12)$$

From Eq. (3) we have

$$\mathbf{B} \cdot \nabla = \mathcal{J}^{-1}(\partial_\theta + q\partial_\zeta) . \qquad (13)$$

Also useful is the identity, for any vector field $\mathbf{f}$,

$$\mathcal{J}\nabla \cdot \mathbf{f} = \partial_\psi(\mathcal{J}\mathbf{f} \cdot \nabla\psi) + \partial_\theta(\mathcal{J}\mathbf{f} \cdot \nabla\theta) + \partial_\zeta(\mathcal{J}\mathbf{f} \cdot \nabla\zeta) , \qquad (14)$$

which follows from the representation

$$\mathbf{f} = \mathcal{J}(\nabla\theta \times\nabla\zeta\nabla\psi + \nabla\zeta \times\nabla\psi\nabla\theta + \nabla\psi\times\nabla\theta\nabla\zeta) \cdot \mathbf{f} , \qquad (15)$$

and the fact that $\nabla \cdot (\nabla\mu\times\nabla\nu) = 0$, where μ and ν are arbitrary scalars.

III. BASIS VECTORS

Rather than use $\nabla\psi$, $\nabla\theta$, and $\nabla\zeta$ as basis vectors, it is more physical to use the set $\mathbf{s}$, $\nabla\psi$, and $\mathbf{B}$, where $\mathbf{s}$ is a surface vector defined by

$$\mathbf{s} \equiv \nabla_s\zeta - q\nabla_s\theta . \tag{16}$$

From Eq. (3) we see that

$$\mathbf{B} = \mathbf{s}\times\nabla\psi , \tag{17}$$

whence it is clear that $\mathbf{s}$, $\nabla\psi$, and $\mathbf{B}$ form an orthogonal (though not orthonormal) basis. By crossing Eq. (17) with $\nabla\psi$, we see that $\mathbf{s}$ may alternatively be defined in a coordinate-free way

$$\mathbf{s} = (\nabla\psi\times\mathbf{B})/|\nabla\psi|^2 . \tag{18}$$

Note that $\mathbf{s}$ as defined here differs by a component in the $\nabla\psi$ direction from the vector $\mathbf{s}$ used in Refs. 3 and 4. The present definition, being invariant under coordinate change seems more physical. By squaring Eq. (18), we find the normalization

$$|\mathbf{s}|^2 = B^2/|\nabla\psi|^2 . \tag{19}$$

Also, we observe that $\mathbf{s}$ can be written

$$\mathbf{s} = \nabla_s\alpha = \nabla\alpha - [(\nabla\alpha\cdot\nabla\psi)/|\nabla\psi|^2]\nabla\psi , \tag{20}$$

where

$$\alpha \equiv \zeta - q\theta , \tag{21}$$

so that

$$(\nabla\alpha\cdot\nabla\psi)/|\nabla\psi|^2 = -\mathscr{R}(\psi,\theta) - q'(\psi)\theta , \tag{22}$$

with

$$\mathscr{R} \equiv (q\nabla\psi\cdot\nabla\theta - \nabla\psi\cdot\nabla\zeta)/|\nabla\psi|^2 . \tag{23}$$

Since

$$\mathbf{B} = \nabla\alpha\times\nabla\psi , \tag{24}$$

we see that the magnetic field lines are formed by the intersection of surfaces of constant α and constant ψ. For short-wavelength (ballooning) modes, α forms a convenient coordinate,[15] but the aperiodic component in $\nabla\alpha$, evidenced by the last term in Eq. (22) makes it inconvenient for numerical work in general.

Calling q' the *global magnetic shear*, we see from Eq. (22) that a natural definition for the *local magnetic shear*[2,16,17] $S(\psi,\theta)$ is

$$S = -\mathbf{B}\cdot\nabla[(\nabla\alpha\cdot\nabla\psi)/|\nabla\psi|^2]$$
$$= \mathscr{F}^{-1}[q' + (\partial_\theta + q\partial_\zeta)\mathscr{R}] . \tag{25}$$

We need the Jacobian factor in Eq. (25) to make S an invariant, as can also be seen by taking the curl of Eq. (20) to show

$$S = -\mathbf{s}\cdot\nabla\times\mathbf{s} . \tag{26}$$

We term $\mathscr{R}$ the *integrated residual shear*.

IV. RESOLUTION OF $\nabla_s\theta$ AND $\nabla_s\zeta$

So far we have not used the ignorability of ζ, except perhaps to ensure the existence of magnetic surfaces. However, as we now show, it can be used quite powerfully to derive the existence of an invariant which allows us to express $\nabla_s\theta$ and $\nabla_s\zeta$ in terms of the basis vectors $\mathbf{B}$ and $\mathbf{s}$. This is because ignorability implies that the last term of Eq. (14)

vanishes if f is any equilibrium quantity. In particular, noting that Eq. (9) can be written

$$\nabla\cdot(|\nabla\psi|^2\mathbf{s}) = 0 , \tag{27}$$

we take $f = |\nabla\psi|^2\mathbf{s}$ in Eq. (14) and find a differential equation for $\mathbf{s}\cdot\nabla\theta$,

$$\partial_\theta(\mathscr{F}|\nabla\psi|^2\mathbf{s}\cdot\nabla_s\theta) = 0 . \tag{28}$$

Integrating, we have

$$\mathbf{s}\cdot\nabla_s\theta = -g(\psi)/\mathscr{F}|\nabla\psi|^2 , \tag{29}$$

where $g(\psi)$ is a constant of integration, which is essentially the $f(\psi)$ of the toroidal case[5] or the $H(\psi)$ of the helical case.[8,9] By considering the most general change of variable consistent with ignorability of ζ,

$$\theta = \theta' + h(\theta',\psi) , \tag{30}$$

where h is a 2π-periodic function of θ', we can easily show that $\mathscr{F}\nabla_s\theta$ is invariant under such coordinate changes. Thus $g(\psi)$ is also an invariant.

From Eqs. (16), (19), and (29) we have

$$\mathbf{s}\cdot\nabla_s\zeta = (\mathscr{F}B^2 - gq)/\mathscr{F}|\nabla\psi|^2 . \tag{31}$$

To find the parallel components we use Eq. (13) to show

$$\mathbf{B}\cdot\nabla_s\theta = \mathscr{F}^{-1} , \tag{32}$$

and

$$\mathbf{B}\cdot\nabla_s\zeta = q\mathscr{F}^{-1} . \tag{33}$$

From Eqs. (29)–(33) we readily find

$$|\nabla_s\theta|^2 = \frac{g^2 + |\nabla\psi|^2}{\mathscr{F}^2B^2|\nabla\psi|^2} , \tag{34}$$

$$\nabla_s\theta\cdot\nabla_s\zeta = \frac{q|\nabla\psi|^2 - g(\mathscr{F}B^2 - gq)}{\mathscr{F}^2B^2|\nabla\psi|^2} , \tag{35}$$

$$|\nabla_s\zeta|^2 = \frac{(\mathscr{F}B^2 - gq)^2 + q^2|\nabla\psi|^2}{\mathscr{F}^2B^2|\nabla\psi|^2} . \tag{36}$$

V. RESOLUTION OF j

Since $\mathbf{s}\times\mathbf{B} = -|\mathbf{s}|^2\nabla\psi$, it is readily verified that the expression

$$\mathbf{j} = \sigma\mathbf{B} - (|\nabla\psi|^2/B^2)p'(\psi)\mathbf{s} \tag{37}$$

satisfies the equilibrium equation, Eq. (6), where $\sigma \equiv \mathbf{j}\cdot\mathbf{B}/B^2$ is yet to be determined.

We can alternatively determine $\mathbf{j}$ from Eq. (5). It is easiest to resolve $\mathbf{j}$ onto $\nabla\theta$ and $\nabla\zeta$,

$$\mathbf{j}\cdot\nabla_s\theta = -g'\mathscr{F}^{-1} , \tag{38}$$

$$\mathbf{j}\cdot\nabla_s\zeta = \mathscr{F}^{-1}\partial_\theta[(\mathscr{F}B^2\nabla\psi\cdot\nabla\theta)/|\nabla\psi|^2 - g\mathscr{R}]$$
$$+ \mathscr{F}^{-1}\partial_\psi(\mathscr{F}B^2 - gq) , \tag{39}$$

where we have used Eqs. (23), (29), and (31). Comparing Eq. (38) with the $\nabla_s\theta$ component of (37) we determine σ,

$$\sigma = -g' - gp'/B^2 . \tag{40}$$

Comparing Eq. (39) with the $\nabla_s\zeta$ component of (37) we also find the interesting identity

$$\partial_\psi(\mathscr{F}B^2) + \mathscr{F}p' - gq'$$
$$= \partial_\theta[g\mathscr{R} - (\mathscr{F}B^2\nabla\psi\cdot\nabla\theta)/|\nabla\psi|^2] . \tag{41}$$

All terms of Eq. (41) are $O(\epsilon^{-1})$ in the high-beta ordering of Appendix A except for the term $\mathscr{F}p'$. It is therefore inadvis-

able to use Eq. (41) to substitute for p' as cancellations of large terms can arise, leading to numerical errors.

We take the primitive set of quantities, in terms of which the stability formulas of the subsequent sections can be written as algebraic expressions, to be the surface quantities $p(\psi)$, $p'(\psi)$, $q(\psi)$, $q'(\psi)$, $g(\psi)$, $g'(\psi)$ and the poloidally varying quantities $|\nabla\psi|^2$, $\nabla\psi \cdot \nabla\theta$, B^2, $\partial_\theta B^2$, $\partial_\psi B^2$, $\mathscr{J}$, $\partial_\psi \mathscr{J}$, $\mathscr{R}$, and $\partial_\theta \mathscr{R}$. We show in Appendix B that, in order to represent the gradient of an arbitrary vector, we should add $\partial_\psi |\nabla\psi|^2$ and $\partial_\theta |\nabla\psi|^2$ to the primitive set. It will be shown in Sec. VIII how these quantities can be calculated numerically. In the other sections the quantities of the primitive set will be taken as known and no further information about the equilibrium will be needed.

VI. FLUXES

Just as Eq. (1) gives the poloidal magnetic flux $2\pi\psi$, the toroidal magnetic flux $2\pi\Phi(\psi)$ is given by

$$\Phi = (2\pi)^{-2}\int_V d\tau\, \mathbf{B}\cdot\nabla\zeta\,. \tag{42}$$

From Eq. (13) we see that

$$\Phi = \int_0^\psi q(\psi')d\psi'\,, \tag{43}$$

so that q can also be interpreted as $\Phi'(\psi)$.

The poloidal current $2\pi I(\psi)$ is defined by replacing $\mathbf{B}$ with $\mathbf{j}$ in Eq. (1). From Eq. (38) we then get

$$I(\psi) = (-g)_0^\psi \equiv g(0) - g(\psi)\,. \tag{44}$$

Clearly, for a vacuum magnetic field, g must be constant.

Similarly, the toroidal current $2\pi J(\psi)$ is defined by Eq. (42) with $\mathbf{B}$ replaced by $\mathbf{j}$. From Eq. (39) we find

$$J(\psi) = [\langle \mathscr{J}B^2\rangle - gq]_0^\psi\,, \tag{45}$$

where $\langle\cdot\rangle \equiv (2\pi)^{-1}\int_0^{2\pi}\cdot\,d\theta$. Note that a stellarator with zero toroidal current must have $q = \langle \mathscr{J}B^2\rangle/g$. This is also approximately true for a large-aspect-ratio tokamak (see Appendix A). An alternative expression for J may be had by using Eq. (41);

$$J(\psi) = -\int_0^\psi (\langle\mathscr{J}\rangle p' + qg')d\psi\,. \tag{46}$$

VII. CURVATURE

We calculate the magnetic field line curvature $\boldsymbol{\kappa} \equiv \mathbf{e}_\parallel \cdot \nabla\mathbf{e}_\parallel$ (where $\mathbf{e}_\parallel \equiv \mathbf{B}/B$) from the identity[2]

$$\boldsymbol{\kappa} = [\nabla_\perp(2p + B^2)]/2B^2\,, \tag{47}$$

where $\nabla_\perp \equiv (\mathbf{I} - \mathbf{e}_\parallel\mathbf{e}_\parallel)\cdot\nabla$. Clearly $\boldsymbol{\kappa}\cdot\mathbf{B}\equiv 0$, so the most general form of $\boldsymbol{\kappa}$ is

$$\boldsymbol{\kappa} = \kappa_\psi\nabla\psi + \kappa_s\mathbf{s}\,, \tag{48}$$

where the normal curvature is found by dotting Eq. (47) with $\nabla\psi$,

$$\kappa_\psi = \frac{2p' + \partial_\psi B^2}{2B^2} + \frac{\nabla\psi\cdot\nabla\theta\,\partial_\theta B^2}{2|\nabla\psi|^2 B^2}\,, \tag{49}$$

and the geodesic curvature is found by dotting with $\mathbf{s}$,

$$\kappa_s = -(g/2\mathscr{J}B^4)\partial_\theta B^2\,. \tag{50}$$

Comparing Eq. (50) with Eq. (40) we also see that

$$\kappa_s = -\frac{\partial_\theta\sigma}{2p'}\mathscr{J}^{-1}\,, \tag{51}$$

which holds even in nonaxisymmetric geometries[2] if $\mathscr{J}^{-1}\partial_\theta$ is replaced by $\mathbf{B}\cdot\nabla$.

VIII. EQUILIBRIUM AND MAPPING

The purpose of this section is to outline the modifications that have been made to the flux coordinate code of DeLucia *et al.*[18] in order to allow for either helical or toroidal symmetry. We take the shape of the plasma boundary to be specified in an appropriate surface of section. We have called the new code FEQ2.5.

Consider the r, ϕ, z cylindrical coordinates of Fig. 1 or 2. Helical symmetry[7] implies that all equilibrium quantities are functions only of r and u where

$$u = l\phi - hz\,, \tag{52}$$

l being the dimension of the discrete symmetry group associated with rotations about the z axis. Scalar equilibrium quantities are 2π-periodic in u. Thus $2\pi/h$ is the helical period measured along the z axis, but the length along the z axis for a complete turn of the flux tube is l periods, or $2\pi l/h$.

Formally, the helically invariant system includes the axisymmetric system as the special case $l = 1$, $h\to\infty$, but this limit makes sense only if we hold the plasma shape constant (as h varies) in a cross-sectional surface (Fig. 2) which does not have the z axis as its normal. We take this surface of section to be the half-plane $\phi = 0$ (S_ϕ of Fig. 1), and provide the option in the code for the plasma shape to be specified in this plane so that axisymmetric equilibria can be treated. To visualize the limit $h\to\infty$, imagine the compression of a single coil of a helical spring. When completely compressed ($h = \infty$), the ends of the coil join continuously, and the coil becomes a toroid. The equivalent limit $l = 0$, $h = 1$ is incompatible with our interpretation of l, cf. Eq. (62a).

On the other hand, the plane $z = 0$ is often more convenient for specifying the plasma shape of helical systems and we also provide the option of working in this surface of section. If this plane were used for the $h\to\infty$ case the plasma would be reduced to a disk of zero thickness.

There is another fundamental difference between the axisymmetric case, $h\to\infty$, and the helical case, h finite, taken as a straight, periodic model for toroidal systems with a helical distortion imposed (e.g., stellarators). To treat axisymmetric systems we make a topological identification between the sections $\phi = 0$ and $\phi = 2\pi$. But to model helically deformed toroids, we identify the sections $z = 0$ and $z = 2\pi m/h$, where m is the number of field periods around the machine the long way. It can easily be seen that these two identifications are topologically different: In the helical case the magnetic axis and the curve $r = $ const, $u = $ const are topologically linked (Fig. 3), while they are not in the axisymmetric case. This explains how it can be that the term $1/l$ is present in Eq. (4) in the helical case, but not in the axisymmetric case.

The limit of a cylinder with arbitrary cross section can be obtained by taking the large-aspect-ratio limit ($r\to\infty$) of the axisymmetric case ($h\to\infty$), provided care is taken to order terms correctly (Appendix A).
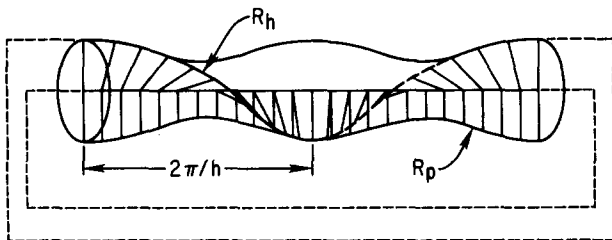
FIG. 3. Two helical periods of an $l = 2$ system, showing the ribbon R_h used for defining the helical flux ψ, and the ribbon R_p used for defining the poloidal flux $\psi_p = \psi + \Phi / l$. The dashed lines show a possible topological identification of the intersections of the magnetic axis and a constant-ψ, θ line with two $z =$ constant surfaces an integral number of periods apart. These lines are topologically linked (in contrast to the similarly identified lines intersecting a $\phi =$ constant surface in the toroidal case), illustrating the fact that a field line with $q = \infty$ would have a rotational transform of 2π every l periods.

Helical symmetry and $\nabla \cdot \mathbf{B} = 0$ imply that

$$\mathbf{B} = lh\mathbf{u} \times \nabla\psi + lhg\mathbf{u} \tag{53}$$

where

$$\mathbf{u} = (l\hat{z} + hr\hat{\phi})/(l^2 + h^2 r^2) , \tag{54}$$

where $\hat{z} \equiv \nabla z$ and $\hat{\phi} \equiv r\nabla\phi$. The field lines for a typical helical equilibrium are shown in Fig. 4. The coordinate-free representation of the equilibrium equation for helically symmetric systems is

$$\nabla \cdot (K\nabla\psi) = \frac{-2K^2}{lh} g - Kg \frac{\partial g}{\partial \psi} - \frac{\partial p}{\partial \psi} , \tag{55}$$

where

$$K = l^2 h^2/(l^2 + h^2 r^2) . \tag{56}$$

It is easily shown that $2\pi l\psi$ is the flux through a helical ribbon with length along the z axis of $2\pi l /h$ (Fig. 3), with the magnetic axis as one edge and with the same symmetry as the equilibrium. To do this we use

$$\mathbf{B} = h\hat{z} \times \nabla\psi + B_z \mathbf{h} , \tag{57}$$

where

$$\mathbf{h} = \hat{z} + hr\hat{\phi}/l \tag{58}$$

is a vector pointing in the symmetry direction, i.e., $\mathbf{h} \cdot \nabla f(r,u) = 0$. Since $\nabla \cdot \mathbf{B} = 0$, $2\pi l\psi$ is also the flux through any ribbon that is obtained by continuous deformation; that is, which has one edge on the magnetic axis and the other edge on the flux surface and winds around the magnetic axis once in l helical periods. However, this helical flux must be
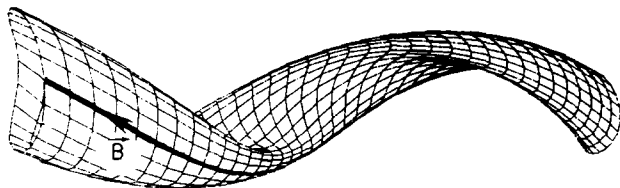

FIG. 4. Helical magnetic surface with magnetic field lines rather than $\theta =$ const lines. Note that the field lines twist more slowly than the constant-θ lines, corresponding to $q < 0$.

distinguished from the poloidal flux when we model a helically deformed toroid. The flux per helical period through any ribbon which does not wrap around the magnetic axis but repeats after l helical periods is called the poloidal flux (per period) and denoted as $2\pi\psi_p$. Thus

$$l\psi_p = \Phi + l\psi , \tag{59}$$

where $2\pi\Phi$ is the toroidal flux; that is, the flux through any cross section of the flux tube. This relationship then gives Eq. (4) [from the use of $\iota/2\pi = d\psi_p/d\Phi$ and Eq. (43)]. We again note that in the axisymmetric toroidal case, the $1/l$ term should be dropped because the way the ends of the tube are topologically identified means that $2\pi\psi$ itself is the poloidal flux.

In order to allow an arbitrary twisting surface of section we consider a coordinate system X, Z, Y where

$$X = X(r,u) , \tag{60a}$$

and

$$Z = Z(r,u) \tag{60b}$$

are Cartesian coordinates in the surface of section. We suppose these equations to be invertible: $\nabla X \times \nabla Z \cdot \nabla Y \neq 0$. We take Y to be an ignorable coordinate

$$\left. \frac{\partial}{\partial Y} \right|_{XZ} = 0 \tag{61}$$

when acting on equilibrium scalars. For definiteness we will consider here only two such surfaces of section. The $\phi =$ const plane, where

$$X = r , \quad Z = u/h , \quad Y = l\phi , \tag{62a}$$

and the $z =$ const plane, where

$$X = r\cos(u/l) , \quad Z = r\sin(u/l) , \quad Y = hz . \tag{62b}$$

We now transform to flux coordinates[17] Ψ, Θ, Y, where

$$X = X(\Psi,\Theta) , \tag{63}$$

$$Z = Z(\Psi,\Theta) , \tag{64}$$

and

$$\left. \frac{\partial}{\partial Y} \right|_{\Psi,\Theta} = 0 . \tag{65}$$

The angle Θ has been chosen to give equal arc lengths in the surface of section for equal increments of Θ at fixed Ψ. The intermediate coordinate Ψ is approximately equal to ψ, and the mapping equations (63) and (64) are adjusted iteratively [by solving Eq. (55) at each step] until ψ is equal to Ψ to within a specified tolerance. These requirements replace Eq. (36) of DeLucia et al.[18] Adequate resolution near the magnetic axis is obtained by using an unequally spaced mesh in Ψ, giving constant increments of $\Psi^{1/2}$.

We now indicate how the added level of coordinates (X,Z) modifies the calculation of the metric quantities of DeLucia et al.[18] The left-hand side of the equilibrium equation can be written

$$\nabla \cdot (K\nabla\Psi) = (1/\mathscr{J}_{\Psi,\Theta,Y})[(\psi_\Psi h^{\Psi\Psi} + \psi_\Theta h^{\Theta\Psi})_\Psi$$
$$+ (\psi_\Psi h^{\Theta\Psi} + \psi_\Theta h^{\Theta\Theta})_\Theta] , \tag{66}$$

where a single subscript Ψ or Θ denotes partial differentiation with respect to that variable,

$$\mathscr{J}_{\Psi,\Theta,Y} \equiv 1/(\nabla\Psi\times\nabla\Theta\cdot\nabla Y) = \mathscr{J}_{\Psi,\Theta}J\,, \tag{67}$$

$$\mathscr{J}_{\Psi,\Theta} \equiv X_\Psi Z_\Theta - X_\Theta Z_\Psi\,, \tag{68}$$

$$J \equiv 1/(\nabla X\times\nabla Z\cdot\nabla Y)\,, \tag{69}$$

$$h^{\Psi\Psi} = (KJ/\mathscr{J}_{\Psi,\Theta})(Z_\Theta^2|\nabla X|^2$$
$$- 2Z_\Theta X_\Theta\nabla X\cdot\nabla Z + X_\Theta^2|\nabla Z|^2)\,, \tag{70}$$

$$h^{\Theta\Psi} = (-KJ/\mathscr{J}_{\Psi,\Theta})[Z_\Theta Z_\Psi|\nabla X|^2$$
$$- (Z_\Theta X_\Psi + Z_\Psi X_\Theta)\nabla X\cdot\nabla Z + X_\Theta X_\Psi|\nabla Z|^2]\,,$$

and

$$h^{\Theta\Theta} = (KJ/\mathscr{J}_{\Psi,\Theta})(Z_\Psi^2|\nabla X|^2$$
$$- 2ZX\,\nabla X\cdot\nabla Z + X_\Psi^2|\nabla Z|^2)\,. \tag{71}$$

The new quantities evaluated for the $\phi = \mathrm{const}$ surface of section are

$$K = l^2/(l^2/h^2 + X^2)\,, \tag{72}$$

$$J = X/l\,, \tag{73}$$

$$|\nabla X|^2 = 1\,, \tag{74}$$

$$|\nabla Z|^2 = [(l^2/h^2) + X^2]/X^2\,, \tag{75}$$

$$\nabla X\cdot\nabla Z = 0\,, \tag{76}$$

and for the $z = \mathrm{const}$ surface of section are

$$K = l^2 h^2/[l^2 + h^2(X^2 + Z^2)]\,, \tag{72'}$$

$$J = 1/h\,, \tag{73'}$$

$$|\nabla X|^2 = 1 + h^2 Z^2/l^2\,, \tag{74'}$$

$$|\nabla Z|^2 = 1 + h^2 X^2/l^2\,, \tag{75'}$$

$$\nabla X\cdot\nabla Z = -h^2 XZ/l^2\,. \tag{76'}$$

The expression for the safety factor is also modified. Using

$$q = \frac{d\Phi}{d\Psi}\,, \tag{77}$$

$$\Phi = \frac{1}{2\pi}\int \mathbf{B}\cdot d\mathbf{S}\,, \tag{78}$$

with

$$d\mathbf{S} = \nabla Y\mathscr{J}_{\Psi,\Theta,Y}d\Psi\,d\Theta\,, \tag{79}$$

we find

$$q = \frac{1}{2\pi}\oint \mathbf{B}\cdot\nabla Y\mathscr{J}_{\Psi,\Theta,Y}\,d\Theta\,, \tag{80}$$

where

$$\mathbf{B}\cdot\nabla Y = -KC/\mathscr{J}_{\Psi,\Theta,Y} + Kg\,, \tag{81}$$

and

$$C = -Z_\Theta/h \qquad (\phi = \mathrm{const\ plane}) \tag{82a}$$

or

$$C = (XZ_\Theta - ZX_\Theta)/l \qquad (z = \mathrm{const\ plane})\,. \tag{82b}$$

When the iterations have converged, FEQ2.5 tabulates the scalar quantities $p(\psi)$, $q(\psi)$, $g(\psi)$, and the coordinates $X(\psi,\Theta)$, $Z(\psi,\Theta)$. These are written to a disk file EQDSK.

Finally, we illustrate how to transform to magnetic coordinates (required for the stability analysis) by calculating the primitive set of quantities listed in Sec. V. This step is performed by a postprocessing code FMAP2.5, which pro-

duces tables of the primitive set in a disk file METDSK. It is this file which is used by the stability codes BAL2.5 and PEST2.5.

We write the magnetic field $\mathbf{B}$ as in Sec. II,

$$\mathbf{B} = \nabla\zeta\times\nabla\psi + \nabla\Phi\times\nabla\theta\,, \tag{83}$$

and find θ as a function of Θ from

$$\left.\frac{\partial\theta}{\partial\Theta}\right|_\psi = J\mathscr{J}_{\psi,\theta}/\mathscr{J}\,, \tag{84}$$

where $\mathscr{J}$ is assumed given up to some multiplicative function of ψ. Specifically, we allow Jacobians of the form

$$\mathscr{J} = \alpha(\psi)\mathscr{J}_E^{n_E}\mathscr{J}_P^{n_P}/(B^2)^{n_B}\,. \tag{85}$$

where $\mathscr{J}_E$ is the Jacobian for "equal arc" θ (i.e., $\theta\equiv\Theta$), $\mathscr{J}_P$ is the Jacobian for $\theta\equiv\theta_P$ [see Eq. (92)], and the B^2 factor is included because Jacobians with $1/B^2$ poloidal variation have been found to arise naturally in some analytic formulations.[16,19] Hamada coordinates can be obtained by setting $n_E = n_P = n_B = 0$. The coefficient $\alpha(\psi)$ is adjusted so that θ has period 2π. Derivatives with respect to θ at constant ψ and derivatives with respect to ψ at constant θ are now easily evaluated using the chain rule.

$$\left.\frac{\partial}{\partial\theta}\right|_\psi = \left.\frac{\partial\Theta}{\partial\theta}\right|_\psi\left.\frac{\partial}{\partial\Theta}\right|_\psi\,, \tag{86}$$

$$\left.\frac{\partial}{\partial\psi}\right|_\theta = -\left.\frac{\partial\Theta}{\partial\psi}\right|_\theta\left.\frac{\partial\Theta}{\partial\theta}\right|_\psi\left.\frac{\partial}{\partial\Theta}\right|_\psi + \left.\frac{\partial}{\partial\psi}\right|_\Theta\,. \tag{87}$$

The metric quantities $|\nabla\psi|^2$, $\nabla\psi\cdot\cdot\nabla\theta$ and $\mathscr{J}$ are given by

$$|\nabla\psi|^2 = \frac{(Z_\Theta^2|\nabla X|^2 + X_\Theta^2|\nabla Z|^2 - 2Z_\Theta X_\Theta\nabla X\cdot\nabla Z)}{\mathscr{J}_{\psi,\Theta}^2}\,, \tag{88}$$

$$\nabla\psi\cdot\cdot\nabla\theta = -[Z_\Theta Z_\psi|\nabla X|^2 - (Z_\Theta X_\psi + Z_\psi X_\Theta)\nabla X\cdot\cdot\nabla Z$$
$$+ X_\Theta X_\psi|\nabla Z|^2]J^2/\mathscr{J}^2\,, \tag{89}$$

$$\mathscr{J} = -\frac{[\nabla Y\cdot\nabla\psi + q(1 + \delta_\Theta)\nabla\Theta\cdot\nabla\psi + (q\delta)_\Theta|\nabla\psi|^2]}{|\nabla\psi|^2}\,, \tag{90}$$

where

$$\delta = \theta_P - \theta\,, \tag{91}$$

with θ_P being defined by

$$\frac{\partial\theta_P}{\partial\Theta} = \frac{J\mathscr{J}_{\psi,\Theta}}{q/B\cdot\nabla Y}\,. \tag{92}$$

Also

$$\nabla Y\cdot\nabla\psi = [Z_\Theta\nabla X\cdot\nabla Y - X_\Theta\nabla Z\cdot\nabla X]/\mathscr{J}_{\psi,\Theta}\,, \tag{93}$$

and

$$\nabla Y\cdot\cdot\nabla X = 0 \qquad (\phi = \mathrm{const\ plane})\,, \tag{94a}$$

or

$$\nabla Y\cdot\nabla X = h^2 Z/l \qquad (z = \mathrm{const\ plane})\,, \tag{94b}$$

$$\nabla Z\cdot\cdot\nabla Y = l^2/hX^2 \qquad (\phi = \mathrm{const\ plane})\,, \tag{95a}$$

or

$$\nabla Z\cdot\nabla Y = -h^2 X/l \qquad (z = \mathrm{const\ plane})\,. \tag{95b}$$

We have verified the new code by checking that it agrees with previous codes in the axisymmetric limit, and also by checking that the same results are obtained if the same helical equilibrium is calculated using the two surfaces of section (constant ϕ and constant z).

IX. BALLOONING AND INTERCHANGE STABILITY

Using the model density tensor $\rho\nabla\psi\nabla\psi$ of the PEST2 code,[3,4] the line averaged Lagrangian[15] for ballooning modes is given by

$$\mathcal{L} = \frac{1}{2}\int_{-\infty}^{\infty} ds \left[\frac{k^2\xi^2}{k_\alpha^2 B^2} - \left(2p'\frac{\mathbf{\kappa}\cdot\mathbf{B}\times\mathbf{k}}{B^2 k_\alpha} + \lambda\rho\right)\xi^2\right],$$

(96)

where $ds \equiv \mathcal{J}\, d\theta$, $\dot\xi \equiv d\xi/ds$,

$$\mathbf{k}/k_\alpha = \mathbf{s} - [\mathcal{R} + q'(\theta - \theta_k)]\nabla\psi,$$

(97)

with θ_k a constant and λ the eigenvalue of the Euler–Lagrange equation which makes $\xi(\theta)\to 0$ as $|\theta|\to\infty$. The local dispersion relation $\omega^2 = \lambda\,(\psi,\theta_k)$, where ω is the frequency of the mode, can be used to construct a WKB approximation to the global eigenmode.[20]

In this section we present a simple and efficient method for finding the eigenvalue λ which has been implemented as a computer code, BAL2.5, to study the ballooning stability of "heliac" equilibria.[13] First note that Eq. (96) can, using Eq. (51) and judicious integration by parts, be written in the form

$$\mathcal{L} = \frac{1}{2}\int_{-\infty}^{\infty} ds\,[\alpha\dot\xi^2 - 2\beta\xi\dot\xi - (\lambda\rho + \gamma)\xi^2],$$

(98)

where

$$\alpha = 1/|\nabla\psi|^2 + (|\nabla\psi|^2/B^2)[\mathcal{R} + q'(\theta - \theta_k)]^2$$

(99)

[not to be confused with the α of Eq. (21)],

$$\beta = [\mathcal{R} + q'(\theta - \theta_k)](\sigma - \sigma_0),$$

(100)

and

$$\gamma = 2p'\kappa_\psi + (\sigma - \sigma_0)S,$$

(101)

where S is the local shear, Eq. (25). The integration by parts avoids having an oscillating secular term in γ. The subtraction of $\sigma_0(\psi)$, the value of σ at the point on each surface where the curvature is worst ($p'\kappa_\psi$ is maximal), has been found empirically to increase the numerical accuracy. A further integration by parts can be done to remove the $\partial_\theta\mathcal{R}$ term in S if desired. We now introduce the conjugate "momentum" $\eta \equiv \partial\mathcal{L}/\partial\dot\xi$, and proceed to the Hamiltonian equations of motion in the standard way[21]

$$\dot\xi = (\eta + \beta\xi)/\alpha,$$

(102)

$$\dot\eta = -\beta(\eta + \beta\xi)/\alpha - (\lambda\rho + \gamma)\xi.$$

(103)

This coupled pair of ordinary differential equations was first derived by Greene[22] and is convenient for numerical integration and also for constructing the asymptotic series for the large $|\theta|$ behavior. In BAL2.5 the integration is started at $\theta = \theta_0 \pm \pi\theta_\pi$, where θ_π is much greater than unity (typically, θ_π around ten is adequate), and runs inward with an initial guess, $\lambda = \lambda_j$, to the matching point θ_0, where the next guess, λ_{j+1}, is computed from the jump in the logarithmic derivative at $\theta = \theta_0$ using the recursion relation

$$\lambda_{j+1} - \lambda_j = -\frac{[\![\alpha\dot\xi_j/\xi_j]\!]_{\theta_0}}{D_j},$$

(104)

where

$$D_j \equiv \sum_\pm \pm \int_{\theta_0}^{\theta_0 \pm \pi\theta_\pi} \frac{\rho\xi_j^2\,ds}{\xi_j^2(\theta_0\pm 0)}.$$

(105)

This formula is a variant of the Rayleigh–Ritz variational principle and converges stably and quadratically to the lowest eigenvalue.

The asymptotic behavior of ξ for our model density can easily be shown to be

$$\xi \sim |\theta|^{p_\lambda},$$

(106)

where

$$p_\lambda = -\tfrac{1}{2} - (-D_I - \langle\mathcal{J}/\alpha_2\rangle\langle\rho\mathcal{J}\rangle\lambda)^{1/2},$$

(107)

with $\langle\ \rangle$ denoting averages over θ. Here

$$D_I \equiv -\frac{1}{4} - \left\langle\frac{\mathcal{J}\beta_1}{\alpha_2}\right\rangle\left(1 + \left\langle\frac{\mathcal{J}\beta_1}{\alpha_2}\right\rangle\right)$$
$$+ \left\langle\frac{\mathcal{J}}{\alpha_2}\right\rangle\left\langle\frac{\mathcal{J}\beta_1^2}{\alpha_2}\right\rangle + \langle\mathcal{J}\gamma\rangle,$$

(108)

with $\beta_1 = q'(\sigma - \sigma_0)$ and $\alpha_2 = q'^2|\nabla\psi|^2/B^2$ being the coefficients of the highest powers of θ occurring in the coefficients β and α.

The eigenfunctions are square integrable (implying the possibility of a discrete spectrum) provided

$$\lambda < \frac{-D_I}{\langle\mathcal{J}/\alpha_2\rangle\langle\rho\mathcal{J}\rangle}.$$

(109)

Thus, as long as $D_I < 0$, marginal or unstable modes ($\lambda \leqslant 0$) are discrete. This corresponds to the Mercier criterion for stability against ideal interchange modes,[2] but otherwise has no physical significance owing to our artificial density model. Resistive interchange stability (within the subsidiary ordering of Glasser et al.[23]) can likewise be determined by showing $D_R < 0$, where[23]

$$D_R = D_I + (H - \tfrac{1}{2})^2,$$

(110)

with

$$H \equiv \left\langle\frac{\mathcal{J}}{\alpha_2}\right\rangle\left(\frac{\langle\mathcal{J}\beta_1|\nabla\psi|^2/\alpha_2\rangle}{\langle\mathcal{J}|\nabla\psi|^2/\alpha_2\rangle} - \frac{\langle\mathcal{J}\beta_1/\alpha_2\rangle}{\langle\mathcal{J}/\alpha_2\rangle}\right).$$

(111)

In the high-β tokamak ordering of Appendix A, the terms contributing to the magnetic well parameter

$$\frac{d^2V}{d\Phi^2} = \frac{(2\pi)^2}{q}\left(\frac{\langle\mathcal{J}\rangle}{q}\right)'$$

(112)

are not dominant, and we conclude that a more appropriate design parameter is D_R.

X. GLOBAL STABILITY

The quadratic energy functional δW_p for small oscillations of amplitude ξ about an equilibrium state can be written[24–26]

$$2\delta W_p = \int d\tau\left(\left|\mathbf{Q} + \xi\cdot\nabla\psi\frac{\mathbf{j}\times\nabla\psi}{|\nabla\psi|^2}\right|^2 + \gamma p|\nabla\cdot\xi|^2 - 2U|\xi\cdot\nabla\psi|^2\right),$$

(113)

where

$$2U = 2p'\kappa_\psi + \sigma^2|\mathbf{s}|^2 + \sigma S . \tag{114}$$

The form of U given here (see Appendix B) is more practical for application than the usual one[24-26] since it is in terms of the normal curvature, parallel current, and local shear introduced in previous sections.

As described in the Appendix of Ref. 4, we can reduce the stability problem to a minimization of δW_p over the *scalar* field $\xi = \boldsymbol{\xi} \cdot \boldsymbol{\nabla}\psi$ by a prior analytic minimization over the other two components of $\boldsymbol{\xi}$ using a method due to Bineau.[25] The resulting form of δW_p is

$$
2\delta W_p = (2\pi)^2 \int_0^{\psi_a} \Big\langle -(\,\mathscr{P}\xi\,)^*(\mathscr{F}\Delta_s)^{-1}\mathscr{P}\xi
$$
$$
+ \mathscr{F}\,|\mathbf{B}\cdot\boldsymbol{\nabla}\xi\,|^2 - 2\mathscr{F}U\,|\xi\,|^2\Big\rangle + (2\pi)^2\delta_{n,0}
$$
$$
\times \int_0^{\psi_a} d\psi \left(\frac{\gamma p|\langle \mathscr{F}\xi\,\rangle'|^2}{\langle \mathscr{F}\rangle} + \langle \mathscr{F}|\boldsymbol{\nabla}_s v|^2\rangle \right), \tag{115}
$$

where

$$\mathscr{P}\xi \equiv \boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\psi\mathbf{B}\cdot\boldsymbol{\nabla}\xi - \mathbf{j}\times\boldsymbol{\nabla}\psi\xi)/|\boldsymbol{\nabla}\psi|^2 , \tag{116}$$

$$\Delta_s \equiv \boldsymbol{\nabla}\cdot\boldsymbol{\nabla}_s , \tag{117}$$

and v is a multivalued potential satisfying

$$\Delta_s v = 0 , \tag{118}$$

and

$$\boldsymbol{\nabla}_s v = [\boldsymbol{\nabla}\times(\boldsymbol{\xi}\times\mathbf{B}) + (\xi\mathbf{j}\times\boldsymbol{\nabla}\psi)/|\boldsymbol{\nabla}\psi|^2]_s - \boldsymbol{\nabla}_s u , \tag{119}$$

where u is single-valued. As in Sec. VI, $\langle\ \rangle$ denotes averaging over θ. We have assumed that ξ depends on ζ through a factor $\exp(-in\zeta)$, indicating through the Kronecker delta factor $\delta_{n,0}$ that the last two terms of Eq. (115) contribute only to the modes having the same symmetry as the equilibrium ($n = 0$).

When Eqs. (40) and (49) are taken into account, Eq. (114) already gives U in terms of the primitive set of equilibrium quantities. With the assumed ζ-dependence, $\mathbf{B}\cdot\boldsymbol{\nabla}\xi = \mathscr{F}^{-1}(\partial_\theta - inq)\xi$, and

$$
\mathscr{P}\xi = (\partial_\theta - inq)\partial_\psi\xi
$$
$$
+ (\partial_\theta - inq)[(\boldsymbol{\nabla}\psi\cdot\boldsymbol{\nabla}\theta)/|\boldsymbol{\nabla}\psi|^2](\partial_\theta - inq)\xi
$$
$$
+ (\partial_\theta - inq)(\,p'/B^2 - g\sigma/|\boldsymbol{\nabla}\psi|^2 + in\mathscr{R})\xi
$$
$$
- in\mathscr{F}(\sigma|\mathbf{s}|^2 + S)\xi . \tag{120}
$$

We also have

$$
\mathscr{F}\Delta_s = \partial_\theta(\mathscr{F}\,|\boldsymbol{\nabla}_s\theta\,|^2)\partial_\theta - in(\mathscr{F}\boldsymbol{\nabla}_s\theta\cdot\boldsymbol{\nabla}_s\zeta)\partial_\theta
$$
$$
- in\partial_\theta(\mathscr{F}\boldsymbol{\nabla}_s\theta\cdot\boldsymbol{\nabla}_s\zeta) - n^2\mathscr{F}\,|\boldsymbol{\nabla}_s\zeta|^2 . \tag{121}
$$

In the above two equations, ∂_θ acts on everything to its right. The coefficients are given by Eqs. (34)–(36).

To evaluate the second $n = 0$ term we write

$$\boldsymbol{\nabla}_s v = \sum_{i=1}^2 I_i(\psi)\mathbf{Y}_s^{(i)} , \tag{122}$$

where $\boldsymbol{\nabla}\times\mathbf{Y}^{(i)} = 0$ and $\boldsymbol{\nabla}\cdot\mathbf{Y}_s^i = 0$. We normalize $\mathbf{Y}^{(1)}$ [$\mathbf{Y}^{(2)}$] so that the line integral around any closed circuit is the same as that of $-\boldsymbol{\nabla}\zeta/2\pi$ ($\boldsymbol{\nabla}\theta/2\pi$). By representing the $\mathbf{Y}^{(i)}$ as gradients of multivalued scalar potentials,[4] we can easily show

$$\mathbf{Y}_s^{(1)} = -\frac{\boldsymbol{\nabla}_s\zeta}{2\pi} - \frac{(C_1 - \mathscr{F}\boldsymbol{\nabla}_s\zeta\cdot\boldsymbol{\nabla}_s\theta)}{\mathscr{F}|\boldsymbol{\nabla}_s\theta|^2}\frac{\boldsymbol{\nabla}_s\theta}{2\pi} , \tag{123}$$

and

$$\mathbf{Y}_s^{(2)} = \frac{C_2}{\mathscr{F}|\boldsymbol{\nabla}_s\theta|^2}\frac{\boldsymbol{\nabla}_s\theta}{2\pi} , \tag{124}$$

where

$$C_1(\psi) = \langle\boldsymbol{\nabla}_s\zeta\cdot\boldsymbol{\nabla}_s\theta/|\boldsymbol{\nabla}_s\theta|^2\rangle C_2(\psi) , \tag{125}$$

$$C_2(\psi) = \langle(\mathscr{F}|\boldsymbol{\nabla}_s\theta|^2)^{-1}\rangle^{-1} . \tag{126}$$

We can eliminate ξ_s from Eq. (119) by dotting with $\mathbf{Y}^{(i)}$ and integrating over a magnetic surface, thus obtaining

$$\sum_{j=1}^2 L'_{ij}I_j = V'_i[\xi] , \tag{127}$$

where

$$L'_{ij}(\psi) = (2\pi)^2\langle\mathscr{F}\mathbf{Y}_s^{(i)}\cdot\mathbf{Y}_s^{(j)}\rangle , \tag{128}$$

$$V'_i[\xi] = (2\pi)^2\left\langle\frac{\mathscr{F}\mathbf{Y}_s^{(i)}\cdot\boldsymbol{\nabla}\times(\xi\mathbf{s}) + \mathscr{F}\xi\,\mathbf{Y}_s^{(i)}\times\mathbf{j}\cdot\boldsymbol{\nabla}\psi}{|\boldsymbol{\nabla}\psi|^2}\right\rangle , \tag{129}$$

so that

$$L'_{11} = \langle(\mathscr{F}|\boldsymbol{\nabla}\psi|^2|\boldsymbol{\nabla}_s\theta|^2)^{-1}\rangle + C_1^2/C_2 , \tag{130}$$

$$L'_{12} = -C_1 , \tag{131}$$

$$L'_{22} = C_2 , \tag{132}$$

and

$$
V'_1 = 2\pi\left\langle\frac{(C_1 - \mathscr{F}\mathbf{s}\cdot\boldsymbol{\nabla}_s\theta)}{\mathscr{F}|\boldsymbol{\nabla}_s\theta|^2}\partial_\psi\xi\right\rangle
$$
$$
+ 2\pi\left\langle\frac{(C_1 - \mathscr{F}\mathbf{s}\cdot\boldsymbol{\nabla}_s\theta)}{\mathscr{F}|\boldsymbol{\nabla}_s\theta|^2}\frac{\boldsymbol{\nabla}\psi\cdot\boldsymbol{\nabla}\theta}{|\boldsymbol{\nabla}\psi|^2}\partial_\theta\xi\right\rangle
$$
$$
+ 2\pi\left\langle\left[\frac{C_1}{\mathscr{F}|\boldsymbol{\nabla}_s\theta|^2}\left(\frac{p'}{B^2} - \frac{g\sigma}{|\boldsymbol{\nabla}\psi|^2}\right)\right.\right.
$$
$$
\left.\left. - \frac{g'}{\mathscr{F}|\boldsymbol{\nabla}\psi|^2|\boldsymbol{\nabla}_s\theta|^2} + \mathscr{F}S\right]\xi\right\rangle , \tag{133}
$$

$$
V'_2 = -2\pi C_2\left\langle\frac{\partial_\psi\xi}{\mathscr{F}|\boldsymbol{\nabla}_s\theta|^2} + \frac{\boldsymbol{\nabla}\psi\cdot\boldsymbol{\nabla}\theta}{\mathscr{F}|\boldsymbol{\nabla}\psi|^2|\boldsymbol{\nabla}_s\theta|^2}\partial_\theta\xi\right.
$$
$$
\left. + \left(\frac{p'}{B^2} - \frac{g\sigma}{|\boldsymbol{\nabla}\psi|^2}\right)\frac{\xi}{\mathscr{F}|\boldsymbol{\nabla}_s\theta|^2}\right\rangle . \tag{134}
$$

Solving Eq. (127), and using Eq. (122), we find the last term of δW_p in Eq. (115) to be

$$\delta_{n,0}\sum_{i,j=1}^2\int_0^{\psi_a}d\psi\,V'_i[\xi]^*(\mathbf{L}'^{-1})_{ij}V'_j[\xi] . \tag{135}$$

At the time of writing, the formulation given above had been used to modify the PEST2 code to a new version PEST2.5, except that the $n = 0$ terms had not been included. The vacuum energy remains to be formulated.

XI. CONCLUSIONS

We have found a formulation of magnetic coordinates appropriate to ideal, isotropic pressure hydromagnetic equilibria, which is both concise and practical. Since such equilibria can form the starting point for investigating a variety of

physical effects, such as kinetic stability and wave propagation, as well as the hydromagnetic stability calculations described here, we anticipate that this formulation will be found widely useful.

APPENDIX A: HIGH-BETA ORDERING

In order to choose between different, but analytically identical [e.g., due to use of Eq. (41)], expressions we use the criterion that they must be well-ordered in a large-aspect-ratio ordering appropriate to axisymmetric systems with high $\beta_p \equiv 2p/B_p^2$, where p and B_p are a typical pressure and poloidal magnetic field, respectively. This also ensures that equations are well-ordered near the magnetic axis. By "well-ordered" we mean that the left-hand side should not be formally smaller in the expansion parameter ϵ than the largest term on the right-hand side, so that numerical errors are not introduced unnecessarily by the near cancellation of large quantities.

For definiteness we refer to axisymmetric systems (Sec. VIII) and take

$$a/R \sim \epsilon , \tag{A1}$$

where a and R are typical minor and major radii, respectively. We order

$$q \sim 1 , \tag{A2}$$

so it follows from $q \sim aB/RB_p$ that $B_p/B \sim \epsilon$.

We take units so that

$$a \sim 1 , \quad B \sim 1 , \tag{A3}$$

whence $R \sim \epsilon^{-1}, B_p \sim \epsilon$. It follows from $g \sim RB$ that $g \sim \epsilon^{-1}$.

We define the high-β ordering by taking

$$\beta_p \sim \epsilon^{-1} , \tag{A4}$$

so that $p \sim \epsilon$. Since $\psi \sim aRB_p$, we have $\psi \sim 1$, so that $p' \sim 1$, and $q' \sim 1$.

However g' is not ordered the same as g, but rather

$$g' \sim 1 . \tag{A5}$$

That is, g is constant to leading order in an expansion about the magnetic axis. This may be seen by considering that the Grad–Shafranov equation implies

$$gg' = -X^2 p' + O(1) ,$$

where X is defined in Eq. (62a).

Also, since $B^2 = g^2/X^2 + O(\epsilon^2)$, we see from Eq. (40) that the terms making up σ cancel to leading order, and we have

$$\sigma \sim \epsilon . \tag{A6}$$

Furthermore, the leading term of X is constant in an expansion about the magnetic axis, so $X'/X \sim \bullet$ and

$$\partial_\psi B^2 \sim \epsilon , \quad \partial_\theta B^2 \sim \epsilon . \tag{A7}$$

Finally, we see from $\mathscr{J} = X/|\nabla\psi||\nabla_s\theta|$ that $\mathscr{J} \sim \epsilon^{-1}$.

In summary, we list the orderings of the primitive set of equilibrium quantities and a few others;

$$g \sim \mathscr{J}' \sim \mathscr{J} \sim \epsilon^{-1} , \tag{A8}$$

$$q \sim g' \sim \nabla\psi \sim \nabla\theta \sim \nabla\zeta \sim \mathscr{R} \sim \partial_\theta \mathscr{R} \sim 1 , \tag{A9}$$

$$\partial_\psi B^2 \sim \partial_\theta B^2 \sim \kappa_s \sim \kappa_\psi \sim \sigma \sim \epsilon . \tag{A10}$$

APPENDIX B: VECTOR CALCULUS IN MAGNETIC COORDINATES

The straight-field-line coordinate system is neither orthogonal nor unique. We have thus been led to depart from the standard tensor calculus approach of using $\nabla\psi$, $\nabla\theta$, and $\nabla\zeta$ as a covariant basis set, and instead have used the more physical basis

$$\mathbf{e}^\psi \equiv \nabla\psi , \quad \mathbf{e}^\| \equiv \mathbf{B} , \quad \mathbf{e}^s \equiv \mathbf{s} . \tag{B1}$$

Since magnetic coordinates are useful in applications other than those treated in this paper, we have felt it desirable to show how to perform arbitrary vector calculus operations in our basis.

In this Appendix we show how to evaluate the dyadic $\nabla\mathbf{f}$, where $\mathbf{f}$ is an arbitrary vector, in terms of the primitive set augmented with the components of $\nabla|\nabla\psi|^2$. This result includes $\nabla \cdot \mathbf{f}$ and $\nabla \times \mathbf{f}$. As an application we evaluate the quantity U in Eq. (115). The results of this section are valid in nonsymmetric systems as well.

Resolving $\mathbf{f}$ onto our basis we have

$$\mathbf{f} = \sum_i f_i \mathbf{e}^i , \tag{B2}$$

where $i \in \{\psi, \|, s\}$. We also define the contravariant basis vectors

$$\mathbf{e}_i \equiv \mathbf{e}^i / |\mathbf{e}^i|^2 , \tag{B3}$$

so that the unit dyadic $\mathbf{I}$ can be represented

$$\mathbf{I} = \sum_i \mathbf{e}_i \mathbf{e}^i = \sum_i \mathbf{e}^i \mathbf{e}_i . \tag{B4}$$

From Eq. (B2) we have

$$\nabla \cdot \mathbf{f} = \sum_i \mathbf{e}^i \cdot \nabla f_i + \sum_i f_i \nabla \cdot \mathbf{e}^i , \tag{B5}$$

$$\nabla \times \mathbf{f} = \sum_i (\nabla f_i) \times \mathbf{e}^i + \sum_i f_i \nabla \times \mathbf{e}^i . \tag{B6}$$

The quantities $\nabla \cdot \mathbf{e}^i$ and $\nabla \times \mathbf{e}^i$ can all be evaluated in terms of the primitive set. First we need the result, following from Eqs. (20) and (22),

$$\nabla \times \mathbf{s} = -\nabla\psi \times \nabla(\mathscr{R} + q'\theta) . \tag{B7}$$

This gives $\nabla \times \mathbf{e}^s$, but also, from the identity $|\mathbf{s}|^2 \nabla\psi = \mathbf{B} \times \mathbf{s}$, we have

$$\nabla \cdot (|\mathbf{s}|^2 \nabla\psi) = \mathbf{j} \cdot \mathbf{s} - \mathbf{B} \cdot \nabla \times \mathbf{s}$$
$$= -p' - |\nabla\psi|^2 \mathbf{s} \cdot \nabla(\mathscr{R} + q'\theta) . \tag{B8}$$

Since $|s|^2 = B^2/|\nabla\psi|^2$, we can calculate $\nabla\psi \cdot \nabla|s|^2$ in terms of the augmented primitive set, and thus Eq. (B8) gives $\nabla \cdot e^\psi$. We also have $\nabla \cdot e^\| = 0$, and, from Eq. (27), we have $\nabla \cdot (|\nabla\psi|^2 e^s) = 0$. Finally we note that $\nabla \times e^\psi = 0$ and $\nabla \times e^\| = j$.

The most general vector calculus expression involving only first derivatives is ∇f, which we calculate using the identity[27]

$$2a \cdot \nabla b \cdot c = a \cdot \nabla(b \cdot c) + c \cdot \nabla(b \cdot a) - b \cdot \nabla(a \cdot c)$$

$$+ (b \times c) \cdot \nabla \times a + (b \times a) \cdot \nabla \times c$$

$$+ (a \times c) \cdot \nabla \times b . \qquad (B9)$$

By dotting ∇f on both sides with I, we find

$$2\nabla f = 2 \sum_i (\nabla f_i)e^i + f \cdot T + (f \cdot T)^T - T \cdot f$$

$$+ D \times f + (D \times f)^T - f \cdot D \times I , \qquad (B10)$$

where T is the triadic

$$T \equiv \sum_i e_i e_i \nabla |e^i|^2 , \qquad (B11)$$

D is the dyadic,

$$D \equiv \sum_i e_i (\nabla \times e^i) , \qquad (B12)$$

and superscript T denotes the transposed dyadic.

We observe that D involves no $\nabla\psi$ components. It is readily verified that Eq. (B10) is consistent with Eqs. (B5) and (B6). We can also use it to show

$$2B \cdot (\nabla n) \cdot s = -|s|^2\sigma - |\nabla\psi|S , \qquad (B13)$$

and

$$2B \cdot \nabla n \cdot B = -2|\nabla\psi|p' - (\nabla\psi \cdot \nabla B^2)/|\nabla\psi| , \qquad (B14)$$

where $n \equiv \nabla\psi/|\nabla\psi|$. We can then easily derive Eq. (114) from the form[24-26]

$$U = [B \cdot (\nabla n) \cdot j \times n]/|\nabla\psi|^2 , \qquad (B15)$$

although it is actually easier to derive U directly from first principles by manipulating the form of δW due to Furth et al.[2,28] into the form of Eq. (114).

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