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MAGNETO-THERMO-ELASTICITY—LARGE-TIME BEHAVIOR FOR LINEAR SYSTEMS *

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Abstract. Initial and initial-boundary value problems for linearized magneto-thermo-elastic models are considered. For the Cauchy problem in three space dimensions, a polynomial rate of decay as time tends to infinity is proved. In bounded domains a boundary condition of memory type is considered for the displacement. When the relaxation function satisfies dissipative properties and decays exponentially, we show that the solution of the magneto-thermo-elastic system decays exponentially. When the relaxation function decays polynomially, it is proved that the solution decays polynomially. Energy methods are used.

1. Introduction. We consider initial and initial-boundary value problems for some linear magneto-thermo-elastic models describing elastic materials where reciprocal effects of the temperature, the magnetic field and the elastic displacement are taken into account. The linear differential equations—for the homogeneous, isotropic case—which govern these models in three space dimensions, are the following:

$$u_{tt} - Eu - \alpha[\nabla \times h] \times \vec{H} + \gamma \nabla\theta = 0, \tag{1.1}$$

$$h_t - \Delta h - \beta \nabla \times [u_t \times \vec{H}] = 0, \tag{1.2}$$

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$$\theta_t - \kappa\Delta\theta + \gamma \operatorname{div} u_t = 0. \tag{1.3}$$

Here $u = (u^1, u^2, u^3)' = u(t,x)$ is the displacement vector depending on the time variable $t \geq 0$ and on $x \in \mathbb{R}^3$, $h = (h^1, h^2, h^3)' = h(t,x)$ is the magnetic field, $\theta = \theta(t,x)$ is the temperature difference with respect to a fixed reference temperature, E is the elasticity operator

$$Eu = \left[(C_{ijkl}u^k_{,l})_{,j}\right]_{i=1,2,3}, \tag{1.4}$$

where C_{ijkl} $(i,j,k,l = 1,2,3)$ are the elastic moduli being constant here and leading in the homogeneous isotropic case under consideration to

$$C_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{jk}\delta_{il}) \qquad (\delta_{ij}: \text{ Kronecker delta}) \tag{1.5}$$

and hence

$$Eu = \mu\Delta u + (\lambda+\mu)\nabla \operatorname{div} u, \tag{1.6}$$

with positive constants λ and μ. The coupling constants α, β satisfy $\alpha\beta > 0$ and $\gamma \in \mathbb{R}$. κ is a positive constant and $\vec{H} = (0,0,H)'$ is a constant vector with $H \neq 0$, distinguishing the x_3-direction. The notation "$_{,j}$" means differentiation with respect to x_j. Additionally, one has initial conditions

$$u(0,x) = u_0(x), \;\; u_t(0,x) = u_1(x), \;\; h(0,x) = h_0(x), \;\; \theta(0,x) = \theta_0(x). \tag{1.7}$$

The Cauchy problem (1.1)–(1.3), (1.7) will be considered in Section 2 giving a description of the asymptotic behavior of the solution as time tends to infinity in terms of polynomial decay rates depending on the smoothness of the initial data. The Cauchy problem in magneto-elasticity i.e. neglecting thermal effects, was recently studied by Andreou and Dassios [2] using spectral analysis and techniques provided from thermoelasticity; polynomial decay rates were obtained. Here we discuss the more complicated system of magneto-thermo-elasticity—indeed, it might be too complicated for the approach from [2]—and we use energy methods constructing appropriate Lyapunov functionals. Note that our results are also valid for magneto-elasticity, i.e., for $\gamma = 0$, $\theta = 0$.

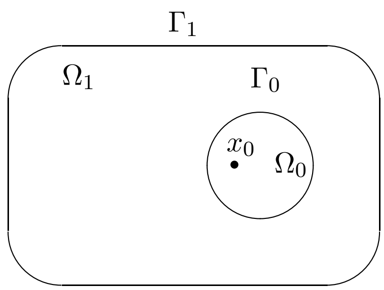
For a derivation of the equations and earlier papers on magneto-thermo-elastic *plane waves* see Paria [12], Willson [17] or Chander [3] and the references in [2].

The bounded domain case will be studied in Section 3. Then x will vary in a bounded domain $\Omega \subset \mathbb{R}^3$ with smooth boundary $\Gamma = \partial\Omega$ and $\Gamma = \Gamma_0 \cup \Gamma_1$ with $\Gamma_0 \cap \Gamma_1 \neq \emptyset$. We assume that there is $x_0 \in \Omega$ such that

$$\Gamma_0 = \{x \in \Gamma : (x - x_0) \cdot \nu(x) \leq 0\}, \tag{1.8}$$

$$\Gamma_1 = \{x \in \Gamma : (x - x_0) \cdot \nu(x) \geq a > 0\} \tag{1.9}$$

for some $a > 0$ and $\nu = \nu(x)$ denoting the exterior normal vector in $x \in \Gamma$. This means that typically we have the following.



$\Omega = \Omega_1 \setminus \Omega_0$

The boundary conditions for h and θ are

$$\nu \times (\nabla \times h) = 0, \quad \nu \cdot h = 0, \quad \theta = 0 \qquad \text{on } \Gamma \tag{1.10}$$

and the memory-type boundary condition

$$u = 0 \qquad \text{on } \Gamma_0, \qquad u + r * \partial_\nu u = 0 \qquad \text{on } \Gamma_1 \tag{1.11}$$

for u. We denote by $\partial_\nu u$ the abbreviation

$$\partial_\nu u := \underbrace{(C_{ijkl} u^k_{,l} \nu^j)_{i=1,2,3}}_{=: \frac{\partial u}{\partial \nu_A}} - \alpha H h^3 \nu + \alpha H \nu^3 h, \tag{1.12}$$

which is the natural Neumann-type boundary operator for the equations (1.1). By $*$ we denote the convolution in time,

$$(r * f)(t) := \int_0^t r(t-s) f(s)\, ds.$$

Differentiating the boundary condition on Γ_1 we get

$$\partial_\nu u + \frac{r'}{r(0)} * \partial_\nu u = -\frac{u_t}{r(0)}, \tag{1.13}$$

or, in terms of the associated resolvent kernel g,

$$\partial_\nu u = -\tau u_t - \tau g * u_t \tag{1.14}$$

with $\tau := \frac{1}{r(0)} > 0$. First we shall assume that g essentially decays exponentially; i.e., for $t \geq 0$

$$0 < g(t) \leq c_0 e^{-g_0 t},\ -c_1 g(t) \leq g'(t) \leq -c_2 g(t),\ -c_3 g'(t) \leq g''(t) \leq -c_4 g'(t), \tag{1.15}$$

with positive constants $g_0, c_0, c_1, c_2, c_3, c_4$. The classical example $\tilde{g}$ is of course $\tilde{g}(t) = c_0 e^{-g_0 t}$. The exponential-type kernel together with the "damping" boundary condition on Γ_1 will lead to an exponential decay result for $(u, h, \theta)(t)$. Second, we shall consider polynomially decaying kernels satisfying for $t \geq 0$

$$\begin{aligned} &0 < g(t) \leq b_0 (1+t)^{-p}, \quad -b_1 g(t)^{\frac{p+1}{p}} \leq g'(t) \leq -b_2 g(t)^{\frac{p+1}{p}}, \\ &-b_3 |g'(t)|^{\frac{p+2}{p+1}} \leq g''(t) \leq -b_4 |g'(t)|^{\frac{p+2}{p+1}}, \end{aligned} \tag{1.16}$$

with positive constants b_0, b_1, b_2, b_3, b_4 and $p > 1$. The typical example $\bar{g}$ is $\bar{g}(t) = b_0(1+t)^{-p}$. The result will be a polynomial decay for the solution.

We present the first discussion of magneto-thermo-elastic initial *boundary* value problems that leads to a description of the asymptotic behavior in terms of decay rates as time tends to infinity. Perla Menzala and Zuazua studied the asymptotic behavior of magneto-elastic systems with u being zero everywhere on the boundary; see [13]. They proved decay results without giving decay rates, but it should be observed that their system is more complicated to deal with because of the boundary conditions considered. In this connection, it should be mentioned that in the related thermo-elastic system exponential decay can only be expected in special situations like radial symmetry, for example, see Jiang and the authors [7], but not in general, see Koch [8] or Lebeau and Zuazua [9].

In our case, the complexity of the system can be dealt with because of the memory-type boundary condition. For a discussion of this kind of boundary condition in connection with viscoelastic effects see e.g. the papers [14, 15] by Qin. In [6], Fabrizio and Morro also consider a boundary condition of memory type for an electromagnetic system (see also Ciarletta [4]). The boundary condition was also studied for a pure elastic system by Andrade and Muñoz Rivera [1]. Their approach and techniques from Muñoz Rivera and Barreto [11] are adapted here and modified for our purposes.

In Section 2 we discuss the Cauchy problem and in Section 3 the bounded-domain case is considered. In both sections, the existence and uniqueness of solutions with smoothness in relation to that of the initial data is simply assumed, its proof—e.g. via semigroup theory as in [13] for the magneto-elastic system—is rather standard and omitted here.

We denote the Fourier transform by

$$(\mathcal{F}w)(\eta) \equiv \hat{w}(\eta) = \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} e^{-ix\eta} w(x)\, dx$$

for a function $w : \mathbb{R}^3 \to \mathbb{R}^{(3)}$.

2. The Cauchy problem. We consider smooth solutions to the Cauchy problem (1.1)–(1.3), (1.7). The necessary regularity of the initial data will be specified below. Noting that

$$(\nabla \times h)\times \vec{H} = -H\nabla h^3 + H\frac{\partial h}{\partial x_3}, \quad \nabla \times (u_t \times \vec{H}) = H\frac{\partial u_t}{\partial x_3} - H(0,0,\operatorname{div} u_t)'$$

and using (1.6), the system (1.1)–(1.3) is equivalent to

$$u_{tt} - \mu\Delta u - (\mu+\lambda)\nabla \operatorname{div} u + \alpha H \nabla h^3 - \alpha H \frac{\partial h}{\partial x_3} + \gamma\nabla\theta = 0, \tag{2.1}$$

$$h_t - \Delta h - \beta H \frac{\partial u_t}{\partial x_3} + \beta H(0,0,\operatorname{div} u_t)' = 0, \tag{2.2}$$

$$\theta_t - \kappa\Delta\theta + \gamma \operatorname{div} u_t = 0. \tag{2.3}$$

Taking the Fourier transform and denoting $v := \hat{u}$, $w := \hat{h}$, $\psi := \hat{\theta}$, with Fourier variable $\eta = (\eta_1, \eta_2, \eta_3)'$, we obtain from (2.1)–(2.3)

$$v_{tt} + \mu|\eta|^2 v + (\mu+\lambda)(\eta v)\eta - i\alpha H\eta w^3 + i\alpha H\eta_3 w - i\gamma\psi\eta = 0, \tag{2.4}$$

$$w_t + |\eta|^2 w + i\beta H\eta_3 v_t - i\beta H(0,0,v_t\eta)' = 0, \tag{2.5}$$

$$\psi_t + \kappa|\eta|^2\psi - i\gamma v_t\eta = 0. \tag{2.6}$$

Multiplying equation (2.4) by $\overline{u}_t$, and (2.5) by $\frac{\alpha}{\beta}\overline{w}$ and (2.6) by $\overline{\psi}$ we get

$$\frac{d}{dt}\mathcal{E}_1(t) = -\frac{\alpha}{\beta}|\eta|^2|w|^2 - \kappa|\eta|^2|\psi|^2$$

where $\mathcal{E}_1(t) := \frac{1}{2}\{|v_t|^2 + \mu|\eta|^2|v|^2 + (\mu+\lambda)(\eta v)^2 + \frac{\alpha}{\beta}|w|^2 + |\psi|^2\}(t)$. To study the dissipative properties of the above system, we will consider the equations for the components separately. Denoting $z := \eta_1 v^1 + \eta_2 v^2$, we obtain

$$v^1_{tt} + \mu|\eta|^2 v^1 + (\mu+\lambda)\eta_1 z + (\mu+\lambda)\eta_1\eta_3 v^3 - i\alpha H\eta_1 w^3 + i\alpha H\eta_3 w^1 - i\gamma\psi\eta_1 = 0, \tag{2.7}$$

$$v^2_{tt} + \mu|\eta|^2 v^2 + (\mu+\lambda)\eta_2 z + (\mu+\lambda)\eta_2\eta_3 v^3 - i\alpha H\eta_2 w^3 + i\alpha H\eta_3 w^2 - i\gamma\psi\eta_2 = 0, \tag{2.8}$$

$$v^3_{tt} + \mu|\eta|^2 v^3 + (\mu+\lambda)\eta_3 z + (\mu+\lambda)\eta_3^2 v^3 - i\gamma\psi\eta_3 = 0, \tag{2.9}$$

$$w^1_t + |\eta|^2 w^1 + i\beta H\eta_3 v^1_t = 0, \tag{2.10}$$

$$w^2_t + |\eta|^2 w^2 + i\beta H\eta_3 v^2_t = 0, \tag{2.11}$$

$$w^3_t + |\eta|^2 w^3 - i\beta H z_t = 0. \tag{2.12}$$

Lemma 2.1. *Let* $\Phi_1(t) := -\mathrm{Re}\{\frac{i}{\eta_3}(\overline{v^1_t} w^1 + \overline{v^2_t} w^2)\}(t)$, (Re : real part), $\Phi_2(t) := -\mathrm{Re}\,\{\frac{i}{\eta_3}(\overline{v^1_{tt}} w^1_t + \overline{v^2_{tt}} w^2_t)\}(t)$. *Then we have for* $\eta_3 \neq 0$

$$\frac{d}{dt}\Phi_1 \leq -\frac{\beta H}{2}\left(|v^1_t|^2 + |v^2_t|^2\right) + \frac{c|\eta|^2}{\eta_3^2}(1+|\eta|^2)|w|^2 + \frac{c|\eta|^2}{|\eta_3|}(|v^1| + |v^2|)|w| + c|\eta||v^3||w| + c\frac{|\eta|}{|\eta_3|}|\psi||w|,$$

$$\frac{d}{dt}\Phi_2 \leq -\frac{\beta H}{2}\left(|v^1_{tt}|^2 + |v^2_{tt}|^2\right) + \frac{c|\eta|^2}{\eta_3^2}(1+|\eta|^2)|w_t|^2 + c|\eta||v^3_t||w_t| + \frac{c|\eta|^2}{|\eta_3|}|\overline{v^1_t} w^1_t + \overline{v^2_t} w^2_t|.$$

Here and in the sequel c *will denote positive constants not depending on* η *or on* t.

Proof. Multiplying equation (2.10) by $\frac{-i}{\eta_3}\overline{v^1_t}$ and equation (2.11) by $\frac{-i}{\eta_3}\overline{v^2_t}$ we get

$$\frac{d}{dt}\Phi_1 = \mathrm{Re}\,\frac{i}{\eta_3}|\eta|^2(w^1\overline{v^1_t} + w^2 v^2_t) - \beta H(|v^1_t|^2 + |v^2_t|^2) - \mathrm{Re}\,\frac{i}{\eta_3}(w^1 v^1_{tt} + w^2 v^2_{tt}).$$

Using the equations (2.7), (2.8) the conclusion for Φ_1 follows. The estimate for Φ_2 follows similarly after differentiating the equations with respect to t. □

Introducing the function $\Phi_3(t) := \operatorname{Re}(v_{tt}^1\overline{v_t^3})(t)$ and $\Phi_4(t) := \operatorname{Re}(v_{tt}^2\overline{v_t^3})(t)$ we have

Lemma 2.2. *Let for* $\eta_3 \neq 0 \neq \eta_1,\ J(t) := \frac{\eta_1}{\eta_3}\Phi_3(t) + \frac{\eta_3}{\eta_1}\Phi_3(t) + \frac{\eta_1}{\eta_3}\Phi_4(t)$. *Then*

$$\frac{d}{dt}J \leq c|\eta|^4\Big(\frac{1}{\eta_1^2} + \frac{1}{\eta_3^2}\Big)\big(|v_t^1|^2 + |v_t^2|^2\big) - \frac{\mu+\lambda}{2}|\eta|^2|v_t^3|^2 + c|\eta|^2\Big(\frac{1}{\eta_1^2} + \frac{1}{\eta_3^2}\Big)|w_t|^2$$
$$+ \Big(\frac{\eta_3}{\eta_1} + \frac{\eta_2}{\eta_3} + \frac{\eta_1}{\eta_3}\Big)(|\eta|^2|v| + |\eta|\,|\psi|)(|v_{tt}^1| + |v_{tt}^2|) + c|\eta|^2\Big(\frac{1}{\eta_1^2} + \frac{1}{\eta_3^2}\Big)|\psi_t|^2.$$

Proof. Differentiating Φ_3 we get

$$\frac{d}{dt}\Phi_3 = (v_{ttt}^1\overline{v_t^3} + v_{tt}^1\overline{v_{tt}^3}) = -(\mu|\eta|^2 v_t^1 + (\mu+\lambda)\eta_1 v_t + (\mu+\lambda)\eta_1\eta_3 v_t^3$$
$$- i\alpha H\eta_1 w_t^3 + i\alpha H\eta_3 w_t^1 + \gamma\eta_1\psi_t)\overline{v_t^3} + v_{tt}^1\overline{v_{tt}^3},$$

which implies

$$\frac{d}{dt}\Big(\frac{\eta_1}{\eta_3}\Phi\Big) \leq \frac{c|\eta|^4}{\eta_3^2}|v_t^1|^2 + \frac{c|\eta|^2|z_t|^2}{\eta_3^2} - \frac{(\mu+\lambda)}{2}\eta_1^2|v_t^3|^2$$
$$+ \frac{c|\eta|^2}{\eta_3^2}\big(|w_t|^2 + |\psi_t|^2\big) - \frac{\eta_1}{\eta_3}v_{tt}^1\overline{v_{tt}^3}.$$

Similarly we have

$$\frac{d}{dt}\Big(\frac{\eta_3}{\eta_1}\Phi_3\Big) \leq \frac{c|\eta|^4}{\eta_1^2}|v_t|^2 + \frac{c|\eta|^2|z_t|^2}{\eta_1^2} - \frac{(\mu+\lambda)}{2}\eta_3^2|v_t^3|^2$$
$$+ \frac{c|\eta|^2}{\eta_1^2}\big(|w_t|^2 + |\psi_t|^2\big) + \frac{\eta_3}{\eta_1}v_{tt}^1\overline{v_{tt}^3},$$
$$\frac{d}{dt}\Big(\frac{\eta_2}{\eta_3}\Phi_4\Big) \leq \frac{c|\eta|^4|v_t|^2}{\eta_3^2} + \frac{c|\eta|^2|z_t|^2}{\eta_3^2} - \frac{(\mu+\lambda)}{2}\eta_2^2|v_t^3|^2$$
$$+ \frac{c|\eta|^2}{\eta_3^2}\big(|w_t|^2 + |\psi_t|^2\big) + \frac{\eta_2}{\eta_3}v_{tt}^2\overline{v_{tt}^3}.$$

Summing up these inequalities our conclusion follows. □

As a consequence of Lemma 2.1 and Lemma 2.2 we obtain:

Lemma 2.3. *Let, for* $\eta_1 \neq 0 \neq \eta_3$, $\mathcal{A} \equiv \mathcal{A}(\eta) := |\eta|^2\big(\frac{1}{\eta_1^2} + \frac{1}{\eta_3^2}\big)$, *and let* $\delta > 0$ *be sufficiently small. Then we have the estimates*

$$\begin{aligned}
&\frac{d}{dt}\left\{(|\eta|^2 + |\eta|^2\mathcal{A})\Phi_1 + \delta J\right\} \leq -\frac{\beta H}{4}(|\eta|^2 + |\eta|^2\mathcal{A})(|v_t^1|^2 + |v_t^2|^2)\\
&- \delta k_0|\eta|^2|v_t|^2 + c(|\eta|^2 + |\eta|^2\mathcal{A})\frac{|\eta|^2}{\eta_3^2}(1 + |\eta|^2)(|w|^2 + |\psi|^2)\\
&+ c\mathcal{A}|\eta|^3|v^3|\,|w| + \frac{c|\eta|^4}{|\eta_3|}\mathcal{A}(|v^1| + |v^2|)|w| + c\delta\mathcal{A}(|w_t|^2 + |\psi_t|^2)\\
&+ c\delta\Big|\frac{\eta_1}{\eta_3} + \frac{\eta_3}{\eta_1} + \frac{\eta_2}{\eta_3}\Big|\,\big(|\eta|^2|v| + |\eta|\,|\psi|\big)\,(|v_{tt}^1| + |v_{tt}^2|),
\end{aligned}$$

where $k_0 > 0$ *is a constant depending only on the coefficients of the differential equation.*

Now let $\Phi_5(t) := \mathrm{Re}(v_t\overline{v})(t)$. Using equation (2.1) it is not difficult to prove that

$$\frac{d}{dt}\Phi_5 \leq |v_t|^2 - \frac{\mu}{2}|\eta|^2|v|^2 - \frac{\mu + \lambda}{2}(\eta v)^2 + c|\eta|(|w|^2 + |\psi|^2),$$

which, together with Lemma 2.3, yields

$$\begin{aligned}
&\frac{d}{dt}\big\{(|\eta|^2 + |\eta|^2\mathcal{A})\Phi_1 + \delta J + \frac{\delta k_0}{2}|\eta|^2\Phi_5\big\}\\
&\leq -\frac{\beta H}{4}|\eta|^2\mathcal{A}(|v_t^1|^2 + |v_t^2|^2) - \frac{\delta k_0}{2}|\eta|^2\mathcal{E}_1\\
&+ c\big\{\frac{|\eta|^4}{\eta_3^2}(1 + |\eta|^2)\mathcal{A} + |\eta|^2 + |\eta|^2\mathcal{A} + \frac{|\eta|^6}{\eta_3^2}\mathcal{A}\big\}(|w|^2 + |\psi|^2)\\
&+ c\frac{|\eta|^4}{|\eta_3|}\mathcal{A}(|v^1| + |v^2|)|w| + c\delta\mathcal{A}(|v_{tt}^1|^2 + |v_{tt}^2|^2) + c\delta\mathcal{A}(|w_t|^2 + |\psi_t|^2). \quad (2.13)
\end{aligned}$$

Notice that

$$\begin{aligned}
&\frac{d}{dt}\big\{(|\eta|^2 + |\eta|^2\mathcal{A})\Phi_1 + \delta J + \frac{\delta k_0}{2}|\eta|^2\Phi_5\big\} \leq -\frac{\beta H}{4}|\eta|^2\mathcal{A}(|v_t^1|^2 + |v_t^2|^2)\\
&- \frac{\delta k_0}{2}|\eta|^2\mathcal{E}_1 + c|\eta|^6\frac{1 + |\eta|^2}{\eta_3^2}\big(\frac{1}{\eta_1^2} + \frac{1}{\eta_3^2}\big)(|w|^2 + |\psi|^2)\\
&+ c\delta\mathcal{A}(|v_{tt}^1|^2 + |v_{tt}^2|^2) + c\delta\mathcal{A}(|w_t|^2 + |\psi_t|^2) + c\frac{|\eta|^4}{|\eta_3|}\mathcal{A}(|v^1| + |v^2|)|w| \quad (2.14)
\end{aligned}$$

and that

$$c\frac{|\eta|^4}{|\eta_3|}\mathcal{A}(|v^1|+|v^2|)|w| \le \varepsilon\frac{|\eta|^4}{|\eta_3|^2}(|v^1|^2+|v^2|^2)+c(\varepsilon)|\eta|^4\mathcal{A}^2|w|^2, \tag{2.15}$$

where $\varepsilon > 0$ will be chosen small enough in the sequel and $c(\varepsilon)$ denotes a constant depending on ε. Let $\Phi_6(t) := \mathrm{Re}(v_t^1\overline{v^1}+v_t^2\overline{v^2})(t)$. Then, by

$$\begin{aligned}\frac{d}{dt}\Phi_6 &= |v_t^1|^2+|v_t^2|^2+v_{tt}^1\overline{v^1}+v_{tt}^2\overline{v^2}\\ &= |v_t^1|^2+|v_t^2|^2-\mu|\eta|^2|v^1|^2-\mu|\eta|^2|v^2|^2-(\mu+\lambda)|z|^2\\ &\quad-(\mu+\lambda)\eta_3v^3\overline{z}+i\alpha H\overline{z}w^3-i\alpha H\eta_3(w^1\overline{v^1}+w^2\overline{v^2})-i\gamma z\psi,\end{aligned}$$

we get

$$\begin{aligned}\frac{d}{dt}\Phi_6 \le{}& |v_t^1|^2+|v_t^2|^2-\frac{\mu}{2}|\eta|^2(|v^1|^2+|v^2|^2)-\frac{\mu+\lambda}{2}|z|^2\\ &+c(|w|^2+(|\psi|^2)+c\eta_3^2|v^3|^2.\end{aligned} \tag{2.16}$$

Taking ε small enough we conclude from (2.13)–(2.16)

$$\begin{aligned}&\frac{d}{dt}\big\{(|\eta|^2+|\eta|^2\mathcal{A})\Phi_1+\delta J+\frac{\delta k_0}{2}|\eta|^2\Phi_5+\frac{\delta k_0}{\psi}\frac{|\eta|^2}{\eta_3^2}\Phi_6\big\}\\ &\le -\frac{\beta H}{8}|\eta|^2\mathcal{A}(|v_t^1|^2+|v_t^2|^2)-\frac{\delta k_0}{8}|\eta|^2\mathcal{E}_1\\ &+|\eta|^6\frac{1+|\eta|^2}{\eta_3^2}\big(\frac{1}{\eta_1^2}+\frac{1}{\eta_3^2}\big)(|w|^2+|\psi|^2)+c\delta\mathcal{A}(|v_{tt}^1|^2+|v_{tt}^2|^2)\\ &+c\delta\mathcal{A}(|w_t|^2+|\psi_t|^2)+c(\varepsilon)|\eta|^4\mathcal{A}^2|w|^2-\frac{k_0}{4}\frac{|\eta|^4}{\eta_3^2}(|v^1|^2+|v^2|^2).\end{aligned} \tag{2.17}$$

Note that

$$\begin{aligned}\frac{d}{dt}(\mathcal{A}\Phi_2) \le{}& -\frac{\beta H}{2}\mathcal{A}(|v_{tt}^1|^2+|v_{tt}^2|^2)+\frac{|\eta|^2}{\eta_3^2}\mathcal{A}(1+|\eta|^2)|w_t|^2\\ &+\underbrace{c|\eta|\mathcal{A}|v_t^3|\,|w_t|}_{=:I_1}+\underbrace{c\frac{|\eta|^2}{\eta_3^2}\mathcal{A}|v_t^1\overline{w_t^1}+v_t^2w_t^2|}_{=:I_2}+\frac{|\eta|}{|\eta_3|}\mathcal{A}|\psi_tw_t|,\end{aligned} \tag{2.18}$$

and

$$I_1 \leq \frac{\delta k_0}{4}|\eta|^2|v_t^3|^2 + c(\delta)\mathcal{A}^2|w_t|^2, \tag{2.19}$$

$$I_2 \leq \delta\mathcal{A}|\eta|^2(|v_t^1|^2 + |v_t^2|^2) + c(\delta)\frac{|\eta|^2}{\eta_3^2}\mathcal{A}|w_t|^2. \tag{2.20}$$

Denoting by Φ the following function,

$$\Phi(t) := (|\eta|^2 + |\eta|^2\mathcal{A})\Phi_1(t)(+\delta J(t) + \frac{\delta k_0}{2}|\eta|^2\Phi_5(t) + \frac{\delta k_0|\eta|^2}{4|\eta_3|^2}\Phi_6(t) + \mathcal{A}\Phi_2(t),$$

we obtain from (2.13)–(2.20)

$$\begin{aligned}\frac{d}{dt}\Phi \leq& -\frac{\beta H}{8}|\eta|^2\mathcal{A}(|v_t^1|^2 + |v_t^2|^2) - \frac{\delta k_0}{4}|\eta|^2\mathcal{E}_1 - \frac{\beta H}{4}\mathcal{A}(|v_{tt}^1|^2 + |v_{tt}^2|^2)\\ &+ c|\eta|^6\frac{1+|\eta|^2}{\eta_3^2}\big(\frac{1}{\eta_1^2} + \frac{1}{\eta_3^2}\big)(|w|^2 + |\psi|^2)c(\delta)\underbrace{\big(\frac{|\eta|^2}{\eta_3^2}\mathcal{A} + \mathcal{A}^2\big)}_{=:I_3}(|w_t|^2 + |\psi_t|^2)\\ &+ c(\varepsilon)|\eta|^4\mathcal{A}^2|w|^2 - \frac{k_0|\eta|^4}{4\eta_3^2}(|v^1|^2 + |v^2|^2).\end{aligned}$$

Using $I_3 \leq 2\mathcal{A}^2$, we conclude

$$\begin{aligned}\frac{d}{dt}\Phi \leq& -\frac{\beta H}{8}|\eta|^2\mathcal{A}(|v_t^1|^2 + |v_t^2|^2) - \frac{\delta k_0}{4}|\eta|^2\mathcal{E}_1 - \frac{\beta H}{4}\mathcal{A}(|v_{tt}^1|^2 + |v_{tt}^2|^2) \quad (2.21)\\ &+c|\eta|^2(1+|\eta|^2)\mathcal{A}^2(|w|^2+|\psi|^2) + c(\delta)\mathcal{A}^2(|w_t|^2+|\psi_t|^2) - \frac{k_0|\eta|^4}{\psi\eta_3^2}(|v^1|^2+|v^2|^2).\end{aligned}$$

Let

$$\mathcal{L}(t) := \Phi(t) + N_1(1+|\eta|^2)\mathcal{A}^2\mathcal{E}_1(t) + N_2\frac{\mathcal{A}^2}{|\eta|^2}\mathcal{E}_2(t),$$

where $\mathcal{E}_2(t) \equiv \mathcal{E}_1(t; v_t, w_t, \psi_t)$ if $\mathcal{E}_1(t) = \mathcal{E}_1(t; v, w, \psi)$, and N_1, N_2 are positive constants to be determined below. Observe that $\mathcal{L}$ is nonnegative for large N_1, N_2. From the inequalities up to (2.21) we know that there is $d_0 > 0$ such that

$$\frac{d}{dt}\mathcal{L} \leq -d_0\mathcal{N} \tag{2.22}$$

where $d_0, d_1, \ldots$ will denote positive constants and

$$\mathcal{N}(t) := |\eta|^2\mathcal{A}(|v_t^1|^2 + |v_t^2|^2)(t) + |\eta|^2\mathcal{E}_1(t) + \mathcal{A}(|v_{tt}^1|^2 + |v_{tt}^2|^2) + \frac{|\eta|^4}{\eta_3^2}(|v^1|^2 + |v^2|^2) + (1+|\eta|^2)\mathcal{A}^2(|w|^2 + |\psi|^2) + \frac{\mathcal{A}^2}{|\eta|^2}(|w_t|^2 + |\psi_t|^2).$$

Since

$$|\Phi| \leq \tilde{c}\big((1+|\eta|^2)\mathcal{A}^2\mathcal{E}_1 + \frac{\mathcal{A}^2}{|\eta|^2}\mathcal{E}_2\big), \tag{2.23}$$

for some $\tilde{c} > 0$, and

$$\mathcal{E}_2(t) \leq \tilde{c}|\eta|^2(1+|\eta|^2)\mathcal{E}_1(t) \tag{2.24}$$

we have

$$\mathcal{L}(t) \leq d_1(1+|\eta|^2)\mathcal{A}^2\mathcal{E}_1(t). \tag{2.25}$$

The relations (2.22)–(2.25) imply

$$\frac{d}{dt}\mathcal{L} \leq -d_0\mathcal{N} \leq -d_0|\eta|^2\mathcal{E}_1 \leq -\underbrace{\frac{d_0}{d_1}}_{=:d_2}\underbrace{\frac{|\eta|^2}{1+|\eta|^2\mathcal{A}^2}}_{=:k(\eta)}\mathcal{L},$$

and hence

$$\mathcal{L}(t) \leq \mathcal{L}(0)e^{-d_2k(\eta)t}. \tag{2.26}$$

Using (2.23), (2.24) and choosing $N_2 := \frac{\tilde{c}}{2}$, we obtain

$$\begin{aligned}\mathcal{L} &\geq (N_1 - 2N_2)(1+|\eta|^2)\mathcal{A}\mathcal{E}_1 - \frac{N_2}{2}\frac{\mathcal{A}^2}{|\eta|^2}\mathcal{E}_2 \\ &\geq \big(N_1 - 2N_2 - \frac{N_2}{2} - \frac{N_2}{2}\big)(1+|\eta|^2)\mathcal{A}^2\mathcal{E}_1;\end{aligned}$$

that is, choosing $N_1 := \frac{5}{2}N_2$,

$$\mathcal{L} \geq N_2(1+|\eta|^2)\mathcal{A}^2\mathcal{E}_1. \tag{2.27}$$

From (2.25) and (2.27) we have

$$N_2(1+|\eta|^2)\mathcal{A}^2\mathcal{E}_1(t) \leq \mathcal{L}(t) \leq d_1(1+|\eta|^2)\mathcal{A}^2\mathcal{E}_1(t).$$

Therefore, (2.25) implies for the "energy" term $\mathcal{E}_1$, $\mathcal{E}_1(t) \le \frac{d_1}{N_2}\mathcal{E}_1(0)e^{-d_2 k(\eta)t}$, which implies for $m \in \mathbb{N}_0$, $t \ge 0$:

$$\int_0^t s^m \mathcal{E}_1 \, ds \le c(m)\big(\frac{1}{k(\eta)}\big)^{m+1}\mathcal{E}_1(0). \tag{2.28}$$

This gives the desired polynomial decay in a standard manner:

$$\frac{d}{dt} t\mathcal{E}_1(t) = \mathcal{E}_1(t) + t\frac{d}{dt}\mathcal{E}_1(t) \le \mathcal{E}_1(t)$$

implies using (2.28) for $m = 0$: $\mathcal{E}_1(t) = \mathcal{O}(t^{-1})$ as $t \to \infty$. The estimate

$$\frac{d}{dt} t^2\mathcal{E}_1(t) \le 2t\mathcal{E}_1(t)$$

implies $\mathcal{E}_1(t) = \mathcal{O}(t^{-2})$, and so on. Integrating (2.28) with respect to η—for appropriate initial data yielding a finite value of the integral on the right-hand side—gives the final result. Observing $\mathcal{E}_1(0) = \mathcal{E}_1(0,\eta)$, $\mathcal{A} = \mathcal{A}(\eta)$ we have

Theorem 2.4. *If the initial data satisfy*

$$\int_{\mathbb{R}^n} \big[\frac{(1+|\eta|^2)\mathcal{A}^2}{|\eta|^2}\big]^{m+1}\mathcal{E}_1(0,\eta)\, d\eta < \infty,$$

$m \in \mathbb{N}_0$, *then the energy term*

$$E(t) := \frac{1}{2}\int_{\mathbb{R}^n} \big\{|u_t|^2 + \mu|\nabla u|^2 + (\mu+\lambda)|\operatorname{div} u|^2 + \frac{\alpha}{\beta}|h|^2 + |\theta|^2\big\}(t,x)\, dx$$

associated with a solution (u,h,θ) to the Cauchy problem (1.1)–(1.3), (1.7) *decays polynomially: $E(t) = \mathcal{O}(t^{-m})$ as $t \to \infty$.*

3. The bounded domain case. In a bounded domain $\Omega \subset \mathbb{R}^3$ described by (1.8), (1.9) we consider the differential equations (1.1)–(1.3) together with the initial conditions (1.7) and the boundary conditions (1.10), (1.11). The initial magnetic field h_0 is assumed to satisfy

$$\operatorname{div} h_0 = 0, \tag{3.1}$$

which implies by (1.2) $\operatorname{div} h(t,\cdot) = 0$ for all $t \ge 0$. Either the condition (1.15) or (1.16) on the resolvent kernel g is assumed to hold. Since we have formulated conditions in terms of the resolvent kernel, we point out the following relationship between the decay of a kernel and the decay of the associated resolvent kernel.

Let $b(t) := -\frac{r'(t)}{r(0)}$. Then b and g satisfy $b + g = -b * g$.

Lemma 3.1. (i) *If g satisfies*

$$\exists\,\gamma > 0 \;\; \exists\, c_g > 0 \;\; \forall\, t \geq 0: \quad |g(t)| \leq c_g e^{-\gamma t},$$

and if for some $0 < \varepsilon < \gamma$, $c_g < \gamma - \varepsilon$ holds, then we have

$$\forall\, t \geq 0: \quad |b(t)| \leq \frac{c_g(\gamma-\varepsilon)}{\gamma - \varepsilon - c_g} e^{-\varepsilon t}.$$

(ii) *If g satisfies*

$$\exists\, p > 1 \;\; \exists c_g > 0 \;\; \forall\, t \geq 0: \quad |g(t)| \leq c_g(1+t)^{-p},$$

and if

$$\frac{1}{c_g} > c_p := \sup_{0 \leq t < \infty} \int_0^t (1+t)^p (1+t-\tau)^{-p} (1+\tau)^{-p}\, d\tau$$

holds, then we have

$$\forall\, t \geq 0: \quad |b(t)| \leq \frac{c_g}{1 - c_g c_p}(1+t)^{-p}.$$

Proof. (i): Let $\tilde{g}(t) := e^{\varepsilon t} g(t)$, $\tilde{b}(t) := e^{\varepsilon t} b(t)$. Then $\tilde{b} + \tilde{g} = -\tilde{b} * \tilde{g}$. The operator G given by $G(h) := \tilde{g} * h$, acting on $C^0([0,T])$, $T > 0$ arbitrary, but fixed, has norm less than or equal to $\frac{c_g}{\gamma - \varepsilon}$; hence,

$$\sup_{0 \leq t \leq T} |\tilde{b}(t)| \leq \frac{1}{1 - c_g/(\gamma - \varepsilon)} \sup_{0 \leq t \leq T} |\tilde{g}(t)| \leq \frac{c_g(\gamma - \varepsilon)}{\gamma - \varepsilon - c_g},$$

which implies the assertion in (i).

(ii): Let $\tilde{g}(t) := (1+t)^p g(t)$, $\tilde{b}(t) := (1+t)^p b(t)$. Then $\tilde{b} + \tilde{g} = -k[g] * b$ with kernel

$$k[g](t,\tau) := \tilde{g}(t-\tau)(1+t)^p (1+t-\tau)^{-p} (1+\tau)^{-p}.$$

The operator $K[g]$ acting on $C^0([0,T])$ as $K[g](h) := k[g] * h$ has norm less than or equal to

$$c_g \sup_{0 \leq t < \infty} \int_0^t (1+t)^p (1+t-\tau)^{-p} (1+\tau)^{-p}\, d\tau \leq c_g c_p;$$

hence

$$\sup_{0 \leq t \leq T} |\tilde{b}(t)| \leq \frac{1}{1 - c_g c_p} \sup_{0 \leq t \leq T} |\tilde{g}(t)| \leq \frac{c_g}{1 - c_g c_p},$$

which gives the assertion. □

Remarks. 1. For the finiteness of c_p compare Lemma 7.4 in [16] in a more general setting.

2. Since the resolvent kernel of the resolvent kernel is the orinigal kernel, it is clear that in (ii) of Lemma 3.1 no stronger uniform polynomial decay can be obtained. In this sense the characterization is sharp and shows that exponentially decaying kernels correspond to exponentially decaying resolvents, and polynomially decaying kernels correspond to polynomially decaying resolvents.

Taking the boundary condition (1.11) for u on Γ_1 in the form (1.14), and performing an integration by parts, we get

$$\partial_\nu u = -\tau u_t - \tau g * u_t = -\tau u_t - \tau g(0)u - \tau g' * u + \tau g u_0. \tag{3.2}$$

As a nonnegative energy function we define

$$\begin{aligned} F(t) := \frac{1}{2}\int_\Omega (|u_t|^2 + C_{ijkl}u^k_{,l}u^i_{,j} + \frac{\alpha}{\beta}|h|^2 + |\theta|^2)(t,x)\,dx \\ - \frac{\tau}{2}\int_{\Gamma_1}(g'\Box u)(t,z)\,dz + \frac{\tau}{2}g(t)\int_{\Gamma_1}|u|^2(t,z)\,dz, \end{aligned} \tag{3.3}$$

where

$$(f\Box\varphi)(t) := \int_0^t f(t-s)|\varphi(t)-\varphi(s)|^2\,ds.$$

Lemma 3.2. *For $f,\varphi \in C^1([0,\infty),\mathbb{R})$, we have*

$$2(f*\varphi)(t)\varphi_t(t) = (f'\Box\varphi)(t) + \frac{d}{dt}\Big\{\int_0^t f(s)\,ds|\varphi(t)|^2 - (f\Box\varphi)(t)\Big\} - f(t)|\varphi(t)|^2.$$

Proof.

$$\begin{aligned} \frac{d}{dt}(f\Box\varphi) &= f'\Box\varphi + 2\int_0^t f(t-s)(\varphi(t)-\varphi(s))\,ds\,\varphi_t \\ &= f'\Box\varphi - 2f*\varphi\varphi_t + 2\int_0^t f(s)\,ds\,\varphi\varphi_t \\ &= f'\Box\varphi - 2f*\varphi\varphi_t + \frac{d}{dt}\Big\{\int_0^t f(s)\,ds|\varphi|^2\Big\} - f|\varphi|^2. \end{aligned}$$

Lemma 3.3.

$$\frac{d}{dt}F \leq -\frac{\alpha}{\beta}\int_\Omega |\nabla \times h|^2\,dx - \kappa\int_\Omega (\nabla\theta)^2\,dx - \frac{\tau}{2}\int_{\Gamma_1}|u_t|^2\,dz \\ - \frac{\tau}{2}\int_{\Gamma_1} g''\Box u\,dz + \frac{\tau g'}{2}\int_{\Gamma_1}|u|^2\,dz + \tau g^2\int_{\Gamma_1}|u_0|^2\,dz.$$

Proof. Multiplying equation (1.1) by u_t, equation (1.2) by $\frac{\alpha}{\beta}h$ and (1.3) by θ, we obtain after integration and summation

$$\frac{1}{2}\frac{d}{dt}\underbrace{\int_\Omega (|u_t|^2 + C_{ijkl}u^k_{,l}u^i_{,j} + \frac{\alpha}{\beta}|h|^2 + |\theta|^2)\,dx}_{=:M=M(t)} \\ = \int_{\Gamma_1}\partial_\nu u u_t\,dz - \frac{\alpha}{\beta}\int_\Omega |\nabla\times h|^2\,dx - \kappa\int_\Omega |\nabla\theta|^2 dz. \tag{3.4}$$

Using (3.2) and Lemma 3 we get

$$\frac{d}{dt}M = -\frac{\alpha}{\beta}\int_\Omega |\nabla\times h|^2\,dx - \kappa\int_\Omega |\nabla\theta|^2\,dx - \tau\int_{\Gamma_1}|u_t|^2\,dz \\ - \tau g(0)\int_{\Gamma_1} u u_t\,dz - \tau\int_{\Gamma_1}(g' * u)u_t + \tau g\int_{\Gamma_1} u_0 u_t\,dz \\ = -\frac{\alpha}{\beta}\int_\Omega |\nabla\times h|^2\,dx - \kappa\int_\Omega |\nabla\theta|^2\,dx - \tau\int_{\Gamma_1}|u_t|^2\,dz \\ - \frac{1}{2}\frac{d}{dt}\{\tau g\int_{\Gamma_1}|u|^2\,dz - \tau\int_{\Gamma_1} g'\Box u\,dz\} \\ - \frac{\tau}{2}\int_{\Gamma_1} g''\Box u\,dz + \frac{\tau}{2}g'\int_{\Gamma_1}|u|^2\,dz + \tau g\int_{\Gamma_1} u_0 u_t\,dz,$$

which yields the assertion. □

Define $q(x) := x - x_0$.

Lemma 3.4. *Let* $f := \alpha(\nabla\times h)\times \overrightarrow{H} - \gamma\nabla\theta$. *Then*

$$\frac{d}{dt}\int_\Omega u_t q^k u_{,k}dx = \int_\Gamma \frac{\partial u}{\partial\nu_A} q^k u_{,k}dz - \int_\Omega C_{ijml}u^m_{,l}q^k_{,j}u^i_{,k}dx - \tfrac{1}{2}\int_\Gamma q^k\nu^k C_{ijml}u^m_{,l}u^i_{,j}dz \\ + \tfrac{1}{2}\int_\Gamma q^k\nu^k|u_t|^2\,dz + \tfrac{1}{2}\int_\Omega q^k_{,k}C_{ijml}u^m_{,l}u^i_{,j}dx - \tfrac{1}{2}\int_\Omega q^k_{,k}|u_t|^2 dx + \int_\Omega f q^k u_{,k}dx.$$

Proof. Multiplying (1.1) by $q^k u_{,k}$ and integrating yields

$$\begin{aligned}\frac{d}{dt}\int_\Omega u_t q^k u_{,k}\,dx &= \int_\Omega u_{tt} q^k u_{,k}\,dx + \int_\Omega u_t q^k u_{t,k}\,dx\\ &= \int_\Omega (C_{ijml}u^m_{,l})_{,j} q^k u^i_{,k}\,dx + \frac{1}{2}\int_\Omega q^k |u_t|^2_{,k}\,dx + \int_\Omega f q^k u_{,k}\,dx\\ &= \int_\Gamma C_{ijml}u^m_{,l}\nu^j q^k u^i_{,k}\,dz - \int_\Omega C_{ijml}u^m_{,l}\partial_j(q^k u^i_{,k})\,dx\\ &+ \frac{1}{2}\int_\Gamma q^k\nu^k|u_t|^2 dz - \frac{1}{2}\int_\Omega q^k_{,k}|u_t|^2\,dx + \int_\Omega f q^k u_{,k}\,dx.\end{aligned}$$

Using the symmetry of the moduli C_{ijml}, we conclude

$$\begin{aligned}&\frac{d}{dt}\int u_t q^k u_{,k}\,dx = \int_\Gamma \frac{\partial u}{\partial \nu_A} q^k u_{,k}\,dz - \int_\Omega C_{ijml}u^m_{,l} q^k_{,j} u^i_{,k}\,dx\\ &- \tfrac{1}{2}\int_\Omega q^k (C_{ijml}u^m_{,l}u^i_{,j})_{,k} dx + \tfrac{1}{2}\int_\Gamma q^k\nu^k|u_t|^2 dz - \tfrac{1}{2}\int_\Omega q^k_{,k}|u_t|^2 dx + \int_\Omega f q^k u_{,k} dx.\end{aligned}$$

Proof is complete. □

From Lemma 3.4 we get using $q^k = x^k - x^k_0$

$$\begin{aligned}&\frac{d}{dt}\int_\Omega u_t q^k u_{,k} dx = \int_\Gamma \frac{\partial u}{\partial \nu_A} q^k u_{,k} dz - \int_\Omega C_{ijml}u^m_{,l}u^i_{,j} dx - \tfrac{1}{2}\int_\Gamma q^k\nu^k C_{ijml}u^m_{,l}u^i_{,j} dz\\ &+ \tfrac{1}{2}\int_\Gamma q^k\nu^k|u_t|^2\,dz + \tfrac{3}{2}\int_\Omega C_{ijml}u^m_{,l}u^i_{,j}\,dx - \tfrac{3}{2}\int_\Omega |u_t|^2\,dx + \int_\Omega f q^k u_{,k}\,dx,\end{aligned}$$

which implies

$$\begin{aligned}\frac{d}{dt}\int_\Omega u_t q^k u_{,k}\,dx &= \int_\Gamma \frac{\partial u}{\partial \nu_A} q^k u_{,k}\,dz - \frac{1}{2}\int_\Omega (|u_t|^2 + C_{ijml}u^m_{,l}u^i_{,k})_k\\ &+ \int_\Omega (C_{ijml}u^m_{,l}u^i_{,k} - |u_t|^2)\,dx - \frac{1}{2}\int_\Gamma q^k\nu^k C_{ijml}u^m_{,l}u^i_{,k} dz\\ &+ \frac{1}{2}\int_\Gamma q^k\nu^k|u_t|^2\,dz + \int_\Omega f q^k u_{,k}\,dx. \qquad (3.5)\end{aligned}$$

Using the differential equation (1.1) we obtain

$$\begin{aligned}\frac{d}{dt}\int_\Omega u u_t\,dx &= \int_\Omega u u_{tt}\,dx + \int_\Omega |u_t|^2\,dx\\ &= \int_\Omega u^i\left\{(C_{ijml}u^m_{,l})_{,j} + f^i\right\}\,dx + \int_\Omega |u_t|^2\,dx\\ &= \int_\Gamma u\frac{\partial u}{\partial \nu_A}\,dz - \int_\Omega (C_{ijml}u^m_{,l}u^i_{,j} - |u_t|^2)\,dx + \int_\Omega u f\,dx,\end{aligned}$$

which implies

$$\int_\Omega (C_{ijml} u^m_{,l} u^i_{,j} - |u_t|^2)\, dx = -\frac{d}{dt}\int_\Omega u u_t\, dx + \int_\Gamma u \frac{\partial u}{\partial \nu_A}\, dz + \int_\Omega u f\, dx.$$

Substituting this identity into (3.5) yields

$$\begin{aligned}\frac{d}{dt}\underbrace{\left\{\int_\Omega u_t q^k u_{,k}\, dx + \int_\Omega u u_t dx\right\}}_{=:\chi(t)} &= \int_\Gamma \frac{\partial u}{\partial \nu_A} q^k u_{,k}\, dx \\ &- \frac{1}{2}\int_\Omega (|u_t|^2 + C_{ijml} u^m_{,l} u^i_{,j})\, dx + \int_\Gamma u \frac{\partial u}{\partial \nu_A}\, dz + \int_\Omega u f\, dx \\ &- \frac{1}{2}\int_\Gamma q^k \nu^k C_{ijml} u^m_{,l} u^i_{,j}\, dz + \frac{1}{2}\int_\Gamma q^k \nu^k |u_t|^2\, dz + \int_\Omega f q^k u_{,k}\, dx.\end{aligned}$$

Since $u = 0$ on Γ_0, we have

$$\int_{\Gamma_0} \frac{\partial u}{\partial \nu} q^k u_{,k}\, dz = \int_{\Gamma_0} q^k \nu^k \left|\frac{\partial u}{\partial \nu}\right|^2 dz, \tag{3.6}$$

$$\int_{\Gamma_0} \operatorname{div} u\, \nu q^k u_{,k}\, dz = \int_{\Gamma_0} q^k \nu^k |\operatorname{div} u|^2\, dz, \tag{3.7}$$

$$\int_{\Gamma_0} (\nabla u)\nu q^k u_{,k}\, dz = \int_{\Gamma_0} q^k \nu^k |\operatorname{div} u|^2. \tag{3.8}$$

Observing that (1.5) yields

$$\frac{\partial u}{\partial \nu_A} = \lambda \operatorname{div} u\, \nu + \mu \frac{\partial u}{\partial \nu} + \mu(\nabla u)\nu,$$

we conclude from (3.6)–(3.8)

$$\int_\Gamma \frac{\partial u}{\partial \nu_A} q^k u_{,k} dz = \int_{\Gamma_0} q^k \nu^k \left\{\mu \left|\frac{\partial u}{\partial \nu}\right|^2 + (\mu+\lambda)|\operatorname{div} u|^2\right\} dz + \int_{\Gamma_1} \frac{\partial u}{\partial \nu_A} q^k u_{,k}\, dz,$$

which implies

$$\begin{aligned}\frac{d}{dt}\chi(t) &= \int_{\Gamma_0} q^k \nu^k \left\{\mu \left|\frac{\partial u}{\partial \nu}\right|^2 + (\mu+\lambda)|\operatorname{div} u|^2\right\} dz + \int_{\Gamma_1} \frac{\partial u}{\partial \nu_A} q^k u_{,k}\, dz \\ &- \frac{1}{2}\int_\Omega \{|u_t|^2 + C_{ijml} u^m_{,l} u^i_{,j}\}\, dx - \frac{1}{2}\int_\Gamma q^k \nu^k C_{ijml} u^m_{,l} u^i_{,j}\, dz \\ &+ \int_\Gamma u \frac{\partial u}{\partial \nu_A}\, dz + \int_\Omega u f\, dx + \frac{1}{2}\int_{\Gamma_1} q^k \nu^k |u_t|^2\, dz + \int_\Omega f q^k u_{,k}\, dx.\end{aligned} \tag{3.9}$$

Since on Γ_1, we have $q^k\nu^k \geq a > 0$, we get

$$\int_{\Gamma_1} \frac{\partial u}{\partial \nu_A} q^k u_{,k}\, dz \leq c \int_{\Gamma_1} \Big|\frac{\partial u}{\partial \nu_A}\Big|^2 dz + \frac{1}{8}\int_{\Gamma_1} q^k\nu^k C_{ijml} u^m_{,l} u^i_{,j}\, dz, \tag{3.10}$$

$$\int_{\Gamma_1} u \frac{\partial u}{\partial \nu_A}\, dz \leq c \int_{\Gamma_1} \Big|\frac{\partial u}{\partial \nu_A}\Big|^2 dz + \frac{1}{8}\int_{\Omega} C_{ijml} u^m_{,l} u^i_{,j}\, dx, \tag{3.11}$$

where $c > 0$ denotes various positive constants. Observing that on Γ_0

$$C_{ijml} u^m_{,l} u^i_{,j} = \mu \Big|\frac{\partial u}{\partial \nu}\Big|^2 + (\lambda + \mu)(\operatorname{div} \mathrm{u})^2$$

and using (3.9)–(3.11) we obtain

$$\begin{aligned}\frac{d}{dt}\chi(t) &\leq -\frac{1}{4}\int_\Omega (|u_t|^2 + C_{ijml} u^m_{,l} u^i_{,j})\, dx + c\int_{\Gamma_1} \Big|\frac{\partial u}{\partial \nu_A}\Big|^2 dz \\ &+ \frac{1}{2}\int_{\Gamma_1} q^k\nu^k |u_t|^2\, dz - \frac{1}{4}\int_{\Gamma_1} q^k \nu^k C_{ijml} u^m_{,l} u^i_{,j}\, dz + \int_\Omega uf\, dx + \int_\Omega f q^k u_{,k}\, dx.\end{aligned}$$

On Γ_1 we have using (3.2)

$$\begin{aligned}\frac{\partial u}{\partial \nu_A} &= \partial_\nu u + \alpha H h^3 \nu + \alpha H h \nu^3 = -\tau u_t - \tau g u \\ &- \tau \int_0^t g'(t-s)(u(s,\cdot) - u(t,\cdot))\, ds + \tau g u_0 + \alpha H h^3 \nu - \alpha H h \nu^3.\end{aligned}$$

Together with

$$\int_0^t g'(t-s)(u(s,\cdot) - u(0,\cdot))\, ds \leq \Big(\int_0^t |g'(s)|\, ds\Big)^{1/2} (|g'|\Box u)^{1/2}$$

we arrive at

$$\begin{aligned}&\int_\Omega |f|^2\, dx \leq c\int_\Omega |\nabla \times h|^2\, dx + c\int_\Omega |\nabla\theta|^2\, dx, \\ &\int_{\Gamma_1} \Big|\frac{\partial u}{\partial \nu_A}\Big|^2 dz \leq c\int_{\Gamma_1} (|u_t|^2 + g^2|u|^2 + |g'|\Box u) dx + cg^2 F(0) + c\int_\Omega |\nabla \times h|^2 dx.\end{aligned}$$

Using

$$\int_\Omega |h|^2\, dx + \int_\Omega |\nabla h|^2\, dx \leq c\int_\Omega |\nabla \times h|^2\, dx$$

in our situation because of the properties of h and the fact that Ω is simply connected (see page 358 in [5] or page 157 in [10]), we thus have proved

Lemma 3.5.

$$\frac{d}{dt}\chi(t) \leq -\frac{1}{2}\int_\Omega (|u_t|^2 + C_{ijml}u^m_{,l}u^i_{,j})\,dx - \frac{1}{4}\int_{\Gamma_1} C_{ijml}u^m_{,l}u^i_{,j}\,dz$$
$$+ c\int_\Omega (|\nabla\times h|^2 + |\nabla\theta|^2)\,dx + c\int_{\Gamma_1}|u_t|^2\,dz + c\int_{\Gamma_1}|g'|\Box u\,dz + cg^2F(0).$$

Now we can formulate and prove the main results of this section. First we assume the case where the kernel decays exponentially.

Theorem 3.6. *Let g be an exponentially decaying resolvent kernel as in* (1.15), *and assume* (3.1). *Then the energy F defined in* (3.3), *which is associated with the solution of the initial-boundary value problem* (1.1)–(1.3), (1.7), (1.10), (1.11) *decays exponentially; i.e.,*

$$\exists d_0, d_1 > 0 \quad \forall t \geq 0: \quad F(t) \leq d_0 e^{-d_1 t}F(0).$$

Proof. Let

$$L(t) := NF(t) + \chi(t), \tag{3.12}$$

$N > 0$ sufficiently large. Then there are positive constants k_0, k_1 such that for all $t \geq 0$

$$k_0F(t) \leq L(t) \leq k_1F(t). \tag{3.13}$$

Moreover, for N large enough, using the Lemmas 3.3 and 3.5,

$$\frac{d}{dt}L(t) \leq -k_2F(t) + cg^2(t)F(0)$$

with a constant $k_2 > 0$. Here we used the assumption (1.15) in order to conclude the following estimates:

$$-\frac{\tau}{2}\int_{\Gamma_1} g''\Box u\,dz \leq \tilde{c}\int_{\Gamma_1} g'\Box u\,dz$$

and

$$\frac{\tau}{2}\int_{\Gamma_1} g'|u|^2dz \leq -\tilde{c}\int_{\Gamma_1} g|u|^2dz$$

for the corresponding two terms appearing in Lemma 3.3, where $\tilde{c}$ is a positive constant.

Thus we obtain

$$\frac{d}{dt}L(t) \leq -\frac{k_2}{k_1}L(t) + cg^2(t)F(0).$$

Using the exponential decay of g we conclude

$$\exists \tilde{d}_0, \tilde{d}_1 > 0 \quad \forall t \geq 0: \quad L(t) \leq \tilde{d}_0 e^{-\tilde{d}_1 t} L(0),$$

which implies the assertion by using (3.13) again. □

Finally, we consider the case where g decays polynomially as in (1.16).

Theorem 3.7. *Let g be a polynomially decaying resolvent kernel as in* (1.16), *and assume* (3.1). *Then the energy F defined in* (3.3), *which is associated to the initial boundary value problem* (1.1)–(1.3), (1.7), (1.10), (1.11), *decays polynomially, i.e.,*

$$\exists\, d_2 > 0 \ \forall\, t \geq 0: \quad F(t) \leq \frac{d_2}{(1+t)^p}F(0).$$

Proof. We define the functional L as in (3.12), and we have the equivalence to the energy term F as given in (3.13) again. A negative term

$$-cg(t)\int_{\Gamma_1} |u|^2(t,z)\,dz$$

can be obtained from Lemma 3.5 and the estimate

$$g(t)\int_{\Gamma_1} |u|^2(t,z)\,dz \leq c\int_\Omega C_{ijml}u^m_{,l}u^i_{,j}(t,x)\,dx.$$

From Lemma 3.3 and Lemma 3.5, using the properties of g'' from the assumption (1.16) for the term

$$-\frac{\tau}{2}\int_{\Gamma_1} g''\Box u\,dz,$$

we obtain

$$\begin{aligned}\frac{d}{dt}L(t) \leq &-k_3\Big(M(t) + g(t)\int_{\Gamma_1}|u|^2\,dz + N\int_{\Gamma_1}|g'|^{1+\frac{1}{p+1}}\Box u\,dz\Big)\\ &+ k_4\int_{\Gamma_1}|g'|\Box u\,dz + cg^2(t)F(0),\end{aligned} \tag{3.14}$$

where k_3, k_4 denote positive constants and $M = M(t)$ was defined in (3.4).

In the sequel we need several inequalities collected in the next three lemmas, which are based on those from [11], partially extending those.

Lemma 3.8. *Let m and h be integrable functions, and let $0 \le r < 1$ and $q > 0$. Then, for $t \ge 0$,*

$$\int_0^t |m(t-\tau)\, h(\tau)|\, d\tau \le \Big(\int_0^t |m(t-\tau)|^{1+\frac{1-r}{q}} |h(\tau)|\, d\tau \Big)^{\frac{q}{q+1}} \Big(\int_0^t |m(t-\tau)|^r |h(\tau)|\, d\tau \Big)^{\frac{1}{q+1}}.$$

Proof. Define

$$v(\tau) := |m(t-\tau)|^{1-\frac{r}{q+1}} |h(\tau)|^{\frac{q}{q+1}}, \qquad w(\tau) := |m(t-\tau)|^{\frac{r}{q+1}} |h(\tau)|^{\frac{1}{q+1}}.$$

Then, for fixed $t \ge 0$, $|m(t-\tau)h(\tau)| = |v(\tau)\, h(\tau)|$. An application of Hölder's inequality with exponents $\delta = \frac{q}{q+1}$ for v, $\delta^* = q+1$ for w, gives the assertion of Lemma 3.8. □

Lemma 3.9. *Let $p > 1$, $0 \le r < 1$ and $t \ge 0$. Then we have for $r > 0$*

$$\int_{\Gamma_1} |g'| \square u\, dz \le 2 \Big(\int_0^t |g'(\tau)|^r d\tau \|u\|^2_{L^\infty((0,t),L^2(\Gamma_1))} \Big)^{\frac{1}{1+(1-r)(p+1)}} \times \Big(\int_{\Gamma_1} |g'|^{1+\frac{1}{p+1}} \square u\, dz \Big)^{\frac{(1-r)(p+1)}{1+(1-r)(p+1)}},$$

and for $r = 0$

$$\int_{\Gamma_1} |g'| \square u\, dz \le 2 \Big(\int_0^t \|u(\tau,\cdot)\|^2_{L^2(\Gamma_1)}\, d\tau + t \|u(t,\cdot)\|^2_{L^2(\Gamma_1)} \Big)^{\frac{1}{p+2}} \times \Big(\int_{\Gamma_1} |g'|^{1+\frac{1}{p+1}} \square u\, dz \Big)^{\frac{p+1}{p+2}}.$$

Proof. Apply Lemma 3.8 with $m(\tau) := |g'(\tau)|$, $h(\tau) := \int_{\Gamma_1} |u(t) - u(\tau)|^2\, dz$ and $q := (1-r)(p+1)$ (t fixed). This proves Lemma 3.9. □

Lemma 3.10. *Let $f \ge 0$ be differentiable, let $\alpha > 0$ and let f satisfy*

$$f'(t) \le \frac{-\bar{c}_1}{f(0)^{1/\alpha}} f(t)^{1+\frac{1}{\alpha}} + \frac{\bar{c}_2}{(1+t)^\beta} f(0)$$

for $t \ge 0$, positive constants $\bar{c}_1, \bar{c}_2$ and $\beta \ge \alpha+1$. Then there exists a constant $\bar{c}_3 > 0$ such that for $t \ge 0$

$$f(t) \le \frac{\bar{c}_3}{(1+t)^\alpha} f(0).$$

Proof. Let $t \geq 0$ and $F(t) := f(t) + \frac{2\bar{c}_2}{\alpha}(1+t)^{-\alpha} f(0)$. Then

$$F' = f' - 2\bar{c}_2(1+t)^{-(\alpha+1)} f(0) \leq \frac{-\bar{c}_1}{f(0)^{1/\alpha}} f^{1+\frac{1}{\alpha}} - \bar{c}_2(1+t)^{-(\alpha+1)} f(0),$$

where we used $\beta \geq \alpha + 1$. Hence

$$F' \leq \frac{-c}{f(0)^{1/\alpha}} \left(f^{1+\frac{1}{\alpha}} + (1+t)^{-(\alpha+1)} f(0)^{1+\frac{1}{\alpha}} \right) \leq \frac{-c}{F(0)^{1/\alpha}} F^{1+\frac{1}{\alpha}}.$$

Integration yields $F(t) \leq \frac{F(0)}{(1+ct)^{\alpha}} \leq \frac{c}{(1+t)^{\alpha}} f(0)$, whence $f(t) \leq \frac{\bar{c}_3}{(1+t)^{\alpha}} f(0)$ follows for some $\bar{c}_3 > 0$, which proves Lemma 3.10. □

Applying Lemma 3.9 with $r > 0$ we get

$$\int_{\Gamma_1} |g'|^{1+\frac{1}{p+1}} \Box u \, dz \geq \frac{c}{\left(\int_0^t |g'|^r(\tau)\, d\tau \right)^{1+\frac{1}{(1-r)(p+1)}} F(0)^{\frac{1}{(1-r)(p+1)}}} \left(\int_{\Gamma_1} |g'| \Box u \, dz \right)^{1+\frac{1}{(1-r)(p+1)}}$$

(with $c = c(r)$ as long as r is not yet fixed). On the other hand we have

$$\left(cg \int_{\Gamma_1} |u|^2 \, dz + M \right)^{1+\frac{1}{(1-r)(p+1)}} \leq cF(0)^{\frac{1}{(1-r)(p+1)}} \left(M + cg \int_{\Gamma_1} |u|^2 \, dz \right).$$

We conclude from (3.14) using the last two inequalities:

$$\frac{d}{dt} L(t) \leq \frac{-c}{F(0)^{\frac{1}{(1-r)(p+1)}}} \left[\left(M + cg \int_{\Gamma_1} |u|^2 \, dz \right)^{1+\frac{1}{(1-r)(p+1)}} + \left(\int_{\Gamma_1} |g'| \Box u \, dz \right)^{1+\frac{1}{(1-r)(p+1)}} \right] + cg^2 F(0)$$

if $r > \frac{1}{p+1}$ (such that $\int_0^\infty |g'(\tau)|^r d\tau < \infty$). This implies, using (3.13),

$$\frac{d}{dt} L(t) \leq \frac{-\tilde{c}}{L(0)^{\frac{1}{(1-r)(p+1)}}} L(t)^{1+\frac{1}{(1-r)(p+1)}} + cg^2(t) L(0) \tag{3.15}$$

with some $\tilde{c} > 0$. Lemma 3.10 with $f = L$, $\alpha = (1-r)(p+1)$ and $\beta = p^2$ gives

$$L(t) \leq \frac{c}{(1+t)^{(1-r)(p+1)}} L(0). \tag{3.16}$$

Choosing $1 \geq r > \frac{1}{p+1}$ such that $(1-r)(p+1) > 1$ or, equivalently, $\frac{1}{p+1} < r < \frac{p}{p+1}$, we obtain from the inequality (3.16)

$$\int_0^\infty F(\tau)\,d\tau \leq c\int_0^\infty L(\tau)\,d\tau \leq cL(0) \tag{3.17}$$

and

$$t\|u(t,\cdot)\|^2_{L^2(\Gamma_1)} \leq ctL(t) \leq cL(0) \tag{3.18}$$

as well as

$$\int_0^t \|u(\tau,\cdot)\|^2_{L^2(\Gamma_1)}\,d\tau \leq c\int_0^\infty L(\tau)\,d\tau \leq cL(0). \tag{3.19}$$

With the estimates (3.17)–(3.19) we conclude, using Lemma 3.9 again, now with $r = 0$,

$$\int_{\Gamma_1} |g'|^{1+\frac{1}{p+1}} \Box u\,dz \geq \frac{c}{F(0)^{\frac{1}{p+1}}}\left(\int_{\Gamma_1} |g'|\Box u\,dz\right)^{1+\frac{1}{p+1}},$$

and hence, with the same arguments as in the derivation of (3.15),

$$\frac{d}{dt}L(t) \leq \frac{-\bar{c}}{L(0)^{\frac{1}{p+1}}}L(t)^{1+\frac{1}{p+1}} + cg^2(t)L(0).$$

This implies by applying Lemma 3.10 again

$$L(t) \leq \frac{c}{(1+t)^p}L(0),$$

and hence, by (3.13) and for some $d_2 > 0$,

$$F(t) \leq \frac{c}{(1+t)^p}F(0),$$

which completes the proof of Theorem 3.7. $\Box$

Remark. For a thermoviscoelastic system discussed in [11] it was shown that a polynomial relaxation function cannot lead to an exponential decay. This gives a hint for the conjecture that the polynomial decay rate obtained here can not be replaced by an exponential decay result. Indeed, for the system of pure elasticity, with the memory-type boundary as discussed here, the exponential decay for *exponential* kernels was shown in [1]. A merely *polynomial* kernel there can *not* lead to a general exponential decay result, which can be seen as follows.

In one space dimension the system of equations of elasticity in $\Omega := (0,1)$ with memory-type boundary condition reduces to

$$u_{tt} - \alpha u_{xx} = 0, \tag{3.20}$$

$$u(t=0) = u_0, \qquad u_t(t=0) = u_1, \tag{3.21}$$

$$u(x=0) = 0, \qquad (u + (u_x * r))(x=1) = 0, \tag{3.22}$$

where α is a positive constant. The energy is given by

$$\mathcal{E}(t) := \int_0^1 \left(|u_t|^2 + \alpha |u_x|^2\right)(t,x)\,dx.$$

We will argue by contradiction. Assume

$$\exists c > 0 \;\; \exists \delta > 0 \;\; \forall t \geq 0: \quad \mathcal{E}(t) \leq c e^{-\delta t} \mathcal{E}(0), \tag{3.23}$$

and let us take

$$u_0 = 0, \qquad u_1 \in C_0^\infty(\Omega) \setminus \{0\} \tag{3.24}$$

and for the kernel b with $b(t) = -r'(t)/r(0)$: $b(t) = \frac{1}{(1+t)^p}$ for some $p > 1$.

With the conditions above, $v := u_t$ satisfies the same system (3.20), (3.22) as u, due to the choice of the initial data in (3.24). Hence also the energy associated with v decays exponentially, which implies, using the differential equation and Sobolev's imbedding theorem, that there is a constant c_0 depending on the initial data such that for all $t \geq 0$

$$|u_t(t,1)| + |u_x(t,1)| \leq c_0 e^{-\delta t}. \tag{3.25}$$

The boundary condition (3.22) can also be stated as (cp. (1.13))

$$(-b * u_x + u_x + \tau u)(x=1) = 0,$$

which implies by (3.25)

$$\left| \int_0^t \frac{1}{(1+t-s)^p} u_x(s,1)\,ds \right| \leq c_0 e^{-\delta t}. \tag{3.26}$$

On the other hand, by dividing the integral from 0 to t into two parts from 0 to $t/2$ and from $t/2$ to t, it can be easily seen that for any $m > 1$

$$\Big|\int_0^t \frac{1}{(1+t-s)^m} u_x(s,1)\,ds\Big| \leq \frac{c_m}{(1+t)^m}. \tag{3.27}$$

For $t \geq 0$ and $\beta \geq 0$, let

$$G_\beta(t) := \int_{t+\beta}^\infty u_x(s,1)\,ds.$$

Then

$$\begin{aligned}
&\int_0^t \frac{1}{(1+t-s)^p} u_x(s,1)\,ds \\
&= \Big[\frac{1}{(1+s)^p} G_\beta(t-s)\Big]_{s=0}^{s=t} + p\int_0^t \frac{1}{(1+s)^{p+1}} G_\beta(t-s)\,ds \\
&= \frac{G_\beta(0)}{(1+t)^p} - G_\beta(t) + \mathcal{O}\Big(\frac{1}{(1+t)^{p+1}}\Big),
\end{aligned} \tag{3.28}$$

where we used (3.27) for $m = p+1$.

Case 1: $\exists\tilde{\beta} \in [0,\infty] : G_{\tilde{\beta}}(0) \neq 0$. Thus, from (3.28),

$$\lim_{t\to\infty} \Big|\int_0^t \frac{1}{(1+t-s)^p} u_x(s,1)\,ds\Big| (1+t)^p = G_{\tilde{\beta}}(0) \neq 0,$$

which is a contradiction to (3.26).

Case 2: $\forall\beta \in [0,\infty] : G_\beta(0) = 0$. This implies $\forall t \geq 0 : u_x(t,1) = 0$, and hence, using the boundary condition and the initial condition $u_0 = 0$, we see that u satisfies

$$\begin{aligned}
&u_{tt} - \alpha u_{xx} = 0, \\
&u(t=0) = 0, \quad u_t(t=0) = u_1, \quad u(x=0) = 0, \quad u(x=1) = 0,
\end{aligned}$$

which implies that the energy

$$\mathcal{E}(t) = \int_0^1 |u_1|^2(x)\,dx$$

is constant, being a contradiction to the assumption (3.23).

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