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MAJORIZATION IN MULTIVARIATE DISTRIBUTIONS

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In case the joint density f of $X = (X_1, \cdots, X_n)$ is Schur-concave (is an order-reversing function for the partial ordering of majorization), it is shown that $P(X \in A + \theta)$ is a Schur-concave function of θ whenever A has a Schur-concave indicator function. More generally, the convolution of Schur-concave functions is Schur-concave.

The condition that f is Schur-concave implies that $X_1, \cdots, X_n$ are exchangeable. With exchangeability, the multivariate normal and certain multivariate "t", beta, chi-square, "F" and gamma distributions have Schur-concave densities. These facts lead to a number of useful inequalities.

In addition, the main result of this paper can also be used to show that various non-central distributions (chi-square, "t", "F") are Schur-concave in the noncentrality parameter.

1. Introduction. For exchangeable random variables $X_1, X_2, \cdots, X_n$, probabilities of the form

$$P\{(X_1 - \theta_1, \cdots, X_n - \theta_n) \in A\} \equiv P\{X \in A + \theta\}$$

often exhibit a monotonicity property in values of θ partially ordered according to majorization. We obtain some theory for such monotonicity which is sufficiently general to yield a number of interesting examples. Included in the study are some standard models for dependence and discussions of several non-central distributions.

In n dimensions the vector a is said to be majorized by the vector b (written $a \prec b$) if, upon reordering components to achieve

$$a_1 \geqq a_2 \geqq \cdots \geqq a_n, \qquad b_1 \geqq b_2 \geqq \cdots \geqq b_n,$$

it follows that

$$\textstyle\sum_{i=1}^k a_i \leqq \sum_{i=1}^k b_i, \quad i = 1, 2, \cdots, n-1, \qquad \sum_{i=1}^n a_i = \sum_{i=1}^n b_i.$$

See, e.g., Hardy, Littlewood and Pólya (1952), page 49, or Berge (1963), page 184. Functions φ for which $a \prec b$ implies $\varphi(a) \leqq \varphi(b)$ are said to be Schur-convex; if $\varphi(a) \geqq \varphi(b)$ they are called Schur-concave. Such functions are permutation-symmetric, i.e., invariant under permutations of components of the argument. A necessary and sufficient condition that a permutation-symmetric

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differentiable function be Schur-concave is that

$$\left(\frac{\partial\varphi(x)}{\partial x_i} - \frac{\partial\varphi(x)}{\partial x_j}\right)(x_i - x_j) \leqq 0 \tag{1}$$

for all $i \neq j$ (Schur (1923), and Ostrowski (1952)). The inequality is reversed for Schur-convex functions.

It is easily verified that

$$(a_1, a_2, \cdots, a_n) \succ (\sum a_i/n)(1, 1, \cdots, 1)$$

for all vectors a. Thus, when $\sum a_i$ is fixed, a Schur-concave function achieves a maximum at the point where components are equal. Another useful observation is that

$$(1, 0, \cdots, 0) \succ \tfrac{1}{2}(1, 1, 0, \cdots, 0) \succ \tfrac{1}{3}(1, 1, 1, 0, \cdots, 0) \succ \cdots \succ \frac{1}{n}(1, 1, \cdots, 1).$$

Most of the results of this paper identify specific Schur-concave functions; from this, the reader can construct various inequalities by evaluating the functions at pairs of points ordered by majorization. The results are nearly all consequences of one basic theorem. Thus, we shall identify Schur-concave functions by indicating situations under which the conditions of the theorem are satisfied.

2. The main theorem.

Most results of this paper are obtained from the following theorem.

2.1. THEOREM. *Suppose that the random variables $X_1, X_2, \cdots, X_n$ have a joint density f that is Schur-concave. If $A \subset R^n$ is a Lebesgue-measurable set which satisfies*

$$y \in A \quad \text{and} \quad x \prec y \Rightarrow x \in A, \tag{2}$$

then

$$\int_{A+\theta} f(x)\, dx = P\{X \in A + \theta\}$$

is a Schur-concave function of θ.

The condition that f is Schur-concave implies that f is permutation-symmetric; this is just the condition that the random variables $X_1, X_2, \cdots, X_n$ are exchangeable.

Condition (2) is satisfied whenever A is a permutation-symmetric convex set. This follows from the fact that (i) $x \prec y$ implies $x = yD$ for some doubly stochastic matrix D (Hardy, Littlewood and Pólya (1952) page 49), and (ii) the set of doubly stochastic matrices is the convex hull of the permutation matrices (Birkhoff's Theorem—see, e.g., Mirsky (1963)). But (2) does not imply that A is convex; indeed, if A_1 and A_2 satisfy (2), then so does $A_1 \cup A_2$. Condition (2) does not even imply measurability.

The condition that f is Schur-concave is equivalent to the condition that for each constant c, $\{y : f(y) \geqq c\}$ satisfies (2). Thus, the condition is satisfied if $\{y : f(y) \geqq c\}$ is convex (called *unimodal* by Anderson (1955)), and f is permutation-symmetric.

We mention these facts partly to point out the distinction between the above theorem and a result of Mudholkar (1966), which generalizes a theorem of Anderson (1955). From this generalization, Mudholkar obtains as a special case the conclusion of the above theorem, but with the additional requirements that A and sets of the form $\{y : f(y) \geqq c\}$ are convex. An advantage of our weaker condition is that it is often much easier to check than is convexity. Moreover, the convexity condition is too strong for certain applications.

Theorem 2.1 is of special interest in the case that $X_1, X_2, \cdots, X_n$ are independent and have a common marginal density g. In this case, the joint density f is given by $f(x) = \prod g(x_i)$.

2.2. REMARK. If $f(x) = \prod_{i=1}^n g(x_i)$, then f is Schur-concave if and only if $\log g$ is concave. When g is a differentiable function, the Schur-concavity of f can easily be checked using condition (1). This result is of particular interest in view of a result of Mudholkar (1969) to the effect that $f(x) = \prod g(x_i)$ is unimodal (i.e., sets of the form $\{y : f(y) \geqq c\}$ are convex) if g is log-concave. Consequently, the converse of Mudholkar's result is true. Moreover, in the case of independence, our condition that f is Schur-concave coincides with the condition of Mudholkar (1966) that f is unimodal.

There is a corollary to Theorem 2.1 worth mentioning.

2.3. COROLLARY. *If f_1 and f_2 are nonnegative integrable Schur-concave functions defined on $\mathscr{R}^n$ then their convolution*

$$f(\theta) = \int_{\mathscr{R}^n} f_1(x) f_2(\theta - x)\, dx$$

is Schur-concave.

PROOF. Since f_2 is Schur-concave, $\hat{f}_2(x) \equiv f_2(-x)$ is Schur-concave, and by Theorem 2.1,

$$\int_{A+\theta} f_2(-x)\, dx = \int_{\mathscr{R}^n} I_A(x) f_2(\theta - x)\, dx$$

is a Schur-concave function of θ. To complete the proof, approximate f_1 by an increasing sequence of simple functions $\varphi_k = \sum \alpha_i I_{A_i}$, where the sets A_i satisfy the conditions of Theorem 2.1, and then use Lebesgue's monotone convergence theorem. □

The above corollary shows that the class $\mathscr{F}$ of all nonnegative integrable Schur-concave functions on $\mathscr{R}^n$ is closed under convolutions. The class $\mathscr{F}$ is a convex cone closed also under the formations of maxima, minima and products as well as monotone nonnegative transformations.

2.4. COROLLARY. *If the exchangeable random variables $X_1, X_2, \cdots, X_n$ have a joint density which is Schur-concave, and if φ is also permutation-symmetric, nonnegative and Schur-concave, then*

$$E\varphi(X - \theta) \quad \text{and} \quad P\{\varphi(X - \theta) \geqq c\}$$

are Schur-concave functions of θ.

To prove that $P\{\varphi(X-\theta) \geqq c\}$ is Schur-concave, let $A = \{u\colon -\varphi(u) \leqq c\}$. Then, because $-\varphi$ is Schur-convex, $x \prec y$ and $y \in A$ implies $x \in A$. Thus, by Theorem 2.1,

$$\int_{A+\theta} f(x)\,dx = P\{(X_1, \cdots, X_n) \in A + \theta\} = P\{X - \theta \in A\}$$
$$= P\{-\varphi(X-\theta) \leqq c\} = P\{\varphi(X-\theta) \geqq c\}$$

is Schur-concave. □

Of course, the condition that $P\{\varphi(X-\theta) \geqq c\} \leqq P\{\varphi(X-\xi) \geqq c\}$ for all c implies $E\varphi(X-\theta) \leqq E\varphi(X-\xi)$, so that there is no need to offer an independent proof that $E\varphi(X-\theta)$ is Schur-concave. Although the Schur-concavity of $E\varphi(X-\theta)$ follows from that of $P\{\varphi(X-\theta) \geqq c\}$, one can take φ to be the indicator function I_A of the set A to obtain Theorem 2.1 from the fact that $E\varphi(X-\theta)$ is Schur-concave.

Corollary 2.3 was also obtained by Mudholkar (1969) with the stronger hypothesis that f is permutation-symmetric and unimodal, and $\{u\colon \varphi(u) \geqq c\}$ is convex for all c.

Proof of Theorem 2.1. We must show that if $\theta \prec \xi$, then

$$\int_{A+\theta} f(x)\,dx \geqq \int_{A+\xi} f(x)\,dx\,. \tag{3}$$

Because f and A are permutation-symmetric,

$$\int_{A+\theta} f(x)\,dx = \int_{A+\pi(\theta)} f(x)\,dx$$

for all permutations π. Thus, we can assume that

$$\theta_1 \geqq \theta_2 \geqq \cdots \geqq \theta_n \qquad \text{and} \qquad \xi_1 \geqq \xi_2 \geqq \cdots \geqq \xi_n\,.$$

According to a result of Hardy, Littlewood and Pólya (1952) θ can be derived from ξ by a finite number of pairwise averages of components. Consequently, we can assume that $\theta_i = \xi_i$ if $i \neq j$ or k, where $j < k$. Of course, $\theta_j + \theta_k = \xi_j + \xi_k = 2\delta$, say. If $\xi_j = \delta + \alpha$ and $\theta_j = \delta + \beta$, then $\xi_k = \delta - \alpha$, $\theta_k = \delta - \beta$ and $\alpha > \beta > 0$.

Let $u = x_j + x_k$, and $v = x_j - x_k$. To obtain (3), integrate first on v, conditionally with the other variables held fixed. Observe that

$$\left\{v\colon \left(x_1, \cdots, x_{j-1}, \frac{u+v}{2}, x_{j+1}, \cdots, x_{k-1}, \frac{u-v}{2}, x_{k+1}, \cdots, x_n\right) \in A + \xi\right\}$$
$$= \left\{v\colon \left(x_1 - \xi_1, \cdots, x_{j-1} - \xi_{j-1}, \frac{(u-2\delta)+(v-2\alpha)}{2}, x_{j+1} - \xi_{j+1}, \cdots,\right.\right.$$
$$\left.\left. x_{k-1} - \xi_{k-1}, \frac{(u-2\delta)-(v-2\alpha)}{2}, x_{k+1} - \xi_{k+1}, \cdots, x_n - \xi_n\right) \in A\right\}$$
$$\equiv B_\alpha\,.$$

The set B_0 is symmetric and convex. Moreover, as a function of v,

$$f(x) \equiv f\left(x_1, \cdots, x_{j-1}, \frac{u+v}{2}, x_{j+1}, \cdots, x_{k-1}, \frac{u-v}{2}, x_{k+1}, \cdots, x_n\right)$$

is symmetric and unimodal. Thus by the theorem of Anderson (1955) or of Wintner (1938),

$$\int_{B_\beta} f(x)\, dv \geqq \int_{B_\alpha} f(x)\, dv .$$

The inequality is preserved upon integrating out the remaining variables to yield (3). □

3. Bounds for distribution functions. Interesting applications of Theorem 2.1 are obtained with the sets

$$A_* = \{x : x_1 \leqq 0, \cdots, x_n \leqq 0\} \qquad \text{and} \qquad A^* = \{x : x_1 > 0, \cdots, x_n > 0\} .$$

In this way, we conclude that if the joint density is Schur-concave, then

$$F(x) = P(X_1 \leqq x_1, \cdots, X_n \leqq x_n) \qquad \text{and} \qquad \bar{F}(x) = P(X_1 > x_1, \cdots, X_n > x_n)$$

are Schur-concave in x.

Since $(x_1, \cdots, x_n) \succ (1, 1, \cdots, 1)\alpha/n$ where $\alpha = \sum x_i$ we obtain

Bound I.

$$F(\alpha, \alpha, \cdots, \alpha) \geqq F(x) \qquad \text{and} \qquad \bar{F}(\alpha, \alpha, \cdots, \alpha) \geqq \bar{F}(x) .$$

In case $x_i \geqq 0$ for all i, it is also true that $(\sum x_i, 0, \cdots, 0) \succ (x_1, \cdots, x_n)$. Thus, for nonnegative random variables X_i with a Schur-concave joint density,

Bound II.

$$F(x) \geqq F(\textstyle\sum x_i, 0, \cdots, 0) \qquad \text{and} \qquad \bar{F}(x) \geqq \bar{F}(\textstyle\sum x_i, 0, \cdots, 0) ;$$

equivalently, for $x_1 \geqq 0, \cdots, x_n \geqq 0$, and nonnegative random variables X_i,

$$P\{X_1 > x_1, \cdots, X_n > x_n\} \geqq P\{X_1 > \textstyle\sum x_i\} .$$

If attention is confined to the set A_* or A^*, conditions weaker than that of Theorem 2.1 can be obtained; Section 7 is devoted to a discussion of more general conditions under which F and $\bar{F}$ are Schur-concave.

In Section 4, we consider certain special cases where another useful bound on the distribution function can be obtained.

4. A direct verification. In some instances, one can directly verify the condition of Theorem 2.1 that a joint density is Schur-concave by using condition (1). One such case is given in the following proposition.

4.1. PROPOSITION. *If the joint density f has the form*

$$f(x) = g(x\Lambda x') ,$$

where g is a decreasing function and $\Lambda = (\lambda_{ij})$ is positive definite with $\lambda_{11} = \cdots = \lambda_{nn}$ and $\lambda_{ij} = \lambda$ when $i \neq j$, then f is Schur-concave.

Although this result is easily checked from (1), it can also be easily verified that $f(x) = g(x\Lambda x')$ is unimodal in the sense that sets of the form $\{x : g(x\Lambda x') \geqq c\}$ are convex. Consequently the result of Mudholkar (1966) applies in place of Theorem 2.1 for sets A that are permutation-symmetric and convex.

To check that $\{x: g(x\Lambda x') \geqq c\}$ is convex, suppose that $g(y\Lambda y') \geqq c$ and $g(z\Lambda z') \geqq c$. Then $y\Lambda y' \leqq g^{-1}(c)$ and $z\Lambda z' \leqq g^{-1}(c)$. Because $x\Lambda x'$ is convex in x, this implies $[\alpha y + (1-\alpha)z]\Lambda[\alpha y + (1-\alpha)z]' \leqq g^{-1}(c)$ and hence $g([\alpha y + (1-\alpha)z]\Lambda[\alpha y + (1-\alpha)z]') \geqq c$.

Multivariate normal distributions. If $X_1, \cdots, X_n$ are exchangeable and are jointly normally distributed, then Proposition 4.1 shows that their joint density is Schur-concave.

Multivariate "t" distribution. If $U_1, \cdots, U_n$ are exchangeable and jointly normally distributed, and if Z^2 is chi-square distributed ($Z \geqq 0$), then $X_1 = U_1/Z, \cdots, X_n = U_n/Z$ have a joint density f of the form $g(X\Lambda X')$ where $g(w)$ is proportional to $(1+w)^{-a}$, $a > 0$. Thus Proposition 4.1 again applies to show that f is Schur-concave.

Multivariate beta distribution. If $U_1, \cdots, U_n$ and Z are independent random variables with a chi-square distribution and $U_1, \cdots, U_n$ are identically distributed, then from Olkin and Rubin (1964), Theorem 3.3,

$$X_i = U_i/(\sum U_i + Z)\,, \qquad i = 1, 2, \cdots, n$$

have a multivariate beta distribution with joint density of the form

$$f(x) = k(\prod x_i^{r-1})(1 - \sum x_j)^{s-1}\,.$$

Here, condition (1) is easily verified directly, so that Theorem 2.1 applies.

5. Applications to mixtures of distributions and some models for dependency. Suppose that for each z in some set Ω, the density $f_z(x)$ is Schur-concave, i.e., f_z satisfies the conditions of Theorem 2.1. Then for any random variable Z taking values in Ω and having distribution G, the mixture

$$f(x) = \int_\Omega f_z(x)\, dG(z)$$

also satisfies the conditions of Theorem 2.1. This is an immediate consequence of the observation that the class of Schur-concave functions forms a convex cone. Some interesting examples arise directly from the fact that the class of densities satisfying the conditions of Theorem 2.1 is closed under the formation of mixtures. (In contrast, this closure does not hold for unimodal densities.)

A multivariate chi-square distribution. Suppose $S = (s_{ij})$ is the sample covariance matrix based on a sample of size $N \geqq p$ from a p-variate normal distribution with covariance matrix $\Sigma = (\sigma_{ij})$, $\sigma_{ii} = \sigma^2$, $\sigma_{ij} = \sigma^2\rho$, $i \neq j = 1, \cdots, p$. The joint density of $(s_{11}, \cdots, s_{pp})$ has the form of a mixture of independent chi-square densities (which are log concave). Consequently, the joint density of $s_{11}, \cdots, s_{pp}$ is Schur-concave. (The density can be expressed in terms of a multiple infinite series, and does not have a simple expression. When $p = 2$, the density was first obtained by Bose (1935), and expressed as a mixture by Siotani (1940).)

A particularly interesting case is that in which $f_z(x) = \prod g_z(x_i)$ for some univariate density g_z. Here, a random vector $X_{1,Z}, \cdots, X_{n,Z}$ with density f has components which are conditionally independent, given Z. We have already observed (Remark 2.2) that f_z satisfies the conditions of Theorem 2.1 if and only if $\log g_z$ is concave. Of course, conditional independence serves as a source of various models for dependence.

5.1. PROPOSITION. *Let $\varphi_z(u)$ be linear and increasing in u for all z. If $U_1, U_2, \cdots, U_n$ and Z are independent random variables, and if the U_i have a common log-concave density, then the joint density f of*

$$X_1 = \varphi_Z(U_1), \cdots, X_n = \varphi_Z(U_n)$$

satisfies the conditions of Theorem 2.1.

If the random variables $X_1, X_2, \cdots, X_n$ are generated as in Proposition 5.1, then one can give the following lower bounds for $F(x) = P(X_1 \leqq x_1, \cdots, X_n \leqq x_n)$ and $\bar{F}(x) = P(X_1 > x_1, \cdots, X_n > x_n)$:

Bounds III.

$$P(X_1 \leqq x_1, \cdots, X_n \leqq x_n) \geqq \prod_{i=1}^n P(X_i \leqq x_i) ,$$
$$P(X_1 > x_1, \cdots, X_n > x_n) \geqq \prod_{i=1}^n P(X_i > x_i) .$$

Before comparing the bounds II with III, we give a proof of III.

Esary, Proschan and Walkup (1967) show that III holds if $X_1, X_2, \cdots, X_n$ are *associated*, in the sense that for all real-valued increasing functions φ, ψ of n arguments,

$$\text{Cov}\,(\varphi(X_1, \cdots, X_n), \psi(X_1, \cdots, X_n)) \geqq 0 .$$

Since the random variables $X_1, \cdots, X_n$ of Proposition 5.1 are conditionally independent, given Z, the conditional covariances are all nonnegative (Theorem 2.1, Esary, Proschan and Walkup (1967)); this property is preserved in unconditioning so that $X_1, \cdots, X_n$ are associated. Hence III holds.

Now let us compare the lower bounds $P\{X_1 > \sum_{i=1}^n x_i\}$ and $\prod_{i=1}^n P\{X_i > x_i\}$ of II and III. If the inequality

$$P\{X_1 > \textstyle\sum_{i=1}^n x_i\} \geqq \prod_{i=1}^n P\{X_i > x_i\}$$

holds for $n = 2$ and all x_1, x_2, then one can show by iteration that it holds for all n. For $n = 2$, the property

$$P\{X_1 > x_1 + x_2\} \geqq P\{X_1 > x_1\}P\{X_2 > x_2\}$$

arises in a reliability context and is discussed by Marshall and Proschan (1972) in the case X_1, X_2 are identically distributed as they are here. They term the property "new worse than used," and the reverse inequality is called "new better than used." In some of the examples which follow neither property holds, so that neither inequality is always better then the other. On the other hand, the multivariate gamma discussed below includes cases for which the "new better than used" and "new worse than used" properties hold.

5.2. *Ratios of random variables.* We consider some examples below in which

$\varphi_z(x) = x/z, z > 0$. $U_1, \cdots, U_n$ and Z are independent random variables, the U_i have a common log-concave density, and Z is a univariate random variable such that $P(Z > 0) = 1$.

The multivariate "F" distribution. A particular multivariate "F" distribution which arises in statistical contexts is generated as follows: $U_1, \cdots, U_n$ each have a chi-square distribution with $r \geqq 2$ degrees of freedom, and Z has a chi-square distribution with s degrees of freedom. Since $U_1, \cdots, U_n$ and Z have log-concave densities,

$$X_1 = U_1/Z, \cdots, X_n = U_n/Z$$

have a joint density f which satisfies the conditions of Theorem 2.1. Moreover, the marginals of f are "F" densities. Here, one has the Bounds I and II. For this application, Bounds III were obtained by Kimball (1951). Bounds I and II were obtained by Olkin (1973), who shows by verifying (1) that for this example $\bar{F}$ is Schur-concave.

If $Y_1, \cdots, Y_n$ has a multivariate beta distribution (Section 4), and $X_1, \cdots, X_n$ has a multivariate F-distribution, then one may obtain one distribution from the other by the transformations $Y_i = X_i/(1 + \sum X_j)$ and $X_i = Y_i/(1 - \sum Y_j)$. Consequently, some results for one distribution may be obtained from the other.

Multivariate "t" distribution. As shown in Section 4, the multivariate "t" density is Schur-concave. If we assume that $U_1, \cdots, U_n$ are independent (rather than exchangeable as in Section 4), then we obtain from Proposition 5.1 the same conclusion because $U_1, \cdots, U_n$ have log-concave densities. However, with independence we have the additional Bounds III which do not necessarily hold for the exchangeable case.

5.3. *Sum of random variables.* A further example of Proposition 5.1 is the case in which $\varphi_z(x) = x + z$.

Multivariate normal distribution. If $U_1, \cdots, U_n$ and Z are independent normally distributed random variables and the U_i are identically distributed, then $X_1 = U_1 + Z, \cdots, X_n = U_n + Z$ have a Schur-concave joint density as already observed more generally in Section 4. This special case is of interest because Bounds III apply.

A multivariate gamma distribution. If $U_1, \cdots, U_n$ and Z are independent U_i have a gamma distribution with density $f(u; \alpha) = \lambda(\lambda u)^{\alpha-1}e^{-\lambda u}/\Gamma(\alpha)$, $\alpha \geqq 1$, and Z has a gamma distribution with density $f(u; \beta)$ then

$$X_1 = U_1 + Z, \cdots, X_n = U_n + Z$$

are jointly distributed with marginal densities $f(u; \alpha + \beta)$. Here the marginal densities of the X_i's are "new better than used" if $\alpha + \beta \geqq 1$, and they are "new worse than used" if $\alpha + \beta \leqq 1$. Consequently, the bound of III is better than II when $\alpha + \beta \geqq 1$, and the bound of III is weaker than II when $\alpha + \beta < 1$.

6. Non-central distributions. There are several "non-central" distributions which are derived from multivariate distributions and have vector-valued parameters. Such distributions are often Schur-concave in their parameters.

Non-central chi-square distribution. This distribution function is given by

$$F_\theta(t) = \frac{1}{(2\pi)^{n/2}} \int_{\{\sum x_i^2 \leq t\}} \exp[-\tfrac{1}{2} \sum (x_i - \theta)^2]\, dx$$
$$= \frac{1}{(2\pi)^{n/2}} \int_{x \in A-\theta} \exp(-\tfrac{1}{2} \sum x_i^2)\, dx\,,$$

where $A = \{x: \sum x_i^2 \leqq t\}$. Since A is convex and since the integrand is a product of log-concave densities, it follows from Theorem 2.1 that F_θ is Schur-concave in $-\theta$, which is equivalent to being Schur-concave in θ.

It is well known that F_θ can be written as a mixture of central chi-square distribution functions where the mixing distribution is Poisson with parameter $\sum \theta_i^2$. Condition (1) is not easily checked directly when F_θ is written in this form or in the form above. However, one can show that F_θ is decreasing in $\beta = \sum \theta_i^2$ either by direct differentiation or via total positivity. Since $\varphi(\theta) = \sum \theta_i^2$ is Schur-convex, this implies that F_θ is Schur-concave in θ.

Non-central "t" distribution. The distribution function of the non-central "t" is given by

$$F_\theta(t) = \int_{\{\sum x_i \leq t\sqrt{s}\}} \exp[-\tfrac{1}{2} \sum (x_i - \theta_i)^2] s^{\frac{1}{2}n-1} e^{-\frac{1}{2}s}\, dx\, ds\,.$$

The sets $A_s = \{x: \sum x_i \leqq t\sqrt{s}\}$ are all convex in x, and we can write

$$F_\theta(t) = \int_{s \geq 0} \{\int_{x \in A_s} \exp[-\tfrac{1}{2} \sum (x_i - \theta_i)^2]\, dx\} s^{\frac{1}{2}n-1} e^{-\frac{1}{2}s}\, ds$$
$$= \int_{s \geq 0} \{\int_{x \in A_s - \theta} \exp[-\tfrac{1}{2} \sum x_i^2]\, dx\} s^{\frac{1}{2}n-1} e^{-\frac{1}{2}s}\, ds\,.$$

The inner integral is Schur-concave in θ for each fixed s by Theorem 2.1, and hence the mixture $F_\theta(t)$ is Schur-concave in θ.

Non-central "F" distribution. The distribution function of the non-central "F" distribution can be written as

$$F_\theta(t) = \int_{\{\sum x_i^2 \leq st\}} \exp[-\tfrac{1}{2} \sum (x_i - \theta_i)^2] s^{\frac{1}{2}n-1} e^{-\frac{1}{2}s}\, dx\, ds$$
$$= \int_{s \geq 0} \{\int_{x \in \{\sum x_i^2 \leq st\}} \exp[-\tfrac{1}{2} \sum (x_i - \theta_i)^2]\, dx\} s^{\frac{1}{2}n-1} e^{-\frac{1}{2}s}\, ds\,.$$

Here the argument used for the non-central "t" requires little modification to show that the non-central "F" distribution function is Schur-concave in θ.

7. Distribution functions and survival functions. Here we consider exclusively the two special sets

$$A_* = \{x: x_i \leqq 0 \;\text{ for all }\; i\}$$

and

$$A^* = \{x: x_i > 0 \;\text{ for all }\; i\}\,.$$

These sets satisfy the conditions of Theorem 2.1, but

$$F(a) = \int_{A_*+a} f(x)\,dx$$

and

$$\bar{F}(a) = \int_{A^*+a} f(x)\,dx$$

may be Schur-concave under weaker conditions on f than those required in Theorem 2.1.

7.1. PROPOSITION. *In the independent identically distributed case,* i.e., $F(a) = \prod_{i=1}^n G(a_i)$ *and* $\bar{F}(a) = \prod_{i=1}^n \bar{G}(a_i)$ *for some univariate distribution function* $G = 1 - \bar{G}$,

F *is Schur-concave if and only if* $\log G$ *is concave,*

$\bar{F}$ *is Schur-concave if and only if* $\log \bar{G}$ *is concave.*

PROOF. We apply condition (1) to prove the assertion for F. For $a_j > a_k$,

$$\frac{\partial F(a)}{\partial a_j} - \frac{\partial F(a)}{\partial a_k} = \prod_{i \neq j,k} G(a_i)[g(a_j)G(a_k) - g(a_k)G(a_j)] \leqq 0 ,$$

which holds if and only if

$$\frac{g(a_j)}{G(a_j)} \leqq \frac{g(a_k)}{G(a_k)} ,$$

or $d/dz \log G(z)$ is decreasing in z, i.e., $\log G$ is concave. The proof for $\bar{F}$ is similar. □

The condition of Proposition 7.1 that $\log G$ is concave has not received much attention in the literature. By contrast, the condition that $\log \bar{G}$ is concave has been extensively studied in the context of reliability theory (see, e.g., Barlow and Proschan (1965)). Both conditions are implied by the condition that the density g of G is log-concave.

The following result is to be compared with Proposition 3.1.

7.2. PROPOSITION. *Let* $\varphi_z(u)$ *be concave and increasing in* u *for all* z. *If* $U_1, U_2, \cdots, U_n$ *and* Z *are independent random variables, and the* U_i *have a common distribution function* G *such that* $\bar{G} = 1 - G$ *is log-concave, then*

$$X_1 = \varphi_Z(U_1), \cdots, X_n = \varphi_Z(U_n)$$

are such that $P(X_1 > x_1, \cdots, X_n > x_n)$ *is a Schur-concave function of* x.

PROOF. We assume that φ_z is strictly increasing and differentiable so that it has an increasing differentiable inverse, and we also assume that G has a density g. Then

$$\begin{aligned} P(X_1 > x_1, \cdots, X_n > x_n) &= P(\varphi_Z(U_1) > x_1, \cdots, \varphi_Z(U_n) > x_n) \\ &= EP(\varphi_Z(U_1) > x_1, \cdots, \varphi_Z(U_n) > x_n \mid Z) \\ &= EP(U_1 > \varphi_Z^{-1}(x_1), \cdots, U_n > \varphi_Z^{-1}(x_n) \mid Z) \\ &= E \prod_{i=1}^n P(U_i > \varphi_Z^{-1}(x_i) \mid Z) = E \prod_{i=1}^n \bar{G}(\varphi_Z^{-1}(x_i)) . \end{aligned}$$

Now, apply condition (1) to $h_z(x) = \prod_{i=1}^n \bar{G}(\varphi_Z^{-1}(x_i))$. If $x_j > x_k$,

$$(4) \qquad \frac{\partial h_z(x)}{\partial x_j} - \frac{\partial h_z(x)}{\partial x_k} = \prod_{i \neq j \text{ or } k} \bar{G}(\varphi_Z^{-1}(x_i)) \left[\frac{-g(\varphi_Z^{-1}(x_j))}{\varphi_Z'(\varphi_Z^{-1}(x_j))} \bar{G}(\varphi_Z^{-1}(x_k)) + \frac{g(\varphi_Z^{-1}(x_k))}{\varphi_Z'(\varphi_Z^{-1}(x_k))} \bar{G}(\varphi_Z^{-1}(x_j)) \right].$$

If $y_i = \varphi_Z^{-1}(x_i)$, then $y_j > y_k$, and the difference (4) is nonnegative if and only if

$$(5) \qquad \frac{g(y_k)}{\varphi_Z'(y_k)} \bar{G}(y_j) \geqq \frac{g(y_j)}{\varphi_Z'(y_j)} \bar{G}(y_k).$$

But $\log \bar{G}$ is concave so $g(y_k)/\bar{G}(y_k) \geqq g(y_j)/\bar{G}(y_j)$, and φ_Z is concave so $\varphi_Z'(y_j) \geqq \varphi_Z'(y_k)$. Consequently (5) holds. To complete the proof, we need only remark that we have shown $P(X_1 > x_1, \cdots, X_n > x_n)$ to be a mixture of Schur-concave functions. □

7.3. PROPOSITION. *Let $\varphi_z(u)$ be convex and increasing in u for all z. If $X_1, \cdots, X_n$ are defined as in Proposition* 7.2, *and* $\log G$ *is concave, then $P(X_1 \leqq x_1, \cdots, X_n \leqq x_n)$ is a Schur-concave function of x.*

We omit the proof of this result, which is analogous to the proof of Proposition 7.2.

7.4. *The minimum of random variables.* The function $\varphi_z(u) = \min(u, z)$ is concave and increasing in u for each z, so it satisfies the condition of Proposition 7.2. If U_1, U_2 and Z are independently distributed having an exponential distribution with parameters λ, λ and λ_{12}, respectively, then

$$X_1 = \min(U_1, Z) \qquad \text{and} \qquad X_2 = \min(U_2, Z)$$

have the bivariate exponential distribution given by

$$\bar{F}(x_1, x_2) \equiv P(X_1 > x_1, X_2 > x_2) = \exp\{-\lambda x_1 - \lambda x_2 - \lambda_{12} \max(x_1, x_2)\}.$$

By Proposition 7.2, $\bar{F}$ is Schur-concave.

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