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Majorizing multiplicative cascades for directed polymers in random media

Francis Comets and Vincent Vargas

Université Paris 7, Mathématiques, case 7012, 2, place Jussieu, 75251 Paris, France
E-mail address: comets@math.jussieu.fr, vargas@math.jussieu.fr

Abstract. In this note we give upper bounds for the free energy of discrete directed polymers in random media. The bounds are given by the so-called generalized multiplicative cascades from the statistical theory of turbulence. For the polymer model, we derive that the quenched free energy is different from the annealed one in dimension 1, for any finite temperature and general environment. This implies localization of the polymer.

1. Introduction

Let $\omega = (\omega_n)_{n\in\mathbb{N}}$ be the simple random walk on the d-dimensional integer lattice $\mathbb{Z}^d$ starting at 0, defined on a probability space $(\Omega, \mathcal{F}, P)$. We also consider a sequence $\eta = (\eta(n,x))_{(n,x)\in\mathbb{N}\times\mathbb{Z}^d}$ of real valued, non-constant and i.i.d. random variables defined on another probability space $(H, \mathcal{G}, Q)$ with finite exponential moments. The path ω represents the directed polymer and η the random environment.

For any $n > 0$, we define the (random) polymer measure μ_n on the path space $(\Omega, \mathcal{F})$ by:

$$\mu_n(d\omega) = \frac{1}{Z_n}\exp(\beta H_n(\omega))P(d\omega)$$

where $\beta \in \mathbb{R}^+$ is the inverse temperature, where

$$H_n(\omega) \stackrel{\text{def.}}{=} \sum_{j=1}^{n} \eta(j, \omega_j)$$

and where

$$Z_n = P[\exp(\beta H_n(\omega))]$$

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is the partition function. We use the notation $P[X]$ for the expectation of a random variable X. By symmetry, we can – and we will – restrict to $\beta \geqslant 0$.

The free energy of the polymer is defined as the limit

$$p(\beta) = \lim_{n\to\infty} \frac{1}{n} \ln(Z_n(\beta)/Q[Z_n(\beta)]), \tag{1.1}$$

where the limit exists Q-a.s. and in L^p for all $p \geqslant 1$ and is non-random (cf. Comets et al., 2003). An application of Jensen's inequality to the concave function $\ln(\cdot)$ yields $p(\beta) \leqslant 0$. As shown in theorem 3.2 (b) in Comets and Yoshida (2006), there exists a $\beta_c \in [0,\infty]$ such that

$$p(\beta) \quad \begin{cases} = 0 & \text{if} \quad \beta \in [0, \beta_c], \\ < 0 & \text{if} \quad \beta > \beta_c. \end{cases}$$

Determining the critical value β_c is an important question in the study of directed polymers. Indeed, one can show that the negativity of $p(\beta)$ is equivalent to a localization property for $(\omega_n)_{n\in\mathbb{N}}$,$(\widetilde{\omega}_n)_{n\in\mathbb{N}}$ two independent random walks under the polymer measure μ_n (cf Corollary 2.2 in Comets et al. (2003)):

$$p(\beta) < 0 \quad \Longleftrightarrow \quad \exists c > 0 \ : \ \varliminf_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \mu_{k-1}^{\otimes 2}(\omega_k = \widetilde{\omega}_k) \geqslant c \quad Q-a.s.$$

The statement in the right-hand side means that the polymer localizes in narrow corridors with positive probability. It is easily seen to be equivalent to

$$\exists c > 0 \ : \ \varliminf_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \max_{x\in\mathbb{Z}^d} \mu_{k-1}(\omega_k = x) \geqslant c \quad Q-a.s. \tag{1.2}$$

It is not known how to characterize directly these corridors, and therefore this criterion for the transition localization/delocalization is rather efficient since it does not require any knowledge on them. Hence, it is of primer importance to get good upper bounds on p in order to spot the transition. Our main result is the following.

Theorem 1.1. *In dimension $d = 1$, $\beta_c = 0$.*

There is a clear consensus on this fact in the physics literature, but no proof for it, except via the replica method or in the (different) case of a space-periodic environment where much more computations can be performed Brunet and Derrida (2000).

This result follows from a family of upper bounds, given by the free energies $p_m^{\text{tree}}(\beta)$ of models on trees depending on an integer parameter m ($m \geqslant 1$). These trees are deterministic and regular, with random weights, they fall in the scope of the generalized multiplicative cascades Liu (2000) or smoothing transformations Durrett and Liggett (1983) which are well known generalizations of the random cascades introduced in Mandelbrot (1974) for a statistical description of turbulence. When the environment variables have nice concentration properties – e.g., gaussian or bounded η's –, we prove in theorem 3.6 that the polymer free energy is the infimum over m of the one of the m-tree model. For general environmental distribution we only have an upper bound from theorem 3.3, but it is enough to show the above theorem. This also explains the title of the present paper. In order to highlight our results, we summarize theorems 3.3 and 3.6.

Theorem 1.2. *Let $d \geqslant 1$ and $p_m^{\text{tree}}(\beta)$ be given by (3.1–3.2). We have*

$$p(\beta) \leqslant \inf_{m \geqslant 1} \frac{1}{m} p_m^{\text{tree}}(\beta).$$

If the environment η is bounded or gaussian, the equality holds.

Recall at this point that directed polymers in a Bernoulli random environment are positive temperature versions of oriented percolation. Our bounds here have a flavor similar to the lower bounds for the critical threshold in 2-dimensional oriented percolation (i.e., $d = 1$ in our notations) in section 6 of Durrett Durrett (1984). In that paper, percolation is compared to Galton-Watson processes obtained in running oriented percolation for m steps ($m \geqslant 1$), and then using the distribution of wet sites as offspring distribution.

We now analyse the localization phenomenon in terms of energy-entropy balance, in the framework of supercritical 1-dimensional oriented percolation. Assume that η is Bernoulli distributed with parameter $p > \vec{p}_c(1)$. The infinite cluster is the set of points (t, x) with $t \in \mathbb{N}, x \in \mathbb{Z}, P(\omega_t = x) > 0$, which are connected to ∞ by an open oriented path – i.e., a path ω with $\eta(s, \omega_s) = 1 \ \forall s \geqslant t$. It is known that this cluster, at large scale, is approximatively a cone with vertex $(0, 0)$, direction $[0, x)$ and positive angle, and it has a positive density. In words, there is a huge number of oriented paths of length n with energy $H_n = n - \mathcal{O}(1)$. Also, there is an even larger number of paths with energy $H_n \sim cn$ for $c \in [p, 1)$, which are of interest when we move to non-zero temperature, i.e., when we consider the polymer measure μ_n. With exponentially many suitable paths on the energetic level, one could expect the polymer endpoint to be more or less uniformly distributed in a large interval. However, according to theorem 1.1, the polymer measure has a strong localization property (1.2). Hence, localization is essentially an entropic phenomenon, due to large spatial fluctuations in the number of paths with suitable energy.

For numerics, our upper bounds do not seem very efficient: on the basis of preliminary numerical simulations they converge quite slowly as $m \to \infty$. Finally we mention that lower bounds for the polymer free energy can be obtained from a well-known super-additivity property, see formula (2.3).

2. Notations and preliminaries

We first introduce some further notations.

Let $((\omega_n)_{n \in \mathbb{N}}, (P^x)_{x \in \mathbb{Z}^d})$ denote the simple random walk on the d-dimensional integer lattice $\mathbb{Z}^d$, defined on a probability space $(\Omega, \mathcal{F})$: for x in $\mathbb{Z}^d$, under the measure P^x, $(\omega_n - \omega_{n-1})_{n \geqslant 1}$ are independent and

$$P^x(\omega_0 = x) = 1, \ \ P^x(\omega_n - \omega_{n-1} = \pm\delta_j) = \frac{1}{2d}, \ \ j = 1, \ldots, d,$$

where $(\delta_j)_{1 \leqslant j \leqslant d}$ is the j-th vector of the canonical basis of $\mathbb{Z}^d$. Like in the introduction, we will use the notation P for P^0.

For the environment, we assume that for all $\beta \in \mathbb{R}$,

$$\lambda(\beta) \stackrel{\text{def.}}{=} \ln Q(e^{\beta\eta(n,x)}) < \infty.$$

It is convenient to consider the normalized partition function

$$W_n = Z_n / Q[Z_n] = P[\exp(\beta H_n(\omega) - n\lambda(\beta))].$$

We define for $k < n$, $x, y \in \mathbb{Z}^d$,

$$H_{k,n}(\omega) = \sum_{j=1}^{n-k} \eta(k+j, \omega_j)$$

and

$$W^x_{k,n}(y) = P^x(e^{\beta H_{k,n}(\omega)-(n-k)\lambda(\beta)} 1_{\omega_{n-k}=y}). \qquad (2.1)$$

In the sequel, $W_n(x)$ will stand for $W^0_{0,n}(x)$. The Markov property of the simple random walk yields

$$W_n = \sum_{x,y \in \mathbb{Z}^d} W_k(x) W^x_{k,n}(y). \qquad (2.2)$$

This identity will be extensively used in the sequel.

Finally, we recall (Comets et al., 2003) that with p defined by (1.1) it holds

$$p(\beta) = \lim_{n\to\infty} \frac{1}{n} Q(\ln(W_n(\beta))) = \sup_{n \geqslant 1} \frac{1}{n} Q(\ln(W_n(\beta))), \qquad (2.3)$$

where the last equality is a consequence of super-additivity arguments.

2.1. *Definition and well known facts on generalized multiplicative cascades.* In this section, we introduce a model of generalized multiplicative cascades on a tree. For an overview of results, we refer to Liu (2000). Let $N \geqslant 2$ be a fixed integer and

$$U = \bigcup_{k \in \mathbb{N}} [|\ 1, N\ |]^k$$

be the set of all finite sequences $u = u_1 \ldots u_k$ of elements in $[|\ 1, N\ |]$. With the previous notation, we write $|\ u\ | = k$ for its length. For $u = u_1 \ldots u_k$,$v = v_1 \ldots v_k$ two finite sequences, let uv denote the sequence $u_1 \ldots u_k v_1 \ldots v_k$. Let q be a non degenerate probability distribution on $(\mathbb{R}^*_+)^N$. It is known (cf. Liu, 2000) that there exist a probability space with probability measure denoted by $\mathtt{P}$ (and expectation $\mathtt{E}$), and random variables $(A_u)_{u \in U}$ defined on this space, such that the random vectors $(A_{u1}, \ldots, A_{uN})_{u \in U}$ form an i.i.d. sequence with common distribution q. We set the root variable $A_\varnothing$ constant and equal to 1. We assume that the $(A_i)_{1 \leqslant i \leqslant N}$ are normalized:

$$\mathtt{E}(\sum_{i=1}^N A_i) = 1$$

and that they have moments of all order: $\mathtt{E}[\sum_{i=1}^N A_i^p] < \infty\ \forall p \in \mathbb{R}$. Consider the process $(W^{\text{casc}}_n)_{n \in \mathbb{N}}$ defined by

$$W^{\text{casc}}_n = \sum_{u_1, \ldots, u_n \in [|1,N|]} A_{u_1} A_{u_1 u_2} \ldots A_{u_1 \ldots u_n} \qquad (2.4)$$

and the filtration

$$\mathcal{G}_n := \sigma\{A_u; |\ u\ | \ \leqslant n\}, \qquad n \geqslant 1.$$

Then $(W^{\text{casc}}_n, \mathcal{G}_n)_{n \geqslant 1}$ is a non negative martingale so the limit

$$W^{\text{casc}}_\infty = \lim_{n\to\infty} W^{\text{casc}}_n$$

exists. We are interested in the behavior of the associated free energy:

$$p_n = \frac{1}{n} \ln W^{\text{casc}}_n\ .$$

In the case where the $(A_i)_{i \leqslant N}$ are i.i.d, the exact limit of p_n as n goes to infinity was derived in Franchi (2000). In the general case, the proofs in Franchi (2000) can easily be adapted to show the following result.

Theorem 2.1. *The following convergence holds* $\mathtt{P}$*-a.s. and in* L^p *for all* $p \geqslant 1$*:*

$$p_n \underset{n\to\infty}{\longrightarrow} \inf_{\theta \in]0,1]} \frac{1}{\theta} \ln(\mathtt{E}\sum_{i=1}^{N} A_i^{\theta}) \leqslant 0.$$

The above inequality is a consequence of the normalization. Finding the limit of p_n as n tends to infinity amounts to studying the function v defined by

$$\forall \theta \in]0,1], \qquad v(\theta) = \frac{1}{\theta} \ln(\mathtt{E}\sum_{i=1}^{N} A_i^{\theta}) ,$$

which has derivative

$$v'(1) = \mathtt{E}\sum_{i=1}^{N} A_i \ln(A_i) .$$

Lemma 2.2. *If* $\mathtt{E}\sum_{i=1}^{N} A_i \ln(A_i) \leqslant 0$*, the function* v *is strictly decreasing on* $]0,1]$ *and thus*

$$\inf_{\theta \in]0,1]} v(\theta) = v(1) = 0.$$

If $\mathtt{E}\sum_{i=1}^{N} A_i \ln(A_i) > 0$*, there exists a unique* $\theta^* \in]0,1[$ *such that*

$$\inf_{\theta \in]0,1]} v(\theta) = v(\theta^*) < 0.$$

Proof : For all $\theta \in]0,1]$, we have the following expression for the derivative of v:

$$v'(\theta) = \frac{g(\theta)}{\theta^2}$$

where g is given by

$$g(\theta) = \theta \frac{\mathtt{E}\sum_{i=1}^{N} A_i^{\theta} \ln(A_i)}{\mathtt{E}\sum_{i=1}^{N} A_i^{\theta}} - \ln(\mathtt{E}\sum_{i=1}^{N} A_i^{\theta}).$$

In particular, we obtain the value of $v'(1)$ given above. By direct computation, one can obtain the following expression for g'

$$\forall \theta > 0 \qquad g'(\theta) = \theta \frac{\mathtt{E}(\sum_{i=1}^{N} A_i^{\theta}(\ln(A_i) - \mathtt{E}(\ln(A) \mid A^{\theta}))^2)}{\mathtt{E}(\sum_{i=1}^{N} A_i^{\theta})}$$

where $\mathtt{E}(\ln(A) \mid A^{\theta})$ is a notation for

$$\mathtt{E}(\ln(A) \mid A^{\theta}) = \frac{\mathtt{E}(\sum_{i=1}^{N} A_i^{\theta} \ln(A_i))}{\mathtt{E}(\sum_{i=1}^{N} A_i^{\theta})}.$$

In particular, g is strictly increasing and we have

$$g(1) = \mathtt{E}(\sum_{i=1}^{N} A_i \ln(A_i)).$$

By considering the two cases $g(1) \leqslant 0$ and $g(1) > 0$, we can easily conclude. □

3. Majorizing polymers with cascades

Let us fix an integer $m \geqslant 1$ and define L_m to be the set of points visited by the simple random walk at time m:

$$L_m \stackrel{def}{=} \{x \in \mathbb{Z}^d; P(w_m = x) > 0\}.$$

We introduce $(W^{\text{tree}}_{m,n})_{n \geqslant 1} \equiv (W^{\text{casc}}_n)_{n \geqslant 1}$ the martingale of the multiplicative cascade associated to the random vector $(W_m(x))_{x \in L_m}$, i.e., defined by (2.4) when $N = |L_m|$ and q is the law of $(W_m(x))_{x\in L_m}$; we remind that $W_m(x)$ stands for $W^0_{0,m}(x)$ with $W^0_{0,m}(x)$ given by (2.1). Let $p^{\text{tree}}_m(\beta)$ denote the associated free energy. In view of theorem 2.1, $p^{\text{tree}}_m(\beta)$ is given by

$$p^{\text{tree}}_m(\beta) = \inf_{\theta \in]0,1]} v_m(\theta) \tag{3.1}$$

where v_m is given by the expression

$$\forall \theta \in]0,1] \qquad v_m(\theta) = \frac{1}{\theta} \ln(Q \sum_{x \in L_m} W_m(x)^\theta). \tag{3.2}$$

We will first need the following monotonicity lemma.

Lemma 3.1. *Assume that $\phi :]0,\infty[\longrightarrow \mathbb{R}$ is $\mathcal{C}^1$ and that there are constants $C, p \in [1,\infty[$ such that*

$$\forall u > 0 \qquad |\ \phi'(u)\ | \ \leqslant Cu^p + Cu^{-p}.$$

Then for all $x \in L_m$ $\phi(W_m(x))$, $\frac{\partial \phi(W_m(x))}{\partial\beta} \in L^1(Q)$, $Q\phi(W_m(x))$ is $\mathcal{C}^1$ in $\beta \in \mathbb{R}$ and

$$\frac{\partial}{\partial\beta} Q\phi(W_m(x)) = Q\frac{\partial}{\partial\beta}\phi(W_m(x)).$$

Suppose in addition that ϕ is concave. Then ,

$$\forall \beta \geqslant 0 \qquad Q\frac{\partial}{\partial\beta}\phi(W_m(x)) \leqslant 0.$$

Proof : The proof is an immediate adaptation of the proof of lemma 3.3 in Comets and Yoshida (2006). □

As a consequence we can define the following

Proposition 3.2. *The function p^{tree}_m is non-increasing in β. There exists a critical value $\beta^m_c \in (0,\infty]$ such that*

$$p^{\text{tree}}_m(\beta) \begin{cases} = 0 & \text{if} \quad \beta \in [0, \beta^m_c], \\ < 0 & \text{if} \quad \beta > \beta^m_c. \end{cases}$$

Proof : For all $\theta \in]0,1]$, the function $x \to x^\theta$ is concave so by lemma 3.1, we see from expression (3.2) that $v_m(\theta)$ is non-increasing as a function of β. Therefore, we see from (3.1) that p^{tree}_m is itself non-increasing in β and we obtain the existence of β^m_c ($\beta^m_c \in [0,\infty]$). Since

$$\begin{aligned} v'_m(1) &= Q \sum_{x\in L_m} W_m(x) \ln W_m(x) \\ &\longrightarrow \sum_{x \in L_m} P(\omega_m = x) \ln P(\omega_m = x) < 0 \end{aligned}$$

as $\beta \searrow 0$, we conclude that β_c^m is strictly positive by continuity of $\partial_\theta v_m(\theta,\beta)_{|\theta=1}$ in β and by lemma 3.1.

□

Theorem 3.3. *We have the following inequality*

$$p(\beta) \leqslant \inf_{m \geqslant 1} \frac{1}{m} p_m^{\text{tree}}(\beta). \tag{3.3}$$

Proof : Let $\theta \in (0,1)$ and m be a positive integer. By using the subadditive estimate

$$\forall u, v > 0, \qquad (u+v)^\theta < u^\theta + v^\theta, \tag{3.4}$$

we have for all $n \geqslant 1$

$$\begin{aligned}
Q\frac{1}{n}\ln W_{nm} &= Q\frac{1}{\theta n}\ln W_{nm}^\theta \\
&\overset{(2.2)}{=} Q\frac{1}{\theta n}\ln\left(\sum_{x_1,\dots,x_n} W_m(x_1)\dots W_{(n-1)m,nm}^{x_{n-1}}(x_n)\right)^\theta \\
&\overset{(3.4)}{\leqslant} Q\frac{1}{\theta n}\ln\sum_{x_1,\dots,x_n} W_m(x_1)^\theta\dots W_{(n-1)m,nm}^{x_{n-1}}(x_n)^\theta \\
&\overset{(\text{Jensen})}{\leqslant} \frac{1}{\theta n}\ln Q\sum_{x_1,\dots,x_n} W_m(x_1)^\theta\dots W_{(n-1)m,nm}^{x_{n-1}}(x_n)^\theta \\
&= \frac{1}{\theta n}\ln\left(Q\sum_x W_m(x)^\theta\right)^n \\
&= \frac{1}{\theta}\ln Q\sum_x W_m(x)^\theta.
\end{aligned}$$

The proof is complete by taking the limit as $n \to \infty$ and then by taking the infimum over all $\theta \in]0,1]$ and $m \geqslant 1$. □

In particular, to prove $p(\beta) < 0$ it suffices to find $m \geqslant 1$ (in fact, $m \geqslant 2$) and $\theta \in (0,1)$ such that $Q\sum_x W_m(x)^\theta < 1$. The theorem is a handy way to obtain upper bounds on the critical β.

Remark 3.4. *For the sequence of cascade models, the authors do not know if the sequence $(p_m^{\text{tree}}(\beta))_{m \geqslant 1}$ is subadditive. However we can show that:*

$$\inf_{m \geqslant 1} \frac{1}{m} p_m^{\text{tree}}(\beta) = \lim_{m\to\infty} \frac{1}{m} p_m^{\text{tree}}(\beta). \tag{3.5}$$

Proof : Let $\theta \in]0,1[$ and $m \geqslant 1$. Using (3.4), we find by a similar computation that for all $k \geqslant 2$

$$\begin{aligned}
Q\sum_y W_{km}(y)^\theta &= Q\sum_y \big(\sum_{x_1,\dots,x_{k-1}} W_m(x_1)\dots W_{(k-1)m,km}^{x_{k-1}}(y)\big)^\theta \\
&< Q\sum_y \sum_{x_1,\dots,x_{k-1}} W_m(x_1)^\theta\dots W_{(k-1)m,km}^{x_{k-1}}(y)^\theta \\
&= \big(Q\sum_x W_m(x)^\theta\big)^k.
\end{aligned} \tag{3.6}$$

In view of (3.1) and of the smoothness of $v_m(\cdot)$, we conclude that

$$\frac{1}{km} p_{km}^{\rm tree}(\beta) \leqslant \frac{1}{m} p_m^{\rm tree}(\beta).$$

Observe that when $p_m^{\rm tree}(\beta) < 0$, the infimum in (3.1) is achieved for some $\theta \in (0,1)$, and therefore the above inequality is strict. In particular,

$$\inf_{m \geqslant 1} \frac{1}{m} p_m^{\rm tree}(\beta) = \varliminf_{m\to\infty} \frac{1}{m} p_m^{\rm tree}(\beta). \tag{3.7}$$

By repeating the steps in (3.6), we see that, for $0 \leqslant \ell < m, k \geqslant 1$ and $\theta \in (0,1]$,

$$v_{km+\ell}(\theta) \leqslant k v_m(\theta) + v_\ell(\theta) ,$$

whereas, by concavity,

$$v_\ell(\theta) \leqslant \frac{1}{\theta} \sum_x \big(QW_\ell(\theta)\big)^\theta = v_\ell(\theta, 0) ,$$

where $v_\ell(\theta,0) = v_\ell(\theta,\beta)_{|\beta=0} \in (0,\infty)$. Therefore,

$$\max_{km \leqslant n<(k+1)m} \frac{v_n(\theta)}{n} \leqslant \frac{k}{(k+\varepsilon)m} v_m(\theta) + \frac{1}{km} v_\ell(\theta,0) ,$$

where $\varepsilon = 0$ or 1 according to the sign of $v_m(\theta)$. Now, recalling that $v_m(\theta) \geqslant p_m^{\rm tree}(\beta)$ and taking the limit $k \to \infty$, leads to

$$\limsup_n \frac{p_n^{\rm tree}(\beta)}{n} \leqslant \frac{v_m(\theta)}{m}, \quad m \geqslant 1, \theta \in (0,1].$$

Combined with (3.7), this implies (3.5). □

We add another

Remark 3.5. *Suppose that there exists $m \geqslant 1$ such that*

$$Q \sum_x W_m(x) \ln W_m(x) = 0,$$

then we have $p(\beta) < 0$.

Proof : We have

$$\begin{aligned} Q \sum_y W_{2m}(y) \ln W_{2m}(y) &= Q \sum_{x,y} W_m(x) W_{m,2m}^x(y) \ln W_{2m}(y) \\ &> \sum_{x,y} QW_m(x) W_{m,2m}^x(y) \ln \big(W_m(x) W_{m,2m}^x(y)\big) \\ &= \sum_x \big(QW_m(x) \ln W_m(x)\big) \sum_y QW_{m,2m}^x(y) \\ &\quad + \sum_x \big(QW_m(x)\big) \sum_y QW_{m,2m}^x(y) \ln W_{m,2m}^x(y) \\ &= 2 \sum_x QW_m(x) \ln W_m(x) \\ &= 0. \end{aligned}$$

Hence, by lemma 2.2, $p_{2m}^{\rm tree}(\beta) < 0$ and finally $p(\beta) < 0$. □

As a consequence of theorem 3.3, we get our main result

Proof of theorem 1.1: Let $d = 1$, $\theta \in]0,1]$ and $\beta > 0$. By using lemma 4.1 in Comets et al. (2003), there exists a $c(\theta) > 0$ such that

$$\forall m \geqslant 1 \qquad Q(W_m^\theta) \leqslant e^{-c(\theta)m^{\frac{1}{3}}}.$$

Therefore

$$\begin{aligned} Q(\sum_{x \in L_m} (W_m(x))^\theta) &\leqslant \mid L_m \mid Q(W_m^\theta) \\ &\leqslant \mid L_m \mid e^{-c(\theta)m^{\frac{1}{3}}} \underset{m\to\infty}{\longrightarrow} 0, \end{aligned}$$

where we have used the fact that $\mid L_m \mid = O(m)$. In particular, there exists $m \geqslant 1$ such that

$$Q(\sum_{x \in L_m} (W_m(x))^\theta) < 1.$$

We have $p_m^{\text{tree}}(\beta) < 0$ and so by theorem 3.3 $p(\beta) < 0$. □

Theorem 3.6. *Suppose the environment η is bounded or gaussian. Then the inequality (3.3) is in fact an equality*

$$p(\beta) = \inf_{m \geqslant 1} \frac{1}{m} p_m^{\text{tree}}(\beta).$$

Proof : The inequality $p(\beta) \leqslant \inf_{m \geqslant 1} \frac{1}{m} p_m^{\text{tree}}(\beta)$ is in fact the conclusion of theorem 3.3 and thus is true for all environments.

We must show that $\inf_{m \geqslant 1} \frac{1}{m} p_m^{\text{tree}}(\beta) \leqslant p(\beta)$. We treat the gaussian case, the bounded case being similar. If $\beta \leqslant \beta_c$, we have by definition $p(\beta) = 0$ and since for all $m \geqslant 1$, $p_m^{\text{tree}}(\beta) \leqslant 0$, the result is obvious. Suppose that β is such that $\beta > \beta_c$. By definition of β_c, $p(\beta) < 0$. Let $\theta \in]0,1]$. We have by the concentration result (4.5)

$$\begin{aligned} Q(W_m^\theta) &= e^{\theta Q(\ln(W_m))} Q(e^{\theta(\ln W_m - Q(\ln(W_m))}) \\ &\leqslant e^{\theta p(\beta) m + \frac{\beta^2 \theta^2 m}{2}}. \end{aligned}$$

For all $m \geqslant 1$,

$$\begin{aligned} \frac{1}{m} p_m^{\text{tree}}(\beta) &\leqslant \frac{1}{\theta m} \ln(Q(\sum_{x \in L_m} (W_m(x))^\theta)) \\ &\leqslant \frac{1}{\theta m} \ln(\mid L_m \mid) + \frac{1}{\theta m} \ln(Q(W_m^\theta)) \\ &\leqslant \frac{1}{\theta m} \ln(\mid L_m \mid) + p(\beta) + \frac{\beta^2 \theta}{2} \\ &\underset{m\to\infty}{\longrightarrow} p(\beta) + \frac{\beta^2 \theta}{2}, \end{aligned}$$

where we have used the fact that $\mid L_m \mid = O(m^d)$. Thus, by remark 3.4

$$\inf_{m \geqslant 1} \frac{1}{m} p_m^{\text{tree}}(\beta) = \lim_{m\to\infty} \frac{1}{m} p_m^{\text{tree}}(\beta) \leqslant p(\beta) + \frac{\beta^2 \theta}{2}.$$

The proof is complete by letting $\theta \downarrow 0$. □

4. Appendix

In this appendix, we recall concentration of measure results we used in the proof of theorem 3.6. For a complete survey on the concentration of measure phenomenon, we refer to Ledoux (1999). In the gaussian case, we have

Theorem 4.1. *Let $M \geqslant 1$ be an integer. We consider $\mathbb{R}^M$ equipped with the usual euclidian norm $\| \cdot \|$. If X_M is a standard gaussian vector on some probability space (with a probability measure* $\mathtt{P}$*) and F is a C-lipschitzian function ($|F(x) - F(y)| \leqslant C\|x-y\|$) from $\mathbb{R}^M$ to $\mathbb{R}$ then*

$$\mathtt{E}(e^{\lambda(F(X_M)-\mathtt{E}(F(X_M)))}) \leqslant e^{\frac{C^2\lambda^2}{2}}. \tag{4.1}$$

Therefore, we have the following concentration result

$$\mathtt{P}(|\ F(X_M) - \mathtt{E}(F(X_M))\ |\ \geqslant r) \leqslant 2e^{-\frac{r^2}{2C^2}}. \tag{4.2}$$

In the bounded case, we get a similar concentration result (cf. Corollary 3.3 in Ledoux, 1999) .

Theorem 4.2. *Let $M \geqslant 1$ be an integer and $a < b$ be two real numbers. If X_M is a random vector in $[a,b]^M$ with i.i.d. components on some probability space and F is a convex and C-lipschitzian function from $[a,b]^M$ to $\mathbb{R}$ for the euclidian norm, then*

$$\mathtt{E}(e^{\lambda(F(X_M)-\mathtt{E}(F(X_M)))}) \leqslant e^{C^2(b-a)^2\lambda^2}. \tag{4.3}$$

Therefore, we have the following concentration result

$$\mathtt{P}(F(X_M) - \mathtt{E}(F(X_M)) \geqslant r) \leqslant e^{-\frac{r^2}{4C^2(b-a)^2}} \tag{4.4}$$

We can derive from the above theorems a concentration result for the free energy at time n:

Corollary 4.3. *If the environment η is standard gaussian then for all $\lambda \geqslant 0$,*

$$Q(e^{\lambda(\ln(W_n)-Q(\ln(W_n)))}) \leqslant e^{\frac{\beta^2\lambda^2 n}{2}}. \tag{4.5}$$

If the environment η belongs to $[a,b]$ for $a < b$ two real numbers, then for all $\lambda \geqslant 0$,

$$Q(e^{\lambda(\ln(W_n)-Q(\ln(W_n)))}) \leqslant e^{\beta^2(b-a)^2\lambda^2 n}. \tag{4.6}$$

Proof : As a function of the environment, $\ln(W_n)$ is convex and $\beta\sqrt{n}$-lipschitzian (cf. the proof of proposition 1.4 in Carmona and Hu, 2002). Therefore, in the gaussian case, the result is a direct application of (4.1) and, in the bounded case, simply (4.3). □

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