Magnetic helicity tensor for an anisotropic turbulence

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The evolution of the magnetic helicity tensor for a nonzero mean magnetic field and for large magnetic Reynolds numbers in an anisotropic turbulence is studied. It is shown that the isotropic and anisotropic parts of the magnetic helicity tensor have different characteristic times of evolution. The time of variation of the isotropic part of the magnetic helicity tensor is much larger than the correlation time of the turbulent velocity field. The anisotropic part of the magnetic helicity tensor changes for the correlation time of the turbulent velocity field. The mean turbulent flux of the magnetic helicity is calculated as well. It is shown that even a small anisotropy of turbulence strongly modifies the flux of the magnetic helicity. It is demonstrated that the tensor of the magnetic part of the α effect for weakly inhomogeneous turbulence is determined only by the isotropic part of the magnetic helicity tensor. [S1063-651X(99)05506-3]

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I. INTRODUCTION

The magnetic helicity $\mathbf{A}^{(t)}\cdot\mathbf{H}$ is a fundamental quantity in magnetohydrodynamics because it is conserved in the limit of infinite electrical conductivity of the medium, where $\mathbf{H}=\boldsymbol{\nabla}\times\mathbf{A}^{(t)}$ is the magnetic field and $\mathbf{A}^{(t)}$ is the magnetic vector potential. In addition, the topological properties of magnetic field are determined by the magnetic helicity (see, e.g., [1,2]). In developed magnetohydrodynamic turbulence the mean magnetic helicity $\langle\mathbf{a}\cdot\mathbf{h}\rangle$ is conserved as well in the limit of infinite magnetic Reynolds numbers and zero mean magnetic field, where $\mathbf{h}$ and $\mathbf{a}$ are fluctuations of the magnetic field and the magnetic vector potential, respectively (see, e.g., [1,2]). The magnetic helicity tensor $\chi_{ij}=\langle a_i(\mathbf{x})h_j(\mathbf{x})\rangle$ determines the tensor of the magnetic part of the α effect. The latter is of fundamental importance in view of magnetic dynamo (see, e.g., [1–3]). In spite of the great importance of this quantity, a dynamics of the magnetic helicity tensor for an anisotropic turbulence is poorly understood.

In the present paper the equation for the magnetic helicity tensor for an anisotropic turbulence and a nonzero mean magnetic field, and for large magnetic Reynolds numbers is derived. It is shown that the isotropic and anisotropic parts of the magnetic helicity tensor have different characteristic times of evolution. The time of variation of the isotropic part of the magnetic helicity tensor is much longer than the correlation time of the turbulent velocity field. On the other hand, the anisotropic part of the magnetic helicity tensor changes for the correlation time of the turbulent velocity field. This anisotropic part is determined only by the turbulent magnetic diffusion tensor. The mean turbulent flux of the magnetic helicity is calculated as well. It is shown that even small anisotropy of turbulence strongly modifies the flux of the magnetic helicity.

II. THE EQUATION FOR THE MAGNETIC HELICITY: SIMPLE APPROACH

First, we derive an equation for the magnetic helicity for an anisotropic turbulence by a simple consideration. The induction equation for the magnetic field $\mathbf{H}$ is given by

$$\partial\mathbf{H}/\partial t=\boldsymbol{\nabla}\times(\mathbf{v}\times\mathbf{H}-\eta\boldsymbol{\nabla}\times\mathbf{H}), \tag{1}$$

where $\mathbf{H}=\mathbf{B}+\mathbf{h}$, $\mathbf{B}=\langle\mathbf{H}\rangle$ is the mean magnetic field, $\mathbf{v}=\mathbf{V}+\mathbf{u}$, $\mathbf{V}=\langle\mathbf{v}\rangle$ is the mean fluid velocity field, and η is the magnetic diffusion due to electrical conductivity of fluid. The equation for the vector potential $\mathbf{A}^{(t)}$ follows from the induction equation (1),

$$\partial\mathbf{A}^{(t)}/\partial t=\mathbf{v}\times\mathbf{H}-\eta\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\mathbf{A}^{(t)})+\boldsymbol{\nabla}\varphi, \tag{2}$$

where $\mathbf{H}=\boldsymbol{\nabla}\times\mathbf{A}^{(t)}$, $\mathbf{A}^{(t)}=\mathbf{A}+\mathbf{a}$, and $\mathbf{A}=\langle\mathbf{A}^{(t)}\rangle$ is the mean vector potential, $\varphi=\tilde{\Phi}+\phi$ is an arbitrary scalar function, and $\tilde{\Phi}=\langle\varphi\rangle$. Now we multiply Eq. (1) by $\mathbf{a}$ and Eq. (2) by $\mathbf{h}$, add them, and average over the ensemble of turbulent fields. This yields an equation for the magnetic helicity $\chi=\langle a_p(\mathbf{x})h_p(\mathbf{x})\rangle$:

$$\partial\chi/\partial t=-2\langle\mathbf{u}\times\mathbf{h}\rangle\cdot\mathbf{B}-2\eta\langle\mathbf{h}\cdot(\boldsymbol{\nabla}\times\mathbf{h})\rangle-\boldsymbol{\nabla}\cdot\tilde{\mathbf{F}} \tag{3}$$

where $\tilde{F}_p=V_p\chi-\chi_{pn}V_n+\langle\mathbf{a}\times\mathbf{u}\rangle\times\mathbf{B}-\eta\langle\mathbf{a}\times(\boldsymbol{\nabla}\times\mathbf{h})\rangle+\langle\mathbf{a}\times(\mathbf{u}\times\mathbf{h})\rangle-\langle\mathbf{h}\phi\rangle$. Electromotive force for an anisotropic turbulence is given by

$$\langle\mathbf{u}\times\mathbf{h}\rangle=\mathbf{U}\times\mathbf{B}+\hat{\alpha}\mathbf{B}-\hat{\eta}\boldsymbol{\nabla}\times\mathbf{B} \tag{4}$$

(see, e.g., [4,5]), where $\hat{\eta}\equiv\hat{\eta}_{mn}=(\eta_{pp}\delta_{mn}-\eta_{mn})/2$, $\eta_{mn}=\eta\delta_{mn}+\tilde{\eta}_{mn}$, $\tilde{\eta}_{mn}=\langle\tau u_m u_n\rangle$, $(\mathbf{U})_n=-\boldsymbol{\nabla}_m\tilde{\eta}_{mn}/2$ is the velocity caused by the turbulent diamagnetism, $\hat{\alpha}=\alpha_{mn}=\alpha_{mn}^{(v)}+\alpha_{mn}^{(B)}$, and the tensors $\alpha_{mn}^{(v)}$ and $\alpha_{mn}^{(B)}$ are given by

$$\alpha_{mn}^{(v)}=-[\varepsilon_{mji}\langle\tau u_i(\mathbf{x})\boldsymbol{\nabla}_n u_j(\mathbf{x})\rangle+\varepsilon_{nji}\langle\tau u_i(\mathbf{x})\boldsymbol{\nabla}_m u_j(\mathbf{x})\rangle]/2, \tag{5}$$

$$\alpha_{mn}^{(B)}=[\varepsilon_{mji}\langle\tau h_i(\mathbf{x})\boldsymbol{\nabla}_n h_j(\mathbf{x})\rangle+\varepsilon_{nji}\langle\tau h_i(\mathbf{x})\boldsymbol{\nabla}_m h_j(\mathbf{x})\rangle]/(2\mu_0\rho). \tag{6}$$

Substituting Eq. (4) into Eq. (3) we obtain after simple manipulations an equation for the magnetic helicity:

$$\frac{\partial\chi}{\partial t} = -2\eta\left(\frac{\partial^2\chi}{\partial x_p\partial y_p}\right)_{r\to 0} + 2\eta_{mn}B_m(\nabla\times\mathbf{B})_n - 2\hat{\alpha}_{mn}B_mB_n - \nabla\cdot\mathbf{F}, \tag{7}$$

where we used an identity $\langle\mathbf{h}\cdot(\nabla\times\mathbf{h})\rangle = (\partial^2\chi/\partial x_p\partial y_p)_{r\to 0}$, and $\mathbf{r}=\mathbf{y}-\mathbf{x}$. The second and third terms in Eq. (7) describe the sources of the magnetic helicity. Therefore, the mean magnetic field $\mathbf{B}$, the mean electric current $\propto\nabla\times\mathbf{B}$, and the hydrodynamic helicity are the sources of the magnetic helicity. The first term in Eq. (7) determines the relaxation of the magnetic helicity with the characteristic time T which depends on the molecular magnetic diffusion η. This time is given by

$$T^{-1} = \frac{2\eta}{\chi}\left(\frac{\partial^2\chi}{\partial x_p\partial y_p}\right)_{r=0}. \tag{8}$$

The characteristic relaxation time T of the magnetic helicity is $T\sim\tau_0\mathrm{Rm}$ (Rm denotes ''magnetic Reynolds number''), i.e., it is much longer than the correlation time $\tau_0 = l_0/u_0$ of the turbulent velocity field, where u_0 is the characteristic turbulent velocity in the maximum scale of turbulent motions l_0. The last term in Eq. (7) describes the turbulent flux $\mathbf{F}$ of the magnetic helicity which will be calculated in Sec. III. Equation (7) in the case of an isotropic turbulence coincides with that derived in [6] (see also [7,8]).

III. THE EQUATION FOR THE MAGNETIC HELICITY TENSOR: METHOD OF PATH INTEGRALS

In this section we derive an equation for the magnetic helicity tensor. To this purpose we use a method of path integrals (see, e.g., [5,9–11]). This method allows us to derive the equation for the tensor $\chi_{ij} = \langle a_i(\mathbf{x})h_j(\mathbf{y})\rangle_{\mathbf{r}\to\mathbf{0}}$:

$$\frac{\partial\chi_{ij}}{\partial t} = -2\eta\left(\frac{\partial^2\chi_{ij}}{\partial x_p\partial y_p}\right)_{r=0} - 2\tilde{\eta}_{np}\left(\frac{\partial^2\chi_{nj}}{\partial x_p\partial y_i}\right)_{r=0} + \frac{\partial}{\partial R_p}(\varepsilon_{jpl}\alpha^{(v)}_{ls}\chi_{is} - V_p\chi_{ij} + V_l\chi_{lj}\delta_{ip}) + \frac{\partial V_j}{\partial R_p}\chi_{ip} - \frac{\partial V_p}{\partial R_i}\chi_{pj} + 2\alpha^{(v)}_{is}h_{sj} - \alpha^{(v)}_{ks}h_{sk}\delta_{ij} + \varepsilon_{isp}S_ph_{sj} + I_{ij} \tag{9}$$

(for details, see Appendix A), where $\mathbf{R}=(\mathbf{x}+\mathbf{y})/2$,

$$I_{ij} = \alpha^{(v)}_{is}B_jB_s - \alpha^{(v)}_{ks}B_kB_s\delta_{ij} + \varepsilon_{ikl}\langle\tau u_l b\rangle B_kB_j + 2\varepsilon_{kli}\tilde{\eta}_{lp}B_k(\partial B_j/\partial R_p) + J_{ij}, \tag{10}$$

$h_{ij} = \langle h_i(\mathbf{x})h_j(\mathbf{x})\rangle$, $\tilde{\eta}_{ij} = \langle u_i(\mathbf{x})u_j(\mathbf{x})\rangle$, $S_i = \langle u_i(\mathbf{x})b(\mathbf{x})\rangle$, $\tilde{\varphi}_j = \langle\phi(\mathbf{x})h_j(\mathbf{x})\rangle$, $J_{ij} = \varepsilon_{lpj}\alpha^{(v)}_{ls}\langle(\partial a_i/\partial x_p)h_s\rangle + \partial\tilde{\varphi}_j/\partial R_i - \langle(\partial h_j/\partial x_i)\phi\rangle$, and $b=\nabla\cdot\mathbf{u}$. Equation (9) is derived for the case $\nabla\tilde{\eta}_{ij}=0$. We use here the δ-correlated in time random process to describe a turbulent velocity field. The results remain valid also for the velocity field with a finite correlation time, if the second-order correlation functions of the magnetic field and the magnetic helicity vary slowly in comparison with the correlation time of the turbulent velocity field (see, e.g., [9,12]). We also take into account the dependence of the momentum relaxation time on the scale of turbulent velocity field: $\tau(\mathbf{k}) = 2\tau_0(k/k_0)^{1-p}$, where p is the exponent in the spectrum of kinetic turbulent energy, k is the wave number, $k_0 = l_0^{-1}$. The equation for $\chi=\chi_{pp}$ follows from Eq. (9):

$$\frac{\partial\chi}{\partial t} = -2\eta\left(\frac{\partial^2\chi}{\partial x_p\partial y_p}\right)_{r=0} + 2\hat{\eta}_{mn}B_m(\nabla\times\mathbf{B})_n - 2\alpha^{(v)}_{mn}B_mB_n + \nabla_p[\varepsilon_{pmn}\chi_{ns}\alpha^{(v)}_{ms} + V_m\chi_{mp} - (4/3)V_p\chi] \tag{11}$$

(see Appendix A), where hereafter $\nabla_p = \partial/\partial R_p$, and we used the gauge condition for the mean vector potential $\tilde{\eta}_{sp}\nabla_pA_s = 0$. For an isotropic turbulence ($\tilde{\eta}_{mn} = \tilde{\eta}\delta_{mn}/3$) the gauge condition is given by $\nabla\cdot\mathbf{A}=0$. The last term in Eq. (11) describes the turbulent flux of the magnetic helicity $F_p = \varepsilon_{plt}\chi_{ts}\alpha^{(v)}_{ls} + V_s\chi_{sp} - (4/3)V_p\chi$. The mean turbulent flux of the magnetic helicity depends on the tensor of hydrodynamic helicity $\alpha^{(v)}_{ij}$ and the mean fluid velocity $\mathbf{V}$. Comparison of Eq. (11) (which was derived by the path integral method) with Eq. (7) (which was obtained by the simple consideration) shows that these two approaches arrive at the similar equation after the change $\alpha^{(v)}_{mn}\to\alpha_{mn}$. Note that the mean turbulent flux of the magnetic helicity $\mathbf{F}$ cannot be calculated by the simple consideration.

The tensor χ_{ij} can be presented in the form $\chi_{ij} = \chi\delta_{ij}/3 + \mu_{ij}$, where the anisotropic part of the magnetic helicity tensor μ_{ij} has the following properties: $\mu_{pp}=0$, and $\mu_{ij}=\mu_{ji}$. For the calculation of the second spatial derivative $(\partial^2\chi_{ij}/\partial x_p\partial y_p)_{r=0}$ we use the tensor $\chi_{ij}(\mathbf{k}^{(1)},\mathbf{k}^{(2)})$ in $\mathbf{k}$ space:

$$\chi_{ij}(\mathbf{k}^{(1)},\mathbf{k}^{(2)}) = -5[(k_{pp}\delta_{ij} - k_{ij})(\chi_*/5 - k_{mn}\mu_{nm}/2k_{pp}) - \mu_{im}k_{mj} - k_{im}\mu_{mj} + k_{pp}\mu_{ij} + k_{mn}\mu_{nm}\delta_{ij})]/8\pi k^2, \tag{12}$$

where

$$\chi_{ij}(\mathbf{x},\mathbf{y}) = \int\chi_{ij}(\mathbf{k}^{(1)},\mathbf{k}^{(2)})\exp i(\mathbf{k}^{(1)}\mathbf{x} + \mathbf{k}^{(2)}\mathbf{y})d\mathbf{k}^{(1)}d\mathbf{k}^{(2)},$$

and $k_{ij} = k_i^{(2)}k_j^{(1)}$. The tensor $\chi_{ij}(\mathbf{k}^{(1)},\mathbf{k}^{(2)})$ satisfies the identities $k_i^{(1)}\chi_{ij}(\mathbf{k}^{(1)},\mathbf{k}^{(2)})=0$ and $\chi_{ij}(\mathbf{k}^{(1)},\mathbf{k}^{(2)})k_j^{(2)}=0$. These identities correspond to the conditions $\nabla\cdot\mathbf{a}=0$ and $\nabla\cdot\mathbf{h}=0$, respectively. Using Eqs. (8), (12), and (B1) (see Appendix B) we rewrite Eq. (11) for the magnetic helicity in the form

$$\partial\chi/\partial t + \chi/T + \nabla_p(V_p\chi) + 2\alpha^{(v)}_{mn}B_mB_n - 2\hat{\eta}_{mn}B_m(\nabla\times\mathbf{B})_n = \nabla_p(\mu_{sf}\varepsilon_{fpl}\alpha^{(v)}_{ls} + V_s\mu_{sp}). \tag{13}$$

Equation (13) implies that the characteristic relaxation time T of the isotropic part of the magnetic helicity tensor is $T\sim\tau_0\mathrm{Rm}$, i.e., it is much longer than the correlation time $\tau_0 = l_0/u_0$ of the turbulent velocity field. Equations (9) and (13) yield the equation for the tensor μ_{ij}:

$$\eta^*_{jp}\mu_{pi}+8\,\eta^*_{ip}\mu_{pj}-3\mu_{ij}-3\,\delta_{ij}\,\eta^*_{pm}\mu_{mp} =(7/10)(3\,\eta^*_{ij}-\delta_{ij})\chi+O(\tau_0/T), \tag{14}$$

where $\eta^*_{ij}=\tilde{\eta}_{ij}/\tilde{\eta}_{pp}$. We neglected here small terms $\sim(\tau_0/T)$ and $\sim\tau_0 B^2$. It follows from Eq. (14) that the anisotropic part of the magnetic helicity tensor is determined only by the turbulent diffusion tensor. Therefore, the characteristic time of evolution of the anisotropic part μ_{ij} of the magnetic helicity tensor is of the order of τ_0, i.e., it is very small. Solving Eq. (14) in the complete set of the eigenfunctions of the matrix η^*_{ij} we obtain $\mu_{ij}=0$ when $i\neq j$, $\mu_{11}=\mu_1\chi$, $\mu_{22}=\mu_2\chi$, and $\mu_{33}=-(\mu_1+\mu_2)\chi$, where $\eta^*_{ij}=0$ when $i\neq j$, and $\eta^*_{11}=\eta_1$, $\eta^*_{22}=\eta_2$, $\eta^*_{33}=1-(\eta_1+\eta_2)$, and

$$\mu_1=\left(\frac{7}{30}\right)\frac{\varepsilon_2^2-(\varepsilon_1-\varepsilon_2)^2/3}{\varepsilon_1\varepsilon_2+(\varepsilon_1+\varepsilon_2)^2/3},$$

$$\mu_2=\left(\frac{7}{30}\right)\frac{\varepsilon_1^2-(\varepsilon_1-\varepsilon_2)^2/3}{\varepsilon_1\varepsilon_2+(\varepsilon_1+\varepsilon_2)^2/3},$$

$\eta_1=1/3+\varepsilon_1$, and $\eta_2=1/3+\varepsilon_2$. In the case of one preferential direction ($\varepsilon_1=\varepsilon_2\equiv\varepsilon\neq 0$) we obtain $\mu_1=\mu_2=7/30$. When $\varepsilon=0$ the anisotropic part of the magnetic helicity tensor $\mu_{ij}=0$. In the case of one preferential direction (say, in the direction $\mathbf{e}$), Eqs. (13) and (14) yield

$$\partial\chi/\partial t+\chi/T+\boldsymbol{\nabla}_p(V_p^{\rm eff}\chi)+2\,\alpha^{(v)}_{mn}B_mB_n-2\,\hat{\eta}_{mn}B_m(\boldsymbol{\nabla}\times\mathbf{B})_n =0, \tag{15}$$

where $\mathbf{V}^{\rm eff}=23\mathbf{V}/30+7(\mathbf{e}\cdot\mathbf{V})\mathbf{e}/10-7(\mathbf{e}\times\mathbf{D})/15$, and the vector $D_m=\alpha^{(v)}_{mn}e_n$. Equation (15) implies that even small anisotropy of turbulence ($\mathrm{Rm}^{-1}\ll\varepsilon\ll 1$) strongly modifies the flux of the magnetic helicity.

For a weakly inhomogeneous turbulence the magnetic part of the α tensor is given by

$$\alpha^{(B)}_{mn}(\mathbf{r}=0)\sim\frac{2\chi}{9\,\eta_T\mu_0\rho}\,\delta_{mn}\equiv\alpha^{(B)}\delta_{mn} \tag{16}$$

(see Appendix C), where $\alpha^{(B)}=2\chi/(9\,\eta_T\mu_0\rho)$ and $\chi=\chi(\mathbf{R})$. This implies that the tensor for the magnetic part of the α effect for weakly inhomogeneous turbulence is determined only by the isotropic part of the magnetic helicity tensor. Thus, the evolutionary equation for the magnetic part of the α effect in this case is given by

$$\frac{\partial\alpha^{(B)}}{\partial t}+\frac{\alpha^{(B)}}{T}+\frac{1}{\rho}\boldsymbol{\nabla}_p(V_p^{\rm eff}\alpha^{(B)}\rho) =-\frac{4}{9\,\eta_T\mu_0\rho}[\alpha^{(v)}_{mn}B_mB_n-\hat{\eta}_{mn}B_m(\boldsymbol{\nabla}\times\mathbf{B})_n], \tag{17}$$

where we used Eqs. (15) and (16).

IV. DISCUSSION

We have shown here that an anisotropy of a fluid flow strongly modifies the turbulent transport of the magnetic helicity. In particular, even small anisotropy of turbulence significantly changes the mean flux of the magnetic helicity. It is given by $\mathbf{F}=\mathbf{V}^{\rm eff}\chi$. Indeed, if we consider, e.g., a small anisotropy of turbulence: $\varepsilon\sim\mathrm{Rm}^{-\beta}$ (where $\beta<1$), then the vector $D_m=\alpha^{(v)}_{pp}e_m/3+O(\mathrm{Rm}^{-\beta})$. When the mean velocity $\mathbf{V}$ is normal to the vector $\mathbf{e}$ (which is typical for astrophysical applications) we obtain $\mathbf{V}^{\rm eff}\approx 23\mathbf{V}/30$. Therefore, a very small anisotropy $\sim\mathrm{Rm}^{-\beta}$ changes the mean flux of the magnetic helicity to 25%. This result is associated with an existence of a small parameter Rm^{-1} which is the ratio of the relaxation times of anisotropic and isotropic parts of the magnetic helicity tensor. Note that the mean magnetic field is the main source of the magnetic helicity. For zero mean magnetic field the magnetic helicity is very small [15].

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APPENDIX A: DERIVATION OF THE EQUATION FOR THE MAGNETIC HELICITY TENSOR

We use a method of path integrals (and modified Feynman-Kac formula) (see, e.g., [5,9–11]). The solution of the induction equation (1) with the initial condition $\mathbf{H}(t=t_0,\mathbf{x})=\mathbf{H}_0(\mathbf{x})$ is given by the Feynman-Kac formula $H_i(t,\mathbf{x})=M\{G_{ij}(t,t_0)H_{0j}[\boldsymbol{\xi}(t,t_0)]\}$, where the function G_{ij} is determined by the equation $dG_{ij}(t_s,t_0)/ds=N_{ik}G_{kj}$ with the initial condition $G_{ij}=\delta_{ij}$ for $t_s=t_0$. Here $M\{\cdot\}$ is a mathematical expectation over the ensemble of Wiener paths, $t_s=t+s$, and $N_{ik}=\partial v_i/\partial x_k-\delta_{ik}b$, and the Wiener path $\boldsymbol{\xi}_t\equiv\boldsymbol{\xi}(t,t_0)$ is given by $\boldsymbol{\xi}_t=\mathbf{x}-\int_{t_0}^{t}\mathbf{v}(t_s,\boldsymbol{\xi}_s)ds+\sqrt{2\eta}\mathbf{w}(t)$, where $\mathbf{w}_t$ is a Wiener process. This method allows us to get $H_i(t+\Delta t,\mathbf{x})$:

$$H_i(t+\Delta t,\mathbf{x})\simeq H_i(t,\mathbf{x})+M\left\{q_i(\mathbf{x})\Delta t+p_i(\mathbf{x})(\Delta t)^2 +\sqrt{2\eta}Q_{in}(\mathbf{x})\int_0^{\Delta t}w_n d\sigma\right\} \tag{A1}$$

(see Appendix in [5]), where $Q_{in}=H_j\boldsymbol{\nabla}_nN_{ij}-(\boldsymbol{\nabla}_mH_i)(\boldsymbol{\nabla}_nv_m)$, and

$$q_i=H_m\boldsymbol{\nabla}_mv_i-v_m\boldsymbol{\nabla}_mH_i-bH_i+\eta w_mw_n\boldsymbol{\nabla}_m\boldsymbol{\nabla}_nH_i(\Delta t)^{-1},$$

$$p_i=(1/2)H_n[\boldsymbol{\nabla}_m(v_i\boldsymbol{\nabla}_nv_m-v_m\boldsymbol{\nabla}_nv_i)-\boldsymbol{\nabla}_n(bv_i) +\delta_{in}\boldsymbol{\nabla}_m(bv_m)]+\boldsymbol{\nabla}_mH_n(bv_m\delta_{in}-v_m\boldsymbol{\nabla}_nv_i +v_k\boldsymbol{\nabla}_kv_m\delta_{in}/2)+(1/2)v_mv_n\boldsymbol{\nabla}_m\boldsymbol{\nabla}_nH_i.$$

Now we use the following identity $[\boldsymbol{\nabla}\times(\hat{\eta}\boldsymbol{\nabla}\times\mathbf{H})]_k=\boldsymbol{\nabla}_i(H_n\boldsymbol{\nabla}_n\bar{\eta}_{ki}-H_i\boldsymbol{\nabla}_n\bar{\eta}_{nk}-\bar{\eta}_{in}\boldsymbol{\nabla}_nH_k)$, where $\hat{\eta}\equiv\hat{\eta}_{ij}=(\eta_{pp}\delta_{ij}-\eta_{ij})/2$. This identity can be derived as follows. Consider the vector $E_k=\boldsymbol{\nabla}_i(H_n\boldsymbol{\nabla}_n\bar{\eta}_{ki})=\boldsymbol{\nabla}_i\boldsymbol{\nabla}_n(H_n\bar{\eta}_{ki})$, where $\bar{\eta}_{ki}$ is an arbitrary symmetrical tensor, and we use the condition $\boldsymbol{\nabla}\cdot\mathbf{H}=0$. Now we change $n\to i$ and $i\to n$. This yields $E_k=\boldsymbol{\nabla}_n\boldsymbol{\nabla}_i(H_i\bar{\eta}_{kn})=\boldsymbol{\nabla}_i(H_i\boldsymbol{\nabla}_n\bar{\eta}_{kn}+\bar{\eta}_{kn}\boldsymbol{\nabla}_nH_i)$. Using this equation we calculate the vector $C_k=\boldsymbol{\nabla}_i(H_n\boldsymbol{\nabla}_n\bar{\eta}_{ki}$

$-H_i\nabla_n\bar{\eta}_{nk} - \bar{\eta}_{in}\nabla_n H_k) = \nabla_i(H_i\nabla_n\bar{\eta}_{kn} + \bar{\eta}_{kn}\nabla_n H_i - H_i\nabla_n\bar{\eta}_{nk} - \bar{\eta}_{in}\nabla_n H_k) = \nabla_i[(\bar{\eta}_{nk}\delta_{is} - \bar{\eta}_{in}\delta_{ks})\nabla_n H_s]$. Now we introduce the tensor $\hat{\eta} \equiv \hat{\eta}_{ij} = (\bar{\eta}_{pp}\delta_{ij} - \bar{\eta}_{ij})/2$. Using the identity $\varepsilon_{kim}\varepsilon_{jns}\hat{\eta}_{mj} = \bar{\eta}_{nk}\delta_{is} - \bar{\eta}_{in}\delta_{ks}$ we obtain $C_k = [\nabla\times(\hat{\eta}\nabla\times\mathbf{H})]_k$. Note that the multiplication of the latter identity by $\varepsilon_{kit}\varepsilon_{fns}$ yields the definition of the tensor $\hat{\eta}_{ij}$. Therefore, these calculations yield the above identity. Note that $\bar{\eta}_{mn}$ is an arbitrary symmetrical tensor. When $\bar{\eta}_{mn} = W_{mn}$ (where $\nabla W_{mn} = 0$), these identities yield $[\nabla\times(\hat{W}\nabla\times\mathbf{H})]_k = -W_{in}\nabla_i\nabla_n H_k$. We also use an identity $v_i\nabla_n v_k = [\varepsilon_{ikp}\overline{\alpha}^{(v)}_{pn} + \delta_{kn}\bar{S}_i - \delta_{in}\bar{S}_k + \nabla_n(v_k v_k)]/2$ (see [5]) where $\overline{\alpha}^{(v)}_{mn} = -(\varepsilon_{mji}v_i\nabla_n v_j + \varepsilon_{nji}v_i\nabla_m v_j)/2$, and $\bar{S}_m = v_m(\nabla\cdot\mathbf{v}) - \nabla_n(v_n v_m)/2$. Using these equations we obtain $Q_{in} = \varepsilon_{itm}\nabla_t(\varepsilon_{tps}H_p\nabla_n v_s)$, and

$$q_i = \varepsilon_{itm}\nabla_t[\varepsilon_{mls}v_l H_s - \eta(w_p w_p\delta_{mn} - w_m w_n) \times(\nabla\times\mathbf{H})_n]/(2\Delta t), \tag{A2}$$

$$p_i = (-1/4)\varepsilon_{itm}\nabla_t[\varepsilon_{mls}H_s\nabla_p(v_l v_p) - 2\overline{\alpha}^{(v)}_{mn}H_n + (v_p v_p\delta_{mn} - v_m v_n)(\nabla\times\mathbf{H})_n]. \tag{A3}$$

Equations (A1)–(A3) yield an equation for the vector potential $\mathbf{A}^{(t)}$:

$$A^{(t)}_i(t+\Delta t,\mathbf{x}) \simeq A^{(t)}_i(t,\mathbf{x}) + M\left\{Q_i(\mathbf{x})\Delta t + P_i(\mathbf{x})(\Delta t)^2 + \sqrt{2\eta}S_{in}(\mathbf{x})\int_0^{\Delta t} w_n d\sigma\right\} + \Delta t\nabla_i\varphi, \tag{A4}$$

where $\mathbf{H} = \nabla\times\mathbf{A}^{(t)}$, $S_{in} = \varepsilon_{ips}H_p\nabla_n v_s$,

$$Q_i = \varepsilon_{ifk}v_f H_k - \eta(w_p w_p\delta_{il} - w_i w_l)(\nabla\times\mathbf{H})_l/(2\Delta t), \tag{A5}$$

$$P_i = (-1/4)[\varepsilon_{ils}H_s\nabla_p(v_l v_p) - 2\overline{\alpha}^{(v)}_{in}H_n + (v_p v_p\delta_{in} - v_i v_n)(\nabla\times\mathbf{H})_n], \tag{A6}$$

and φ is an arbitrary scalar function which depends on the gauge condition.

Now we introduce a two-point correlation function $\chi^{(xy)}_{ij} = A_{ij} - A_i(t,\mathbf{x})B_j(t,\mathbf{y})$, where $A_{ij} = \langle A^{(t)}_i(t,\mathbf{x})H_j(t,\mathbf{y})\rangle$, $A^{(t)}_i = A_i + a_i$, $H_i = B_i + h_i$, and $\mathbf{A} = \langle\mathbf{A}^{(t)}\rangle$, $\mathbf{B} = \langle\mathbf{H}\rangle$, where equations for the mean fields $\mathbf{A}$ and $\mathbf{B}$ are given by $\partial B_m/\partial t = L^{(B)}_{mn}B_n$, $\partial A_m/\partial t = L^{(A)}_{mn}B_n + \nabla_m\tilde{\Phi}$, $L^{(A)}_{sj}(\mathbf{x}) = \varepsilon_{smj}V_m + \alpha^{(v)}_{sj} - \hat{\eta}_{sm}\varepsilon_{mpj}\nabla_p$, and $L^{(B)}_{ij}(\mathbf{x}) = \varepsilon_{ips}\nabla_p L^{(A)}_{sj}(\mathbf{x})$. Equations (A1), (A2)–(A6) yield

$$\partial A_{ij}/\partial t = L^{(B)}_{js}(\mathbf{y})A_{is} + L^{(A)}_{ik}(\mathbf{x})H_{kj} + N^{(xy)}_{ijks}H_{ks} + \phi_{ij}, \tag{A7}$$

where $\phi_{ij} = \langle[\nabla_i\varphi(\mathbf{x})]H_j(\mathbf{y})\rangle$, $H_{ij} = \langle H_i(\mathbf{x})H_j(\mathbf{y})\rangle$,

$$N^{(xy)}_{ijks} = \alpha^{(xy)}_{is}\delta_{kj} - \alpha^{(xy)}_{ks}\delta_{ij} + \varepsilon_{ikf}S^{(xy)}_f\delta_{js} - \varepsilon_{isk}S^{(xy)}_j + \varepsilon_{ifk}\left(\frac{\partial\tilde{\eta}_{jf}}{\partial y_s} - \frac{\partial\tilde{\eta}_{pf}}{\partial y_p}\delta_{js} - 2\delta_{js}\tilde{\eta}^{(xy)}_{fp}\frac{\partial}{\partial y_p}\right),$$

and

$$\alpha^{(xy)}_{mn} = -\left[\varepsilon_{mji}\left\langle\tau u_i(\mathbf{x})\frac{\partial u_j(\mathbf{y})}{\partial y_n}\right\rangle + \varepsilon_{nji}\left\langle\tau u_i(\mathbf{x})\frac{\partial u_j(\mathbf{y})}{\partial y_m}\right\rangle\right]\bigg/ 2,$$

$$S^{(xy)}_m = \langle\tau u_m(\mathbf{x})b(\mathbf{y})\rangle - (1/2)(\partial\tilde{\eta}_{mn}/\partial y_n),$$

$$\tilde{\eta}^{(xy)}_{mn} = \langle\tau u_m(\mathbf{x})u_n(\mathbf{y})\rangle, \quad \tilde{\eta}_{mn} = (\tilde{\eta}^{(xy)}_{mn} + \tilde{\eta}^{(yx)}_{mn})/2,$$

$\mathbf{v} = \mathbf{V} + \mathbf{u}$, $\mathbf{V} = \langle\mathbf{v}\rangle$, and $b = \nabla\cdot\mathbf{u}$. These tensors satisfy an identity

$$\left\langle\tau u_i(\mathbf{x})\frac{\partial u_k(\mathbf{y})}{\partial y_n}\right\rangle = \left[\varepsilon_{ikm}\alpha^{(xy)}_{mn} + \delta_{kn}S^{(xy)}_i - \delta_{in}S^{(xy)}_k + \frac{\partial\tilde{\eta}_{ki}}{\partial y_n}\right]\bigg/ 2$$

(see, e.g., [4,5]). Similarly we introduce $\alpha^{(yx)}_{mn}$ and $S^{(yx)}_m$:

$$\alpha^{(yx)}_{mn} = -\left[\varepsilon_{mji}\left\langle\tau\frac{\partial u_i(\mathbf{x})}{\partial x_n}u_j(\mathbf{y})\right\rangle + \varepsilon_{nji}\left\langle\tau\frac{\partial u_i(\mathbf{x})}{\partial x_m}u_j(\mathbf{y})\right\rangle\right]\bigg/ 2,$$

$$S^{(yx)}_m = \langle\tau u_m(\mathbf{y})b(\mathbf{x})\rangle - (1/2)(\partial\tilde{\eta}_{mn}/\partial x_n),$$

which satisfy an identity

$$\left\langle\tau\frac{\partial u_i(\mathbf{x})}{\partial x_n}u_k(\mathbf{y})\right\rangle = \left[\varepsilon_{ikp}\alpha^{(yx)}_{pn} + \delta_{kn}S^{(yx)}_i - \delta_{in}S^{(yx)}_k + \frac{\partial\tilde{\eta}_{ki}}{\partial x_n}\right]\bigg/ 2.$$

By means of Eq. (A7) we derive an equation for the tensor $\chi^{(xy)}_{ij}$,

$$\partial\chi^{(xy)}_{ij}/\partial t = L^{(B)}_{js}(\mathbf{y})\chi^{(xy)}_{is} + L^{(A)}_{ik}(\mathbf{x})h_{kj} + N^{(xy)}_{ijks}h_{ks} + N^{(xy)}_{ijks}B_k(\mathbf{x})B_s(\mathbf{y}) + \langle[\nabla_i\phi(\mathbf{x})]h_j(\mathbf{y})\rangle, \tag{A8}$$

where $h_{ij} = \langle h_i(\mathbf{x}))h_j(\mathbf{y})\rangle$. The equation for the tensor $\chi^{(yx)}_{ij}$ follows from Eq. (A8) by the change $\mathbf{x}\to\mathbf{y}$ and $\mathbf{y}\to\mathbf{x}$. Now we introduce a symmetrical tensor: $\chi_{ij} = (\chi^{(xy)}_{ij} + \chi^{(yx)}_{ij})/2$. Consider the case $\nabla\tilde{\eta}_{mn} = 0$. Now we derive an equation for the tensor $\chi_{ij}(\mathbf{r}=0)$ using Eq. (A8). The result is given by Eq. (9). For derivation of Eq. (9) we use the following identities:

$$N_{ijks}h_{ks}=\alpha_{is}^{(v)}h_{sj}-\alpha_{ks}^{(v)}h_{sk}\delta_{ij}+\varepsilon_{isp}S_p h_{sj}+2\,\tilde{\eta}_{mp}\frac{\partial^2\chi_{mj}}{\partial x_p\partial x_i}-2\,\tilde{\eta}_{pn}\frac{\partial^2\chi_{ij}}{\partial x_p\partial x_n}, \quad \text{(A9)}$$

$$L_{is}^{(A)}(\mathbf{x})h_{sj}+L_{is}^{(A)}(\mathbf{y})h_{js}$$

$$=\alpha_{is}^{(v)}(\mathbf{x})h_{sj}+\alpha_{is}^{(v)}(\mathbf{y})h_{sj}+\tilde{\eta}_{pn}\left(\frac{\partial^2\chi_{ij}^{(xy)}}{\partial x_p\partial x_n}+\frac{\partial^2\chi_{ij}^{(yx)}}{\partial y_p\partial y_n}\right)+V_s\left(\frac{\partial\chi_{sj}^{(xy)}}{\partial x_i}+\frac{\partial\chi_{sj}^{(yx)}}{\partial y_i}\right)-V_s\left(\frac{\partial\chi_{ij}^{(xy)}}{\partial x_s}+\frac{\partial\chi_{ij}^{(yx)}}{\partial y_s}\right), \quad \text{(A10)}$$

$$L_{js}^{(B)}(\mathbf{x})\chi_{is}^{(yx)}+L_{js}^{(B)}(\mathbf{y})\chi_{is}^{(xy)}$$

$$=\varepsilon_{jpl}\left(\frac{\partial}{\partial y_p}(\alpha_{ls}^{(v)}\chi_{ip}^{(xy)})+\frac{\partial}{\partial x_p}(\alpha_{ls}^{(v)}\chi_{is}^{(yx)})\right)-\left(\frac{\partial V_p}{\partial x_p}+V_s\frac{\partial}{\partial x_s}\right)\chi_{ij}^{(yx)}-\left(\frac{\partial V_p}{\partial y_p}+V_s\frac{\partial}{\partial y_s}\right)\chi_{ij}^{(xy)}+\chi_{ip}^{(yx)}\frac{\partial V_j}{\partial x_p}+\chi_{ip}^{(xy)}\frac{\partial V_j}{\partial y_p}+\tilde{\eta}_{pn}\left(\frac{\partial^2\chi_{ij}^{(xy)}}{\partial y_p\partial y_n}+\frac{\partial^2\chi_{ij}^{(yx)}}{\partial x_p\partial x_n}\right). \quad \text{(A11)}$$

The tensor $\alpha_{mn}^{(v)}=(\alpha_{mn}^{(xy)}+\alpha_{mn}^{(yx)})/2$. We used here that

$$(\partial\chi_{is}^{(yx)}/\partial x_p+\partial\chi_{is}^{(xy)}/\partial y_p)_{r=0}=2[\boldsymbol{\nabla}_p\chi_{is}-\langle(\partial a_i/\partial x_p)h_s\rangle].$$

The latter identity can be derived as follows:

$$\left(\frac{\partial\chi_{is}^{(yx)}}{\partial x_p}\right)_{r\to 0}=\left(\left\langle\frac{\partial h_s(\mathbf{x})}{\partial x_p}a_i(\mathbf{y})\right\rangle\right)_{r\to 0}=\left(\frac{\partial}{\partial x_p}\langle h_s(\mathbf{x})a_i(\mathbf{y})\rangle-\left\langle\frac{\partial a_i}{\partial x_p}h_s\right\rangle\right)_{r\to 0}=\boldsymbol{\nabla}_p\chi_{is}-\langle(\partial a_i/\partial x_p)h_s\rangle.$$

For the derivation of Eq. (11) we used the following identities:

$$\varepsilon_{ilk}\tilde{\eta}_{lp}B_k\boldsymbol{\nabla}_pB_i=-\hat{\eta}_{im}B_i(\boldsymbol{\nabla}\times\mathbf{B})_m-B_i\boldsymbol{\nabla}_i(\tilde{\eta}_{sp}\boldsymbol{\nabla}_pA_s),$$

and $\tilde{\varphi}_p=-V_p\chi/3+O(l_0^2/l_B^2)$, where l_B is the characteristic scale of the mean magnetic field variations, l_0 is the maximum scale of turbulent motions, and $l_0\ll l_B$.

APPENDIX B: THE DERIVATION OF EQ. (13)

We use here the two-scale approach (see, e.g., [13,14]). Indeed, let us consider, for example, a correlation function

$$\langle u_i(\mathbf{x})u_j(\mathbf{y})\rangle=\int\langle u_i(\mathbf{k}^{(1)})u_j(\mathbf{k}^{(2)})\rangle\exp[i(\mathbf{k}^{(1)}\mathbf{x}+\mathbf{k}^{(2)}\mathbf{y})]\times d\mathbf{k}^{(1)}d\mathbf{k}^{(2)}=\int\tilde{f}_{ij}(\mathbf{r},\mathbf{K})\exp(i\mathbf{K}\cdot\mathbf{R})d\mathbf{K}=\int f_{ij}(\mathbf{k},\mathbf{R})\exp(i\mathbf{k}\cdot\mathbf{r})d\mathbf{k},$$

where

$$\tilde{f}_{ij}(\mathbf{K},\mathbf{r})=\int\langle u_i(\mathbf{k}+\mathbf{K}/2)u_j(-\mathbf{k}+\mathbf{K}/2)\rangle\exp(i\mathbf{k}\cdot\mathbf{r})d\mathbf{k},$$

$$f_{ij}(\mathbf{k},\mathbf{R})\int\int\langle u_i(\mathbf{k}+\mathbf{K}/2)u_j(-\mathbf{k}+\mathbf{K}/2)\rangle\exp(i\mathbf{K}\cdot\mathbf{R})d\mathbf{K},$$

and $\mathbf{R}=(\mathbf{x}+\mathbf{y})/2$, $\mathbf{r}=\mathbf{y}-\mathbf{x}$, $\mathbf{K}=\mathbf{k}^{(1)}+\mathbf{k}^{(2)}$, $\mathbf{k}=(\mathbf{k}^{(2)}-\mathbf{k}^{(1)})/2$, $\mathbf{R}$ and $\mathbf{K}$ correspond to the large scales, and $\mathbf{r}$ and $\mathbf{k}$ describe the small scales. Using Eq. (12) we obtain

$$\eta_{mp}^{*}\left(\frac{\partial^2\chi_{ij}}{\partial x_m\partial y_p}\right)_{r=0}=\tau_0^{-1}[(3\,\eta_{ij}^{*}-\delta_{ij})\chi/10+(\,\eta_{jp}^{*}\mu_{pi}+8\,\eta_{ip}^{*}\mu_{pj}-3\mu_{ij}-3\,\delta_{ij}\,\eta_{pm}^{*}\mu_{mp})/7], \quad \text{(B1)}$$

where $\chi(\mathbf{r}=0)=\int\chi_{*}\exp(i\mathbf{K}\cdot\mathbf{R})d\mathbf{K}\,d\mathbf{k}$, $\mu_{ij}(\mathbf{r}=0)=\int\mu_{ij}^{*}\exp(i\mathbf{K}\cdot\mathbf{R})d\mathbf{K}\,d\mathbf{k}$, $\chi_{*}=\chi_{*}(k,\mathbf{K})$, and $\mu_{ij}^{*}=\mu_{ij}^{*}(k,\mathbf{K})$. In order to obtain Eq. (B1) in $\mathbf{r}$ space we used the transformations: $i\mathbf{k}^{(1)}\to\partial/\partial\mathbf{x}$ and $i\mathbf{k}^{(2)}\to\partial/\partial\mathbf{y}$, and we assumed a weak inhomogeneity of the magnetic helicity, i.e., we neglected the terms $\propto O(\mathbf{K})$ in Eq. (12). We also used the realizability condition for the magnetic helicity (see, e.g., [1]), i.e., we assumed that the spectral densities χ_{*} and $\mu_{ij}^{*}\propto\chi$ are localized in the vicinity of the maximum scale of turbulent motion l_0. In order to derive Eq. (B1) we used the following integrals:

$$Y_{ijmn}\equiv\int(k_ik_jk_mk_n/k^4)\sin\theta\,d\theta\,d\varphi=(4\pi/15)(\delta_{ij}\delta_{mn}+\delta_{im}\delta_{nj}+\delta_{in}\delta_{mj}),$$

$$\int(k_ik_jk_fk_sk_tk_r/k^6)\sin\theta\,d\theta\,d\varphi=(1/7)(Y_{fstr}\delta_{ij}+Y_{jfsr}\delta_{it}+Y_{ifsr}\delta_{jt}+Y_{jfst}\delta_{ir}+Y_{ifst}\delta_{jr}+Y_{ijfs}\delta_{tr}-Y_{ijtr}\delta_{fs}).$$

Equations (B1) and (11) allow us to obtain Eq. (13).

APPENDIX C: THE MAGNETIC PART OF THE α EFFECT FOR WEAKLY INHOMOGENEOUS TURBULENCE

In this appendix we derive a formula for the magnetic part of α effect for weakly inhomogeneous turbulence. We show that this tensor is determined by the trace of the magnetic helicity tensor. The tensor $\alpha_{mn}^{(B)}$ for the magnetic part of the α effect is determined by Eq. (6). Now we calculate

$$\varepsilon_{mji}\langle \tau h_i(\mathbf{x})\boldsymbol{\nabla}_n h_j(\mathbf{y})\rangle$$

$$= -\varepsilon_{mji}\varepsilon_{lqi}\int \tau(\mathbf{k}^{(2)})k_l^{(2)}k_n^{(1)}\langle a_q(\mathbf{k}^{(2)})h_j(\mathbf{k}^{(1)})\rangle \times \exp[i(\mathbf{k}^{(1)}\cdot\mathbf{x}+\mathbf{k}^{(2)}\cdot\mathbf{y})]d\mathbf{k}^{(1)}d\mathbf{k}^{(2)}$$

$$= \int \tau(\mathbf{k}^{(2)})(k_m^{(2)}k_n^{(1)}\chi_{pp}-k_p^{(2)}k_n^{(1)}\chi_{mp}) \times \exp[i(\mathbf{k}^{(1)}\cdot\mathbf{x}+\mathbf{k}^{(2)}\cdot\mathbf{y})]d\mathbf{k}^{(1)}d\mathbf{k}^{(2)},$$

where $\chi_{mn}=\langle a_m(\mathbf{k}^{(2)})h_n(\mathbf{k}^{(1)})\rangle$. Since $\mathbf{k}^{(2)}=\mathbf{k}+\mathbf{K}/2$ and $\mathbf{k}^{(1)}=-\mathbf{k}+\mathbf{K}/2$, we obtain

$$\alpha_{mn}^{(B)}(\mathbf{r}=0)=\int \tau(\mathbf{k})[k_mk_n\chi_{pp}-K_p(k_n\chi_{mp}+k_m\chi_{np}) - K_mK_n\chi_{pp}/2+K_pK_n\chi_{mp}+K_pK_m\chi_{np}] \times \exp[i\mathbf{K}\cdot\mathbf{R}]\,d\mathbf{k}\,d\mathbf{K}/\mu_0\rho, \tag{C1}$$

where ρ is the fluid density, and μ_0 is the magnetic permeability. Equation (C1) implies that the main contribution to the tensor for the magnetic part of the α effect is from the trace for the magnetic helicity tensor, i.e., $\alpha_{mn}^{(B)}(\mathbf{r}=0) \sim \int \tau(\mathbf{k})k_mk_n\chi_{pp}(\mathbf{k},\mathbf{R})\,d\mathbf{k}/\mu_0\rho$. Now we assume that $\chi_{pp}(\mathbf{k},\mathbf{R})\simeq\chi_{pp}(k,\mathbf{R})$, i.e., the trace of the magnetic helicity tensor in $\mathbf{k}$ space is isotropic (it is independent of the direction of $\mathbf{k}$). Therefore,

$$\alpha_{mn}^{(B)}(\mathbf{r}=0)\sim\delta_{mn}\int\tau(k)k^2\chi_{pp}(k,\mathbf{R})dk/(3\mu_0\rho),$$

where we used that $\int(k_mk_n/k^2)\sin\theta\,d\theta\,d\varphi=(4\pi/3)\delta_{mn}$. The spectrum function of the magnetic helicity is given by

$$\chi(k,\mathbf{R})=\chi(\mathbf{R})\frac{c}{4\pi k^2k_0}\left(\frac{k}{k_0}\right)^{-q},$$

$$c=(q-1)\left[1-\left(\frac{k_0}{k_\chi}\right)^{q-1}\right]^{-1},$$

where the wave number k is within interval $k_0<k<k_\chi$, $\chi(\mathbf{R})=\int\chi(k,\mathbf{R})d\mathbf{k}$, and $k_0=l_0^{-1}$. The correlation time is $\tau(k)=2\tau_0(k/k_0)^{1-q}$. The integration in the equation for $\alpha_{mn}^{(B)}(\mathbf{r}=0)$ yields

$$\alpha_{mn}^{(B)}(\mathbf{r}=0)\sim\frac{\chi(\mathbf{R})(q-1)}{9(2-q)\,\eta_T\mu_0\rho}\left[\left(\frac{k_\chi}{k_0}\right)^{4-2q}-1\right] \times\left[1-\left(\frac{k_0}{k_\chi}\right)^{q-1}\right]^{-1}\delta_{mn}. \tag{C2}$$

The realizability condition causes $k_\chi\simeq k_0$, i.e., the magnetic helicity is localized at the maximum scale of turbulent motions (see, e.g., [1,2]). Therefore Eq. (C2) yields Eq. (16).

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