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Majorization and Doubly Stochastic Operators

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Abstract. We present a close relationship between row, column and doubly stochastic operators and the majorization relation on a Banach space $\ell^p(I)$, where I is an arbitrary non-empty set and $p \in [1, \infty]$. Using majorization, we point out necessary and sufficient conditions that an operator D is doubly stochastic. Also, we prove that if P and P^{-1} are both doubly stochastic then P is a permutation. In the second part we extend the notion of majorization between doubly stochastic operators on $\ell^p(I)$, $p \in [1, \infty)$, and consider relations between this concept and the majorization on $\ell^p(I)$ mentioned above. Moreover, we give conditions that generalized Kakutani's conjecture is true.

1. Introduction

Theory of majorization plays important role in mathematics and statistics as well as in physics and economics. It has a lot of applications to various fields in mathematics such as matrix and operator theory [1, 9, 12, 13], frame theory [2], graph theory, inequalities involving convex functions [21], etc. We also refer the reader to the classical book of majorization and applications by Marshall, Olkin and Arnold [15].

Given two vectors $x, y \in \mathbb{R}^n$, we say that x is majorized by y, and denote it $x \prec y$, if

$$\sum_{i=1}^{k} x_i^{\downarrow} \leq \sum_{i=1}^{k} y_i^{\downarrow} \qquad (k = 1, 2, \ldots, n)$$

and

$$\sum_{i=1}^{n} x_i^{\downarrow} = \sum_{i=1}^{n} y_i^{\downarrow}$$

where $x_1^{\downarrow} \geq x_2^{\downarrow} \geq \ldots \geq x_n^{\downarrow}$ is the decreasing rearrangement of components of a vector x.

There are several equivalents for the notion of majorization in finite dimensions. The next well known theorem, which gives an alternative definition for majorization using doubly stochastic matrices, is proved by Hardy, Littlewood and Polya [11]. Namely, $x \prec y$ if and only if there is doubly stochastic matrix A such that $x = Ay$. Recall that a square matrix with non-negative real entries is called doubly stochastic, if each of its row sums and each of its column sums are equal 1.

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Recently, Bahrami, Bayati and Manjegani in their joint papers [4, 5] used doubly stochastic operators to extend the notion of majorization on Banach spaces ℓ^∞ and $\ell^p(I)$, where I is an arbitrary non-empty set, $p \in [1, \infty)$, and they characterized the structure of all bounded linear maps which preserve majorization. They presented some helpful results that we will use in this paper. Some topological properties of this structure can be found in [6]. Also, the convex majorization on $\ell^p(I)$ was introduced in [7]. The other extensions of majorization and applications to diagonals of self-adjoint, positive and compact operators are considered in [3, 14, 16].

In finite dimensions, there is a very useful close relationship between majorization and doubly stochastic matrices.

Theorem 1.1. [15, Theorem I.2.A.4] *An $n \times n$ matrix A is doubly stochastic if and only if $Ax \prec x$ for all vectors $x \in \mathbb{R}^n$.*

The next theorem is proved by Snijders [20], Berge [8], and Farahat [10] using totally different approaches.

Theorem 1.2. *Let P be an invertible matrix. If P and P^{-1} are both doubly stochastic then P is a permutation matrix.*

In Section 3 we will present generalized versions of Theorem 1.1 and Theorem 1.2. More precisely, let $p \in [1, \infty)$ and $A : \ell^p(I) \longrightarrow \ell^p(I)$ be a bounded linear operator. It will be shown that:

- If $A \in DS(\ell^p(I))$ then $Af \prec f$, $\forall f \in \ell^p(I)$ (Lemma 3.7).
- If $Af \prec f$, $\forall f \in \ell^p(I)$ then $A \in CS(\ell^p(I))$ (Lemma 3.7).
- $A \in DS(\ell^p(I))$ if and only if $Af \prec f$, $\forall f \in \ell^p(I)$, and $A^* g \prec g$, $\forall g \in \ell^q(I)$, where q is the conjugate exponent of p (Theorem 3.8).
- If $P \in DS(\ell^p(I))$ and $P^{-1} \in DS(\ell^p(I))$ then P is permutation (Theorem 3.12).

Sherman [18] introduced a pre-order among $n \times n$ doubly stochastic matrices by defining

$$D_1 \lhd D_2 \tag{1}$$

if there is a doubly stochastic matrix D_3 such that

$$D_1 = D_3 D_2.$$

It is easy to see that $D_1 \lhd D_2$ implies $D_1 x \prec D_2 x$, $\forall x \in \mathbb{R}^n$. Kakutani has raised the conjecture that the opposite direction is also true:

$$If \;\; (\forall x \in \mathbb{R}^n) \;\; D_1 x \prec D_2 x, \;\; then \;\; D_1 \lhd D_2.$$

In [18] it is provided that if doubly stochastic matrix D_2 is invertible then Kakutani's conjecture is true. Horn found counterexample [19] of Kakutani's conjecture and showed that invertibility od operator D_2 can not be omitted. Using a different technique, Schreiber [17] verified Sherman's result.

In Section 4 we will extend the pre-order (1) on doubly stochastic operators on $\ell^p(I)$ and we will provide that Kakutani's conjecture is true if $D_2^{-1} \in RS(\ell^p(I))$. More precisely, let $p \in [1, \infty)$ and $D_1, D_2 \in DS(\ell^p(I))$. If $D_1 \lhd D_2$ then $D_1 f \prec D_2 f$, $\forall f \in \ell^p(I)$. Conversely, if D_2 is invertible and $D_2^{-1} \in RS(\ell^p(I))$ then

- $(\forall f \in \ell^p(I))$ $D_1 f \prec D_2 f$ implies $D_1 \lhd D_2$ (Theorem 4.4).

2. Notations and Preliminaries

We will consider real-valued functions $f : I \longrightarrow \mathbb{R}$, where I is an arbitrary non-empty set. We say that an arbitrary function f is summable if there exists a real number σ with the following property:

Given $\epsilon > 0$, we can find a finite set $J_0 \subseteq I$ such that

$$\left|\sigma - \sum_{j\in J} f(j)\right| \leq \epsilon$$

whenever J is a finite set and $J_0 \subseteq J$. Then σ is called the sum of f and we denote it by $\sigma = \sum_{i\in I} f(i)$.

We will denote by $\ell^p(I)$, where I is non-empty set and $p \in [1, \infty)$, the Banach space of all functions $f : I \longrightarrow \mathbb{R}$ such that $\sum_{i\in I} |f(i)|^p < \infty$, equipped with standard p-norm

$$\|f\|_p := \left(\sum_{i\in I} |f(i)|^p\right)^{\frac{1}{p}} < \infty$$

As we know, every function $f \in \ell^p(I)$, $p \in [1, \infty)$, can be represented in the form $f = \sum_{i\in I} f(i)e_i$, where the function $e_i : I \longrightarrow \mathbb{R}$ is defined by Kronecker delta, i.e., $e_i(j) = \delta_{ij}$, $i \in I$. Let $p, q \in (1, \infty)$. A number q is conjugate (dual) exponent of p if $\frac{1}{p} + \frac{1}{q} = 1$. Furthermore, the exponents 1 and ∞ are considered to be dual exponents to each other. Let $p \in [1, \infty)$ and let q be a dual exponent of p. Then, for every function $g \in \ell^q(I)$, the rule $f \longrightarrow \langle f, g\rangle := \sum_{i\in I} f(i)g(i)$ defines a functional on $\ell^p(I)$. In this way, the dual Banach space $\ell^p(I)^*$ can be identified with $\ell^q(I)$ because $\ell^p(I)^*$ is isometrically isomorphic with $\ell^q(I)$. Also, using the dual pairing $\langle\cdot, \cdot\rangle : \ell^p(I) \times \ell^q(I) \longrightarrow \mathbb{R}$, for functions $f \in \ell^p(I)$ and e_i, (considering e_i as an element of the dual space of $\ell^p(I)$) we have

$$f_i = \langle f, e_i\rangle, \qquad \forall i \in I.$$

Hence,

$$f = \sum_{i\in I} \langle f, e_i\rangle e_i$$

Let $A : \ell^p(I) \longrightarrow \ell^p(I)$ be a bounded linear operator, where $p \in [1, \infty)$. The space $\ell^p(I)$ is an ordered Banach space under the natural partial ordering on the set of real valued functions defined on I, so an operator A is called positive if $Ag \geq 0$ for every $g \geq 0$.

An operator $A^* : \ell^q(I) \longrightarrow \ell^q(I)$ is the adjoint operator of $A : \ell^p(I) \longrightarrow \ell^p(I)$, $p \in [1, \infty)$, if $\langle Af, g\rangle = \langle f, A^*g\rangle$, where $f \in \ell^p(I)$, $g \in \ell^q(I)$ and q is the conjugate exponent of p.

We recall definitions of row, column, doubly stochastic operators and majorization when $p \in [1, \infty)$ introduced by Bahrami, Bayati, Manjegani [4].

Definition 2.1. [4, Definition 2.1.] Let $p \in [1, \infty)$ and $A : \ell^p(I) \longrightarrow \ell^p(I)$ be a bounded linear operator. The operator A is called

- *row stochastic*, if A is positive and $\forall i \in I$ $\sum_{j\in I}\langle Ae_j, e_i\rangle = 1$.
- *column stochastic*, if A is positive and $\forall j \in I$ $\sum_{i\in I}\langle Ae_j, e_i\rangle = 1$.
- *doubly stochastic*, if A is both row and column stochastic.
- a *permutation*, if there exists a bijection $\theta : I \longrightarrow I$ for which $Ae_j = e_{\theta(j)}$, for each $j \in I$.

The set of all row stochastic, column stochastic, doubly stochastic operators and permutations on $\ell^p(I)$, $p \in [1, \infty)$ are denoted, respectively, by $RS(\ell^p(I))$, $CS(\ell^p(I))$, $DS(\ell^p(I))$ and $P(\ell^p(I))$.

Definition 2.2. [4, Definition 3.1.] Let $p \in [1, \infty)$. For two elements $f, g \in \ell^p(I)$, we say that f is *majorized* by g, if there exists a doubly stochastic operator $D \in DS(\ell^p(I))$, such that $f = Dg$, and denote it by $f \prec g$.

Also, we will consider the Banach space of all functions $f : I \longrightarrow \mathbb{R}$ for which $\sup_{i\in I} |f(i)| < \infty$, where I is non-empty set, equipped with supremum norm

$$\|f\|_\infty := \sup_{i\in I} |f(i)|$$

and denote it by $\ell^\infty(I)$.

Recently, Bahrami, Bayati, Manjegani [5] introduced the notions of majorization and doubly stochastic operators on the Banach space of all bounded real sequences ℓ^∞. This Banach space is a particular case of Banach space $\ell^\infty(I)$, when I is a countable set, that is, when $I = \mathbb{N}$. In the Section 3, we will generalize this notions when I may be an uncountable set.

Definition 2.3. [5, Definition 2.1.] Let $A : \ell^\infty \longrightarrow \ell^\infty$ be a linear and bounded operator. The operator A is called *doubly stochastic* if there is a doubly stochastic operator $A_0 \in DS(\ell^1)$ such that $A = A_0^*$.

Definition 2.4. [5, Definition 2.4.] For two elements $f, g \in \ell^\infty$, we say that f is *majorized* by g, if there exists a doubly stochastic operator $D \in DS(\ell^\infty)$, such that $f = Dg$, and denote it by $f \prec g$.

Now, we present useful results that we will use in this paper.

Lemma 2.5. [4, Lemma 2.3.] *Let $p \in [1, \infty)$ and $A : \ell^p(I) \longrightarrow \ell^p(I)$ be a positive bounded linear operator. Then*

- *A is row stochastic, if and only if*

$$\forall f \in \ell^1(I), \qquad \sum_{j\in I} \langle Ae_j, f\rangle = \sum_{i\in I} f(i)$$

- *A is column stochastic, if and only if*

$$\forall f \in \ell^1(I), \qquad \sum_{i\in I} \langle Af, e_i\rangle = \sum_{i\in I} f(i)$$

Theorem 2.6. [4, Theorem 2.4.] *Let $p \in [1, \infty)$. If A and B belong to $RS(\ell^p(I))$, then so does AB, i.e. the set $RS(\ell^p(I))$ is closed under the composition. The same conclusion holds for sets $CS(\ell^p(I))$ and $DS(\ell^p(I))$.*

Theorem 2.7. [4, Theorem 3.5.] *For $f, g \in \ell^p(I)$, $p \in [1, \infty)$ the following conditions are equivalent:*

- *$f \prec g$ and $g \prec f$.*
- *There exists a permutation $P \in P(\ell^p(I))$ such that $f = Pg$.*

3. Majorization and Doubly Stochastic Operators

We prove the following results.

Lemma 3.1. *The majorization relation "$\prec$" in Definition 2.2, when $p \in [1, \infty)$, is reflexive and transitive relation i.e. "$\prec$" is a pre-order. In particular, if we identify all functions which are different up to the permutation then we may consider "$\prec$" as a partial order.*

Proof. For any $f \in \ell^p(I)$, $f = If$ implies $f \prec f$, because the identity operator I is doubly stochastic, so "$\prec$" is reflexive.

Transitivity follows from Theorem 2.6. Precisely, if $f \prec g$ and $g \prec h$ then there are $A, B \in DS(\ell^p(I))$ such that $f = Ag$, $g = Bh$ so $f = ABh$. Since $DS(\ell^p(I))$ is closed under the composition hence $f \prec h$.

If we identify all function which are different up to the permutation, then it follows directly from Theorem 2.7 that "$\prec$" is antisymmetric. □

In the next definitions we will introduce row and column stochastic operators, and we also extend the notions of doubly stochastic operators and majorization on $\ell^\infty(I)$.

Definition 3.2. Let $A : \ell^\infty(I) \longrightarrow \ell^\infty(I)$ be a bounded linear operator. The operator A is called

- *row stochastic*, if there is a column stochastic operator $A_0 \in CS(\ell^1(I))$ such that $A = A_0^*$.
- *column stochastic*, if there is a row stochastic operator $A_0 \in RS(\ell^1(I))$ such that $A = A_0^*$.
- *doubly stochastic*, if there is a doubly stochastic operator $A_0 \in DS(\ell^1(I))$ such that $A = A_0^*$.

As we know, $A = A_0^*$ if for every $f \in \ell^\infty(I)$ and for every $g \in \ell^1(I)$, $\langle g, Af\rangle = \langle A_0 g, f\rangle$, where $\langle \cdot,\cdot\rangle : \ell^1(I) \times \ell^\infty(I) \longrightarrow \mathbb{R}$ denotes the dual pairing between $\ell^1(I)$ and its dual space $\ell^\infty(I)$.

The set of all row stochastic, column stochastic and doubly stochastic operators on $\ell^\infty(I)$ will denote, respectively, by $RS(\ell^\infty(I))$, $CS(\ell^\infty(I))$ and $DS(\ell^\infty(I))$.

Theorem 3.3. *Let $p \in [1,\infty]$. Then, $RS(\ell^p(I))$, $CS(\ell^p(I))$ and $DS(\ell^p(I))$ are convex sets.*

Proof. Let $p \in [1,\infty]$ and $A, C \in RS(\ell^p(I))$. We claim that $B = tA + (1-t)C \in RS(\ell^p(I))$ for each $t \in [0,1]$. For $t = 0$ or $t = 1$, $B \in RS(\ell^p(I))$ obviously. Let $t \in (0,1)$. It is easy to see that B is a bounded linear operator on $\ell^p(I)$.

Firstly, let $p \in [1,\infty)$. Since A and C are positive operators, if $f \in \ell^p(I)$, $f \geq 0$, then $Af \geq 0$, $Cf \geq 0$, so B is also positive. Now, for arbitrary $i \in I$ we have

$$\begin{aligned}\sum_{j\in I}\langle Be_j, e_i\rangle &= \sum_{j\in I}\langle (tA + (1-t)C)e_j, e_i\rangle \\ &= t\sum_{j\in I}\langle Ae_j, e_i\rangle + (1-t)\sum_{j\in I}\langle Ce_j, e_i\rangle = 1\end{aligned} \tag{2}$$

since A and C are row stochastic, hence B is row stochastic, so $RS(\ell^p(I))$ is convex set.

We now turn to the case $p = \infty$. As we know, there are $A_0, C_0 \in CS(\ell^1(I))$ such that $A = A_0^*$ and $C = C_0^*$, by Definition 3.2. Combining these facts and the similar argument as in (2), we obtain

$$\begin{aligned}\sum_{j\in I}\langle Be_j, e_i\rangle &= t\sum_{j\in I}\langle A_0^* e_j, e_i\rangle + (1-t)\sum_{j\in I}\langle C_0^* e_j, e_i\rangle \\ &= t\sum_{j\in I}\langle e_j, A_0 e_i\rangle + (1-t)\sum_{j\in I}\langle e_j, C_0 e_i\rangle = 1,\end{aligned}$$

so $RS(\ell^\infty(I))$ is convex set.

In the same way we conclude that if $A, C \in CS(\ell^p(I))$, $p \in [1,\infty]$, then $B \in CS(\ell^p(I))$ so $CS(\ell^p(I))$ is convex set. Clearly, $DS(\ell^p(I))$ is convex set. □

Theorem 3.4. *The set $RS(\ell^\infty(I))$ is closed under the composition, i.e. if $A, B \in RS(\ell^\infty(I))$, then $AB \in RS(\ell^\infty(I))$. The same conclusion holds for sets $CS(\ell^\infty(I))$ and $DS(\ell^\infty(I))$.*

Proof. Let $A, B \in RS(\ell^\infty(I))$. There exist $A_0, B_0 \in CS(\ell^1(I))$ such that $A = A_0^*$ and $B = B_0^*$ by Definition 3.2. We obtain

$$\sum_{j\in I}\langle ABe_j, e_i\rangle = \sum_{j\in I}\langle A_0^* B_0^* e_j, e_i\rangle = \sum_{j\in I}\langle B_0^* e_j, A_0 e_i\rangle = \sum_{j\in I}\langle e_j, B_0 A_0 e_i\rangle = 1$$

because $B_0 A_0 \in CS(\ell^1(I))$ by Theorem 2.6. The rest is left to the reader. □

Definition 3.5. For two elements $f, g \in \ell^\infty(I)$, we say that f is *majorized* by g, if there exists a doubly stochastic operator $D \in DS(\ell^\infty(I))$, such that $f = Dg$, and denote it by $f \prec g$.

Lemma 3.6. *The majorization relation "$\prec$" introduced in Definition 3.5, is reflexive and transitive relation i.e. "$\prec$" is a pre-order.*

Proof. Obviously, reflexivity holds. Transitivity follows from Theorem 3.4. □

We can not consider majorization relation "$\prec$" on $\ell^\infty(I)$ as a partial order, because in [5, Example 2.6.] there are specific functions such that $f \prec g$ and $g \prec h$ do not follow that f and g are permutations of each other.

Lemma 3.7. *Let $p \in [1,\infty)$ and $A : \ell^p(I) \longrightarrow \ell^p(I)$ be a bounded linear operator. If $A \in DS(\ell^p(I))$ then $Af \prec f$, $\forall f \in \ell^p(I)$. Conversely, if $Af \prec f$, $\forall f \in \ell^p(I)$, then $A \in CS(\ell^p(I))$.*

Proof. Let $A \in DS(\ell^p(I))$. If we define $D := A$ it is easy to see that $Af = Df$ and $D \in DS(\ell^p(I))$, therefore $Af \prec f, \forall f \in \ell^p(I)$.

On the other hand, let $Af \prec f$, $\forall f \in \ell^p(I)$. Firstly, we will show that the operator A is positive. Suppose that there exists $0 \leq f \in \ell^p(I)$ such that $Af \geq 0$ is not true. If f has a representation $f = \sum_{i\in I}\langle f, e_i\rangle e_i$, then $Af = \sum_{i\in I}\langle f, e_i\rangle Ae_i$ by continuity of A. Then, from $\langle f, e_i\rangle \geq 0$, $\forall i \in I$, there exists $m \in I$, such that $Ae_m \geq 0$ is not true. However, for e_m exists $D_m \in DS(\ell^p(I))$ such that $Ae_m = De_m \geq 0$ because $e_m \geq 0$ and D is positive. This is contradiction, so the operator A is positive.

Again, let $Af \prec f$, $\forall f \in \ell^p(I)$. Therefore, $Ae_j \prec e_j$, $\forall j \in I$, and there exist $D_j \in DS(\ell^p(I))$ such that $D_je_j = Ae_j$, $\forall j \in I$. Let $j \in I$ is arbitrary chosen. Then, $\sum_{i\in I}\langle Ae_j, e_i\rangle = \sum_{i\in I}\langle D_je_j, e_i\rangle = 1$, because $D_j \in DS(\ell^p(I)) \subset CS(\ell^p(I))$, so $A \in CS(\ell^p(I))$. □

The next theorem gives the necessary and sufficient conditions for a bounded linear operator to be doubly stochastic.

Theorem 3.8. *Let $p \in [1,\infty)$ and $A : \ell^p(I) \longrightarrow \ell^p(I)$ be a bounded linear operator. $A \in DS(\ell^p(I))$ if and only if $Af \prec f$, $\forall f \in \ell^p(I)$ and $A^*g \prec g$, $\forall g \in \ell^q(I)$, where q is the conjugate exponent of p.*

Proof. Let $A \in DS(\ell^p(I))$ where $p \in [1,\infty)$ is arbitrary chosen. Obviously, $Af \prec f$, $\forall f \in \ell^p(I)$, by Lemma 3.7.

If $p \in (1,\infty)$ then $\langle A^*e_i, e_j\rangle = \langle Ae_j, e_i\rangle \geq 0$, $\forall i, j \in I$, and

$$A^*g = \sum_{i\in I}\langle g, e_i\rangle A^*e_i \geq 0,$$

where $0 \leq g \in \ell^q(I)$. Thus, the operator A^* is positive. Using definitions of adjoint operator and doubly stochastic operators it is easy to see that $A^* \in DS(\ell^q(I))$. Hence $A^*g \prec g$, $\forall g \in \ell^q(I)$ by Lemma 3.7. If $p = 1$ and $q = \infty$ then $A^* \in DS(\ell^\infty(I))$ by Definition 3.2 of doubly stochastic operators on $\ell^\infty(I)$ and fact that $A \in DS(\ell^1(I))$ by assumption. Now, for $D := A^*$ is $A^*g = Dg$, $\forall g \in \ell^\infty(I)$, so $A^*g \prec g$, $\forall g \in \ell^\infty(I)$, by Definition 3.5.

Conversely, let $Af \prec f$, $\forall f \in \ell^p(I)$, $p \in [1,\infty)$, and $A^*g \prec g$, $\forall g \in \ell^q(I)$. Again, using Lemma 3.7 is $A \in CS(\ell^p(I))$. It remains to be shown that $A \in RS(\ell^p(I))$. If $p \neq 1$ then $q \neq \infty$, hence $A^* \in CS(\ell^q(I))$, by Lemma 3.7. Moreover, $\sum_{j\in I}\langle Ae_j, e_i\rangle = \sum_{j\in I}\langle e_j, A^*e_i\rangle = 1$, $\forall i \in I$ so $A \in RS(\ell^p(I))$. If $p = 1$ and $q = \infty$, then there are $D_i \in DS(\ell^\infty(I))$ and $B_i \in DS(\ell^1(I))$, $\forall i \in I$, such that $D_ie_i = A^*e_i$ and $D_i = B_i^*$, $\forall i \in I$. Therefore,

$$\sum_{j\in I}\langle Ae_j, e_i\rangle = \sum_{j\in I}\langle e_j, A^*e_i\rangle = \sum_{j\in I}\langle e_j, D_ie_i\rangle = \sum_{j\in I}\langle B_ie_j, e_i\rangle = 1$$

$\forall i \in I$, so $A \in RS(\ell^1(I))$. It follows that $A \in DS(\ell^p(I))$, $p \in [1,\infty)$. □

Corollary 3.9. *Let $p \in [1,\infty)$ and $A : \ell^p(I) \longrightarrow \ell^p(I)$ be a bounded linear operator. If $A \in DS(\ell^p(I))$ then $A^* \in DS(\ell^q(I))$, where q is the conjugate exponent of p.*

Proof. Follows directly from proof of Theorem 3.8. □

Corollary 3.10. *Let $p \in [1,\infty)$ and $A : \ell^p(I) \longrightarrow \ell^p(I)$ be a bounded linear operator. If $Af \prec f$, $\forall f \in \{e_i, i \in I\}$, then $A \in CS(\ell^p(I))$. If additionally, $A^*f \prec f$, $\forall f \in \{e_i, i \in I\}$, then $A \in DS(\ell^p(I))$.*

Proof. Analysing the proofs in above results, Lemma 3.7 and Theorem 3.8, it is easy to check that $Af \prec f$ and $A^*f \prec f$, where $f \in \{e_i, i \in I\}$ are sufficient conditions that operator A is doubly stochastic. □

In finite dimensional case, when $card(I) = n \in \mathbb{N}$ and when $p = 2$, a doubly stochastic operator A becomes a doubly stochastic matrix $A = [a_{ij}] \in M_n$ and $A^* = A^T$. If $Ax \prec x$, for all $x \in \mathbb{R}^n$, applying the majorization relation to $e = (1, 1, \ldots, 1)$, we know that $Ae \prec e$, so $Ae = e$ and $\sum_{j=1}^{n} a_{ij} = 1$, $\forall i \in \mathbb{N}$. Because of this fact and using Lemma 3.7, A is a doubly stochastic matrix so A^* is a doubly stochastic, too, by Corollary 3.9. Now using Lemma 3.7 is $A^*x \prec x$, for all $x \in \mathbb{R}^n$. Thus, the following Corollary 3.11 is a consequence of Theorem 3.8. Notice that Theorem 1.1 is the same as Corollary 3.11.

Corollary 3.11. *An $n \times n$ matrix A is doubly stochastic if and only if $Ax \prec x$ for all vectors $x \in \mathbb{R}^n$.*

Theorem 3.12. *Let $p \in [1,\infty)$ and $P : \ell^p(I) \longrightarrow \ell^p(I) \in DS(\ell^p(I))$. If P is invertible operator and $P^{-1} \in DS(\ell^p(I))$ then P is permutation.*

Proof. Since $P \in DS(\ell^p(I))$ hence $Pf \prec f$ by Lemma 3.7, for every $f \in \ell^p(I)$. Similarly, $P^{-1} \in DS(\ell^p(I))$ and $f = P^{-1}Pf$ implies $f \prec Pf$, for every $f \in \ell^p(I)$. It follows by Theorem 2.7 that there exist permutations $Q_f \in DS(\ell^p(I))$ such that $Q_f f = Pf$, $\forall f \in \ell^p(I)$.

Suppose that there are $r, s \in I$ such that $\langle Pe_s, e_r\rangle \in (0, 1)$. For e_s, there exists a permutation $Q_{e_s} \in P(\ell^p(I))$ for which $Pe_s = Q_{e_s}e_s = e_k$, for any $k \in I$.

Now, we obtain

$$e_k(r) = (Pe_s)(r) = \sum_{j\in I} e_s(j)\langle Pe_j, e_r\rangle = \langle Pe_s, e_r\rangle \in (0, 1),$$

which is a contradiction. Thus, $\langle Pe_j, e_i\rangle \in \{0, 1\}$, $\forall i, j \in I$. Since $P \in DS(\ell^p(I))$ we get for every $i \in I$ that there is only one $j \in I$ such that $\langle Pe_j, e_i\rangle = 1$, and conversely for every $j \in I$ there is only one $i \in I$ such that $\langle Pe_j, e_i\rangle = 1$. We define the function $\theta : I \longrightarrow I$ to be $\theta(j) = i$, if $\langle Pe_j, e_i\rangle = 1$. It is easy to check that θ is a bijection and $Pe_j = e_{\theta(j)}$, hence P is a permutation. □

4. Generalized Kakutani's Conjecture

We will introduce a majorization relation between doubly stochastic operators on $\ell^p(I)$, where $p \in [1,\infty)$, and thus restate Kakutani's conjecture for these operators.

Definition 4.1. Let $p \in [1,\infty)$, and let $DS(\ell^p(I))$ be the set of all doubly stochastic operators. For two operators $D_1, D_2 \in DS(\ell^p(I))$ we say that D_1 is *majorized* by D_2 and denote it by $D_1 \lhd D_2$, if there exists a doubly stochastic operator $D_3 \in DS(\ell^p(I))$ such that $D_1 = D_3D_2$.

Lemma 4.2. *The majorization relation "$\lhd$" introduced in* Definition 4.1 *is reflexive and transitive relation i.e. "$\lhd$" is a pre-order. In particular, if we consider only surjective operators and identify all operators which are different up to the permutation then "$\lhd$" is antisymmetric relation on this subset of $DS(\ell^p(I))$, so we may consider "$\lhd$" as a partial order.*

Proof. Notice that $D = ID$, $\forall D \in DS(\ell^p(I))$, where I is the identity operator, which is obviously doubly stochastic, hence "$\lhd$" is reflexive. Let $D_1, D_2, D_3 \in DS(\ell^p(I))$. If $D_1 \lhd D_2$ and $D_2 \lhd D_3$ then there are $D_4, D_5 \in DS(\ell^p(I))$ such that $D_1 = D_4D_2$ and $D_2 = D_5D_3$. Now, $D_1 = D_4D_5D_3$ so $D_1 \lhd D_3$ by Theorem 2.6. Similarly, if $D_1 \lhd D_2$ and $D_2 \lhd D_1$ then $D_1 = ABD_1$ and $D_2 = BAD_2$, where $D_1 = AD_2$, $D_2 = BD_1$ and $A, B \in DS(\ell^p(I))$. If $R(D_1) = R(D_2) = \ell^p(I)$ then $A = B^{-1}$, so $A, B \in P(\ell^p(I))$ by Theorem 3.12. □

If $D_1 \lhd D_2$, then by definition there exists $D_3 \in DS(\ell^p(I))$ such that $D_1 = D_3D_2$ i.e. $D_1f = D_3D_2f$, $\forall f \in \ell^p(I)$, so $D_1f \prec D_2f$, $\forall f \in \ell^p(I)$. However, the opposite direction is more complicated and it does not hold in general for an arbitrary operator $D_2 \in DS(\ell^p(I))$.

Conjecture 4.3. *Let $p \in [1,\infty)$ and $D_1, D_2 \in DS(\ell^p(I))$. If for every $f \in \ell^p(I)$ $D_1 f \prec D_2 f$, then $D_1 \lhd D_2$.*

This conjecture is not true in general, because the invertibility of operator D_2 is essential, and necessity of this condition in finite dimensional case is presented in [19] by an counterexample. In accordance with that, the operator D_2 has to be invertible in our generalization, too. If we show that the existence of the doubly stochastic operators $D_f^3 \in DS(\ell^p(I))$, $\forall f \in \ell^p(I)$, such that $D_1 f = D_f^3 D_2 f$ implies the existence of the operator $D_3 \in DS(\ell^p(I))$ for which $D_1 = D_3 D_2$ is true, then the work is done. In the next theorem we will prove our conjecture if the operator D_2^{-1} is row stochastic.

Theorem 4.4. *Let $p \in [1,\infty)$, $D_1, D_2 \in DS(\ell^p(I))$, D_2 be invertible operator and $D_2^{-1} \in RS(\ell^p(I))$. If for every $f \in \ell^p(I)$ $D_1 f \prec D_2 f$, then $D_1 \lhd D_2$.*

Proof. Assume that $D_1 f \prec D_2 f$, for every $f \in \ell^p(I)$. We know that for arbitrary $g \in R(D_2)$ there exists $f \in \ell^p(I)$ such that $g = D_2 f$.

Let the map $\Psi : R(D_2) \longrightarrow \ell^p(I)$, be defined by

$$\Psi g = D_1 f, \ \ \forall f \in \ell^p(I)$$

where $g = D_2 f$. The domain of the map Ψ is the entire Banach space $\ell^p(I)$, because the operator D_2 is invertible. Thus, $R(D_2) = \ell^p(I)$ and $\Psi : \ell^p(I) \longrightarrow \ell^p(I)$, $p \in [1,\infty)$. Choose arbitrary $f, f_1, f_2 \in \ell^p(I)$ and fix $g = D_2 f$, $g_1 = D_2 f_1$ and $g_2 = D_2 f_2$.

Let us first check that the map Ψ is well-defined. Suppose that $g_1 = g_2$. We need to show that $\Psi g_1 = D_1 f_1 = D_1 f_2 = \Psi g_2$. From $D_2 f_1 = D_2 f_2$, it follows $D_2(f_1 - f_2) = 0$ by linearity of D_2, and $D_1(f_1 - f_2) \prec D_2(f_1 - f_2)$ by our assumption. Now, $D_1(f_1 - f_2) = 0$ so $\Psi g_1 = \Psi g_2$.

Using linearity of D_1 and D_2 we obtain

$$\Psi(\alpha g) = \Psi(D_2(\alpha f)) = D_1(\alpha f) = \alpha \Psi g$$

where $\alpha \in \mathbb{R}$.

$$\Psi(g_1 + g_2) = \Psi(D_2(f_1 + f_2)) = D_1(f_1 + f_2) = D_1 f_1 + D_1 f_2 = \Psi g_1 + \Psi g_2$$

so we conclude that the map Ψ is linear.

$$\begin{aligned} ||\Psi(g)|| &= ||\Psi(D_2 f)|| = ||D_1 f|| \le ||D_1|| ||f|| = ||D_1|| ||D_2^{-1} g|| \\ &\le ||D_1|| ||D_2^{-1}|| ||g||. \end{aligned}$$

It follows that $\frac{||\Psi(g)||}{||g||} \le ||D_1|| ||D_2^{-1}||$ and $||\Psi|| \le ||D_1|| ||D_2^{-1}||$, so we obtain that the map Ψ is bounded. Moreover, $||\Psi|| \le ||D_2^{-1}||$ because the norm of a doubly stochastic operator is at most 1, by [4, Lemma 2.5].

If additionally $g = D_2 f \ge 0$, it follows that exists $D_f^3 \in DS(\ell^p(I))$ such that $\Psi(g) = D_1 f = D_f^3 D_2 f = D_f^3 g \ge 0$, because the operator D_f^3 is positive, so Ψ is positive.

Let $j \in I$ be arbitrary chosen, and $a_j = D_2^{-1} e_j \in \ell^p(I)$. Then for this a_j, there exists $D_{a_j}^3 \in DS(\ell^p(I))$ such that $D_1 a_j = D_{a_j}^3 D_2 a_j$, i.e. $D_1 D_2^{-1} e_j = D_{a_j}^3 e_j$. Now,

$$\sum_{i \in I} \langle \Psi e_j, e_i \rangle = \sum_{i \in I} \langle D_1 D_2^{-1} e_j, e_i \rangle = \sum_{i \in I} \langle D_{a_j}^3 e_j, e_i \rangle = 1$$

since $D_{a_j}^3 \in DS(\ell^p(I))$, hence $\Psi \in CS(\ell^p(I))$. The operator D_1^* is doubly stochastic by Corollary 3.9, and $\sum_{k \in I} D_1^* e_i(k) = \sum_{k \in I} \langle D_1^* e_i, e_k \rangle = \sum_{k \in I} \langle e_i, D_1 e_k \rangle = 1$, so $D_1^* e_i \in \ell^1(I)$, $\forall i \in I$.

$$\begin{aligned} \sum_{j \in I} \langle \Psi e_j, e_i \rangle &= \sum_{j \in I} \langle D_1 D_2^{-1} e_j, e_i \rangle = \sum_{j \in I} \langle D_2^{-1} e_j, D_1^* e_i \rangle \\ &= \sum_{k \in I} D_1^* e_i(k) = \sum_{k \in I} \langle D_1^* e_i, e_k \rangle \\ &= \sum_{k \in I} \langle e_i, D_1 e_k \rangle = 1, \ \ \forall i \in I, \end{aligned}$$

where

$$\sum_{j\in I}\langle D_2^{-1}e_j, D_1^*e_i\rangle = \sum_{k\in I} D_1^*e_i(k) \tag{3}$$

by Lemma 2.5, because $D_1^*e_i \in \ell^1(I)$ and D_2^{-1} is a row stochastic operator. Finally, $\Psi \in DS(\ell^p(I))$, thus $D_3 := \Psi$ is a desired operator for which $D_1 = D_3D_2$. Is follows that $D_1 \triangleleft D_2$. □

Suppose that $D_2^{-1} \notin RS(\ell^p(I))$. In this case, there exists $g \in \ell^1(I)$ such that $\sum_{j\in I}\langle D_2^{-1}e_j, g\rangle \neq \sum_{k\in I} g(k)$ by Lemma 2.5. If there is a doubly stochastic operator D_1 such that $D_1^*e_i = g$ (or $(D_1^{-1})^*g = e_i$ if D_1 is invertible) for any $i \in I$, then $\sum_{j\in I}\langle \Psi e_j, e_i\rangle \neq 1$ by (3). Therefore, under the last assumption, the condition $D_2^{-1} \in RS(\ell^p(I))$ can not be omitted in the last theorem. In accordance with the above consideration, we pose the following question:

Question: Is it true that for any given $D_1, D_2 \in DS(\ell^p(I))$ where D_2 is invertible, the majorization $D_1 f \prec D_2 f$ for all $f \in \ell^p(I)$, implies $D_1 \triangleleft D_2$?

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