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Magnitude Knowledge:

The Common Core of Numerical Development

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Highlights

1. The integrated theory of numerical development posits that the unifying theme of numerical development is children's growing understanding of numerical magnitudes.
2. The theory focuses on four aspects of numerical development: representing increasingly precisely the magnitudes of non-symbolic numbers, connecting small symbolic numbers to their non-symbolic referents, extending understanding from smaller to larger whole numbers, and accurately representing the magnitudes of rational numbers.
3. For both whole and rational numbers, knowledge of numerical magnitudes is correlated with, predictive of, and causally related to other crucial aspects of mathematics, including arithmetic and overall math achievement.
4. Interventions that emphasize acquisition of numerical magnitude knowledge have yielded a broad range of positive effects with both whole and rational numbers.

Abstract

The integrated theory of numerical development posits that a central theme of numerical development from infancy to adulthood is progressive broadening of the types and ranges of numbers whose magnitudes are accurately represented. The process includes four overlapping trends: 1) representing increasingly precisely the magnitudes of non-symbolic numbers, 2) connecting small symbolic numbers to their non-symbolic referents, 3) extending understanding from smaller to larger whole numbers, and 4) accurately representing the magnitudes of rational numbers. The present review identifies substantial commonalities, as well as differences, in these four aspects of numerical development. With both whole and rational numbers, numerical magnitude knowledge is concurrently correlated with, longitudinally predictive of, and causally related to multiple aspects of mathematical understanding, including arithmetic and overall math achievement. Moreover, interventions focused on increasing numerical magnitude knowledge often generalize to other aspects of mathematics. The cognitive processes of association and analogy seem to play especially large roles in this development. Thus, acquisition of numerical magnitude knowledge can be seen as the common core of numerical development.

Magnitude Knowledge:

The Common Core of Numerical Development

(Introduction)

Understanding of numerical development is expanding rapidly. Thriving literatures have arisen on the growth of numerical understanding in infancy, childhood, and adolescence; on development of subitizing, counting, estimation, and arithmetic; on knowledge of whole numbers, fractions, and decimals; on neural representations; on relations between conceptual and procedural knowledge; on the roles of working memory, executive functioning, anxiety, and other mental processes; on longitudinal stability of individual differences; on variations in knowledge among children whose families differ in wealth, language, culture, and the numerical input they provide; on relations to other quantitative concepts, especially space and time; on effects of instructional interventions; and so on.

Reflecting the popularity of this research area, a web search of the program of the most recent meeting of the Society for Research in Child Development (2015) indicated that “math” was a keyword or appeared in the abstract of 216 presentations. This number of appearances exceeded that for many other popular areas including “perception” (117), “attention” (166), “memory” (141), “reasoning” (67), “space” or “spatial” (87), “moral” (110), “reading” (91), and “executive function” (180). (We used “math” rather than “number” as a search term to avoid counting the many abstracts in which the word “number” does not indicate a focus on mathematics.)

One source of the prominence of this research area is the inherent importance of numerical knowledge. Kant (1781) proposed that number is an *a priori* concept, that is, an idea that must be present from birth for people and other animals to function. How could people or other animals survive if they did not at least roughly represent the number of predators or prey in their environment? Consistent with this idea, rats, pigeons, monkeys, fish, lions, and other animals have been found to represent the approximate numbers of objects and events that they encounter (Dehaene, 2011; McComb, 1994; Piffer, Petrazzini, & Agrillo, 2013).

Piaget (e.g., 1952) followed Kant in proposing that a rudimentary form of numerical understanding is present from birth. However, Piaget went beyond Kant in demonstrating that this early understanding is limited and grows considerably during infancy and childhood. Major contemporary theories, including neo-Piagetian (Case & Okamoto, 1996), information processing (e.g., Klahr & MacWhinney, 1998), sociocultural (Gauvain, 2001), core knowledge (Feigenson, Dehaene, & Spelke, 2004), evolutionary (Geary, 2008), and dynamic systems (Cantrell & Smith, 2013) also emphasize numerical understanding as a crucial part of cognitive development.

Practical considerations also contribute to the popularity of numerical development research. Numerical knowledge is a major determinant of academic, occupational, and financial success in contemporary society. To cite just one illustration, numerical knowledge at age 7 predicts SES at age 42, even after controlling for the SES of the family into which the person was born, as well as the person's own IQ and reading achievement (Ritchie & Bates, 2013). Desires to bridge the large gaps separating the numerical knowledge of children from East Asian and Western societies, and from low and high-income families within each society, create additional motivation for research in this area.

Yet another reason for the proliferation of research on numerical development is the many fascinating empirical phenomena that have been discovered in recent years. Consider three examples: 1) Newborns who are habituated to a given number of tones subsequently look longer at a matching number of objects than at a different number of objects (Izard, et al., 2009); 2) 4-year-olds' numerical knowledge predicts their mathematical knowledge at age 15, even after statistically controlling for their IQ, SES, reading comprehension, working memory, and a host of other variables (Watts, Duncan, Siegler, & Davis-Kean, 2014); 3) individual neurons in monkeys' parietal lobe respond preferentially to specific numerosities for sets as large as 32 objects (Nieder & Merten, 2007). Researchers are as subject to reinforcement effects as anyone else; numerical development has been, and continues to be, a very rewarding research area.

The diversity of issues and discoveries regarding numerical development is exciting, but it also has created a large problem: how to integrate the varied phenomena into a unified framework. As Arnold Toynbee (1954), p. 195) asked about history: Are there unifying themes, or is it just one damned thing after another?

The present article proposes that numerical development does have a unifying theme: generation of increasingly precise magnitude representations for an increasingly broad range of numbers. This developmental process produces four types of changes: (1) generating increasingly precise representations of numbers expressed non-symbolically; (2) connecting symbolic to non-symbolic representations of the magnitudes of small whole numbers; (3) extending the range of whole numbers whose magnitudes can be accurately represented; (4) progressing beyond whole numbers to accurately represent the magnitudes of an increasing range of rational numbers. These trends are overlapping rather than successive; they start at different times, but their subsequent development overlaps considerably.

The Integrated Theory of Numerical Development

Based on the view that the growth of numerical magnitude knowledge is the core of numerical development, Siegler, Thompson, and Schneider (2011) formulated the integrated theory of numerical development. The current theory subsumes earlier versions but includes additional hypotheses as well. It can be summarized as follows:

1) The magnitudes of all rational numbers are represented on a mental number line, a dynamic structure that begins with small whole numbers and over the course of development expands rightward to include larger whole numbers, leftward to include negative numbers, and interstitially to include fractions and decimals.

2) Within specific ranges of whole numbers (e.g., 0-10, 0-100, 0-1000), magnitude representations progress from a compressive, approximately logarithmic distribution to an approximately linear one. Transitions occur earlier for smaller than for larger numerical ranges, corresponding to when children gain experience with the numbers in the range.

3) Development of rational numbers involves learning that many properties of whole numbers do not characterize other types of numbers but that all real numbers have magnitudes and can be represented on number lines.

4) Numerous processes influence development of numerical magnitude knowledge, but two that play especially large roles are association and analogy.

5) Because magnitude knowledge is central to numerical development, as posited by the theory, knowledge of the magnitudes of both whole and rational numbers should be both correlated with and causally related to other aspects of mathematics, including arithmetic and mathematics achievement test scores.

6) Because magnitude knowledge is central to numerical development, interventions designed to improve knowledge of both whole and rational number magnitudes should have substantial positive effects on a wide range of mathematical outcomes.

The remainder of this section elaborates these hypotheses and provides evidence relevant to them.

The integrated theory begins with the proposal that numerical magnitudes are represented along a mental number line. This widely accepted hypothesis has roots in psychometrics (Galton, 1880), animal behavior (Mechner, 1958; Platt & Johnson, 1971), and adult cognition research (Moyer & Landauer, 1967; Restle, 1970). In Western and Far Eastern cultures, the mental number line is usually horizontally oriented, with smaller numbers on the left and larger ones on the right (Dehaene, 2011). This spatial organization of numerical magnitudes is not unique to humans. After being familiarized with a target number of dots (e.g., 5), newborn chicks spontaneously associate smaller sets of dots (e.g., 2) with the left side of space and larger sets of dots (e.g., 8) with the right side (Rugani, Vallortigara, Priftis, & Regolin, 2015). With chicks as with people, numerical-spatial mappings reflect the numerical context: After familiarization to 20 rather than 5 dots, the chicks associated 8 dots with the left side of the space. People can represent numbers vertically (Simms, Muldoon, & Towse, 2013) or circularly (e.g., Bächtold, Baumüller, & Brugger, 1998) if asked to do so, but the left-to-right horizontal orientation is the default option, at least in Western and East Asian cultures.

The hypothesis that numerical magnitudes are represented along a mental number line is supported by many types of evidence. One is the *distance effect*: the farther two numbers are from each other, the faster and more accurately the numbers' sizes can be compared (Moyer & Landauer, 1967). Another type of evidence is the *SNARC* (Spatial-Numerical Association of Response Codes) *effect*. When asked to indicate whether a number is larger or smaller than another number, people are faster when asked to indicate "smaller" by pressing a button on their left and "larger" by pressing a button on their right than when the pairing is reversed (e.g., Dehaene, et al., 1990; Wood, Willmes, Nuerk, & Fischer, 2008). The SNARC effect emerges even when numerical magnitude is irrelevant to the task; odd/even judgments are made more quickly for numbers that are small within the context of the task when the relevant button is on the left than when it is on the right, and vice versa for large numbers (Hubbard, Piazza, Pinel, & Dehaene, 2005). SNARC effects have been found to emerge between 5 and 9 years of age, with the exact age of emergence varying with whether magnitudes are relevant to the task (effects are found at earlier ages when magnitudes are relevant) and other aspects of the experimental procedure (Berch, Foley, Hill, & McDonough-Ryan, 1999; Hoffmann, Hornung, Martin, & Schiltz, 2013; van Galen & Reitsma, 2008; White, Szücs, & Soltész, 2011).

Viewing the mental number line as a dynamic structure that can represent all numerical magnitudes allows the integrated theory to extend from infants' representations of non-symbolic numerical magnitudes to young children's representations of symbolic whole number magnitudes to older children's, adolescents' and adults' representations of symbolic fraction, decimal, and negative magnitudes. Prominent alternative theories of numerical development, such as those of Geary (2006), Leslie, Gelman, and Gallistel (2008), and Wynn (1995), have focused on growth of whole number knowledge from infancy to middle childhood.

The integrated theory incorporates a much greater range of acquisitions over a much longer period of development.

Another basic claim of the integrated theory is that a central challenge of numerical development is learning which properties of whole numbers apply to all numbers and which do not. Each whole number is expressed by a unique symbol, but each fraction can be expressed in infinite ways; each whole numbers has a unique predecessor and successor, but no fraction or decimal does; multiplying natural numbers never yields a product less than either multiplicand, but multiplying decimals or fractions from 0-1 always does; adding whole numbers never yields an answer smaller than either addend, but adding negatives always does; etc. On the other hand, whole numbers, fractions, decimals, and negatives share the property of representing magnitudes that can be located on number lines.

The current version of the integrated theory proposes that the mechanisms of association and analogy play particularly large roles in numerical development. Associating numerical symbols with discrete non-symbolic stimuli, such as sets of discrete objects, sounds, and pointing gestures, and with continuous dimensions such as distance and time, seems particularly important for learning the magnitudes of small symbolic whole numbers. For example, children may gain a sense of the magnitudes of the numbers 1-10 through counting sets of objects, because the larger the number, the more time it takes to reach it, the greater the amount of space the objects occupy, the greater the number of counting words stated, and the greater the number of pointing gestures.

Analogy, in particular structure mapping, is hypothesized to play a central role in extending knowledge about the magnitudes of relatively small whole numbers with which we have substantial experience (e.g., numbers in the 0-10 and 0-100 ranges) to large whole numbers and rational numbers with which we have minimal experience (e.g., whole numbers above 1000 and fractions whose denominators are neither single-digits nor multiples of two). The analogizing is piecemeal, however; children who draw analogies between the 0-100 and 0-1,000 range do not necessarily extend the analogy to larger numbers, and children who analogize from whole numbers to fractions from 0-1 do not necessarily do so for larger fractions. Specifying the processes through which children draw these analogies is beyond the scope of this article; at a general level, Holyoak's (2012) and Gentner and Markman's (1997) theories of analogical reasoning resemble the current perspective. The focus here will be on presenting evidence that analogy is crucial to extending numerical knowledge from smaller to larger numbers, rather than on specifying the analogical reasoning process.

The remainder of this article summarizes what is known about numerical magnitude development and how the integrated theory lends coherence to its many particulars. The article focuses on four main developments: generating increasingly precise representations of non-symbolic magnitudes, connecting symbolic to non-symbolic magnitudes for small whole numbers, accurately representing increasingly large whole numbers, and extending knowledge of whole number to rational number magnitudes.

The Innate Kernel: Approximate Representations of Non-Symbolic Numerical Magnitudes

Piaget (e.g., 1952) hypothesized that infants' numerical understanding is limited to an undifferentiated concept of quantity that does not distinguish number from other quantitative dimensions, such as space and time. Subsequent research, however, indicated that this view was too conservative. For example, cross-modal matching data suggest that for small numbers, infants perceive similarities among stimuli that only share number in common, such as three objects and three tones (Jordan & Brannon, 2006; Kobayashi, Hiraki, & Hasegawa, 2005).

Non-symbolic number, like many quantitative dimensions, exhibits ratio dependence (Murray, 1993). That is, ability to discriminate between sets with differing numbers of objects depends on the ratio of the number of objects in the sets. With development, children become able to discriminate smaller ratios. Newborns in Izard, et al. (2009) who discriminated 3:1 numerical ratios did not show similar competence when the ratio was 2:1. By age six months, infants discriminate between 2:1 ratios -- for example, they discriminate sets of 32 versus 16 and 16 versus 8 objects – and by 9 months, they discriminate 3:2 ratios (Cordes & Brannon, 2008; Xu & Spelke, 2000). The increasing precision of discrimination continues well beyond infancy (Figure 1). By 3 years, children reliably discriminate 4:3 ratios; by 6 years, 6:5 ratios; and by adulthood, 8:7 ratios (Bonny & Lourenco, 2014; Halberda & Feigenson, 2008; Holloway & Ansari, 2008).

Although precision increases with age, there is considerable continuity in the brain areas that most actively process non-symbolic numerical magnitudes. Areas in the intraparietal sulcus (IPS) and dorsolateral prefrontal cortex (DPFC) play major roles in processing numerical stimuli from infancy to adulthood, as indicated by single-cell recording of neuronal activity in those areas (Nieder, 2011) and by the activation of those areas increasing as a function of the numerical distance between habituated and novel stimuli in habituation paradigms (Cantlon, Brannon, Carter, & Pelphrey, 2006; Jacob & Nieder, 2009; Piazza, 2011). Based on the faster activation of neurons in the IPS in response to numerical stimuli, it appears that real-time numerical processing starts in the IPS and rapidly moves to functionally connected areas of the DPFC, where the signal gains control over behavior (Nieder & Miller, 2004; Nieder, 2011).

At the cognitive level, two mechanisms have been proposed to account for this ratio dependence and for one notable exception to it. The Approximate Number System (ANS), proposed by Dehaene (2003), Piazza, et al. (2004), and Verguts and Fias (2004), posits logarithmically spaced, approximate, amodal, internal representations of numerical magnitudes with fixed amounts of noise around the mean value for each number. A second mechanism, the Object Tracking System (OTS), in which 1-4 objects can be tracked in parallel within a display, has been proposed to account for features of numerical processing with very small sets that differ from those with larger sets: superior speed and accuracy, lower variability, greater learning of non-numerical features of such sets, and lack of correlation between Weber Fractions for the very small sets and larger sets (Feigenson, Carey, & Spelke, 2002; Libertus & Brannon, 2010; Piazza, 2011; Revkin, et al., 2008). Interestingly, the same differentiation in mechanisms for numerically representing non-symbolic sets of 1-4 objects and larger sets seems to operate in guppies as in people (Agrillo, Pfiffer, Bisazza, & Butterworth, 2012).

The shared feature of ratio dependence, together with parallels across different modalities in the ages at which infants can discriminate a given ratio (Lipton & Spelke, 2003), have re-opened an issue that seemed for many years to be "settled science": whether infants possess an undifferentiated quantity concept that subsumes number, space, time, and other quantitative dimensions. Piaget's (1952) classic number conservation task was designed to illustrate that children did not cleanly separate number and space until middle childhood, and his time/speed/distance tasks (Piaget, 1971) were designed to make the same point for those quantitative dimensions.

Piaget's view that infants and young children cannot represent number separately from other quantitative concepts was clearly incorrect (Gelman & Gallistel, 1978). However, recent studies indicate that Piaget had identified an important phenomenon: Infants form general quantitative representations that include numerical, spatial, and temporal magnitudes.

Lourenco and Longo (2010; 2011) demonstrated that for length, duration, and number, if infants repeatedly saw the larger of two stimuli decorated in a particular way, they dishabituated to seeing that decoration associated with the stimulus that had the smaller value *on a different dimension*. Thus, if the longer of two bars was always striped during the habituation phase, infants subsequently dishabituated more when objects in the less numerous collection had stripes than when objects in the more numerous collection did. Such transfer among length, number, and duration was present for all six combinations of dimensions in the habituation and dishabituation phases. This general quantitative representation is not unique to infants; older children and adults also show transfer across quantitative dimensions (Dormal, Seron, & Pesenti, 2006; Henik & Tzelgov, 1982; Rousselle & Noël, 2008).

Consistent with the view that improvement in non-symbolic numerical representations reflects in large part development of a more general quantitative concept, the ratios that infants can discriminate at a given age are similar across quantitative dimensions. Thus, 6-month-olds can discriminate between 2:1 ratios for number, spatial extent, and duration, but they cannot discriminate between 3:2 ratios for any of them; 9-month-olds can discriminate between 3:2 ratios for all of them but not between 5:4 ratios for any of them, and so on (Brannon, Lutz, & Cordes, 2006; Lipton & Spelke, 2003; van Marle & Wynn, 2006; Xu & Spelke, 2000).

Also consistent with the presence of a general quantitative concept, neuroimaging data from people and electrophysiological data from other animals indicate that number, spatial extent, and temporal duration are represented in close proximity in a fronto-parietal network (Cohen Kadosh, Lammertyn, & Izard, 2008). Thus, from infancy onward, non-symbolic number seems to be represented not only as a specific dimension but also within a more general magnitude system (Newcombe, Levine, & Mix, 2015; Lourenco, 2015).

From Non-Symbolic to Symbolic Representations

Early developing non-symbolic numerical magnitude knowledge provides potentially useful referents for learning the magnitudes of numbers expressed symbolically, for example the spoken word "six" and the Arabic numeral "6." This led Dehaene (2008, p. 552) to hypothesize, "When we learn number symbols, we simply attach their arbitrary shapes to the relevant non-symbolic quantity." Dehaene's hypothesis suggested a plausible explanation of individual differences in understanding of symbolic numbers: Relatively precise non-symbolic numerical representations might lead to relatively precise symbolic numerical representations (Halberda, Mazzocco, & Feigenson, 2008).

These intuitively appealing hypotheses that non-symbolic numerical magnitude representations are foundational to symbolic representations and that individual differences in the one underlie individual differences in the other have generated a very large literature (see Feigenson, Libertus, & Halberda, 2013, and Lyons & Ansari, 2015, for reviews). This section first presents evidence supportive of the hypotheses, then evidence calling them into question, and then an alternative explanation.

First consider data supportive of the ideas that non-symbolic numerical magnitude representations play important roles in acquiring symbolic numerical magnitude representations and in explaining individual differences in them.

Distance and magnitude effects similar to those with non-symbolic magnitudes are present with symbolic numbers (e.g., Sekuler & Mierkiewicz, 1977). Individual differences in ability to discriminate non-symbolic magnitudes are predictive of concurrent and subsequent individual differences in discrimination between symbolic magnitudes, arithmetic learning, and math achievement test scores (e.g., Gilmore, McCarthy, & Spelke, 2010; Libertus, Feigenson, & Halberda, 2011). Brief training in estimating the sum of dot arrays can improve the speed of children's symbolic arithmetic (Hyde, Khanum, & Spelke, 2014) and the number of arithmetic problems that adults can solve in a fixed amount of time (Park & Brannon, 2013; 2014). Brain areas used to process non-symbolic and symbolic numbers overlap considerably (Nieder & Dehaene, 2009). Finally, habituation to non-symbolically presented numbers produces habituation to symbolically presented numbers and vice versa, as measured by fMRI activations (Piazza, Pinel, Le Bihan, & Dehaene, 2007).

Now consider data and logical arguments that have called into question the hypothesized causal relation between non-symbolic and symbolic numerical magnitude representations, or at least its strength. The ratio and distance effects that are present with both non-symbolic and symbolic numerical stimuli are also present with many non-numerical tasks including odor discrimination (Parnas, Lin, Huetteroth, & Misenbock, 2013) and ordering of letters of the alphabet (Van Opstal, Gevers, De Moor, & Verguts, 2008). Thus, the fit of both non-symbolic and symbolic numbers to Weber's Law does not mean that they share any specifically numerical representation.

Individual differences in the precision of non-symbolic and symbolic number representations, and in the relation between non-symbolic representations and overall math achievement, have proved far weaker than they originally appeared. For example, two meta-analyses (Chen & Li, 2014; Fazio, et al., 2014) that examined partially overlapping sets of studies found that the average weighted correlation between non-symbolic magnitude discrimination and overall math achievement was quite weak, $r = .20$ and $r = .22$, respectively. Chen and Li (2014) conducted a separate meta-analysis of longitudinal studies of predictive relations between non-symbolic numerical magnitude discrimination and math achievement test scores and obtained a weighted mean value of $r = .17$. Because the tasks and procedures examined in these studies varied in many ways, Lyons, et al. (2014) presented identical procedures and tasks to large samples of first to sixth graders (200 children in each grade). They obtained a mean relation of $r = .24$ between non-symbolic numerical representations and mental arithmetic ability. Similarly, an internet study with more than 10,000 participants yielded a relation of $r = .21$ between non-symbolic magnitude discrimination and mathematics achievement (Halberda, Ly, Wilmer, Naiman, & Germine, 2012). All of these correlations are significant, but all also indicate that the relation between non-symbolic and symbolic math knowledge is weak.

Another type of evidence, effects of training non-symbolic numerical magnitude discrimination on knowledge of symbolic magnitudes, has proven to be inconsistent. Although the three previously cited interventions using non-symbolic stimuli yielded positive effects on symbolic arithmetic, an equal number of similar training regimens failed to yield effects (Dewind & Brannon, 2012; Park & Brannon, 2014; Wilson, Revkin, Cohen, Cohen, & Dehaene, 2006).

The neural data relevant to the connection between non-symbolic and symbolic numerical representations have also proved more complex than they originally appeared. Although these data indicate that symbolic and non-symbolic numbers are processed in the same general brain areas, the distribution of processing activity within those areas elicited by identical non-symbolic and symbolic numbers seem to be minimally related or unrelated (Bulthé, de Smedt, & Op de Beeck, 2014; Damarla & Just, 2013; Eger, et al., 2009; Lyons, Ansari, & Beilock, 2014). As with the behavioral data, it is hard to know exactly what it means to say that non-symbolic and symbolic numbers share a common representation.

Perhaps most important, it is unclear how approximate non-symbolic representations could contribute to creating precise symbolic understanding of large numbers. No one can consistently discriminate 100 from 101 dots, but everyone who understands the decimal system knows with absolute certainty that “101” is larger than “100.” Moreover, it is unclear how ability to discriminate between 5:4 rather than 4:3 ratios could help a child learn that 8*7= 56 rather than 54.

These data and analyses do not imply that there is no causal connection between knowledge of the magnitudes of non-symbolic and symbolic numbers. It seems likely that children learn the meaning of small symbolically expressed single-digit numbers by associating them with non-symbolic representations of the corresponding sets. Consistent with this possibility, Carey (2009), Le Corre & Carey (2007), and Lyons and Ansari (2015) have proposed that children learn symbolic number words in the subitizing range (1-4) through a slow process occurring over a period of roughly six months, in which they associate first the counting word “1,” then “2,” then “3,” and then “4,” with the non-symbolic quantities produced by subitizing or parallel individuation. This counting experience leads them to the cardinality principle, an abstract understanding that the number of objects in a set corresponds to the last number used to count the objects in it, which in turn contributes to understanding of symbolic numbers.

Although this account is appealing, it raises the question of why formation of non-symbolic to symbolic numerical associations would stop at four objects. Young children frequently put up fingers and count them. Why wouldn’t the kinesthetic, visual, auditory, and temporal cues that accompany such counting lead to formation of associations between non-symbolic and symbolic representations of the numbers that young children count and add most frequently, in particular the numbers between 1 and 10? By this account, connections between non-symbolic and symbolic numbers greater than 10 would also be anticipated, but would be expected to be weaker, due to less frequent counting beyond 10 (for similar perspectives, see Andres, Michaux, & Pesenti, 2012, and Fischer & Brugger, 2011).

Consistent with this alternative explanation, behavioral data with 4- to 6-year-olds indicate that putting up fingers activates answers to addition problems with sums of 10 or less (Siegler & Robinson, 1982; Siegler & Shrager, 1984). Similarly, fMRI data from 8- to 13-year-olds indicate that judging the correctness of single digit subtraction problems activates finger control areas of the brain (Berteletti & Booth, 2015), and fMRI data from adults indicate a common substrate for mental arithmetic and finger representations for the numbers 1-10 (Andres, Michaux, & Pesenti, 2012). The influence of non-symbolic finger representations on access to symbolic numerical representations is also reflected in adults accessing symbolic numbers more quickly when they see sets of canonical counting fingers raised (e.g., left thumb, forefinger, and middle finger with “3”) than when they see other sets of fingers (Di Luca, Lefevre, & Pesenti, 2010). The hypothesized role of finger representations does not contradict the object-file hypothesis; both seem likely to contribute to knowledge of non-symbolic magnitudes getting knowledge of symbolic magnitudes off the ground, through providing concrete referents for small whole numbers.

From Small to Large Symbolic Whole Numbers

Development of Whole Number Magnitude Knowledge. Learning symbolic whole number magnitudes is a surprisingly slow and piecemeal process. It might seem obvious that children who can count from 1-10 would know that numbers later in the count have greater magnitudes than ones earlier in it, but this is not apparent to young preschoolers. For roughly a year after preschoolers flawlessly count from 1-10, they have limited knowledge of the magnitudes of symbolic numbers in that range (Briars & Siegler, 1984; Le Corre & Carey, 2007; Schaeffer, Eggleston, & Scott, 1974; Wynn, 1992).

Even after children know the relative magnitudes of the symbolic numbers 1-10, they view their magnitudes as being distributed in a compressive, approximately logarithmic, form. This is particularly apparent on the number line estimation task introduced by Siegler and Opfer (2003) and used by many investigators since then (Figure 2). On it, children are presented a series of lines with a constant pair of numbers at the two ends (e.g., 0 and 10 or 0 and 100) and asked to locate on the line a series of other numbers that fall between the endpoints (one number per line).

When asked to locate symbolically expressed numbers on a 0-10 number line, 3- and 4-year-olds space their estimates of 2 and 3 much farther apart than their estimates of 7 and 8, whereas 5- and 6-year-olds space the two pairs of numbers equally (Bertelletti, et al., 2010). Thus, children's representations of whole number magnitudes undergo a transition from approximately logarithmic to approximately linear distributions between ages 3 and 6.

This developmental sequence repeats itself at older ages with larger numbers (Figure 3). In the 0-100 range, 5- and 6-year-olds generate logarithmically increasing patterns of estimates whereas 7- and 8-year-olds' estimates increase linearly (Laski & Siegler, 2007; Siegler & Booth, 2004). In the 0-1000 range, 7- and 8-year-olds generate logarithmically increasing patterns of estimates, but 9- and 10-year-olds generate linearly increasing patterns (Booth & Siegler, 2006; Thompson & Opfer, 2010). On 0-10,000 number lines, 9-year-olds generate a logarithmic estimation pattern and 12-year-olds a linear one (Thompson & Opfer, 2010). The same estimation patterns and developmental progression appears in Chinese children at younger ages (Siegler & Mu, 2008; Xu, Chen, Pan, & Li, 2013) and in children with math learning difficulties at older ages (Geary, et al., 2007; 2008; Landerl, Fussenegger, Moll, & Willburger, 2009; Reeve, Paul, & Butterworth, 2015; Sella, Berteletti, Brazzolotto, Luncageli, & Zorzi, in press).

The ages of transition for different numerical ranges suggest that for much of childhood, children adopt linear representations for a particular range of numbers only when they gain substantial experience with that range. Roughly speaking, children from 3- to 5- or 6-years gain experience with the 0-10 range through counting; 6- to 8-year-olds gain experience with the 0-100 range through one and two digit addition and subtraction; 8- to 10-year-olds gain experience with the 0-1000 range through one- and two digit multiplication and division; and 9- to 12-year-olds gain experience with the 0-10,000 range, and sometimes with yet larger numbers, through multi-digit multiplication and division.

At some point, many people do understand the decimal system. However, even many adults do not fully understand the magnitudes of very large numbers. When presented a number line with “1 thousand” at the left end and “1 billion” at the right end, one-third of adults greatly overestimated the magnitude of 1, 2, 3, and 4 million, estimating their positions to be almost halfway between the two ends (Landy, Charlesworth, & Ottmar, in press).

The complementary finding was that two-thirds of adults in Landy et al. (in press) were accurate in their estimates. This finding raised the question of how they did so, given that almost no one has much experience with such very large numbers.

Analogizing from smaller to larger orders of magnitude seems to be an important mechanism in broadening the range of whole numbers whose magnitudes can be represented linearly. To test whether children can extend linear patterns through analogy, Thompson and Opfer (2010) presented second graders who already generated linear estimation patterns on 0-100 number lines with parallel problems with larger numerical ranges on which they initially generated logarithmic patterns. Children who were given feedback on pairs of parallel problems -- for example, on their placements of 15 on a 0-100 line and 1500 on a 0-10,000 line -- generalized the linear estimation pattern to numbers that had not been presented on 0-10,000, and 0-100,000 scales more often than children who were presented the same feedback problems but not paired in a way that highlighted the analogous relation. Analogical reasoning also influences learning the magnitudes of smaller numbers, for example numbers in the 0-100 and 0-1000 ranges (Opfer & Siegler, 2007; Sullivan & Barner, 2014).

The claim that children undergo a logarithmic to linear shift in representing numerical magnitudes has been challenged on several grounds. The most prominent challenge is that one- or two-cycle power functions fit the data better than do logarithmic and linear functions (Barth & Palladino, 2011; Slusser, Santiago, & Barth, 2013). The data on which this claim is based, however, stem from a form of the number line task that differs in a crucial way from the form used by Siegler and Opfer (2003) and in almost all subsequent studies. This variant form includes a labeled midpoint (e.g., 50 on a 0-100 line) and instructions that call participants' attention to the importance of the number and its location at the midpoint. Given the well-documented phenomenon of anchoring (e.g., Parducci, 1968), it is unsurprising that placing an anchor at the midpoint of a line and calling attention to its importance leads to children using the anchor to guide their estimates.

To test whether the difference in findings was due to the midpoint anchor and instructions calling attention to it, Opfer, Thompson, and Kim (in press) directly compared 0-1000 number line estimates with the anchor present or absent, using the same numbers as in Slusser, et al. (2013). (Slusser et al. had hypothesized that differences in the number being estimated led to the discrepant results.) However, Opfer et al.'s results demonstrated that anchoring was what led to the two-cycle power function fitting when a midpoint anchor was present. The logarithmic function fit better than the two-cycle power function when Slusser et al.'s numbers were used but no anchor was present.

One compelling type of evidence for the value of the number line estimation task is its strength as a predictor of broader mathematical outcomes. Accuracy and linearity of number line estimation correlate quite strongly with concurrent and future math achievement test scores (Booth & Siegler, 2006; Geary, et al., 2007; Geary, Hoard, Nugent, & Byrd-Craven, 2008; Laski & Siegler, 2007; Sasanguie, De Smedt, Defever, & Reynvoet, 2012; Sella, et al., in press; Siegler & Booth, 2004) and with arithmetic proficiency (Ashcraft & Moore, 2012; Booth & Siegler, 2008; Cowan & Powell, 2014; Gunderson, Ramirez, Beilock, & Levine, 2012; LeFevre, et al., 2013). These relations remain present even when plausible third variables, including reading achievement, parental income and education, race, ethnicity, working memory, IQ, and non-symbolic numerical knowledge are statistically controlled (Bailey, Siegler, & Geary, 2014; Booth & Siegler, 2006; 2008; Cowan & Powell, 2014; Fazio, et al., 2014; Geary, Hoard, Nugent, & Byrd-Craven, 2008).

Applying findings on development of whole number magnitudes to improving numerical understanding. These analyses of the development of whole number magnitudes raised the questions of how children come to form linear representations of numerical magnitude and how the acquisition process can be improved. These questions, in turn, inspired an effective intervention for improving the numerical magnitude knowledge of preschoolers from low-income backgrounds.

Counting is almost certainly an important contributor to learning numerical magnitudes, but the substantial gap in time between when children can count from 1-10 or 1-100 and when they know the magnitudes of the numbers in that range indicates that other activities must also play a role. Siegler and Booth (2004) hypothesized that playing numerical board games such as *Chutes and Ladders*, might well be one such contributor.

Chutes and Ladders is a game in which the whole numbers 1-100 are arranged in a 10 X 10 matrix; players roll a dice, and move their tokens the number of spaces indicated by the number that comes up. The game board in Figure 4, based on the first row of *Chutes and Ladders*, can be used to explain how playing the game could contribute to learning of numerical magnitudes. The greater the number in a square, the greater a) the number of discrete movements of the token the child needs to make to reach it, b) the number of number names the child says before reaching it, c) the distance traveled from the origin to reach it, and d) the time that passes before reaching it. Moreover, each dimension increases proportionally with the numbers in the squares: Reaching "8" rather than "4", for example, requires saying twice as many number words, traveling twice as great a distance from the origin, moving the token with twice as many hand movements, and playing for twice as much time. Thus, the game provides highly valid visual, kinesthetic, temporal and auditory cues to the magnitudes represented by different numbers. These cues seemed likely to be useful for associating symbolically expressed numbers with values of non-symbolic quantitative dimensions.

Consistent with this analysis, four 15-minute sessions of experience playing the number board game improved preschoolers' understanding of the symbolic numbers 1-10 (Figure 5). The preschoolers, who were from low-income backgrounds, improved on two measures of numerical magnitude knowledge -- number line estimation and magnitude comparison – and also on counting, numeral identification, and ability to learn the answers to arithmetic problems. Their gains exceeded those of peers who played a parallel color board game (Ramani, Hitti, & Siegler, 2011; Ramani & Siegler, 2008; Siegler & Ramani, 2008). Preschoolers who engaged in other numerical activities (e.g., counting and identifying numerals) improved as much on those tasks but not as much on numerical magnitude knowledge (Siegler & Ramani, 2009; Whyte & Bull, 2008). The gains of children who played the number board game endured for at least two months (Ramani & Siegler, 2008).

Subsequent studies designed to identify sources of the game's effectiveness yielded results consistent with the underlying theoretical analysis. Playing the game with a linear board, which presumably facilitated formation of a mental number, led to greater learning of numerical magnitudes than did playing the same game on a circular board, which presumably increased the difficulty of mapping numbers onto a linear representation (Siegler & Ramani, 2009). Associating the symbolic numbers with the non-symbolic quantitative dimensions also proved crucial to learning. On a 0-100 version of the game (Laski & Siegler, 2014), kindergartners learned considerably more when required to count-on (e.g., by saying aloud "39, 40, 41, 42"), thus ensuring that they encoded the numbers in the squares, than when they were required to count from one (e.g., by saying "1,2,3,4"). A similar game involving whole body movement (walking along a number line) also has yielded encouraging results for improving young children's knowledge of whole number magnitudes (Fischer, Moeller, Bientzle, Cress, & Nuerk, 2011; Link, Nuerk, & Moeller, 2014) as have more encompassing curricula such as Building Blocks

(Clements & Sarama, 2008) that emphasize magnitude understanding. These successful applications increase confidence in the theories that inspired them.

From Whole to Rational Numbers

Within the integrated theory, rational and whole numbers are viewed as co-equal parts of numerical development. This position might seem extreme, given the obvious great importance of whole numbers, but conceptual, empirical, and practical considerations all argue for it.

Ignoring or minimizing the growth of rational number knowledge deprives theories of numerical development of some of their potentially most interesting content. Understanding rational numbers requires a reorganization of the conceptual knowledge acquired in learning about whole numbers. Whole numbers have unique predecessors and successors, are represented by unique symbols (e.g., "6"), never decrease with multiplication, and never increase with division. None of these qualities are true of rational numbers, however, and therefore none is true of numbers in general. Describing and explaining the transition to understanding of rational numbers, explaining why many children and adults never make this transition or make it only partially, and designing interventions to improve understanding of rational numbers pose exciting challenges and opportunities for theories of numerical development.

Empirical data also attest to the importance of emphasizing rational numbers in theories of numerical development. Analyses of large, longitudinal data sets from the U.S. and the U.K. indicated that in both countries, 5th graders' fraction knowledge predicted 10th graders' mathematics achievement test scores, even after controlling for knowledge of all four whole number arithmetic operations, IQ, reading achievement, working memory, family income, maternal education, race, and ethnicity (Siegler, et al., 2012). Concurrent data from the same study showed even stronger relations: Among the 10th graders, correlations between fraction knowledge and overall math achievement exceeded r = .80 in both the U.S. and the U.K. In contrast, in the same data sets, whole number addition, subtraction, and multiplication did not account for significant variance in high school math achievement after the other variables were statistically controlled (though division did). Other studies have also shown both concurrent and predictive relations between knowledge of rational numbers and math achievement (Bailey, et al., 2012; Booth & Newton, 2012; Booth, Newton, & Twiss-Garrity, 2015; Schneider, Grabner, & Paetsch, 2009). To omit such strong relations from theories of mathematical development is to commit an unforced error.

With regard to application, focusing on rational numbers in general, and rational number magnitudes in particular, poses great opportunities for applying insights from numerical development research to a serious educational problem. Understanding fractions, decimals, negatives, ratios, rates, and proportions is essential to mastering not only more advanced areas of mathematics but physical, biological, and social sciences as well. Despite this pervasive importance, and despite instruction in rational numbers typically starting by third grade and continuing for several years thereafter, innumerable examples indicate that many students fail to grasp rational number magnitudes. On the 2004 National Assessment of Educational Progress (NAEP), 50% of 8th graders could not correctly order the magnitudes of 2/7, 1/12, and 5/9 (NCTM, 2007). The problem is not limited to numbers written in common fraction notation; when asked whether .274 or .83 was larger, most 5th and 6th graders chose .274 (Rittle-Johnson, Siegler, & Alibali, 2001). The difficulty extends to adolescents and adults. On the 2004 NAEP, fewer than 30% of U. S. 11th graders translated .029 into the correct fraction (Kloosterman, 2010) and in several recent experiments, community college students correctly answered only about 70% of fraction magnitude comparison problems (chance was 50%) (DeWolf, Grounds, Bassok, & Holyoak, 2014; Schneider & Siegler, 2010; Stigler, Givvin, & Thompson, 2010). The difficulty is not limited to the U.S.; mathematics educators in high achieving countries such as Japan, China, and Taiwan, have also noted it (Chan, et al., 2007; Ni, 2001). Thus, in addition to the inherent importance of understanding rational numbers to mature knowledge of what numbers are, and in addition to empirical findings showing strong concurrent and predictive relations between rational number knowledge and overall mathematics achievement, rational numbers offer enticing opportunities for applying psychological research to improving education.

Development of rational number knowledge. The most obvious facts about rational number understanding is that it develops more slowly and to a lower asymptotic level than whole number knowledge. This seems especially true of knowledge of common fractions. Illustrative of both facts, when Greek children were asked to order 1/7, 5/6, 1, and 4/3, accuracy only improved from 33% in 5th grade to 58% correct in 9th grade (Stayflidou & Vosniadou, 2004). Among U.S. children, fraction magnitude comparison accuracy for fractions with unequal numerators and denominators was only 68% to in 6th and 79% in 8th grade (Siegler & Pyke, 2013). Strikingly, second graders' whole number estimates on 0-100 lines tends to be more accurate than community college students' fraction estimates on 0-5 lines (Bailey, et al., 2015; Fazio, DeWolf, & Siegler, in press; Siegler & Booth, 2004; Siegler, Thompson, & Schneider, 2011).

Alongside these differences between development of whole and rational numbers, however, are many similarities. One is the early developing base of non-symbolic magnitude knowledge. As with whole numbers, 6-month-olds discriminate 2:1 but not 3:2 ratios of colored dots. After habituating to repeated displays with 2:1 ratios of blue to yellow dots, 6-month-olds dishabituated when shown 4:1 but not 3:1 ratios (McCrink & Wynn, 2007). By age 3 years, children draw analogies between equivalent non-symbolic fractions (1/2 of a square : 1/2 of a circle :: 3/4 of a square is to 3/4 of a circle (Goswami, 1989; 1996).

Neuroscience research also points to similarities in the underpinnings and development of whole and rational number knowledge. Symbolic and non-symbolic rational number magnitudes are processed by a fronto-parietal network closely resembling that used to process whole-number magnitudes (e.g., Jacob, Vallentin, & Nieder, 2012; Vallentin & Nieder, 2008). At least in university students, processing of fraction magnitudes in the IPS is automatic; as with whole numbers, it occurs even when there is no specific task (Jacob & Nieder, 2009). Neural activations show similar distance effects with fractions as with whole numbers on magnitude comparison tasks (Ischebeck, et al., 2009). Also as with whole numbers, specific neurons of monkeys respond most strongly to specific ratios of fractions expressed non-symbolically (Vallentin & Nieder, 2008).

Behavioral data with older children and adults also show many similarities between representations of symbolic rational and whole numbers. When comparing fraction magnitudes, distance effects are consistently present among both children (Fazio, et al., 2014; Meert, Gregoire, & Noel, 2010) and adults (Meert, Gregoire, & Noel, 2009; Schneider & Siegler, 2010). Moreover, as noted earlier, both concurrent and predictive relations are present for symbolic rational and whole numbers between overall mathematics achievement and magnitude knowledge, even after statistically controlling for many relevant variables, and the relations are present in Europe and East Asia, as well as North America (e.g., Bailey, et al., 2015; Hallet, Nunes, & Bryant, 2010; Siegler, et al., 2012; Torbeyns, Schneider, Xin, & Siegler, 2015).

Longitudinal data also indicate that individual differences in early whole number magnitude knowledge predict individual differences in later fraction magnitude and arithmetic knowledge (Hecht, 1998; Hecht, Close, & Santisi, 2003; Hecht & Vagi, 2010; Jordan, et al., 2013; Vukovic, et al., 2014). For example, in Bailey, et al. (2014), knowledge of whole number magnitudes at age 6 predicted knowledge of fraction magnitudes and fraction arithmetic at age 13, even after statistically controlling for IQ, executive functioning, race, gender, parental income, and parental education. Lending specificity to the predictive relations, the 6-year-olds' whole number knowledge was not predictive of reading achievement at age 13 after controlling for the covariates.

Much less is known about development of understanding of negatives than fractions or decimals, but magnitudes seem to play a similarly central role. Brain regions involved in representing negative numbers' magnitudes overlap considerably with those used with positive numbers. Between childhood and adulthood, activations of parietal areas increase, and those of frontal areas decrease (Gullick & Wohlford, 2013; Gullick, Wolford, & Temple, 2012). As with positive numbers, distance effects on magnitude comparison problems are present with negatives (Ganor-Stern, Pinhas, Kallai, & Tzelgov, 2010; Gullick & Wohlford, 2013), and individual differences in the sizes of 10-year-olds' distance effects on magnitude comparisons with positives and negatives are related (Gullick & Wohlford, 2013).

Moreover, the process of analogy seems to play a similarly prominent role with negatives as with positives. The best-specified model of the development of negative number processing (Varma & Schwartz (2011) proposes that children initially process the magnitudes of negative numbers through a simple rule (all negatives are less than any positive), then draw an analogy to whole number magnitudes and represent the magnitudes of negatives as a reflection of their positive number representation, and then integrate the rule and the reflected representation into a unified representation. These models accounted for numerous types of data, including an inverse distance effect in which adults' but not sixth graders', magnitude comparisons of a positive and a negative were faster when the distance between the numbers being compared was small (e.g., -1 versus 2) than when it was large (-1 vs. 7).

Interventions to improve rational number magnitude knowledge. As with whole numbers, some of the most compelling evidence for the causal impact of rational number magnitude knowledge on more general knowledge of rational numbers comes from intervention studies. A common feature of the most successful interventions for improving knowledge of fractions and decimals is that they help children understand how these numbers map onto magnitudes (Fujimura, 2001; Moss & Case, 1999; Fuchs, et al., 2013; 2014; in press a & b; Rittle-Johnson, et al., 2001; Schneider, et al., 2009).

One especially impressive demonstration of how instruction can improve knowledge of fraction magnitudes is Fuchs et al.'s (2013; 2014; in press a & b) *Fraction Face-Off* curriculum. Participants in these studies were fourth graders whose math achievement test scores were in the bottom 35% of the distribution. Three times per week for 12 weeks, children in the intervention group received instruction that emphasized building understanding of fraction magnitudes through comparing and ordering fraction magnitudes, and placing fractions on number lines. Children in the control group were presented instruction from the mathematics textbook used in the district. Like most math textbooks, it emphasized the part-whole representation of fractions – for example, dividing a circle into four equal parts, shading one, and noting that ¼ meant one of four equal parts – and fraction arithmetic procedures.

The experimental intervention consistently led to greater improvement not only in conceptual understanding of fractions but also in proficiency with fraction arithmetic, despite fraction arithmetic receiving less attention in the intervention than in the control curriculum. The greater improvement was apparent on publically available NAEP fraction items as well as on items created by the researchers. Improvement in fraction magnitude knowledge mediated other intervention effects. That is, the children whose fraction number line estimation improved most were the ones who showed the greatest improvement on the NAEP items. Thus, with rational as well as whole numbers, interventions that focus on magnitude understanding can have broad positive effects.

Conclusions

The evidence reviewed in this article supports six major conclusions:

1. The growth of numerical magnitude representations provides a useful unifying theme for understanding numerical development from infancy through adulthood.
2. Development of numerical magnitude knowledge involves representing increasingly precisely an increasingly broad range of numbers on a mental number line. The process involves extension of magnitude knowledge from non-symbolic to symbolic numbers, from smaller to larger whole numbers, from

whole numbers to fractions and decimals, and from positives to positives and negatives. Figure 6 presents a schematic diagram of the development.

3. Despite years of relevant instruction, many children and adults gain only limited knowledge of fraction magnitudes.
4. A variety of mental processes contribute to numerical development, but two that appear to play especially prominent roles are association and analogy.
5. Knowledge of magnitudes of both whole and rational numbers correlates with, predicts, and is causally related to arithmetic and overall mathematics achievement.
6. Interventions that focus on improving numerical magnitude knowledge have been successful in improving a wide variety of mathematical outcomes with both whole and rational numbers.

In addition to these conclusions, the present review points to several high priority issues for future research. One priority is deeper analyses of the processes that lead to improvement in numerical magnitude representations. In both childhood and adulthood, individual differences in the precision of numerical magnitude representations correlate with proficiency in many domain-general cognitive processes, including executive functioning (e.g., Bull & Lee, 2014; Siegler & Pyke, 2013), working memory (e.g., Friso-van den Bos, van der Ven, , Kroesbergen, & Van Luit, 2013; Geary, 2011), attention and non-verbal reasoning (e.g., Jordan, et al., 2013; Vukovic, et al., 2014), spatial abilities (e.g., Gunderson, Ramirez, Beilock, & Levine, 2012; Vukovic, et al., 2014), and language (e.g., Chang, Sandhofer. Adelchanow, & Rottman, 2010; Jordan, et al., 2013). However, how these processes combine with each other, and with domain specific associations and analogies involving numbers, is not well understood. The high quality descriptions of development of numerical magnitude knowledge reviewed in this article provide an excellent base for testing clearly specified process accounts of that development.

A second promising area for numerical development research is uniting it with research on arithmetic. Numerical magnitude knowledge has been shown to be moderately to strongly related to arithmetic knowledge with both whole numbers and fractions. Causal relations have also been established: Manipulations designed to increase magnitude knowledge also increase learning of the answers to arithmetic problems, again for both whole numbers and fractions, with the improvement in magnitude knowledge frequently mediating the improvement in other mathematical skills. Again, however, we do not understand in much detail the processes that produce these beneficial effects of magnitude knowledge.

One possibility is that the process works through numerical magnitude knowledge and conceptual understanding of arithmetic operations together constraining plausible answers to arithmetic problems. With whole numbers, numerical magnitude knowledge could help children learn specific answers to problems (e.g., that $9 \times 4 = 36$ rather than 16 or 56). With fractions, where particular arithmetic problems occur far less often than single-digit whole number arithmetic problems, magnitude knowledge could help children learn procedures that generate plausible rather than implausible answers (e.g., that $1/2 + 1/2 = 1$

rather than 2/4). Regardless of how tests of this account fare, uniting analyses of the development of numerical magnitudes and arithmetic is surely a worthwhile goal.

Another worthwhile goal would be to extend to other concepts the type of domain-specific integrated theory proposed here for numerical development. The categories identified by Kant and Piaget as fundamental to understanding the world -- space, time, causality, morality, mind, etc. – seem especially promising areas for such integrations. Formulating domain-specific integrated theories of development for multiple concepts might also allow some degree of integration across concepts, as is already underway in research on time, space, and number (Dehaene & Brannon, 2011; Newcombe, et al., 2015). The hope is that multiple, integrated, domain-specific theories will allow us to address our version of Toynbee's (1937) question and say with confidence "No, development is not just one damned thing after another."

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Figure Captions

1. The development of knowledge of *nonsymbolic* numerical magnitudes. The sets of black and white dots represent experimental stimuli whose numerosity can be discriminated reliably at the specified ages.
2. The number line task for the 0-1000 range
3. Second and fourth graders' 0-1000 number-line estimation (data from Opfer & Siegler, 2007)
4. The number board used in Ramani and Siegler's (2008) experimental group and the color board used in their control group
5. Effects of numerical board game experience on number line estimation, magnitude comparison, numeral identification, and counting (data from Ramani & Siegler, 2008)
6. A schematic diagram of the development of symbolic numerical magnitude knowledge (after Siegler & Lortie-Forgues, 2014)

(Figure 1)

Precision of Discrimination

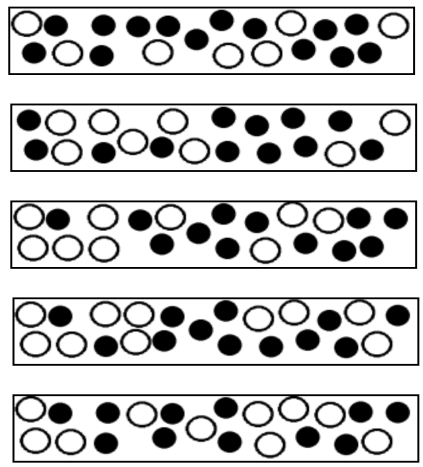
2:1 ratio (≈ 6 months)

3:2 ratio (≈ 9 months)

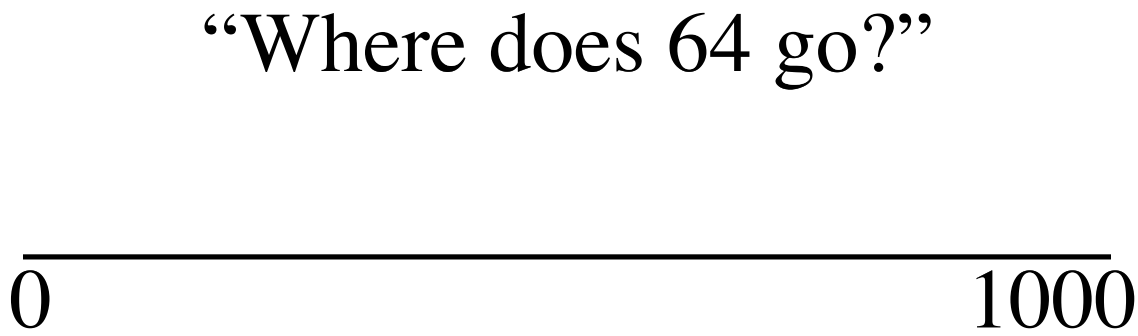
4:3 ratio (≈ 3 years)

6:5 ratio (≈ 6 years)

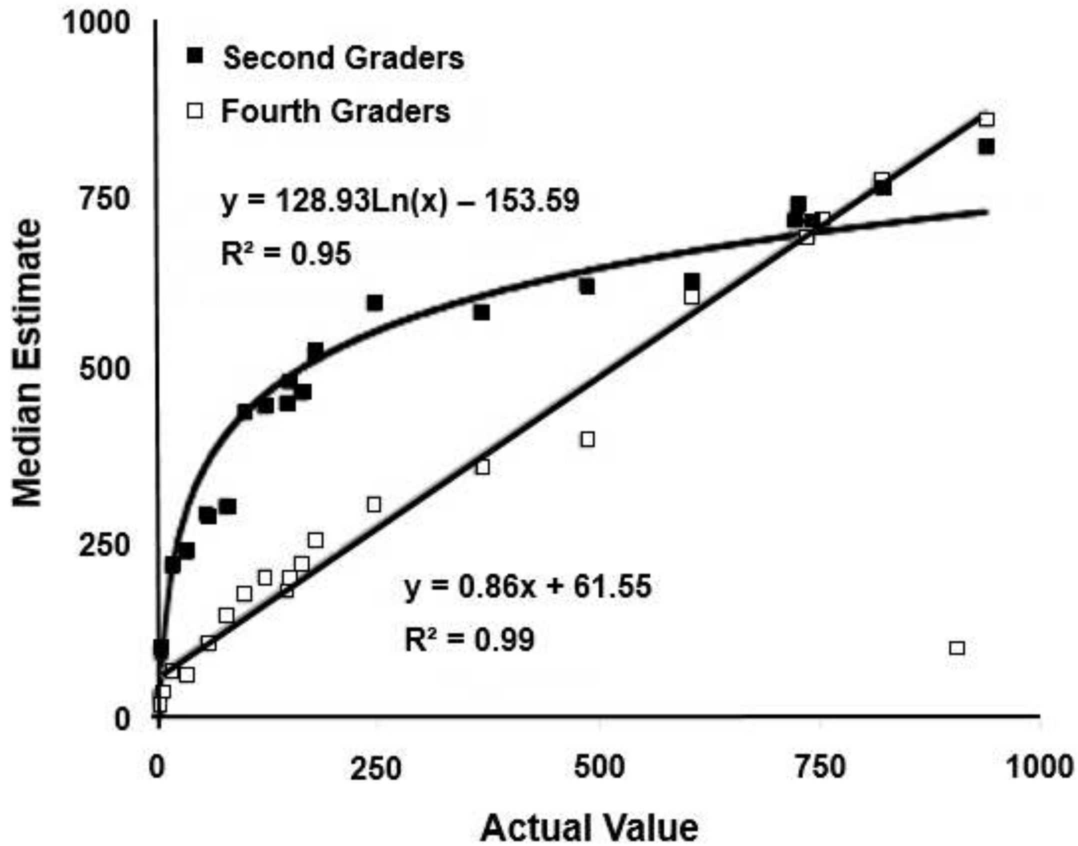
11:10 ratio (some adults)



(Figure 2)



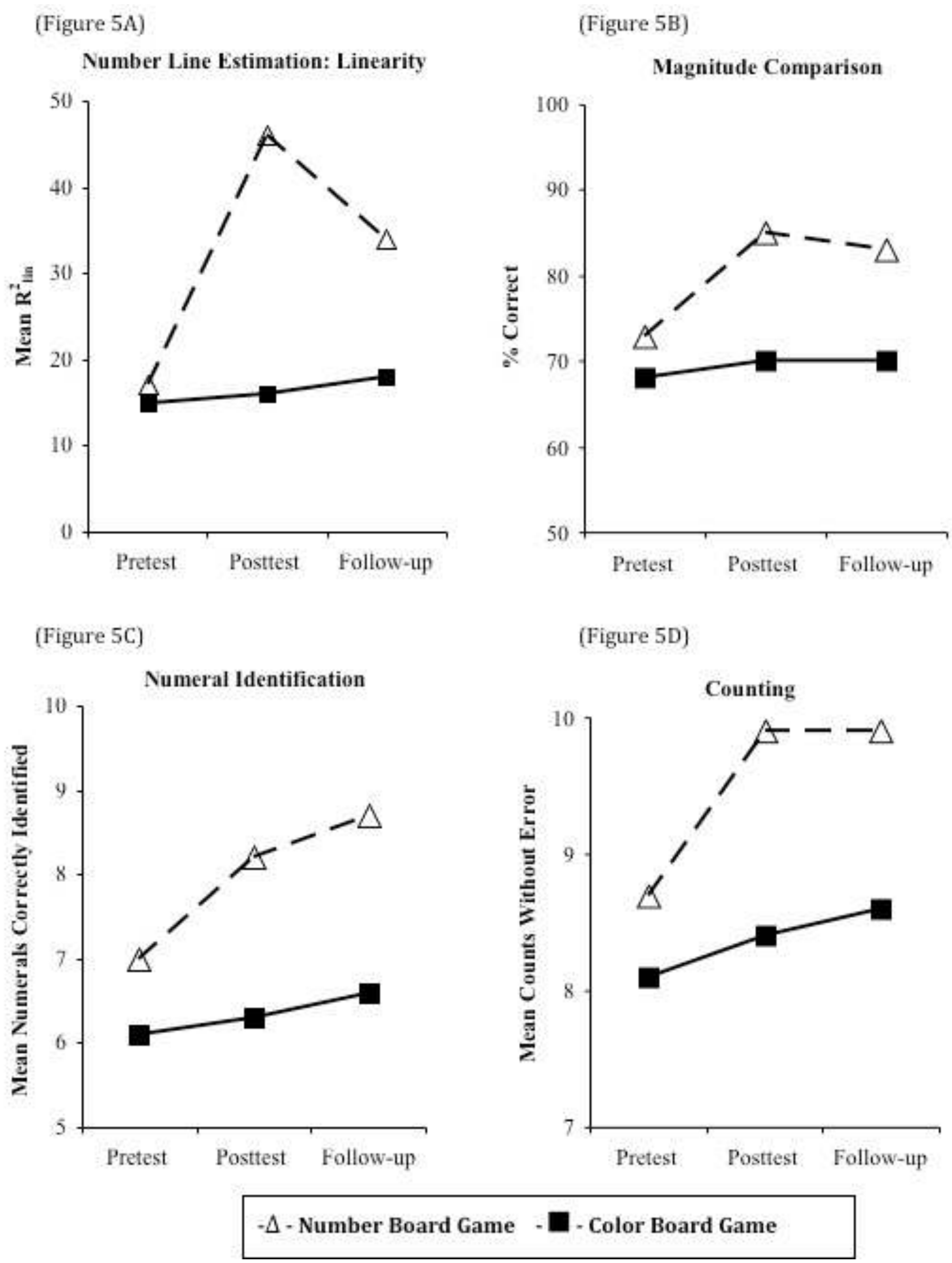
(Figure 3)


Second Graders
Fourth Graders
y = 128.93Ln(x) – 153.59
R² = 0.95
y = 0.86x + 61.55
R² = 0.99
Median Estimate
Actual Value
0
250
500
750
1000

(Figure 4)

(a) Number Board Game

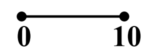
(b) Color Board Game


(Figure 5A)
Number Line Estimation: Linearity
Mean R^2_{lin}
0
10
20
30
40
50
Pretest
Posttest
Follow-up
(Figure 5B)
Magnitude Comparison
% Correct
50
60
70
80
90
100
Pretest
Posttest
Follow-up
(Figure 5C)
Numeral Identification
Mean Numerals Correctly Identified
5
6
7
8
9
10
Pretest
Posttest
Follow-up
(Figure 5D)
Counting
Mean Counts Without Error
7
8
9
10
Pretest
Posttest
Follow-up
-Δ - Number Board Game
- Color Board Game

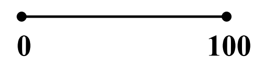
(Figure 6)

<u>Type of Magnitude and Main Acquisition Period</u>

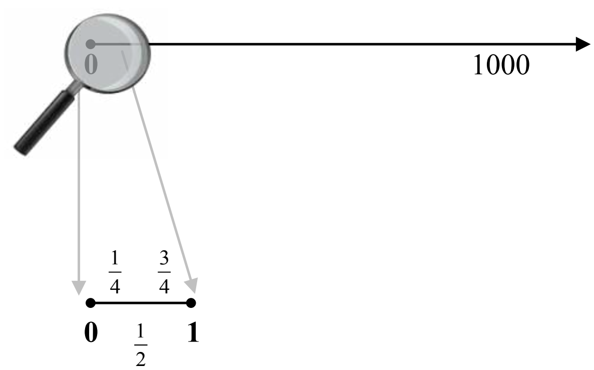
Small whole numbers (≈ 3 to 5 years)



Larger whole numbers (≈ 5 to 7 years)

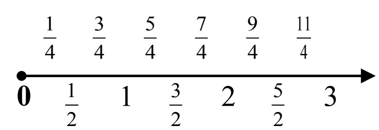


Yet larger whole numbers (≈ 7 to 12 years)



Fractions 0-1 (≈ 8 years to adulthood)

Fractions 0-N (≈ 11 years to adulthood)



Rational numbers (including negatives) (≈ 11 years to adulthood)