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James B. Davies

Michael Hoy

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by

James Davies
and
Michael Hoy

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Department of Economics

Social Science Centre

University of Western Ontario

London, Ontario, CANADA

N6A 5C2

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James Davies and Michael Hoy

Department of Economics
University of Western Ontario
London, Ontario

and

Department of Economics
University of Guelph
Guelph, Ontario

(respectively)

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Please Address Correspondence to:

Prof. J. Davies
Department of Economics
University of Western Ontario
London, Canada N6A 5C2

1. Introduction

An important tool in applied work on income distribution is the Lorenz curve. When one Lorenz curve lies above another, it displays less inequality in an unambiguous sense. Unfortunately, Lorenz curves may intersect. In such cases, taking the observed distributions at face value, the "Lorenz Criterion" does not provide a ranking. Although any specific inequality index will be able to make a comparison, another reasonable index will inevitably go against this determination.

It has often been suggested that at least some of the ambiguity in inequality comparisons could be resolved by a slight strengthening of normative judgements. The Lorenz criterion assumes only that a transfer from richer to poorer reduces inequality, that is, it embodies the *principle of transfers*. Many authors have suggested that for a fixed income gap a transfer of a particular dollar amount from a poorer to a richer person should be considered more disequalizing the lower it occurs in the distribution.[1] (See, e.g. Sen, 1973; Atkinson, 1973; Kolm 1976.) Such a viewpoint has been given various labels. Here we refer to it as *aversion to downside inequality* -- ADI[2] since it parallels precisely the concept of aversion to downside risk proposed by Menezes, Geiss and Tressler (1980) (MGT) in this journal.

Recently, theorists have paid considerable attention to the consequences of assuming aversion to downside inequality. (Shorrocks and Foster, 1987; Dardanoni and Lambert, 1987; Foster and Shorrocks, 1988; and Muliere and Scarsini, 1989). Of greatest relevance to applied researchers, it has been shown that if the true Lorenz curves are known with certainty and cross only once there is a straightforward method of determining when all ADI inequality indices agree. While this is a useful result, it does not help when Lorenz curves intersect more than once.

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[1]For example, we might not care if someone with an income of $1 million lost $1,000 to an individual with income $20,000 higher, but at the same time be quite concerned if someone with only a $10,000 income lost $1,000 to an individual with $30,000.

[2]The ADI concept has been referred to elsewhere as "increasing absolute inequality aversion" (Atkinson), the "principle of diminishing transfers" (Kolm), or "transfer sensitivity" (Shorrocks and Foster).

This paper makes the ADI criterion more fully operational by providing a simple procedure for establishing whether *any* two true distributions which cross a finite number of times can be ranked under ADI. It then illustrates the new criterion using recent family income distribution data for both Canada and the U.S., with sample distributions being treated as if true. The criterion provides rankings in some, but not all, cases of singly and multiply intersecting Lorenz curves. We argue that even where ADI does not provide a ranking, *knowing* this is itself informative since it tells us that even stronger assumptions about inequality indices would have to be made in order to obtain a ranking.

The paper is organized as follows. The next section outlines the background behind the concept of ADI. Section 3 then presents results which make the criterion more fully operational. Finally, in Section 4 we apply ADI to actual data.

2. Background

Inequality Aversion

We will proceed with the help of a simple example. Table 1 shows an initial hypothetical income distribution, Y, among four individuals. The cumulative income and population shares are plotted in the Lorenz curve of Figure 1. Table 1 records the values of two simple inequality measures for this distribution. The Gini coefficient (G) is 0.220 and coefficient of variation, CV, (the standard deviation divided by the mean) is 0.399.

Consider first what would happen if we transferred \$5 from family A to B, thereby generating the Y' income distribution in Table 1. This is a *regressive transfer* or a *mean preserving spread* (*mps*) in the languages of inequality and risk measurement respectively.[3] All those who are *inequality averse* (or who, equivalently, subscribe to the principle of transfers) will regard such a transfer as increasing inequality. Figure 1 shows the resulting

[3]The term "mean preserving spread" was introduced by Rothschild and Stiglitz (1970) to refer a transfer of density from a particular region of a distribution towards the tails of a distribution, leaving the mean unaltered. A "mean preserving contraction" transfers density in the opposite direction. See Appendix B for formal definitions.

downward movement in the lower portion of the Lorenz curve. Referring to Table 1, both the Gini coefficient and *CV* increase - to 0.245 and 0.446 respectively - as would any inequality index which satisfies the principle of transfers.

Now take \$5 away from family *D* and give it to family *B*. This *progressive transfer*, or *mean preserving contraction* (*mpc*), generates the Y'' distribution in Table 1. Figure 2 shows the impact of this unambiguously equalizing operation on the Lorenz curve - - an upward movement of its top portion. Referring to Table 1 we see that this transfer also results in a decline in the Gini coefficient to 0.195, below its original value for the *Y* distribution. The *CV* returns to its original value of 0.399.

The net result of the two changes to the initial distribution *Y*, is made clear in Figure 3. The Y^o and Y'' Lorenz curves have an intersection. Therefore, persons who are simply inequality averse can differ in their inequality rankings of Y^o and Y''.

Our example illustrates the following result for any two income distributions described by the densities *f* and *g*, and with the same mean and population size:[4]

(i) if distribution *f* has a Lorenz curve which is nowhere lower than the *g* Lorenz curve, and which is higher than the *g* curve at some points, then distribution *f* could be obtained from *g* by a series of progressive transfers (*mpc*'s), and
(ii) if the two Lorenz curves intersect, so that the *f* curve is at some points above, and at other points below, the *g* curve, then one must use a mixture of progressive and regressive transfers (*mpc*'s or *mps*'s) to obtain either of the distributions from the other.

To relate the role of the Lorenz curve to the theory of stochastic dominance it is helpful to have some formal structure. Let $X = [\underline{x}, \bar{x}]$ be an interval of real-valued income levels with $\underline{x}$ and $\bar{x}$ non-negative and finite. Let *f* and *g* represent continuous probability (or relative frequency) distributions from a space Ω over *x* with means $E(f)$ and $E(g)$ and variances $V(f)$

[4]See, e.g., Atkinson (1970).

and $V(g)$. If f and g were distributions of lottery outcomes, and all risk averters preferred f to g, then f would display *second degree stochastic dominance* (SSD) over g, or f SSD g. More precisely, all von-Neumann - Morgenstern utility functions, $u(x)$, with $u' > 0$ and $u'' < 0$, would rank f as better than g:[5]

(1) $$\int_{\underline{x}}^{\bar{x}} u(x)f(x)dx > \int_{\underline{x}}^{\bar{x}} u(x)g(x)dx$$

Clearly, f and g may be reinterpreted as income distributions, and u as an elementary *social evaluation function*[6] so that the integrals in (1) represent social welfare. Then an income distribution f is said to display SSD over g, or f SSD g, if equation (1) holds for all additive social welfare functions with $u' > 0$ and $u'' < 0$. In this paper we either assume that the distributions have equal means (as in this section and the next), or standardize all incomes by dividing by the mean (in the applied section). Thus all the welfare comparisons we make will in fact be inequality comparisons, which is a special application of stochastic dominance.

Denote the Lorenz curve ordinate corresponding to the lowest $100P\%$ of income recipients as $L(i;P)$, $i = f, g$. Then, a distribution f *Lorenz dominates* g, or f L g iff: $L(f;P) \geq L(g;P) \quad \forall P \in [0,1]$ and > for some P. Also, $I{:}\Omega \rightarrow R$ is an *inequality index* iff $I(.)$ obeys the principle of transfers, that is if g can be obtained from f by a pure series of

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[5]Riemann-Stieltjes integrals could be used in (1) in order to accommodate discontinuous probability distributions. That is we could write $\int_{\underline{x}}^{\bar{x}} u(x)\, dF(x) > \int_{\underline{x}}^{\bar{x}} u(x)\, dG(x)$ where $F(x)$ and $G(x)$ are distribution functions. We have not adopted this approach, in order to maintain comparability with earlier literature, and in view of the large sample sizes in typical microdata sets (which make continuity not such a bad approximation.)

[6]$u(x)$ may correspond to the individual utility function, in which case $w(x)$ is an additive utilitarian social welfare function, but this is not necessary. See Atkinson (1983), p. 5 for a discussion of this point.

regressive transfers (*mps*'s) then $I(g) > I(f)$. With this additional structure we can now state the well-known result:[7]

Proposition 1 For distributions f and g with equal means, the following statements are equivalent:

i) g can be obtained from f via a series of regressive transfers,

ii) f L g,

iii) f SSD g,

iv) $I(f) < I(g)$ for every inequality index $I(\cdot)$.

Aversion to Downside Inequality

In order to understand the motivation for, and appeal of, ADI let us return to the example considered in Table 1 before presenting a formal definition. Beginning with distribution Y^o, the regressive transfer of \$5 from A to B, to obtain distribution Y', increases the gap between this pair's incomes from \$5 to \$15. We then make a progressive transfer of \$5 from D to B, which reduces the gap between this pair's incomes from \$15 to \$5, to obtain distribution Y''. Although this second transfer has an effect equal in size but opposite in direction to the original regressive transfer, it occurs in a higher income range and so it is appealing in a normative sense to conclude that the two transfers together increase inequality. This is precisely the content of the ADI property; that an income transfer is more important the lower in the distribution that it occurs. Thus, it is sometimes possible to use this criterion to rank distributions which cannot be ranked by the principle of transfers or second degree stochastic dominance alone or, in other words, in some cases where Lorenz curves intersect.

Before giving a formal definition of ADI note the following two points about our example. First, the Gini index indicates less inequality for the Y'' distribution than for the Y

[7]All the elements of this result were demonstrated in Atkinson (1970). See Shorrocks and Foster (1987) for an elegant restatement.

distribution. Thus *the Gini coefficient does not embody aversion to downside inequality.* Second, as a result of the initial regressive transfer (Y to Y') the coefficient of variation rose from .399 to .446, while the progressive transfer (Y' to Y'') brought the value back down to .399. The reason for the return to the starting value is that when the *mps* and *mpc* involve equal changes in income gaps, but in the opposite direction, they have equal but opposite effects on the variance. This is a general property of the *mps-mpc* pairs which have been defined as *mean variance preserving transformations*, or *MVPT*s, by MGT:[8]

Definition 1: A mean-variance preserving transformation (*MVPT*) is a combination of a mean-preserving spread (*mps*) and a mean-preserving contraction (*mpc*) with the properties that (i) the *mps* occurs at lower income levels than does the *mpc* and (ii) the overall effect is to leave the variance unchanged. (See Appendix B for a more formal definition).

The *CV* is unique among inequality measures in being precisely indifferent to *MVPT*s. It thus lies on the boundary between indices which are averse to downside inequality and those which are not.

We can now state:

Definition 2: A distribution f displays less *downside inequality* than another, g, if g can be obtained from f by a series of *MVPT*s.

Definition 3: An inequality index $I(.)$ satisfies aversion to downside inequality (ADI) if $I(f) < I(g)$ wherever f displays less downside inequality than g.

One reason the ADI restriction on inequality measurement has turned out to be analytically fruitful is that when added to the concept of inequality aversion it corresponds to *third degree stochastic dominance* (TSD). When comparing a pair of distributions f and g we

[8]The exact reverse of a MVPT is referred to as a "favorable composite transfer" (*FACT*) by Shorrocks and Foster, 1987.

say f TSD g if all von-Neumann-Morgenstern utility functions, $u(x)$, with $u' > 0$, $u'' < 0$ and $u''' > 0$ would rank f equal to or better than g, as in (1). Thus the stochastic dominance literature tells us that our ADI criterion would, for example, restrict the range of additive social welfare functions we might consider to those with $u''' > 0$.[9]

Formally, the relationship between ADI and TSD can be summed up in the following proposition due to Shorrocks and Foster[10] where attention is again restricted to inequality (rather than general welfare) comparisons:

Proposition 2: For distributions f and g with equal means, the following statements are equivalent:

i) g can be obtained from f by a series of *MVPT*'s and/or *mps*'s,

ii) f TSD g,

iii) $I(f) < I(g)$ for all inequality indices obeying ADI.

Note the absence in this proposition of any condition involving Lorenz curves like (ii) in Proposition 1. How, then, are the insights of Proposition 2 to be implemented in applied work? As we see in the next section, an answer is already available for singly-intersecting Lorenz curves. We extend this to allow one to see whether ADI provides a ranking in comparisons where Lorenz curves cross *any* number of times.

3. Measuring Inequality under Aversion to Downside Inequality - Some Operational Results

The recent theoretical literature on ADI has provided a proposition which is operationally quite helpful. In order to state it we need:

[9]Atkinson (1973) argued that the third derivative restriction was not unattractive. As pointed out by Whitmore (1970), it is a necessary condition for decreasing absolute risk aversion.

[10]See Shorrocks and Foster, 1987, theorem 2.

Definition 4: The Lorenz curve for f crosses that for g n times (and initially from above) if there exists a series of points P_i with $P_0 \equiv 0$, $P_{n+1} \equiv 1$, and $P_0 < P_1 < P_2 < \ldots < P_n < P_{n+1}$ such that for i *odd*:

(1) $L(f;P) \geq L(g;P)$ $\forall P \in [P_{i-1}, P_i]$ and $\exists$ some $\varepsilon > 0$ such that the inequality is strict (>) $\forall P \in (P_i - \varepsilon, P_i)$.

(2) $L(f;P) \leq L(g;P)$ $\forall P \in [P_i, P_{i+1}]$ and < 0 for some $P \in [P_i, P_{i+1}]$.

For i even, switch f and g in (1) and (2).

Now, as shown first by Atkinson (1973):[11]

Proposition 3: Suppose that f and g have the same mean and their Lorenz curves have a single crossing. Then $I(f) < I(g)$ for all inequality indices obeying ADI if and only if (i) the Lorenz curve of f crosses that for g initially from above and (ii) $V(f) \leq V(g)$.

This result allows one to decide whether one distribution can be ranked superior to another according to ADI in cases where Lorenz curves only cross once. However, as discussed below, in practice multiple intersections may occur. So far no one has provided operational procedures for deciding whether ADI will resolve the apparent ambiguity in such comparisons. The following result, which we prove in Appendix A, allows us to determine whether an inequality ranking is provided under ADI in the comparison of any two income distributions which have been standardized to have the same mean:[12]

Proposition 4: If distributions f and g have the same mean and their Lorenz curves cross at least once then $I(f) < I(g)$ for all inequality indices that satisfy both the principle of transfers

[11]See also Dardanoni and Lambert (1987), and Shorrocks and Foster (1987).

[12]For those who are interested only in *relative* inequality such standardization is natural. The popularity of the generalized Lorenz curves introduced by Shorrocks (1983), which do not standardize in this way, indicates that not everyone is content to confine their attention to relative inequality. Generalization to the case where means differ is beyond the scope of this paper.

and ADI iff $V_i(f) \leq V_i(g)$, $\forall\ i = 1,2,...,n+1$ where P_i denotes the population shares (as defined in Definition 4) and $V_i(\cdot)$ denotes the variance within the sub-population defined by $P \in [0, P_i]$.

Proposition 4 indicates that it is necessary and sufficient for f to dominate g under ADI for the variance of f to be less than that of g for every cumulative "sub-population" defined by a point of intersection between Lorenz curves. Notice that $V_1(f) < V_1(g)$ implies that f crosses g initially from above and for $n = 1$ this proposition corresponds to Proposition 3. Also note that since f and g have the same mean, one may compare coefficients of variation rather than variances in implementing Proposition 4.[13]

4. Applications

Crude comparisons of sample Lorenz curves suggest a fairly large number of intersections.[14] However, by using the procedures developed by Beach and Davidson (1983) and Beach and Richmond (1985) one often finds that differences in Lorenz ordinates are not statistically significant. Bishop, Formby and Smith (1991, 1992) and Bishop, Formby and Thistle (1991a, 1991b) for example, characterize many cases of intersecting Lorenz curves as not statistically significant. Roughly speaking, the test procedure generates standard errors for the differences between a pair of Lorenz curves at an arbitrary set of ordinates (eg. every tenth percentile). The power of this test to identify pairs of Lorenz curves which intersect, however, has not been well established. Intuitively, the greater the number of true intersections, the

[13]This procedure can be simplified further, if desired. If the Lorenz curve for a distribution f intersects that for g initially from above, and the CV_i for f is always less than or equal to that for g at even-numbered intersection points, then *a fortiori* the inequality will hold for odd-numbered points. Thus, in practice, it is only necessary to check the CV_i's at every second intersection point.

[14]Typical examples are provided by international comparisons of income distribution. In the data for 23 countries assembled by Van Ginneken and Park (1984), there are at least 53 cases of intersection in the 253 pairwise comparisons. Atkinson (1973) found that in 66 pairwise comparisons with the data of Kuznets (1963) for 12 countries Lorenz curves intersected in 50 cases. Thus the frequency of intersections varies across different data sets, but can be very high.

larger the ranges where the true curves are close together and the difference between sample Lorenz curves p_s unlikely to be statistically significant. In addition, conventional grids may be too coarse. For example, testing at every tenth percentile, sample intersections occuring in the top or bottom 10% of the distribution would be completely ignored. Thus, those cases in which ADI might be useful in clarifying inequality comparisons are also ones where the statistical comparison of Lorenz curves is inherently difficult, and where existing methods of testing have significant limitations.

While it would be attractive to explore both the role of ADI in inequality comparisons and statistical issues that exercise is beyond the scope of the present paper. We shall proceed as if sample distributions corresponded to true population distributions, paralleling the approach of the SSD literature when it was at a similar stage.

We examine data from the Canadian *Survey of Consumer Finance* (*SCF*), and the U.S. *Current Population Survey* (*CPS*), for selected years.[15] Table 2 shows that the Canadian income distribution was relatively stable over the period 1975-88. However, both the shares of the top *and* bottom 20% trended upwards, so that there was a rotation of the Lorenz curve. The result is that in 43 of the 78 pairwise comparisons that can be made sample Lorenz curves intersect. A good example is provided by the change from 1975 to 1988. The Lorenz curves intersect once, at the 29th percentile. The CV_i's for the sub-populations at the intersection, and overall *CV's*, are as follows:

i	P_i	$CV_i(1975)$	$CV_i(1988)$
1	.290	.393	.338
	Overall *CV's*:	.617	.665

[15]In addition to allowing rich cross-section comparisons, these datasets are of special interest in view of the differing recent trends in income distribution in the two countries. (See, e.g., Blackburn and Bloom [1991].)

Thus, 1988 cannot be ranked as more equal than 1975 simply by appeal to the ADI criterion, despite its higher shares for bottom groups. This typifies many of the intersections in Table 2 which cannot be resolved by ADI because the distribution that "lies above" at the bottom has the *lower* Lorenz curve over most of the range of P.

The 1977 vs. 1981 comparison shows another possibility. From the summary data, 1981 appears to be the "trough year" for inequality in Canada. But, while the shares of bottom groups were higher in 1981 than 1977, the share of the top quintile was also slightly higher. Applying ADI we find that, despite the intersection, 1981 was indeed more equal than 1977 according to these data. This is shown by the CV_i's and CV's:

i	P_i	$CV_i(1977)$	$CV_i(1981)$
1	.767	.456	.433
	Overall CV's:	.591	.590

Intuitively, ADI provides a clear ranking here because the distribution which "lies above initially" (1981) has the upper Lorenz curve over most of the range of P.

The Canadian samples also show multiple intersections. For example, the 1979 Lorenz curve lies initially above the 1975 curve, but the two intersect in the 2nd percentile, and again at the 56th percentile. The comparison is summarized by:

i	P_i	$CV_i(1975)$	$CV_i(1979)$
1	.017	.705	.673
2	.560	.414	.425
	Overall CV's:	.617	.599

Although, overall, the CV is lower for 1979 than 1975 and the Lorenz curve for 1979 is initially above that for 1975, the 1979 distribution cannot be ranked more equal since its CV_2 exceeds the 1975 value. In other words, 1979 has too much inequality between percentiles 2 and 56 for ADI to make possible an unambiguous ranking.

ADI *does* provide a ranking of the 1979 and 1981 *after-tax* sample income distributions, which have a double intersection in percentiles 76 and 97. The 1981 curve lies "initially above." The CV_i's and CV's are as follows:

i	P_i	$CV_i(1979)$	$CV_i(1981)$
1	.760	.402	.390
2	.965	.469	.465
	Overall CV's:	.542	.537

Since 1975 in the U.S. there has been a steady increase in inequality in the overall distribution of income and so a straightforward Lorenz comparison applies.[16] However, there are interesting Lorenz intersections and there have been important changes over time for some sub-groups, as shown in Table 3.[17] This table echoes the overall trend towards increased inequality; for each subgroup the 1980 distribution Lorenz dominates 1988. In cross-section certain comparisons also show simple Lorenz domination. For example, the Lorenz curves for either Black or Hispanic families lie well below those for Whites, and families with a female householder have a much less equal distribution than married couples. Regionally, the Northeast and Midwest both Lorenz dominate the West and South. However, there are four cases of intersecting Lorenz curves -- Blacks vs. Hispanics, young vs. old, the Northeast vs. the Midwest, and the South vs. the West.

The income distribution for Blacks was Lorenz dominated by that for Hispanics in 1980, although shares were fairly close at the upper end. By 1988, however, the Hispanic Lorenz curve crossed the Black curve from above in the 92nd percentile. Can ADI resolve the

[16]There are only three exceptions: 1977 vs. 78, 1979 vs. 80, and 1982 vs. 83. In each of these cases there was a small decrease in the share of the top 5% of families. These declines were minor "blips" in the gradual increase of this top share from 15.5% in 1975 to 17.9% in 1989.

[17]For both 1980 and 1988 some income is missing in the microdata tapes due to topcoding. This was corrected by imputing an equal amount of additional income to all families affected. The resulting cumulative shares are the same as those published by the Census Bureau, but there are minor differences in some of the Gini coefficients.

Appeal to ADI alone does not establish a ranking. (This is also true in the young vs. old comparisons.) However, it is interesting to note that if just the top 1% of families were omitted the Hispanic distribution would still dominate in 1988 under ADI.

As shown by the following indicators, the Midwest is unambiguously more equal than the Northeast under ADI, whereas ADI does not "break the tie" for the West and South:

i	P_i	CV_i(N East)	CV_i(Midwest)
1	.943	.600	.570
Overall *CV's*:		.762	.760

i	P_i	CV_i(South)	CV_i(West)
1	.852	.578	.541
Overall *CV's*:		.822	.836

In conclusion, the examples studied in this section show that the ADI criterion can sometimes resolve the ambiguous rankings associated with sample Lorenz curve intersections. Cases where ADI is decisive are likely to be those where the distribution whose Lorenz curve "lies initially above" also lies in the upper position across most percentiles. Since this description does not always apply, ADI is not a panacea for ranking income distributions, even neglecting statistical problems. In order to resolve the ambiguity in these cases, in addition to performing statistical tests, stronger assumptions may have to be made about attitudes toward inequality.

Use of the procedure outlined in this paper does not just demarcate cases where ADI can and cannot provide an overall ranking. Where ADI does not provide a ranking, our procedure identifies why this is the case, and gives notion of "how much" we fall short of such a ranking. Thus, in our view, the use of ADI is important in assessing all cases of intersecting Lorenz curves.

5. Conclusion

The importance of using the Lorenz curve in making inequality comparisons rather than relying on summary measures of dispersion alone has long been recognized. Besides giving a fuller description of differences in inequality, the normative judgements associated with the Lorenz criterion are both explicit and appealing. When Lorenz curves do not intersect, one need only adhere to the *principle of transfers* to rank one distribution as more equal than another.[18]

In this paper we have shown that by accepting a further normative axiom, aversion to downside inequality (ADI), one can extend the Lorenz partial ordering to rank, in some cases, distributions whose Lorenz curves intersect. ADI requires only that one judge a transfer of a particular dollar amount between persons with a given absolute income gap to be more significant the lower down in the distribution it occurs. One advantage of this approach is that if one distribution is judged as more equal than another according to ADI then it follows that *any* inequality index which obeys ADI will agree with this ranking. Thus, using ADI can be much more powerful than consulting a handful of inequality indexes.

Proposition 4 in this paper shows how to make this criterion fully operational and the examples in Section 4 illustrate its real-world usefulness. Although by no means does ADI resolve all ambiguities of inequality comparisons caused by intersecting Lorenz curves, we have illustrated how the basic approach can be informative even for such "failures."

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[18]Also, implicit to the Lorenz criterion and hence the ADI extension are the principles of mean income independence and population size independence.

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Table 1

Hypothetical Income Distributions and Associated Inequality Measures

Family	Income Distribution Y^o	Y'	Y''
A	13	8	8
B	18	23	28
C	31	31	31
D	38	38	33
Total Income	100	100	100
Gini Coefficient	.220	.245	.195
Coefficient of Variation	.399	.446	.399

Table 2

Cumulative Income Shares and Gini Coefficients for Canadian Families, 1975 - 1989, Selected Years

Year	Cumulative Income Share at Percentile ... 20	40	60	80	Gini
1975	6.2%	19.2%	37.4%	61.3%	.326
1977	5.9	19.0	37.5	61.9	.321
1979	6.1	19.1	37.5	61.8	.322
1980	6.2	19.2	37.5	61.6	.323
1981	6.5	19.4	37.7	61.8	.318
1982	6.4	19.0	37.0	61.0	.326
1983	6.3	18.7	36.5	60.6	.334
1984	6.2	18.6	36.6	60.7	.334
1985	6.4	18.8	36.7	60.8	.330
1986	6.4	18.8	36.7	60.7	.331
1987	6.5	18.9	36.7	60.7	.330
1988	6.5	18.9	36.8	60.8	.328
1989	6.5	19.1	36.9	60.7	.328

Source: Statistics Canada, *Income Distributions by Size in Canada*, Publication No. 13-207, annual.

Table 3

Cumulative Income Shares and Inequality Indexes for U.S. Families, by Subgroups, 1980 and 1988

Subgroup	Cumulative Income Share at Percentile ...					Gini	*CV*
	20	40	60	80	95		
I. 1980							
All	5.1%	16.7%	34.2%	58.6%	84.8%	.364	.707
White	5.6	17.5	35.1	59.2	85.1	.353	.688
Black	4.2	13.8	29.9	55.3	84.5	.410	.773
Hispanic	4.8	15.5	32.2	56.9	84.2	.386	.751
Mrd. Couple	6.1	18.5	36.2	60.1	85.5	.338	.658
Female Hr.	4.4	14.1	30.1	54.8	83.3	.412	.811
Age ≤ 64	5.3	17.6	35.4	59.6	85.3	.350	.677
Age ≥ 65	6.1	16.7	32.0	54.3	81.4	.392	.849
Northeast	5.4	17.3	35.0	59.2	85.2	.355	.694
Midwest	5.5	17.6	35.4	59.8	85.3	.349	.670
South	4.8	15.9	33.1	57.5	84.3	.379	.737
West	5.3	16.8	34.2	58.4	84.9	.364	.710
II. 1988							
All	4.6%	15.3	32.0	56.0	82.8	.395	.800
White	5.1	16.2	33.0	56.7	83.0	.382	.778
Black	3.2	11.7	27.0	52.1	82.3	.451	.885
Hispanic	3.9	13.7	29.6	53.9	81.6	.424	.905
Mrd. Couple	5.7	17.3	34.3	57.9	83.7	.363	.737
Female Hr.	3.4	11.9	27.1	51.7	81.3	.452	.918
Age ≤ 64	4.5	15.7	32.8	56.8	83.2	.387	.779
Age ≥ 65	5.9	16.4	31.6	53.9	81.3	.397	.868
Northeast	4.8	15.9	32.7	56.7	83.8	.385	.762
Midwest	5.0	16.4	33.6	57.6	83.5	.375	.760
South	4.3	14.5	30.9	55.1	82.6	.408	.822
West	4.8	15.4	32.0	55.5	81.9	.397	.836

Note: Families are classified by the characteristics of the householder (Hr.).

Source: Authors' calculations using the March 1981 and 1989 *Consumer Population Survey* microdata tapes. See text for details.

Figure 1

Comparing Lorenz Curves: Y^o vs. Y'

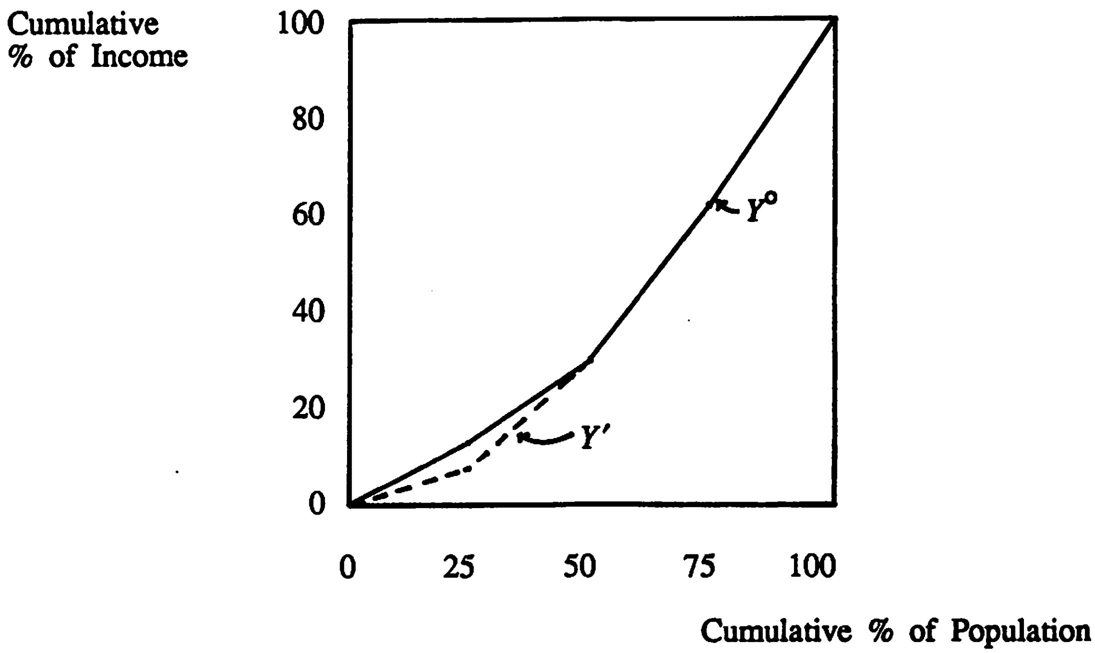


Figure 2

Comparing Lorenz Curves: Y' vs. Y''

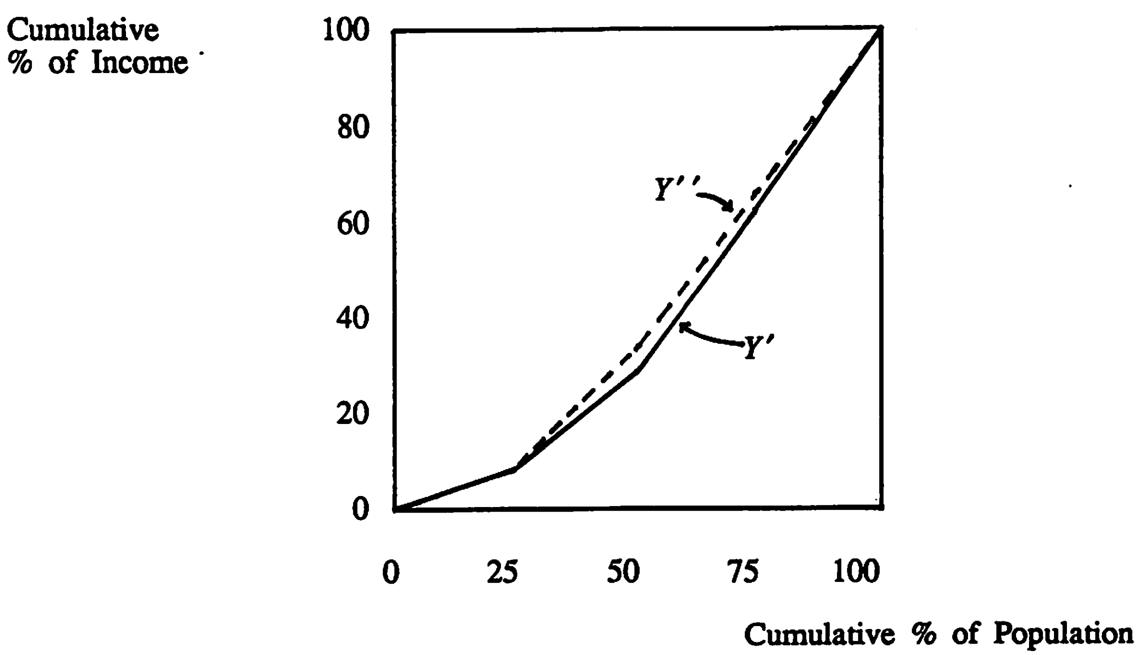
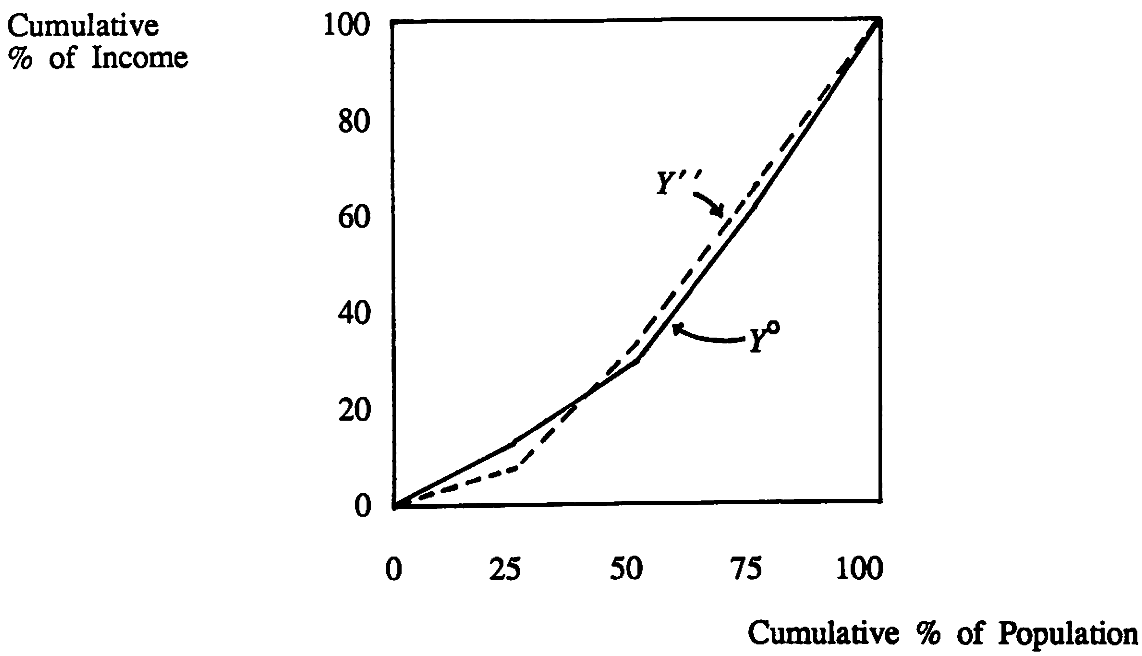


Figure 3

Comparing Lorenz Curves: Y^o vs. Y''



Appendix A

Proof of Proposition 4

We prove sufficiency first. Let Ψ denote the set of inequality indices which satisfy both the principle of transfers and ADI. The strategy of the proof is to employ the results of Propositions 2 and 3 for singly intersecting Lorenz curves iteratively on subpopulations defined by $P \le P_i$, $i = 1,2,\dots,n+1$. Specifically, we apply sets of MVPTs to distribution f on each subpopulation $P \in [P_i, P_{i+1}]$ to generate a series of distributions f^i, $i = 1,2,\dots,n+1$ such that:

$$\text{(A1)} \quad I(f) = I(f^1) \le I(f^2) \le \dots \le I(f^{n+1}) \ \forall\, I(\cdot) \in \Psi$$

with at least one strict inequality. The MVPTs are chosen so that f^{n+1} Lorenz dominates g and so

$$\text{(A2)} \quad I(f^{n+1}) < I(g) \ \forall\, I(\cdot) \in \Psi.$$

The result of the theorem then follows from transitivity.

Step 1: Notice $L(f, P) \ge L(g, P)\ \forall\, P \in [0, P_1]$ and $L(f, P) > L(g, P)$ for some $P \in [0, P_1]$. Hence, set $f^1 \equiv f$ and so $L(f^1, P) \ge L(g, P)\ \forall\, P \in [0, P_1]$ and $V_i(f^1) = V_i(f)$, i = $1,2,\dots,n+1$. It is trivially the case that $I(f^1) = I(f)\ \forall\, I(\cdot) \in \Psi$.

Step 2: On the interval $P \in [0, P_2]$, f^1 intersects g once from above and $V_2(f^1) \le V_2(g)$. Hence, we can apply the results of Propositions 2 and 3. That is, if $V_2(f^1) = V_2(g)$ there would exist a set of MVPTs on f that could be used to obtain g from f. If $V_2(f^1) < V_2(g)$ then some *mps*'s need to be used as well. Hence, $\exists$ a distribution f^2 on $[0, P_2]$ which is obtainable from f

using MVPTs only and which satisfies $L(f^2, P) \geq L(g, P)\ \forall\ P \in [0, P_2]$. Thus, set $f^2 = f^1$ for $P > P_2$ and we have $V_i(f^2) = V_i(f^1)$, $i = 2,3,\ldots,n+1$ and $I(f^2) > I(f^1)\ \forall\ I(\cdot) \in \Psi$.

Step 3: Since $L(f, P) \geq L(g, P)$ on $P \in [P_2, P_3]$ and $L(f, P) > L(g, P)$ for some $P \in [P_2, P_3]$, set $f^3 = f^2$ and it follows that $L(f^3, P) \geq L(g, P)\ \forall\ P \in [0, P_3]$, $L(f^3, P) > L(g, P)$ for some $P \in [0, P_3]$, and $V_i(f^3) = V_i(f^2)$, $i = 3,4,\ldots,n+1$. Trivially, $I(f^3) = I(f^2)\ \forall\ I(\cdot) \in \Psi$.

Step k: $k = 4,5,\ldots,n+1$.

If k is even do as in step 2, i.e., on the interval $P \in [0, P_k]$, f^{k-1} intersects g once from above and $V_k(f^{k-1}) \leq V_k(g)$. Hence we can apply the results of Propositions 2 and 3. Construct f^k from f^{k-1} just as f^2 was constructed from f^1 and so $L(f^k, P) \geq L(g, P)\ \forall\ P \in [0, P_k]$. Set $f^k = f^{k-1}$ for $P > P_k$ and we have $V_i(f^k) = V_i(f^{k-1})$ for $i = k,k+1,\ldots,n+1$ and $I(f^k) > I(f^{k-1})\ \forall\ I(\cdot) \in \Psi$. If k is odd do as in Step 3; i.e., set $f^k = f^{k-1}$ and it follows that $L(f^k, P) \geq L(g, P)\ \forall\ P \in [0, P_k]$, $L(f^k, P) > L(g, P)$ for some $P \in [0, P_k]$ and $V_i(f^k) = V_i(f^{k-1})$, $i = k,k+1,\ldots,n+1$. Trivially, $I(f^k) = I(f^{k-1})\ \forall\ I(\cdot) \in \Psi$.

The series of f^i, $i = 1,2,\ldots,n+1$ thus generated satisfies both (A1) and (A2). In fact, the strict inequality, $I(f^k) > I(f^{k-1})$, applies for k even and equality applies for k odd, so $I(f) < I(g)\ \forall\ I(\cdot) \in \Psi$.

Proof of Necessity

We show that if $V_i(f) > V_i(g)$ for (at least) one of the intersection points then it is not possible to generate the distribution g from f by performing a series of MVPTs and *mps*'s on f. Proposition 2 implies that it therefore cannot be that $I(f) < I(g)\ \forall\ I \in \Psi$.

To see this let $k = \hat{k}$ be the smallest value of $k \in \{1,2,\ldots,n\}$, that is the first point of intersection between the two Lorenz curves, for which $V_{\hat{k}}(f) > V_{\hat{k}}(g)$. Note that $\hat{k}$ must be an even number. Thus, $V_{\hat{k}}(f^{\hat{k}-1}) > V_{\hat{k}}(g)$ where $f^{\hat{k}-1}$ is determined according to the above procedure, and it is *not* possible to find a set of MVPTs applied to $f^{\hat{k}-1}$ that would generate a distribution $f^{\hat{k}}$ with the properties as stipulated in step k above. (This is immediate from the fact that MVPT's do not change the variance.) That is, one cannot generate a set of *mpc*'s to be applied to $f^{\hat{k}-1}$ on $[P_{\hat{k}-1}, P_{\hat{k}}]$ that can be matched by *mps*'s applied to $f^{\hat{k}-1}$ on $[0, P_{\hat{k}-1}]$ which in combination form an MVPT and generate $f^{\hat{k}-1} = g$ on $P \in [0, P_{\hat{k}}]$. Since $V_{\hat{k}}(f^{\hat{k}-1}) > V_{\hat{k}}(g)$, some additional *mpc*'s would be needed to generate g from $f^{\hat{k}-1}$ on $P \in [0, P_{\hat{k}}]$. In order for it to be possible to obtain distribution g from f by performing a series of MVPTs and *mps*'s alone, these additional *mpc*'s would need to be matched with some *mps*'s elsewhere in the distribution to form MVPT's. But, the essential property of a MVPT is that the *mps* occur at incomes below that of the *mpc*. It is not possible to match the required additional *mpc*'s with *mps*'s on $f^{\hat{k}-1}$ and maintain the relationship that $L(f^{\hat{k}}, P) \geq L(g, P)$ for $P \in [0, P_{\hat{k}}]$. This result holds regardless of the relative configuration of $L(f^{\hat{k}}, P)$ and $L(g, P)$ for $P > P_{\hat{k}}$. Thus, there does not exist a set of MVPTs and *mps*'s from which g can be generated from f and so, by Proposition 2, it cannot be that $I(f) < I(g)\ \forall\ I \in \Psi$. qed.

Appendix B

Definitions of *mps*'s, *mpc*'s and *MVPT*'s

This appendix provides formal definitions of mean preserving spreads and contractions (*mps*'s and *mpc*'s) and mean-variance preserving transformations (*MVPT*'s). It is an abridged version of Menezes, Geiss and Tressler (1980, Sec. II, pp. 923-924).

Definition: Let $f(x)$ be any probability or density function. A function $s(x)$ is an *mps* if:

1) $f(x) + s(x) \geq 0$ almost everywhere,

2) $\int_0^1 s(x)\, dx = 0$,

3) $\int_0^1 x\, s(x)\, dx = 0$,

and

4) $s(x)$ $\begin{cases} \leq 0 \text{ for } x \in [a,b] \text{ and } < 0 \text{ for some subinterval,} \\ < 0 \text{ for some subinterval of } [a,b],\ a,b \in (0,1), \\ \geq 0 \text{ for } x \in [a,b]. \end{cases}$

Definition: A function $c(x)$ is an *mpc* if (1), (2) and (3) are satisfied with $c(x)$ substituted for $s(x)$ and if:

5) c(x) $\begin{cases} \geq 0 \text{ for x} \in [d,e] \text{ and } > 0 \text{ for some subinterval} \\ \quad \text{of } [d,e],\ d,e \in (0,1), \\ \leq 0 \text{ for } x \notin [d,e] \end{cases}$

The concept of a mean-variance-preserving transformation follows naturally from an appropriate pairing of an *mps* and an *mpc*. Some additional notation and properties of spreads and contractions are required to define an *MVPT*. Let $S(x) = \int_0^x s(y)\, dy$ and $S^*(x) = \int_0^x S(y)\, dy$. The following properties of $S(x)$ and $S^*(x)$ are easily verified:

6) $S(0) = S(1) = 0$,

7) There exists an x_s in (a,b) such that $S(x) \geq 0$ for all $x \leq x_s$ and $S(x) \leq 0$ for all $x \geq x_s$,

8) $S^*(x) \geq 0$,

9) $S^*(0) = S^*(1) = 0$.

Similarly, let $C(x) = \int_0^x c(y)\, dy$ and $C^*(x) = \int_0^x C(y)\, dy$. The following properties of $C(x)$ and $C^*(x)$ are also easy to verify:

6') $C(0) = C(1) = 0$

7') There exists an x_c and (d,e) such that $C(x) \leq 0$ for all $x \leq x_c$ and $C(x) \geq 0$ for all $x \geq x_c$

8') $C^*(x) \leq 0$

9') $C^*(0) = C^*(1) = 0$

We are now in a position to formally define an *MVPT*.

Definition: A function $t(x)$ is an *MVPT* if

10) $t(x) = s(x) + c(x)$,

11) $\int_0^1 T^*(x)\, dx = \int_0^1 [S^*(x) + C^*(x)]\, dx = 0$,

and

12) $\int_0^x T^*(y)\, dy \geq 0$ for all $x \in [0,1]$ and > 0 for some $x \in (0,1)$.