

Majorization Properties for Subclasses of Analytic p-Valent Functions Defined by Convolution

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ABSTRACT. The object of the present paper is to investigate the majorization properties of certain subclasses of analytic p-valent functions defined by convolution.

1. Introduction

Let f and g be analytic in the open unit disc $U = \{z \in \mathbb{C} : |z| < 1\}$. We say that f is majorized by g in U (see [7]) and write

$$(1.1) \quad f(z) \ll g(z) \quad (z \in U),$$

if there exists a function φ , analytic in U such that

$$(1.2) \quad |\varphi(z)| < 1 \quad \text{and} \quad f(z) = \varphi(z)g(z) \quad (z \in U).$$

It may be noted that (1.1) is closely related to the concept of quasi-subordination between analytic functions.

If f and g are analytic functions in U , we say that f is subordinate to g , written symbolically as $f(z) \prec g(z)$ if there exists a Schwarz function w , which (by definition) is analytic in U with $w(0) = 0$ and $|w(z)| < 1$ for all $z \in U$, such that $f(z) = g(w(z))$, $z \in U$. Furthermore, if the function g is univalent in U , then we have the following equivalence, (see [8, p.4]):

$$f(z) \prec g(z) \Leftrightarrow f(0) = g(0) \quad \text{and} \quad f(U) \subset g(U).$$

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Let $A(p)$ denote the class of functions of the form:

$$(1.3) \quad f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k \quad (p \in \mathbb{N} = \{1, 2, \dots\}),$$

which are analytic and p -valent in the open unit disc U . Also let $A = A(1)$.

For functions $f_i \in A(p)$ ($i = 1, 2$) given by

$$f_i(z) = z^p + \sum_{k=p+1}^{\infty} a_{k,i} z^k \quad (i = 1, 2; p \in \mathbb{N}),$$

we define the Hadamard product (or convolution) of f_1 and f_2 by

$$(1.4) \quad (f_1 * f_2)(z) = z^p + \sum_{k=p+1}^{\infty} a_{k,1} a_{k,2} z^k = (f_2 * f_1)(z).$$

For $h(z)$ given by

$$(1.5) \quad h(z) = z^p + \sum_{k=p+1}^{\infty} \mu_k z^k \quad (\mu_k \geq 0),$$

then a function $f \in A(p)$ is said to be in the class $S_p^j(h; \gamma)$ of p -valent starlike functions of complex order $\gamma \in \mathbb{C}^* = \mathbb{C} \setminus \{0\}$ in U , if and only if

$$\operatorname{Re} \left\{ 1 + \frac{1}{\gamma} \left(\frac{z(f * h)^{(j+1)}(z)}{(f * h)^{(j)}(z)} - p + j \right) \right\} > 0$$

$$(1.6) \quad (z \in U; p \in \mathbb{N}; j \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}; \gamma \in \mathbb{C}^*; |2\gamma - p + j| < (p - j)).$$

Furthermore, a function $f \in A(p)$ is said to be in the class $K_p^j(h, \gamma)$ of p -valent convex functions of complex order $\gamma \in \mathbb{C}^*$ in U , if and only if

$$\operatorname{Re} \left\{ 1 + \frac{1}{\gamma} \left(\frac{z(f * h)^{(j+2)}(z)}{(f * h)^{(j+1)}(z)} - p + j + 1 \right) \right\} > 0$$

$$(1.7) \quad (z \in U; p \in \mathbb{N}; j \in \mathbb{N}_0; \gamma \in \mathbb{C}^*; |\gamma - p + j| < (p - j)).$$

From (1.6) and (1.7) it follows that:

$$(i) \quad S_1^0\left(\frac{z}{1-z}; \gamma\right) = S(\gamma) \quad \text{and} \quad K_1^0\left(\frac{z}{1-z}; \gamma\right) = K(\gamma) \quad (\gamma \in \mathbb{C}^*).$$

The class $S(\gamma)$ was introduced by Nasr and Aouf [9], and the class $K(\gamma)$ was introduced and studied by Nasr and Aouf [9] and Wiatrowski [13].

$$(ii) \quad S_p^0\left(\frac{z^p}{1-z}; p - \alpha\right) = S_p^*(\alpha) \quad \text{and} \quad K_p^0\left(\frac{z^p}{1-z}; p - \alpha\right) = K_p(\alpha) \quad (0 \leq \alpha < p).$$

The class $S_p^*(\alpha)$ was introduced by Patil and Thakare [12], and the class $K_p(\alpha)$ was introduced by Owa [11]. We note that

$$S_p^*(\alpha) \subseteq S_p^*(0) = S_p^* \text{ and } K_p(\alpha) \subseteq K_p(0) = K_p \quad (0 \leq \alpha < p),$$

where S_p^* and K_p denote the subclasses of $A(p)$ consisting of functions which are p -valent starlike in U and p -valent convex in U , respectively (see Goodman [5]).

Definition 1.1. Let $-1 \leq B < A \leq 1, p \in \mathbb{N}, j \in \mathbb{N}_0, \gamma \in \mathbb{C}^*, |\gamma(A - B) + (p - j)B| < (p - j), f \in A(p)$ and $h(z)$ is given by (1.5). Then $f \in S_p^j(h; \gamma; A, B)$, the class of p -valent functions of complex order γ in U if and only if

$$(1.8) \quad \left\{ 1 + \frac{1}{\gamma} \left(\frac{z(f * h)^{(j+1)}(z)}{(f * h)^{(j)}(z)} - p + j \right) \right\} \prec \frac{1 + Az}{1 + Bz}.$$

Clearly, we have the following relationships:

- (i) $S_p^j(h; \gamma; 1, -1) = S_p^j(h; \gamma);$
- (ii) $S_p^j\left(\frac{z^p}{1-z}; \gamma\right) = S_p^j(\gamma).$

We shall need the following lemma.

Lemma 1.2([1]). Let $\gamma \in \mathbb{C}^*$ and $f \in K_p^j(\gamma)$. Then $f \in S_p^j(\frac{1}{2}\gamma)$, that is,

$$(1.9) \quad K_p^j(\gamma) \subset S_p^j\left(\frac{1}{2}\gamma\right) \quad (\gamma \in \mathbb{C}^*).$$

A majorization problem for the classes $S(\gamma)$ and $K(\gamma)$ has recently been investigated by Altintas et al. [2]. Also, majorization problem for the classes $S^* = S^*(0)$ and $K = K(0)$ has been investigated by MacGregor [7]. In this paper we investigate majorization problem for the class $S_p^j(h, \gamma; A, B)$ and some related subclasses.

2. Main Results

Unless otherwise mentioned we shall assume throughout the paper that, $-1 \leq B < A \leq 1, \gamma \in \mathbb{C}^*, p \in \mathbb{N}, j \in \mathbb{N}_0$ and $h(z)$ is given by (1.5).

Theorem 2.1. Let the function $f \in A(p)$ and suppose that $(g * h) \in S_p^j(h, \gamma; A, B)$. If $(f * h)^{(j)}(z)$ is majorized by $(g * h)^{(j)}(z)$ in U , then

$$(2.1) \quad \left| (f * h)^{(j+1)}(z) \right| \leq \left| (g * h)^{(j+1)}(z) \right| \quad (|z| < r_0),$$

where $r_0 = r_0(p, \gamma, j, A, B)$ is the smallest positive root of the equation

$$(2.2) \quad |\gamma(A - B) + (p - j)B| r^3 - [2|B| + (p - j)] r^2 - [2 + |\gamma(A - B) + (p - j)B|] r + (p - j) = 0.$$

Proof. Since $(g * h) \in S_p^j(h, \gamma; A, B)$, we find from (1.8) that

$$(2.3) \quad 1 + \frac{1}{\gamma} \left(\frac{z(g * h)^{(j+1)}(z)}{(g * h)^{(j)}(z)} - p + j \right) = \frac{1 + Aw(z)}{1 + Bw(z)},$$

where w is analytic in U with $w(0) = 0$ and $|w(z)| < 1$ ($z \in U$). From (2.3), we have

$$(2.4) \quad \frac{z(g * h)^{(j+1)}(z)}{(g * h)^{(j)}(z)} = \frac{(p - j) + (\gamma(A - B) + (p - j)B)w(z)}{1 + Bw(z)}.$$

From (2.4), we have

$$(2.5) \quad \left| (g * h)^{(j)}(z) \right| \leq \frac{(1 + |B||z|)|z|}{(p - j) - |\gamma(A - B) + (p - j)B||z|} \left| (g * h)^{(j+1)}(z) \right|.$$

Next, since $(f * h)^{(j)}(z)$ is majorized by $(g * h)^{(j)}(z)$ in U , from (1.2), we have

$$(2.6) \quad (f * h)^{(j)}(z) = \varphi(z)(g * h)^{(j)}(z).$$

Differentiating (2.6) with respect to z , we have

$$(2.7) \quad (f * h)^{(j+1)}(z) = \varphi'(z)(g * h)^{(j)}(z) + \varphi(z)(g * h)^{(j+1)}(z).$$

Thus, by noting that $\varphi(z)$ satisfies the inequality (see [10]),

$$(2.8) \quad \left| \varphi'(z) \right| \leq \frac{1 - |\varphi(z)|^2}{1 - |z|^2} \quad (z \in U),$$

using (2.5) and (2.8), in (2.7), we have

$$(2.9) \quad \left| (f * h)^{(j+1)}(z) \right| \leq \left(|\varphi(z)| + \frac{1 - |\varphi(z)|^2}{1 - |z|^2} \cdot \frac{(1 + |B||z|)|z|}{(p - j) - |\gamma(A - B) + (p - j)B||z|} \right) \left| (g * h)^{(j+1)}(z) \right|,$$

which upon setting

$$|z| = r \quad \text{and} \quad |\varphi(z)| = \rho \quad (0 \leq \rho \leq 1),$$

leads us to the inequality

$$\left| (f * h)^{(j+1)}(z) \right| \leq \frac{\Theta(\rho)}{(1 - r^2)((p - j) - |\gamma(A - B) + (p - j)B|r)} \left| (g * h)^{(j+1)}(z) \right|,$$

where

$$(2.10) \quad \Theta(\rho) = -r(1 + |B|r)\rho^2 + (1 - r^2)[(p - j) - |\gamma(A - B) + (p - j)B|r]\rho + r(1 + |B|r),$$

takes its maximum value at $\rho = 1$, with $r_0 = r_0(p, j, \gamma, A, B)$, where $r_0(p, j, \gamma, A, B)$ is the smallest positive root of (2.2). Therefore the function $\Phi(\rho)$ defined by

$$(2.11) \quad \Phi(\rho) = -\sigma(1 + |B|\sigma)\rho^2 + (1 - \sigma^2)[(p - j) - |\gamma(A - B) + (p - j)B|\sigma]\rho + \sigma(1 + |B|\sigma)$$

is an increasing function on the interval $0 \leq \rho \leq 1$, so that

$$(2.12) \quad \Phi(\rho) \leq \Phi(1) = (1 - \sigma^2)[(p - j) - |\gamma(A - B) + (p - j)B|\sigma] \\ (0 \leq \rho \leq 1; 0 \leq \sigma \leq r_0(p, j, \gamma, A, B)).$$

Hence upon setting $\rho = 1$ in (2.11), we conclude that (2.1) holds true for $|z| \leq r_0 = r_0(p, \gamma, j, A, B)$, where $r_0(p, \gamma, j, A, B)$, is the smallest positive root of (2.2). This completes the proof of Theorem 2.1. \square

Putting $A = 1$ and $B = -1$ in Theorem 2.1, we obtain the following result.

Corollary 2.2. *Let the function $f \in A(p)$ and suppose that $g \in S_p^j(h; \gamma)$. If $(f * h)^{(j)}(z)$ is majorized by $(g * h)^{(j)}(z)$ in U , then*

$$\left| (f * h)^{(j+1)}(z) \right| \leq \left| (g * h)^{(j+1)}(z) \right| \quad (|z| < r_0),$$

where $r_0 = r_0(p, j, \gamma)$ is given by

$$r_0 = r_0(p, j, \gamma) = \frac{k - \sqrt{k^2 - 4|2\gamma - (p - j)|(p - j)}}{2|2\gamma - (p - j)|},$$

where $(k = 2 + (p - j) + |2\gamma - (p - j)|, p \in \mathbb{N}, \gamma \in \mathbb{C}^*)$.

Remark 2.3. Putting $h(z) = \frac{z^p}{1-z}$ in Corollary 2.2, we obtain the result obtained by Altintas and Srivastava [1, Theorem 1].

Putting $h(z) = \frac{z^p}{1-z}$, $A = 1, B = -1$ and $j = 0$ in Theorem 2.1, we obtain the following result.

Corollary 2.4. *Let the function $f \in A(p)$ and suppose that $g \in S_p(\gamma)$ ($\gamma \in \mathbb{C}^*$). If $f(z)$ is majorized by $g(z)$ in U , then*

$$|f'(z)| \leq |g'(z)| \quad (|z| < r_0),$$

where $r_0 = r_0(p; \gamma)$ is given by

$$r_0 = r_0(p; \gamma) = \frac{k - \sqrt{k^2 - 4p|2\gamma - p|}}{2|2\gamma - p|},$$

where $(k = 2 + p + |2\gamma - p|, p \in \mathbb{N}, \gamma \in \mathbb{C}^*)$.

Putting $h(z) = \frac{z^p}{1-z}$, $A = 1, B = -1, p = 1$ and $j = 0$ in Theorem 2.1, we obtain the following result.

Corollary 2.5 ([2,6]). *Let the function $f \in A$ and suppose that $g \in S(\gamma)$ ($\gamma \in \mathbb{C}^*$). If $f(z)$ is majorized by $g(z)$ in U , then*

$$|f'(z)| \leq |g'(z)| \quad (|z| < r_0),$$

where $r_0 = r_0(\gamma)$ is given by

$$r_0 = r_0(\gamma) = \frac{k - \sqrt{k^2 - 4|2\gamma - 1|}}{2|2\gamma - 1|},$$

where $(k = 3 + |2\gamma - 1|, \gamma \in \mathbb{C}^*)$.

Putting $\gamma = 1$ in Corollary 2.5, we obtain the following result.

Corollary 2.6 ([6,7]) *Let the function $f \in A$ and suppose that $g \in S^*$. If $f(z)$ is majorized by $g(z)$ in U , then*

$$|f'(z)| \leq |g'(z)| \quad (|z| < r_0),$$

where r_0 is given by

$$r_0 = 2 - \sqrt{3}.$$

Using Lemma 2.3 it is easy to prove the following lemma.

Lemma 2.7. *Let $\gamma \in \mathbb{C}^*$ and $f \in K_p^j(h; \gamma)$. Then $f \in S_p^j(h; \frac{1}{2}\gamma)$, that is,*

$$(2.13) \quad K_p^j(h; \gamma) \subset S_p^j(h; \frac{1}{2}\gamma) \quad (\gamma \in \mathbb{C}^*).$$

By using Lemma 2.7 and Corollary 2.2, we can prove the following theorem.

Theorem 2.8. *Let the function $f \in A(p)$ and suppose that $(g * h) \in K_p^j(h, \gamma)$. If $(f * h)^{(j)}(z)$ is majorized by $(g * h)^{(j)}(z)$ in U , then*

$$\left| (f * h)^{(j+1)}(z) \right| \leq \left| (g * h)^{(j+1)}(z) \right| \quad (|z| < r_0),$$

where $r_0 = r_0(p, \gamma, j)$ is given by

$$(2.14) \quad r_0 = r_0(p, \gamma, j) = \frac{k - \sqrt{k^2 - 4(p-j)|\gamma - (p-j)|}}{2|\gamma - (p-j)|},$$

where $(k = 2 + (p-j) + |\gamma - (p-j)|, p \in \mathbb{N}, j \in \mathbb{N}_0, \gamma \in \mathbb{C}^*)$.

Proof. In view of the inclusion property (2.13), Theorem 2.8 can be deduced as a simple consequence of Theorem 2.1 with $\gamma \rightarrow \frac{1}{2}\gamma$. \square

Remark 2.9. Putting $h(z) = \frac{z^p}{1-z}$ in Theorem 2.8, we obtain the result obtained by Altintas [1, Theorem 2].

Putting $h(z) = \frac{z^p}{1-z}$ and $j = 0$ in Theorem 2.8, we obtain the following result.

Corollary 2.10. *Let the function $f \in A(p)$ and suppose that $g \in K_p(\gamma)$ ($\gamma \in \mathbb{C}^*$). If $f(z)$ is majorized by $g(z)$ in U , then*

$$|f'(z)| \leq |g'(z)| \quad (|z| < r_0),$$

where $r_0 = r_0(p; \gamma)$ is given by

$$r_0 = r_0(p; \gamma) = \frac{k - \sqrt{k^2 - 4p|\gamma - p|}}{2|\gamma - p|},$$

where $(k = 2 + p + |\gamma - p|, p \in \mathbb{N}, \gamma \in \mathbb{C}^*)$.

Putting $h(z) = \frac{z^p}{1-z}, p = 1$ and $j = 0$ in Theorem 2.8, we obtain the following result.

Corollary 2.11([1]). *Let the function $f \in A$ and suppose that $g \in K(\gamma)$. If $f(z)$ is majorized by $g(z)$ in U , then*

$$|f'(z)| \leq |g'(z)| \quad (|z| < r_0),$$

where $r_0 = r_0(\gamma)$ is given by

$$r_0 = r_0(\gamma) = \frac{k - \sqrt{k^2 - 4|\gamma - 1|}}{2|\gamma - 1|},$$

where $(k = 3 + |\gamma - 1|, \gamma \in \mathbb{C}^*)$.

Letting $\gamma \rightarrow 1$ in Corollary 2.11, we obtain the following result.

Corollary 2.12([7]). *Let the function $f \in A$ and suppose that $g \in K$. If $f(z)$ is majorized by $g(z)$ in U , then*

$$|f'(z)| \leq |g'(z)| \quad \left(|z| \leq \frac{1}{3} \right).$$

Examples

- (i) Putting in (1.6) and (1.7) $h(z) = z^p + \sum_{k=p+1}^{\infty} \Gamma_{k-p}(\alpha_1)z^k$, (see [4]), where
- $$\Gamma_{k-p}(\alpha_1) = \frac{(\alpha_1)_{k-p} \dots (\alpha_q)_{k-p}}{(\beta_1)_{k-p} \dots (\beta_s)_{k-p} (1)_{k-p}}, q \leq s+1, q, s \in \mathbb{N}_0, \alpha_i \in \mathbb{C} (i = 1, 2, \dots, q) \text{ and}$$

$\beta_j \in \mathbb{C} \setminus \mathbb{Z}_0^-$ ($j = 1, 2, \dots, s$), then $(f * h)(z) = H_{p,q,s}(\alpha_1)f(z)$, $S_p^j(h; \gamma) = S_{p,q,s}^j(\alpha_1; \gamma)$ and $K_p^j(h; \gamma) = K_{p,q,s}^j(\alpha_1; \gamma)$. Then we obtain the following results.

Let the function $f \in A(p)$ and $g \in S_{p,q,s}^j(\alpha_1; \gamma)$ ($|2\gamma - \alpha_1| < |\alpha_1|$). If $(H_{p,q,s}(\alpha_1)f(z))^{(j)}$ is majorized by $(H_{p,q,s}(\alpha_1)g(z))^{(j)}$ in U , then

$$\left| (H_{p,q,s}(\alpha_1 + 1)f(z))^{(j)} \right| \leq \left| (H_{p,q,s}(\alpha_1 + 1)g(z))^{(j)} \right| \quad (|z| < r_1),$$

where $r_1 = r_1(\gamma, \alpha_1)$ is given by

$$r_1 = r_1(\gamma, \alpha_1) = \frac{k - \sqrt{k^2 - 4|2\gamma - \alpha_1||\alpha_1|}}{2|2\gamma - \alpha_1|},$$

where $(k = 2 + |\alpha_1| + |2\gamma - \alpha_1|; \gamma, \alpha_1 \in \mathbb{C}^*)$.

Also, let the function $f \in A(p)$ and $g \in K_{p,q,s}^j(\alpha_1; \gamma)$ ($|\gamma - \alpha_1| < |\alpha_1|$). If $(H_{p,q,s}(\alpha_1)f(z))^{(j)}$ is majorized by $(H_{p,q,s}(\alpha_1)g(z))^{(j)}$ in U , then

$$\left| (H_{p,q,s}(\alpha_1 + 1)f(z))^{(j)} \right| \leq \left| (H_{p,q,s}(\alpha_1 + 1)g(z))^{(j)} \right| \quad (|z| < r_2),$$

where $r_2 = r_2(\gamma, \alpha_1)$ is given by

$$r_2 = r_2(\gamma, \alpha_1) = \frac{k - \sqrt{k^2 - 4|\gamma - \alpha_1||\alpha_1|}}{2|\gamma - \alpha_1|},$$

where $(k = 2 + |\alpha_1| + |\gamma - \alpha_1|; \gamma, \alpha_1 \in \mathbb{C}^*)$.

By specializing the parameters p, q, s, α_i ($i = 1, 2, \dots, q$) and β_j ($j = 1, 2, \dots, s$), we obtain various results for different operators.

(ii) Putting in (1.6) and (1.7) $h(z) = z^p + \sum_{k=p+1}^{\infty} \left[\frac{p+\ell+\lambda(k-p)}{p+\ell} \right]^m z^k$, (see, [3]),

where $\ell \geq 0, \lambda > 0, m \in \mathbb{N}_0$, then $(f * h)(z) = I_p^m(\lambda, \ell)f(z)$, $S_p^j(h; \gamma) = S_p^{m,j}(\lambda, \ell; \gamma)$ and $K_p^j(h; \gamma) = K_p^{m,j}(\lambda, \ell; \gamma)$. Also, we obtain the following results.

Let the function $f \in A(p)$ and $g \in S_p^{m,j}(\lambda, \ell; \gamma)$ ($|2\gamma - \left(\frac{p+\ell}{\lambda}\right)| < \left(\frac{p+\ell}{\lambda}\right)$). If $(I_p^m(\lambda, \ell)f(z))^{(j)}$ is majorized by $(I_p^m(\lambda, \ell)g(z))^{(j)}$ in U , then

$$\left| (I_p^{m+1}(\lambda, \ell)f(z))^{(j)} \right| \leq \left| (I_p^{m+1}(\lambda, \ell)g(z))^{(j)} \right| \quad (|z| < r_3),$$

where $r_3 = r_3(p, \gamma, \lambda, \ell)$ is given by

$$r_3 = r_3(p, \gamma, \lambda, \ell) = \frac{k - \sqrt{k^2 - 4\left|2\gamma - \left(\frac{p+\ell}{\lambda}\right)\right|\left(\frac{p+\ell}{\lambda}\right)}}{2\left|2\gamma - \left(\frac{p+\ell}{\lambda}\right)\right|},$$

where $(k = 2 + \left(\frac{p+\ell}{\lambda}\right) + \left|2\gamma - \left(\frac{p+\ell}{\lambda}\right)\right|, p \in \mathbb{N}, \gamma \in \mathbb{C}^*, \ell \geq 0, \lambda > 0, m \in \mathbb{N}_0)$.

Also, let the function $f \in A(p)$ and $g \in K_p^{m,j}(\lambda, \ell; \gamma) \left(\left| \gamma - \left(\frac{p+\ell}{\lambda} \right) \right| < \left(\frac{p+\ell}{\lambda} \right) \right)$. If $(I_p^m(\lambda, \ell)f(z))^{(j)}$ is majorized by $(I_p^m(\lambda, \ell)g(z))^{(j)}$ in U , then

$$\left| (I_p^{m+1}(\lambda, \ell)f(z))^{(j)} \right| \leq \left| (I_p^{m+1}(\lambda, \ell)g(z))^{(j)} \right| \quad (|z| < r_4),$$

where $r_4 = r_4(p, \gamma, \lambda, \ell)$, is given by

$$r_4 = r_4(p, \gamma, \lambda, \ell) = \frac{k - \sqrt{k^2 - 4 \left| \gamma - \left(\frac{p+\ell}{\lambda} \right) \right| \left(\frac{p+\ell}{\lambda} \right)}}{2 \left| \gamma - \left(\frac{p+\ell}{\lambda} \right) \right|},$$

where $(k = 2 + \left(\frac{p+\ell}{\lambda} \right) + \left| \gamma - \left(\frac{p+\ell}{\lambda} \right) \right|, p \in \mathbb{N}, \gamma \in \mathbb{C}^*, \ell \geq 0, \lambda > 0)$.

By specializing the parameters p, λ, ℓ and m we obtain various results for different operators.

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