PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 103(117) (2018), 91–102 DOI: https://doi.org/10.2298/PIM1817091I

MAGNETIC VECTOR FIELDS: NEW EXAMPLES

Jun-ichi Inoguchi and Marian Ioan Munteanu

Abstract. In a previous paper, we introduced the notion of magnetic vector fields. More precisely, we consider a vector field ξ as a map from a Riemannian manifold into its tangent bundle endowed with the usual almost Kählerian structure and we find necessary and sufficient conditions for ξ to be a magnetic map with respect to ξ itself and the Kähler 2-form. In this paper we give new examples of magnetic vector fields.

1. Preliminaries

In [**13**] the authors define the notion of *magnetic maps* with the aim of generalizing the notion of magnetic trajectory on a Riemannian manifold. In fact, both magnetic curves and harmonic maps can be obtained as particular situations of magnetic maps.

Let $f : N \to M$ be a smooth map between two Riemannian manifolds (N, h) of dimension n and (M, g) of dimension m. Suppose that N is compact and let ξ be a global vector field on N having null divergence. Let ω be a 1-form on M. The energy of f is known as $E(f) = \frac{1}{2}\int_N |df|^2 dv_h$, where dv_h is the volume element on N and $|df|$ is the Hilbert-Schmidt norm of the differential df given (in a point $p \in N$) by

$$|df_p|^2 = \sum_{i=1}^{n} g_{f(p)}\left(f_{*,p}e_i, f_{*,p}e_i\right).$$

Here $\{e_i;\ i = 1, \dots, n\}$ is an arbitrary orthonormal basis for T_pN and $f_{*,p} : T_pN \to T_{f(p)}M$ is the tangent map of f at p.

A smooth map $f : (N, h) \to (M, g)$ which is a critical point of $E(f)$ is called a *harmonic map* (see e.g., [**11, 21**]).

Let us now define the following functional for f associated to ξ and ω:

$$\mathcal{P}(f) = \int_N \omega(df(\xi))dv_h.$$

The Landau-Hall functional associated to ξ and ω is defined by

$$LH(f) = E(f) + \mathcal{P}(f).$$

2010 *Mathematics Subject Classification:* Primary 53C43; Secondary 53C07, 53C15, 58E20.
Key words and phrases: magnetic curves; harmonic maps; magnetic maps.
The first author is supported by Kakenhi 15K04834.

Let I be an open interval containing 0. A smooth variation of f is a smooth map $\mathcal{F} : N \times I \to M$, such that $\mathcal{F}(p, 0) = f(p)$. For the sake of simplicity we use the notation $f_\epsilon(p) = \mathcal{F}(p, \epsilon)$. The variation vector field along f is a section in the induced bundle $f^{-1}T(M)$ defined by $V(x) = \frac{\partial f_\epsilon}{\partial \epsilon}\big|_{\epsilon=0}(x)$.

Definition 1.1. [**13**] The map f is called *magnetic* with respect to ξ and ω if it is a critical point of the Landau Hall integral $LH(f)$.

In what follows we compute the first variation $\frac{d}{d\epsilon}LH(f_\epsilon)\big|_{\epsilon=0}$. It is known from the theory of harmonic maps that

$$\frac{d}{d\epsilon}E(f_\epsilon)\Big|_{\epsilon=0} = -\int_N g(\tau(f), V) \circ f \; dv_h,$$

where $\tau(f) := \mathrm{trace}_h \nabla df$ is the *tension field* of f.

Let us focus on the integral $\mathcal{P}$ and compute $\frac{d}{d\epsilon}\mathcal{P}(f_\epsilon)\big|_{\epsilon=0}$. Consider local coordinates $x^1, \ldots, x^n$ on N and $y^1, \ldots, y^m$ local coordinates on M. With respect to this setting, the map f_ϵ may be expressed as $y^\alpha = f^\alpha_\epsilon(x)$, where f^α_ϵ are smooth functions on the domain of coordinates x taking values in $\mathbb{R}$. From now on the indices i, j, k range from 1 to n, while the indices α, β, γ range from 1 to m.

We have

$$\mathcal{P}(f_\epsilon) = \int_N \omega_\alpha(f_\epsilon(x)) \frac{\partial f^\alpha_\epsilon}{\partial x^i}(x)\xi^i(x) \; dv_h.$$

Compute

$$\begin{aligned} (1.1) \quad & \frac{d}{d\epsilon}\mathcal{P}(f_\epsilon)\Big|_{\epsilon=0} \\ &= \int_N \left[\frac{\partial \omega_\alpha}{\partial y^\beta}(f(x))\frac{\partial f^\beta_\epsilon}{\partial \epsilon}(x)\Big|_{\epsilon=0}\frac{\partial f^\alpha}{\partial x^i}(x) + \omega_\alpha(f(x))\frac{\partial}{\partial \epsilon}\Big|_{\epsilon=0}\frac{\partial f^\alpha_\epsilon}{\partial x^i}(x)\right]\xi^i(x) dv_h. \end{aligned}$$

Let us define a vector field X on N by $X(x) = \xi^i(x)\omega_\alpha(f(x))V^\alpha(x)\frac{\partial}{\partial x^i}$ and compute its divergence. We obtain

$$\begin{aligned} \mathrm{div}(X) = (\overset{h}{\nabla}_i \, \xi^i)\omega_\alpha(f(x))V^\alpha(x) &+ \xi^\alpha(x)'\nabla_i\omega_\alpha(f(x))V^\alpha(x) \\ &+ \xi^i(x)\omega_\alpha(f(x))'\nabla_i V^\alpha(x), \end{aligned}$$

where $\overset{h}{\nabla}$ is the Levi-Civita connection on N and $'\nabla$ is the induced connection.

We successively have

$$\begin{aligned} \overset{h}{\nabla}_i \, \xi^i &= \mathrm{div}(\xi), \\ '\nabla_i\omega_\alpha(f(x)) &= \left('\nabla_{\frac{\partial}{\partial x^i}}\omega(f(x))\right)\left(\frac{\partial}{\partial y^\alpha} \circ f\right) = \frac{\partial}{\partial x^i}\omega_\alpha(f(x)) - \omega\left('\nabla_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial y^\alpha} \circ f\right) \\ &= \frac{\partial \omega_\alpha}{\partial y^\beta}(f(x))\frac{\partial f^\beta}{\partial x^i}(x) - \omega_\beta(f(x))\frac{\partial f^\gamma}{\partial x^i}(x)\,\overset{g}{\Gamma}{}^\beta_{\gamma\alpha}(f(x)) \\ &= \frac{\partial f^\gamma}{\partial x^i}(x)\,\overset{g}{\nabla}_\gamma \, \omega_\alpha(f(x)); \end{aligned}$$

$$'\nabla_i V^\alpha = \frac{\partial V^\alpha}{\partial x^i}(x) + \frac{\partial f^\beta}{\partial x^i}(x) \overset{g}{\Gamma}{}^{\alpha}_{\beta\gamma}(f(x))V^\gamma(x).$$

As ξ is divergence free, we get

$$\mathrm{div}(X) = \xi^i(x)\left[\omega_\alpha(f(x))\frac{\partial V^\alpha}{\partial x^i}(x) + \frac{\partial f^\beta}{\partial x^i}(x)\frac{\partial \omega_\alpha}{\partial y^\beta}(f(x))V^\alpha(x)\right].$$

Since $\int_N \mathrm{div}(X)dv_h = 0$, we obtain

$$\int_N \xi^i(x)\omega_\alpha(f(x))\frac{\partial V^\alpha}{\partial x^i}(x)dv_h = -\int_N \xi^i(x)\frac{\partial \omega_\alpha}{\partial y^\beta}(f(x))\frac{\partial f^\beta}{\partial x^i}(x)V^\alpha(x)dv_h. \tag{1.2}$$

Combining (1.1) and (1.2) we find

$$\begin{aligned}
\frac{d}{d\epsilon}\Big|_{\epsilon=0}\mathcal{P}(f_\epsilon) &= \int_N \xi^i(x)\frac{\partial \omega_\alpha}{\partial y^\beta}(f(x))\left(\frac{\partial f^\alpha}{\partial x^i}(x)V^\beta(x) - \frac{\partial f^\beta}{\partial x^i}(x)V^\alpha(x)\right)dv_h \\
&= \int_N \xi^i(x)\frac{\partial f^\alpha}{\partial x^i}(x)\left(\frac{\partial \omega_\alpha}{\partial y^\beta}(f(x)) - \frac{\partial \omega_\beta}{\partial y^\alpha}(f(x))\right)V^\beta(x)dv_h \\
&= \int_N \xi^i(x)\frac{\partial f^\alpha}{\partial x^i}(x)(d\omega)_{\alpha\beta}V^\beta(x)dv_h \\
&= \int_N d\omega(f_*\xi, V)\circ f \; dv_h.
\end{aligned}$$

Define the endomorphism ϕ, called the Lorentz force associated to the potential 1-form ω, by $g(\phi(X), Y) = d\omega(X, Y)$, for all X, Y tangent to M. It follows that

$$\frac{d}{d\epsilon}\Big|_{\epsilon=0}\mathcal{P}(f_\epsilon) = \int_N g(\phi f_*\xi, V)\circ f \; dv_h.$$

We finally obtain

$$\frac{d}{d\epsilon}\Big|_{\epsilon=0}LH(f_\epsilon) = -\int_N g(\tau(f) - \phi f_*\xi, V)\circ f \; dv_h.$$

We state the following.

Theorem 1.1. [**13**] *Let $f : (N, h) \to (M, g)$ be a smooth map. Then f is a magnetic map with respect to ξ and ω if and only if it satisfies the Lorentz equation, that is*

$$\tau(f) = \phi(f_*\xi). \tag{1.3}$$

Sometimes, equation (1.3) will be called the *magnetic equation*. Recall that on a Riemannian manifold (M, g) a *magnetic field* is defined by a closed 2-form F and the *Lorentz force* associated to F is a $(1, 1)$ tensor field ϕ on M given by $g(\phi X, Y) = F(X, Y)$. The *magnetic trajectories* of F are curves γ satisfying the *Lorentz equation* $\nabla_{\gamma'}\gamma' = \phi\gamma'$. This equation is a particular case of equation (1.3) when N is an interval of $\mathbb{R}$ and $\xi = \frac{d}{dt}$, where t is the global coordinate on $\mathbb{R}$. Magnetic curves were intensively studied in the last years by several geometers (including the authors of this article) in different ambient spaces. See for example [**8, 9, 10, 14, 15, 18**].

REMARK 1.1. The Lorentz equation (1.3) was obtained from a variational principle assuming that the domain is compact and the 2-form F is exact. Since it has a tensorial character, one can define a magnetic map $f:(N,h)\to(M,g)$ without the assumptions N compact and F exact (but only closed). Moreover, we will remove also the assumption for ξ to be divergence free.

Let ξ be a global vector field on N and F be a magnetic field on M with the associated Lorentz force ϕ. Similarly to magnetic curves, we may also introduce a *strength* (i.e., a real number) in the equation. Hence, we give the following.

DEFINITION 1.2. We say that f is a *magnetic map* with strength $q\in\mathbb{R}$ associated to ξ and F if the Lorentz equation

$$\tau(f)=q\ \phi(f_*\xi)$$

is satisfied.

2. Vector fields as magnetic maps

In our previous paper [**14**] we ask when a vector field is a magnetic map. More precisely, we consider a Riemannian manifold (M,g) of dimension n and its tangent bundle $(T(M),g_S)$ equipped with the Sasaki metric. On $T(M)$ we also define an almost complex structure J_S by

$$J_SX^H=X^V,\ J_SX^V=-X^H,\ \text{for all } X\in\mathfrak{X}(M).$$

It is known that $(T(M),g_S,J_S)$ is an almost Käherian manifold [**6**]. Hence, the Kähler 2-form $\Omega_S=g_S(J_S\ \cdot,\ \cdot)$ may be considered as a magnetic field on $T(M)$.

A vector field $\xi\in\mathfrak{X}(M)$ will be thought as a map from (M,g) to $(T(M),g_S,J_S)$. In the book of Dragomir and Perrone [**7**], the authors write the following formula

$$\tau(\xi)=-\left\{(\text{trace}_g\,R(\nabla_\bullet\xi,\xi)\bullet)^H+(\Delta_g\xi)^V\right\}\circ\xi.$$

Here Δ_g denotes the rough Laplacian on vector fields, defined by

$$\Delta_gX=-\sum_{k=1}^{n}\left[\nabla_{e_k}\nabla_{e_k}X-\nabla_{\nabla_{e_k}e_k}X\right],$$

where $\{e_k\}_{k=1,\dots,n}$ is an orthonormal frame on M. We also have

$$J_S(\xi_*\xi)=\xi^V-(\nabla_\xi\xi)^H.$$

We state the following.

THEOREM 2.1. [**14**] *Let (M,g) be a Riemannian manifold and $(T(M),g_S,J_S)$ its tangent bundle endowed with the usual almost Kählerian structure. Let ξ be a vector field on M. Then ξ is a magnetic map with strength q associated to ξ itself and the Kähler magnetic field Ω_S if and only if the following conditions hold:*

$$\text{trace}_g\,R(\nabla_\bullet\xi,\xi)\bullet=q\,\nabla_\xi\,\xi, \tag{2.1}$$

$$\Delta_g\xi=-q\xi. \tag{2.2}$$

Consider a Killing vector field ξ on the Riemannian manifold (M,g). We know that:

LEMMA 2.1. *A Killing vector field ξ on a Riemannian manifold (M, g) satisfies the equation $\nabla^2_{XY}\xi = -R(\xi, X)Y$, for all $X, Y \in \mathfrak{X}(M)$.*

We ask now for $\xi : (M, g) \to (T(M), g_S, J_S)$ to be a magnetic map. Then ξ must satisfy (2.2). But $\Delta_g \xi = -\operatorname{trace}_g \nabla^2 \xi$. Using the previous lemma, we get

$$\Delta_g \xi = \operatorname{trace}_g R(\xi, \bullet) \bullet .$$

On the other hand, we have

$$\begin{aligned}\operatorname{Ric}(\xi, X) &= \operatorname{trace}_g\{Z \mapsto R_{Z\xi}X\} = \sum_{i=1}^{n} g(e_i, R_{e_i\xi}X) = -\sum_{i=1}^{n} g(R_{e_i\xi}e_i, X)\\ &= g(\operatorname{trace}_g R(\xi, \bullet)\bullet, X) = -qg(\xi, X), \quad \text{for all } X \in \mathfrak{X}(M).\end{aligned}$$

So, if Q is the Ricci operator, that is $g(QX, Y) = \operatorname{Ric}(X, Y)$, for all X, Y tangent to M, then we get that $Q\xi = -q\xi$. We give the following.

PROPOSITION 2.1. *If a Killing vector field is a magnetic map with strength q, then it is an eigenvector of the Ricci operator corresponding to the eigenfunction $(-q)$.*

REMARK 2.1. In the special case of Einstein manifolds, the strength q is related to the scalar curvature, namely $q = -\frac{\operatorname{scal}}{n}$.

Suppose that M is a real space form $M^n(c)$, case when the curvature tensor is expressed as $R_{XY}Z = c\big(g(Y, Z)X - g(X, Z)Y\big)$, for all $X, Y, Z \in \mathfrak{X}(M)$. We can easily compute $\operatorname{trace}_g R(\nabla_\bullet \xi, \xi)\bullet = c\big(\nabla_\xi \xi - \operatorname{div}(\xi)\big)$. As ξ is Killing, its divergence is zero and thus, the magnetic equation becomes

$$(c - q)\nabla_\xi \xi = 0. \tag{2.3}$$

We obtained the following.

THEOREM 2.2. *Let ξ be a Killing vector field on a real space form $M^n(c)$, $n \geqslant 2$. If ξ is a non-harmonic magnetic map with strength q, then $q = (1-n)c$ and ξ is self parallel, case in which it has constant length.*

PROOF. Note that a real space form $M^n(c)$ is Einstein and its scalar curvature is $\operatorname{scal} = cn(n-1)$. So, as ξ is magnetic, cf. Remark 2.1, we must have $q = (1-n)c$. Obviously, equation (2.3) is satisfied if $q = c$. In this situation we get that M is flat and $q = 0$, that is ξ is a harmonic vector field. If $q \neq c$ then $\nabla_\xi \xi = 0$. As ξ is Killing, we have

$$g(\nabla_\xi \xi, X) + g(\xi, \nabla_X \xi) = 0, \quad \text{for all } X \in \mathfrak{X}(M).$$

It follows that the length of ξ is constant. □

In the end of this section we propose the study of the following problem: *Study non-harmonic magnetic Killing vector fields on the unit sphere $\mathbb{S}^n$.*

3. Magnetic vector fields on almost contact metric manifolds

A (φ, ξ, η)-*structure* on a manifold M is defined by a field φ of endomorphisms of tangent spaces, a vector field ξ and a 1-form η satisfying

$$\eta(\xi) = 1, \quad \varphi^2 = -\mathrm{I} + \eta \otimes \xi, \quad \varphi\xi = 0, \quad \eta \circ \varphi = 0.$$

If (M, φ, ξ, η) admits a compatible Riemannian metric g, namely

$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y), \quad \text{for all } X, Y \in \mathfrak{X}(M),$$

then M is said to have an *almost contact metric structure*, and $(M, \varphi, \xi, \eta, g)$ is called an *almost contact metric manifold*. It follows that $\eta(X) = g(\xi, X)$, for any $X \in \mathfrak{X}(M)$ and ξ is unitary.

The fundamental 2-form Ω is defined by $\Omega(X, Y) = g(\varphi X, Y)$, for any vector fields X and Y. Recall that a *contact metric manifold* is an almost contact metric manifold such that $\Omega = d\eta$. If in addition the structure is normal, that is the normality tensor field $N = [\varphi, \varphi] + 2d\eta \otimes \xi$ vanishes, then the manifold M is called a *Sasakian* manifold. Here $[\varphi, \varphi]$ denotes the Nijenhuis tensor of φ. Denoting by ∇ the Levi-Civita connection associated to g, the Sasakian manifold $(M, \varphi, \xi, \eta, g)$ is characterized by $(\nabla_X \varphi)Y = -g(X, Y)\xi + \eta(Y)X$, for any $X, Y \in \mathfrak{X}(M)$. As a consequence, we have $\nabla_X \xi = \varphi X$, for all $X \in \mathfrak{X}(M)$. A systematic study of these structures is presented in the two books of Blair [**4, 5**]. However, we use the sign convention given by Sasaki, see e.g., [**12**].

On the other hand, a *Kenmotsu manifold* can be defined as a normal almost contact metric manifold such that $d\eta = 0$ and $d\Omega = 2\eta \wedge \Omega$. These manifolds can be characterized using their Levi-Civita connection, by requiring

$$(\nabla_X \varphi)Y = g(\varphi X, Y)\xi - \eta(Y)\varphi X, \quad \text{for every } X, Y \in \mathfrak{X}(M).$$

In our previous paper [**14**], we find some conditions when the Reeb vector field ξ on a Sasakian space form is magnetic, that is satisfies the condition in Theorem 2.1. We obtain that $q = -2n$.

Let us analyze the property of the characteristic vector field ξ on a Kenmotsu manifold to be magnetic. Recall the following two useful formulas:

$$\nabla_X \xi = X - \eta(X)\xi,$$
$$R(X, \xi)Y = g(X, Y)\xi - \eta(Y)X, \quad \text{for every } X, Y \in \mathfrak{X}(M).$$

Compute $\mathrm{trace}_g\, R(\nabla_\bullet \xi, \xi)\bullet$. To do this, consider as usual, a φ adapted orthonormal basis $\{e_i, \varphi e_i, \xi\}$, $i = 1, \ldots, n$. We have $\nabla_{e_i}\xi = e_i$, $\nabla_{\varphi e_i}\xi = \varphi e_i$, $\nabla_\xi \xi = 0$. Hence

$$\begin{aligned}\mathrm{trace}_g\, R(\nabla_\bullet \xi, \xi)\bullet &= \sum_{i=1}^{n} \Big[R(e_i, \xi)e_i + R(\varphi e_i, \xi)\varphi e_i\Big] \\ &= \sum_{i=1}^{n} \Big[g(e_i, e_i)\xi + g(\varphi e_i, \varphi e_i)\xi\Big] = 2n\xi.\end{aligned}$$

Thus, the equation (2.1) becomes $2n\xi = 0$, which is a contradiction.

As a matter of fact, for the second condition of Theorem 2.1, we have

$$\begin{aligned}\Delta_g \xi &= -\sum_{i=1}^{n}\Big[\Big(\nabla_{e_i}\nabla_{e_i}\xi - \nabla_{\nabla_{e_i}e_i}\xi\Big) + \Big(\nabla_{\varphi e_i}\nabla_{\varphi e_i}\xi - \nabla_{\nabla_{\varphi e_i}\varphi e_i}\xi\Big)\Big] \\ &= -\sum_{i=1}^{n}\Big[\eta(\nabla_{e_i}e_i)\xi + \eta(\nabla_{\varphi e_i}\varphi e_i)\xi\Big] = 2n\xi.\end{aligned}$$

Therefore, ξ is an eigenvector of the rough Laplacian with corresponding eigenfunction $q = -2n$. We conclude with the following.

PROPOSITION 3.1. *The characteristic vector field of a Kenmotsu manifold is not magnetic.*

Next we would like to make some comments on the same problem in a cosymplectic manifold. Recall that a *cosymplectic manifold* is an almost contact metric manifold for which the three tensor fields φ, ξ and η are parallel. Therefore, the first condition in the Theorem 2.1 is automatically satisfied. Since $\Delta_g\xi = 0$, the second condition implies $q = 0$, that is ξ is a harmonic map. We conclude with the following.

PROPOSITION 3.2. *If the characteristic vector field of a cosymplectic manifold is magnetic, then it is harmonic.*

At this point we propose another problem:
Study the property of ξ of being a magnetic map on a generalized Sasakian space form. See [**1**].

We end this section with some comments concerning the condition $\operatorname{div}(\xi) = 0$ used in finding the magnetic equation. Because some readers may think that the divergence free condition for ξ is too strong or artificial, we mention that this condition is often satisfied. For example, on almost contact metric manifolds, we know the following:

- The characteristic vector field ξ of a contact metric manifold is divergence free.
- In addition, cosymplectic manifolds have divergence free ξ.
- However, ξ is not always divergence free; e.g. on Kenmotsu manifolds, we have

$$\begin{aligned}\operatorname{div}\,\xi &= \sum_{i=1}^{n} g(\nabla_{e_i}\xi, e_i) + \sum_{i=1}^{n} g(\nabla_{\varphi e_i}\xi, \varphi e_i) \\ &= \sum_{i=1}^{n} g(e_i, e_i) + \sum_{i=1}^{n} g(\varphi e_i, \varphi e_i) = 2n \neq 0.\end{aligned}$$

4. More examples of magnetic maps

4.1. H-minimal submanifolds. Let N be an n-dimensional Lagrangian submanifold in a Kähler manifold M. Then $\zeta := -J\mathsf{H}/n$ is a globally defined tangent vector field on M. Here H is the mean curvature vector field. In our previous paper [**13**], we showed that the inclusion map $\iota : N \to M$ satisfies $\tau(\iota) = J\iota_*\zeta$.

According to Oh [**19**], a Lagrangian submanifold N is said to be *Hamiltonian-minimal* (in short H-minimal) if it is a critical point of the volume functional under compactly supported smooth variations arising from Hamiltonian deformations.

The Euler–Lagrange equation of this variational problem is div $(J\mathsf{H}) = 0$, that is ζ is divergence free.

This implies that every H-minimal Lagrangian submanifold N is magnetic with respect to $\zeta = -J\mathsf{H}/n$ and the Kähler form of M.

4.2. L-minimal submanifolds. In Sasakian geometry, one introduces the notion of L-minimal immersion as follows:

DEFINITION 4.1. [**16**] An n-dimensional Legendrian submanifold N in a Sasakian manifold M is said to be *L-minimal* if it is a critical point of the volume functional under compactly supported smooth variations arising from Legendre deformations.

The Euler–Lagrange equation of this variational problem is $\mathrm{div}(\varphi\mathsf{H}) = 0$.

One can check that every Legendrian submanifold satisfies $\tau(\iota) = \phi\iota_*\zeta$, where the vector field ζ is defined globally on N by $\zeta := -\varphi\mathsf{H}/n$.

Thus every L-minimal Legendrian submanifold in a Sasakian manifold is magnetic with respect to the divergence free vector field ζ and the contact form on M.

4.3. Magnetic hypersurfaces in complex space forms. Let (M, g, J) be a Kähler manifold of complex dimension n and let $f : N \to (M, g, J)$ be an orientable real hypersurface with unit normal vector field ν. Then the Kähler structure (g, J) induces an almost contact metric structure (φ, ξ, η, h) on N as follows. First, define the vector field ξ by $f_*\xi = -J\nu$. Next (φ, η) are defined by the formula

$$Jf_*X = f_*\varphi X + \eta(X)\nu$$

for all tangent vector X on N. Finally, we set $h = f^*g$.

Then the Levi-Civita connections $\widetilde{\nabla}$ of M and ∇ of N are related by the following *Gauss formula* and *Weingarten formula*:

$$\widetilde{\nabla}_X f_*Y = f_*\nabla_X Y + g(AX, Y)\nu, \ ]; \widetilde{\nabla}_X \nu = -f_*AX, \quad X \in \mathfrak{X}(N).$$

The endomorphism field A is called the *shape operator* of N derived from ν. We know that

$$(\nabla_X\varphi)Y = \eta(Y)AX - g(AX, Y)\xi, \quad \nabla_X\xi = \varphi AX.$$

The following result is fundamental (see [**7**]).

PROPOSITION 4.1. *The structure vector field ξ is divergence free.*

PROOF. We can compute

$$\begin{aligned}\mathrm{div}\ \xi &= \sum_{i=1}^{n-1} g(\nabla_{e_i}\xi, e_i) + \sum_{i=1}^{n-1} g(\nabla_{\phi e_i}\xi, \phi e_i) + g(\nabla_\xi\xi, \xi) \\ &= \sum_{i=1}^{n-1} g(\phi Ae_i, e_i) + \sum_{i=1}^{n-1} g(\phi A\phi e_i, \phi e_i) + g(\phi A\xi, \xi).\end{aligned}$$

We note that $g(\phi A\xi, \xi) = 0$. Next we have

$$\langle \phi A\phi e_i, \phi e_i\rangle = -\langle A\phi e_i, \phi^2 e_i\rangle = \langle A\phi e_i, e_i\rangle = \langle \phi e_i, Ae_i\rangle = -\langle e_i, \phi Ae_i\rangle.$$

Thus ξ is divergence free. □

The tension field $\tau(f)$ is given by $\tau(f) = (2n-1)H\nu$. Here H is the mean curvature function. If $\Omega = g(J\cdot, \cdot)$ is considered as a magnetic field on M, then the magnetic equation for the immersion f with respect to $\{\xi, \Omega\}$ and strength q is computed as

$$(2n-1)H\nu = qJ(f_*\xi) = qJ(-J\nu) = q\nu.$$

Thus f is magnetic with respect to $\{\xi, \Omega\}$ if and only if $q = (2n-1)H$.

Proposition 4.2. [**13**] *Let $f : N \to (M, g, J)$ be an orientable real hypersurface of constant mean curvature H with induced almost contact metric structure (φ, ξ, η, h). Then f is a magnetic map with respect to the structure vector field ξ and the Kähler magnetic field Ω with strength $q = (2n-1)H$.*

Now, we add one more example to our previous list of magnetic real hypersurfaces in complex space forms and complex Grassmannian manifolds given in [**13**], namely magnetic real hypersurfaces in complex quadrics.

Example 4.1. In [**3**], Berndt and Suh studied real hypersurfaces in the Grassmannian manifold $\widetilde{\mathrm{Gr}}_2(\mathbb{R}^{m+2})$ of *oriented* 2-planes in Euclidean $(m+2)$-space. As is well known, the Grassmannian manifold $\widetilde{\mathrm{Gr}}_2(\mathbb{R}^{m+2})$ is identified with the complex quadric

$$\mathcal{Q}_m = \{[z_1 : z_2 : \cdots : z_{m+2}] \in \mathbb{C}P^{m+1} \mid z_1^2 + z_2^2 + \cdots + z_{m+2}^2 = 0\}$$

in the complex projective $(m+1)$-space.

When we equip the ambient projective space with the Fubini–Study metric of constant holomorphic sectional curvature 4, then $\mathcal{Q}_m = \mathrm{SO}(m+2)/\mathrm{SO}(2)\times\mathrm{SO}(m)$ is a Hermitian symmetric space of rank 2 and maximal sectional curvature 4 with respect to the induced metric g. The Ricci tensor is given by $\mathrm{Ric} = 2mg$.

Hereafter we assume that $m \geqslant 3$. For $m = 2k$, the map

$$[z_1 : z_2 : \cdots : z_{k+1}] \mapsto [z_1 : z_2 : \cdots : z_{k+1} : iz_1 : iz_2 : \cdots : iz_{k+1}]$$

defines a totally geodesic complex immersion of $\mathbb{C}P^k$ into $\mathcal{Q}_{2k} \subset \mathbb{C}P^{2k+1}$.

For $r \in (0, \pi/2)$, the tube around $\mathbb{C}P^k$ is a homogeneous real hypersurface with principal curvatures $\lambda_1 = 2\cot(2r)$, $\lambda_2 = 0$, $\lambda_3 = -\tan r$, $\lambda_4 = \cot r$ and multiplicities $m_1 = 1$, $m_2 = 2$, $m_3 = m_4 = 2k-2$.

In case $m = 2$, i.e., $k = 1$, we have $\mathbb{C}P^1 \subset \mathcal{Q}_2 = \mathbb{S}^2 \times \mathbb{S}^2$. The principal curvatures of a tube around $\mathbb{C}P^1$ are 0 and $2\cot(2r)$.

The inclusion map of a tube M_r of radius r around $\mathbb{C}P^k$ into $\mathcal{Q}_{2k}$ is a magnetic immersion with respect to the magnetic field $F = \Omega$ with strength

$$q = m_1\lambda_1 + m_2\lambda_2 + m_3\lambda_3 + m_4\lambda_4 = 2(2k-1)\cot 2r.$$

4.4. Harmonic unit vector fields as magnetic maps. A unit vector field ξ on a Riemannian manifold (M, g) is said to be a *harmonic unit vector field* if it is a critical point of the energy functional over the space $\mathfrak{X}_1(M)$ of all smooth unit vector fields on M. The Euler-Lagrange equation of this variational problem is $\Delta_g \xi = |\xi|^2 \xi$. Moreover it is known that ξ is a harmonic map from (M, g) into the unit tangent sphere bundle $U(M)$ with the metric induced from g_S if and only if ξ is a harmonic unit vector field and satisfies $\mathrm{trace}_g\, R(\nabla_\bullet \xi, \xi)\bullet = 0$ (see [**7**]).

Comparing the harmonic map equation for $\xi : M \to U(M)$ and magnetic equation for $\xi : M \to T(M)$ we have

Proposition 4.3. *Let ξ be a unit vector field on a Riemannian manifold (M, g). Assume that ξ satisfies*

- *ξ is divergence free,* (*optional condition*)
- *$\nabla_\xi \xi = 0$,*
- *$|\nabla \xi|$ is constant*
- *$\xi : M \to U(M)$ is a harmonic map.*

Then ξ is a magnetic map into $T(M)$ with strength $q = -|\nabla \xi|^2$.

4.5. Magnetic vector fields on real hypersurfaces. An oriented real hypersurface N of a Kähler manifold M is said to be *Hopf* if the structure vector field ξ introduced in subsection 4.3 is a principal vector field. In that case, if $A\xi = \alpha\xi$, then α is called *the Hopf principal curvature* on N. It is easy to check that ξ satisfies $\nabla_\xi \xi = 0$ if and only if N is Hopf.

The following results are direct consequences of [**20**, Theorem 3.2] due to Perrone.

Proposition 4.4. *Let $N \subset M$ be an oriented Hopf hypersurface of a Kähler-Einstein manifold. Then the structure vector field ξ satisfies:*

(1) *ξ is a harmonic unit vector field if and only if* $\mathrm{grad}\ H = \xi(H)\xi$*, where H is the mean curvature function.*
(2) *If the principal curvature α corresponding to ξ is constant along the trajectories of ξ then $\xi(H) = 0$.*

Corollary 4.1. *Let $N \subset M$ be an oriented Hopf hypersurface of a Kähler–Einstein manifold satisfying $\xi(\alpha) = 0$. Then ξ is a harmonic map into $U(N)$ if and only if the mean curvature is constant.*

Complex space forms are typical examples of Kähler–Einstein manifolds.

Theorem 4.1. *Let N be an oriented Hopf hypersurface with constant principal curvatures in a complex space form M. Then the characteristic vector field ξ of N is a magnetic map with strength $q = -|A|^2 + \alpha^2$.*

Proof. Let N be an oriented Hopf hypersurface with constant principal curvatures in a complex space form M. Then ξ satisfies

$$\Delta_g \xi = |\nabla \xi|^2 \xi, \ \ \mathrm{trace}_g\, R(\nabla_\bullet \xi, \xi)\bullet = 0, \ \ \nabla_\xi \xi = 0.$$

Since all the principal curvatures are constant and $\nabla \xi = \varphi A$, we have $|\nabla \xi|^2 = |A|^2 - \alpha^2$. Hence ξ is a magnetic map with strength $q = -|A|^2 + \alpha^2$. ☐

As is well known, a complete and simply connected complex space form is a *complex projective space* $\mathbb{C}P^n(c)$, a *complex Euclidean space* $\mathbb{C}^n$ or a *complex hyperbolic space* $\mathbb{C}H^n(c)$, according as $c > 0$, $c = 0$ or $c < 0$. Hopf hypersurfaces in $\mathbb{C}P^n(c)$ and $\mathbb{C}H^n(c)$ are classified by Kimura [**17**] and Berndt [**2**], respectively.

Of course, one can check that characteristic vector fields of all homogeneous Hopf real hypersurfaces in $\mathbb{C}P^n(c)$ and $\mathbb{C}H^n(c)$ are magnetic maps into tangent bundles. However, we exhibit here only few examples.

EXAMPLE 4.2 (Type A hypersurfaces). Let us consider

$$\hat{M}_k(r) := \mathbb{S}^{2k+1}(\cos r) \times \mathbb{S}^{2n-1-2k}(\sin r) \subset \mathbb{S}^{2n+1}, \;\; 0 \leqslant k < n, \;\; 0 < r < \frac{\pi}{2}.$$

Then the Hopf projection image $M_k(r)$ of $\hat{M}_k(r)$ is a Hopf hypersurface in the complex projective space $\mathbb{C}P^n(4)$ of constant holomorphic sectional curvature 4. These hypersurfaces $M_k(r)$ are referred as to type A hypersurfaces. Note that type A hypersurfaces are quasi-Sasakian. The type A hypersurface $M_k(r)$ has constant principal curvatures $\lambda_1 = -\tan r$, $\lambda_2 = \cot r$, $\alpha = 2\cot(2r)$, $0 < r < \frac{\pi}{2}$ with multiplicities $m_1 = 2k$, $m_2 = 2(n-k-1)$, $m_\alpha = 1$. Then the characteristic vector field ξ is a magnetic map into $T(M)$ with strength

$$q = -(m_1\lambda_1^2 + m_2\lambda_2^2) = -2\{k\tan^2 r + (n-k-1)\cot^2 r\} < 0.$$

EXAMPLE 4.3 (Horospheres). Let M be a horosphere in the complex hyperbolic n-space $\mathbb{C}H^n(-4)$. It is known that the horosphere in $\mathbb{C}H^n(-4)$ is a Sasakian space form of constant holomorphic sectional curvature -3. The horosphere has constant principal curvatures $\lambda = 1$ with multiplicity $2n-2$ and $\alpha = 2$ with multiplicity 1. Then the strength is $q = -2(n-1)$. This is consistent with Section 2.

EXAMPLE 4.4 (Type B hypersurfaces). Let M be a tube over totally real and totally geodesic real hyperbolic space $\mathbb{H}^n$ in the complex hyperbolic n-space $\mathbb{C}H^n(-4)$ of constant holomorphic sectional curvature -4. Then M is a Hopf hypersurface with constant principal curvatures having the form

$$\lambda_1 = \frac{1}{r}\coth\, u, \;\; \lambda_2 = \frac{1}{r}\tanh\, u, \;\; \alpha = \frac{2}{r}\tanh\,(2u)$$

with multiplicities $m_1 = m_2 = n-1$, $m_\alpha = 1$. Hence the characteristic vector field ξ is a magnetic map into $T(M)$ with strength

$$q = -(m_1\lambda_1^2 + m_2\lambda_2^2) = -\frac{n-1}{r^2}\{\coth^2\, u + \tanh^2\, u\} < 0.$$

Acknowledgments. The second author would like to thank organizers for warm hospitality, cordially atmosphere created in Zlatibor, as well as for supporting him to participate at this Seminar.

References

[1] P. Alegre, A. Carriazo, *Structures on generalized Sasakian-space-forms*, Diff. Geom. Appl. **26** (2008), 656–666.

[2] J. Berndt, *Real hypersurfaces with constant principal curvatures in complex hyperbolic space*, J. Reine Angew. Math. **395** (1989), 132–141.

[3] J. Berndt, Y. J. Suh, *Real hypersurfaces with isometric Reeb flow in complex quadrics*, Internat. J. Math. **24** (2013), 1350050, 18 pages.
[4] D. E. Blair, *Contact Manifolds in Riemannian Geometry*, Lect. Notes Math. 509, Springer-Verlag, Berlin, New York, 1976.
[5] ———, *Riemannian Geometry of Contact and Symplectic Manifolds*, Progress in Math. 203, 2002, Birkhäuser, Boston-Basel-Berlin.
[6] P. Dombrowski, *On the Geometry of the Tangent Bundle*, J. Reine Angew. Math. **210** (1962), 73–88.
[7] S. Dragomir, D. Perrone, *Harmonic Vector Fields. Variational Principles and Differential Geometry*, Elsevier, 2012.
[8] S. L. Druţă-Romaniuc, J. Inoguchi, M. I. Munteanu, A. I. Nistor, *Magnetic curves in Sasakian manifolds*, J. Nonlinear Math. Phys. **22**(3) (2015), 428–447.
[9] ———, *Magnetic curves in cosymplectic manifolds*, Rep. Math. Phys. **78**(1) (2016), 33–48.
[10] S. L. Druţă-Romaniuc, M. I. Munteanu, *Magnetic curves corresponding to Killing magnetic fields in* $\mathbb{E}^3$, J. Math. Phys. **52**(11) (2011), 113506.
[11] J. Eells, L. Lemaire, *Selected Topics in Harmonic Maps*, Conf. Board Math. Sci 50, 1983.
[12] M. Harada, *On Sasakian submanifolds*, (Collection of articles dedicated to Shigeo Sasaki on his sixtieth birthday), Tohoku Math. J. **25**(2) (1973), 103–109.
[13] J. Inoguchi, M. I. Munteanu, *Magnetic maps*, Int. J. Geom. Methods Mod. Phys. **11**(6) (2014), art. 1450058, (22 pages).
[14] ———, *New examples of magnetic maps involving tangent bundles*, Rend. Semin. Mat., Univ. Politec. Torino (spec. issue: Proc. Conf. Geometric Structures on Riemmannian Manifolds) **73**(1-2) (2015), 101–116.
[15] ———, *Periodic magnetic curves in Berger spheres*, Tôhoku Math. J. **69**(1) (2017), 113–128.
[16] H. Iriyeh, *Hamiltonian minimal Lagrangian cones in* $\mathbf{C}^m$, Tokyo J. Math. **28**(1) (2005), 91–107.
[17] M. Kimura, *Real hypersurfaces and complex submanifolds in complex projective space*, Trans. Am. Math. Soc. **296**(1) (1986), 137–149.
[18] M.I. Munteanu, A. I. Nistor, *The classification of Killing magnetic curves in* $\mathbb{S}^2 \times \mathbb{R}$, J. Geom. Phys. **62**(2) (2012), 170–182.
[19] Y. G. Oh, *Volume minimization of Lagrangian submanifolds under Hamiltonian deformations*, Math. Z. **212** (1993), 175–192.
[20] D. Perrone, *The rough Laplacian and harmonicity of Hopf vector fields*, Ann. Global Anal. Geom. **28** (2005), 91–106.
[21] H. Urakawa, *Calculus of Variations and Harmonic Maps*, Translations of Mathematical Monographs 132, 1993.

Institute of Mathematics
University of Tsukuba
1-1-1 Tennodai Tsukuba, Ibaraki
305-8571, Japan
inoguchi@math.tsukuba.ac.jp

Al. I. Cuza University of Iasi
Faculty of Mathematics
Iasi, Romania
marian.ioan.munteanu@gmail.com