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Chapter 12
Making Meaning in Algebra
Examining Students' Understandings and Misconceptions

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Students often get confused and lost when they take Algebra 1.[1] The underpinning of the course is the generalization of the arithmetic they have previously studied and communication about mathematical ideas in a language that is rich in symbolic notation. Therefore it is not surprising for students to find algebra as abstract and unconnected to the real world.

Over the past twenty years, there has been a movement in the United States to make algebra more concrete. Specialized manipulatives have been invented in order to provide "hands-on" materials for students to use. Algebra tiles, positive and negative counters, and balance apparatuses, to name a few, are commercially available materials for "concrete learning" of algebra.

Although concrete materials may be helpful for students to learn algebra, it is not the materials themselves that provide algebraic meaning or understanding. Some educators assume that from the use of hands-on materials, students will automatically jump to abstract understanding of algebra. Research informs us that this is not the case. Physical knowledge is knowledge of objects observable in external reality, while mathematical knowledge is the mental relationships students construct in their heads. The source of mathematical knowledge is thus in the student [Kamii and DeClark 1985].

It is equally true that if students are taught abstract ideas without meaning, there will be no understanding. Students need experiences with a concept to develop meaning for themselves. If we want students to know what mathematics is as a subject, they must understand it. Knowing mathematics — really knowing it — means understanding it. When we memorize rules for moving

[1] In the United States, algebra is often taught in two year-long courses called Algebra 1 and Algebra 2.

symbols around on paper we may be learning something, but we are not learning mathematics [Hiebert et al. 1997].

In order to provide learning experiences that students can use to develop understanding of important algebra concepts, teachers must learn what students understand, and they must be sensitive to possible misconceptions held by the students. The learning experiences should allow students to confront their misconceptions and build upon knowledge they already understand. If students are unfamiliar with symbols, vocabulary, representations, or materials, the meaning students gain might differ significantly from what the instructor intends. Mathematical tools should be seen as support for learning. But this learning does not happen instantly: it requires more than watching demonstrations, but working with tools over extended periods of time, trying them out, and watching what happens. Meaning does not reside in representations and concrete materials; it is constructed by students as they use them [Hiebert et al. 1997].

What is Central to Algebra?

Habits of mind. Identifying what students must learn is paramount. One formulation of important ideas of algebra is in terms of “habits of mind” that students need for algebra. Examples of habits of mind are *doing and undoing, building rules to represent functions*, and *abstracting from computation*.

> *Doing-undoing*. Effective algebraic thinking sometimes involves reversibility: being able to undo mathematical processes as well as do them. In effect, it is the capacity not only to use a process to get to a goal, but also to understand the process well enough to work backward from the answer to the starting point. . . .
>
> *Building rules to represent functions*. Critical to algebraic thinking is the capacity to recognize patterns and organize data to represent situations in which input is related to output by well-defined rules. . . .
>
> *Abstracting from computation*. This is the capacity to think about computations independently of particular numbers that are used. One of the most evident characteristics of algebra has always been its abstractness. But, just what is being abstracted? To answer this, a good case can be made that thinking algebraically involves being able to think about computations freed from the particular numbers they are tied to in arithmetic — that is, abstracting system regularities from computation.
>
> [Driscoll 1999]

For example, if algebra tiles are used to teach factoring of quadratic polynomials, students first need to understand how an area model represents a multiplication, then abstract from arithmetic computation (a habit of mind). This supports students in learning to understand how area is a model for making sense of the relation between factors and product, and how even when actual values are unknown the generalization holds. In the case of factoring, students also need to understand the habits of mind of Doing and Undoing, since factoring is the inverse of multiplying two numbers.

Variable and function. An important concept in algebra is the concept of variable. Often students are first introduced to the term "variable" when they are asked to solve for an unknown in an equation. For example, in the equation $3x + 4 = 19$ there is only one value of x that makes the equation true, namely 5. Although in situations like this, x is often called the variable in the equation, its value does not vary.

It is important that students understand the notion of variable as something that can vary. In prototypical functional relationships, the unknowns vary because a change in the input affects the output. Students can start to understand functions and variables by drawing on their experiences. For example, students can consider a sliding door as an example of a relationship of two variables in which one is a function of the other. The width of the opening varies as the door moves on its track. The distance moved by the door can be considered as the input and the width of the opening as the output. Students should examine other situations in order to elaborate their understanding of variables and functions. When students are asked to make lists of situations that involve variables and functional relationships, they may come up with:

The length of a candle varies with the amount of the time it burns.

The height of a child varies as its age increases.

The temperature of a liquid varies with the amount of time it cools in a freezer.

The distance an object moves varies with the force of a push.

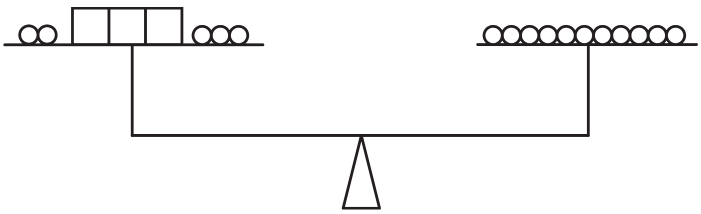
The weight of a piece of a given rope varies with its length.

Students benefit from developing a connection between concrete experiences and the abstract concepts important to algebra.

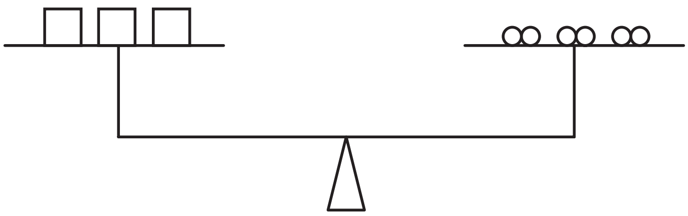
Equality and equation. Another important idea in algebra is equality. In the United States, although students use the equals sign early in their school careers, they often use it to mean "the answer follows." For example, in $45 - 23 = ?$, the equals sign can be and often is interpreted as a signal to execute an arithmetic operation [Siegler 2003]. When used in an equation, the equals sign indicates that the expressions on the left and right sides have the same value. This can be a stumbling block for students who have learned that the equals sign means "the answer follows."

The idea of a balance scale may be used to help students understand how the equals sign is used in equations and what kinds of operations on equations are permissible. Students can connect representations of a balanced scale with operations that preserve equalities in an equation. The equals sign is synonymous with the center of the scale. If a scale is in balance and a weight is added to the pan on the left, an equal weight must be added to the right pan in order for the scale to remain balanced. Experiences of adding or removing "weights" from both sides of a scale representation may help students develop the concept of equality and strategies for solving for unknowns.

For example, suppose a student is asked to solve $3x + 5 = 11$ for x. The student uses a representation of a balanced scale. On the left side are three boxes, each representing the unknown x, and five marbles. On the right side are eleven marbles.



Five marbles may be removed from each side of the scale. This corresponds to the process of subtracting equal quantities from both sides of an equation. When only the unknowns are left on one side of the equation, then the boxes and marbles may be partitioned to determine the unknown.



The importance of using a representation is to emphasize the meaning of the mathematical ideas being investigated. The scale representation is only valuable if students understand that an equation involves the same principles as a balanced scale — that both sides are equal. Students must understand the correspondence between the arithmetic operations and their scale counterparts of adding objects, removing objects, or partitioning objects. With these understandings, students can solidify meanings of solving equations.

An example of a student utilizing the meaning of the equals sign and the power of habits of mind is illustrated by a third-grade student who is determining what number could be substituted for d in:

$$345 + 576 = 342 + 574 + d$$

SAM: There's 300 on each side, so I took them off, and you can do the same thing with the 500s. That gave me 45 plus 76 equals 42 plus 74 plus d. Now you can do the same thing again with the 40s and the 70s, and that leaves 5 plus 6 equals 2 plus 4 plus d. And that's 11 and 6, so d has to be 5 to make that side 11.

MS. V: Sam, how do you know that you can do that?

SAM: If something is the same on both sides of the equal sign, you don't even have to think about it, you can just get rid of it. When you get rid of what's the same, the numbers get smaller and then it gets real easy to tell what d is equal to.

[Carpenter et al. 2003]

Although examples have been presented in this chapter, there is no one good way to present and teach a topic. Often, one student in a class may immediately construct mathematical meaning from an experience while another student actually confuses the meaning. Students need to share ideas and have varied experiences involving important concepts. Multiple representations arise both from mathematics and from students. Because they think differently from one another, students see problems in a range of ways and represent and solve them in multiple ways [Ball 1999]. Experiences with ideas, technology, or materials are important tools teachers need to use to foster students' understanding of algebra.

Assessment Tasks Focused on Algebraic Ideas

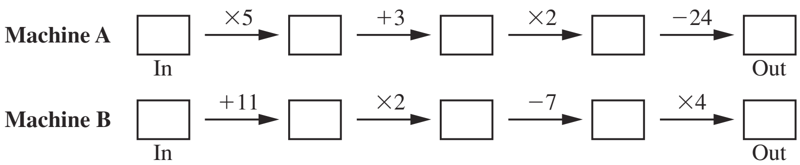
Given the importance of these themes, the Silicon Valley Mathematics Initiative has included in its assessments a number of tasks designed to help teachers understand how well their students are making sense of them. (See Chapter 10 in this volume for more details about this project.) The value of these assessments is found in the responses of students. Through examining student work on these tasks, teachers can focus on important ideas in algebra and identify common misconceptions. This feedback process is a powerful strategy for improving algebra instruction. We give here examples of tasks from the Mathematics Assessment Resource Service aimed at the five major themes discussed in this chapter. Each task is accompanied by a sample of student work and commentary about approaches, understanding, and misconceptions adapted from the Mathematics Assessment Collaborative of the Silicon Valley Mathematics Initiative's *Tools for Teachers*.

Assessment Task Focused on Doing-Undoing

Number Machines

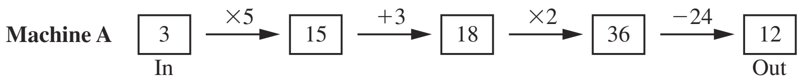
This problem gives you the chance to:
- work with number chains
- explain your reasoning

Here are two number machines:



When you put an integer into the "In" box of a machine and do the operations shown in order, an answer appears in its "Out" box.

For example, if you put 3 into the "In" box of Machine A, 12 appears in its "Out" box.



1. Ray puts 13 into the "In" box of both machines.

(a) Which machine gives the largest number in its "Out" box? ____________

(b) What is the largest answer he gets when he puts in 13? ____________

2. Ray puts an integer into the "In" box of one of the machines. The integer 196 appears in its "Out" box.

(a) Which machine did he use? ________ Which integer did he put in? ________

(b) Explain how you can tell that he did not use the other machine.

3. Leila puts an integer into the "In" box of each machine.
She finds that the answer that appears in the "Out" box of Machine A is the same as the answer that appears in the "Out" box of Machine B.

What was the answer produced by both machines? ____________
Show how you figured it out.

The student whose work is shown on the next page successfully answers question 1 by working his way forward: he fills in the boxes on the top two rows with the appropriate numbers, and concludes that Machine B, with a result of 164, wins out. He can also solve question 2 by using a working backward strategy — specifically, he writes *above* the boxes the result of applying each step backwards, from right to left:

			107		110		220		190
Machine A	13	$\times 5$ →	65	$+3$ →	68	$\times 2$ →	136	-24 →	112
	17		28		56		49		190
Machine B	13	$+11$ →	14	$\times 2$ →	48	-7 →	41	$\times 4$ →	164
	In								Out

As an answer to part (b) of this question, he writes: "I can tell he didn't use the other machine because, while I was working backwards, when I got to the last part I noticed 5 did not go evenly into 107. So it couldn't be that one."

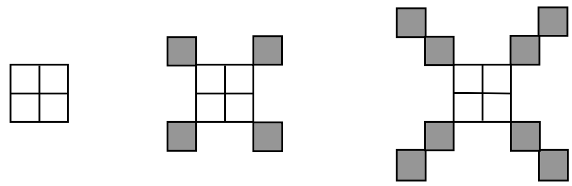
The student does not attempt to answer question 3.

Assessment Task Focused on Building Rules to Represent Functions

Square Patterns

This problem gives you the chance to:
- work with a sequence of tile patterns
- write and use a formula

Mary has some white and gray square tiles.
She uses them to make a series of patterns like these:



Pattern #1 Pattern #2 Pattern #3

1. How many gray tiles does Mary need to make the next pattern?

2. What is the total number of tiles she needs to make pattern number 6?

   Explain how you figured it out.

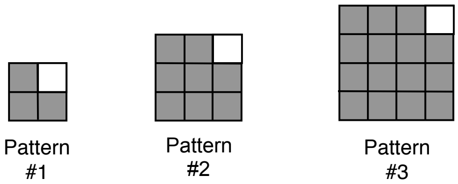
3. Mary uses 48 tiles in all to make one of the patterns.

   What is the number of the pattern she makes?

   Show your work.

4. Write a formula for finding the total number of tiles Mary needs to make pattern # n.

Mary now uses gray and white square tiles to make a different pattern.

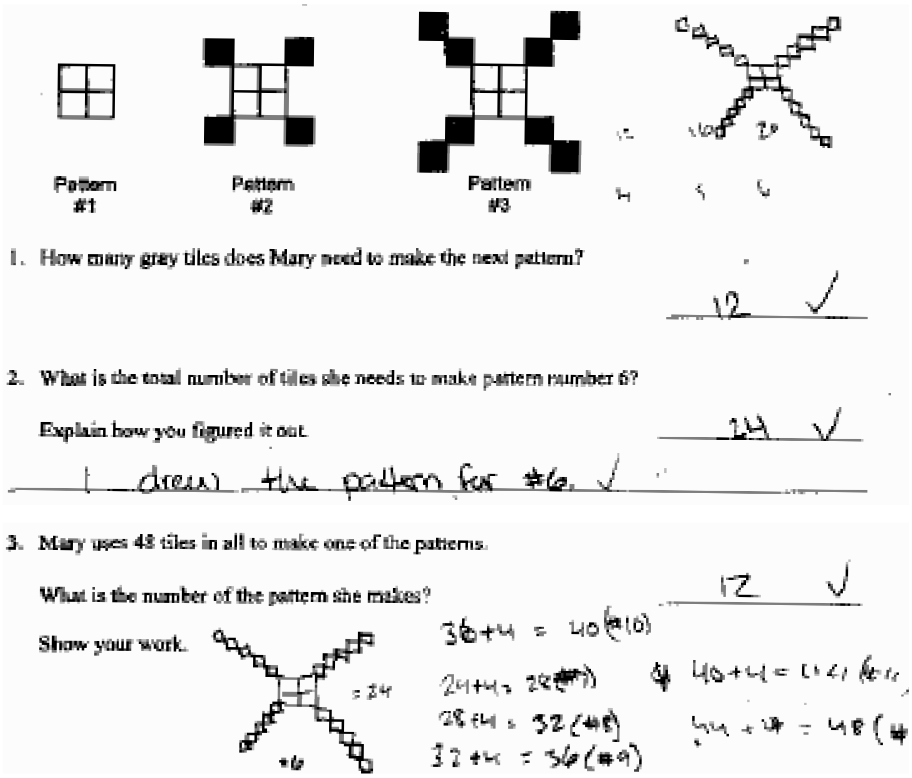


5. How many gray tiles will there be in pattern # 10? ____________

   Explain how you figured it out.

6. Write an algebraic formula linking the pattern number, P, with the number of gray tiles, T.

The student is willing to experiment with patterns in a variety of ways to help make sense of the relationships. In the beginning he uses drawings to solve for parts 2 and 3:



Pattern #1 Pattern #2 Pattern #3

1. How many gray tiles does Mary need to make the next pattern? 12 ✓

2. What is the total number of tiles she needs to make pattern number 6? 24 ✓

   Explain how you figured it out.

   I drew the pattern for #6. ✓

3. Mary uses 48 tiles in all to make one of the patterns.

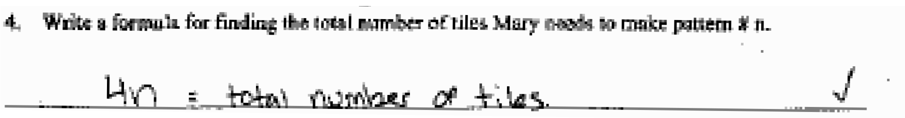
   What is the number of the pattern she makes? 12 ✓

   Show your work.

   = 24
   #6

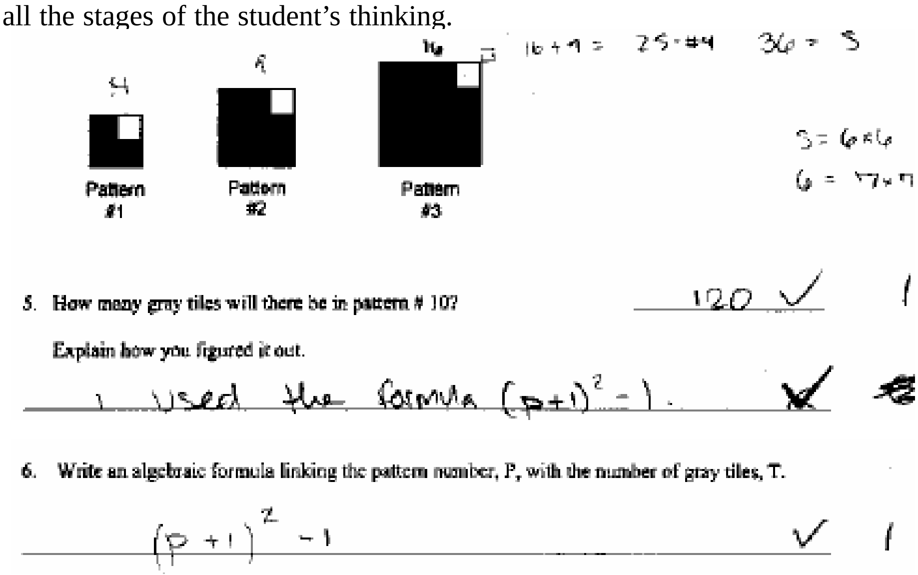
   36+4 = 40(#10)
   24+4 = 28(#7)
   28+4 = 32(#8)
   32+4 = 36(#9)
   40+4 = 44 (#11)
   44+4 = 48 (#

Then he notices a pattern of adding four every time and uses this relationship to verify the drawing. By doing all this thinking about the relationships, the student is then able to come up with an algebraic expression in part 4 that works for any pattern number:


4. Write a formula for finding the total number of tiles Mary needs to make pattern # n.

4n = total number of tiles. ✓

In part 5, the student first notices the differences increases by consecutive odd numbers. With further thinking about the pattern number, he sees that you square one more than the pattern number and subtract 1. It is interesting to see all the stages of the student's thinking.



5. How many gray tiles will there be in pattern # 10?

120 ✓

Explain how you figured it out.

I used the formula $(p+1)^2 - 1$.

6. Write an algebraic formula linking the pattern number, P, with the number of gray tiles, T.

$(p+1)^2 - 1$ ✓

Assessment Task Focused on Abstracting from Computation

Multiples of Three

This problem gives you the chance to:
- test statements to see if they are true
- find examples to match a description
- explain and justify your conclusions

If a number is a multiple of three, its digits add up to a multiple of three.

For example, 15 is a multiple of three (15 = 3 x 5)
and 1 + 5 = 6, which is a multiple of three.

Also, the number 255 is a multiple of three (255 = 3 x 85)
and 2 + 5 + 5 = 12, which is a multiple of three.

1. Use the above rule to test whether 4721 is a multiple of three or not **and** explain how you figured it out.

4721 is not a multiple of 3 because its numbers (4, 7, 2, and 1) do not add up to a multiple of 3 (14), and it cannot be divisible by 3, meaning it cannot be multiplied to get 4721.

2. Use the above rule to find a 5-digit multiple of three **and** explain how you know you are correct.

12345 is a multiple of 3 because its #'s, (1, 2, 3, 4, & 5) when added up, is 15, which is divisible by 3. also, 4115 x 3 = 12345, so it is a multiple of 3.

3. Zara says, "If you add two multiples of three you always get another multiple of three."

Is Zara correct? Yes

Explain how you decided.

If one number is a multiple of 3, and the other one is, then you can say x • 3 = multiple of 3 + y • 3 = multiple of 3 so (x + y) 3 = multiple of 3. Basically you can make them one number and it will still be a multiple of 3.

4 2 2

4. Phil says, "If you add two multiples of three you always get a multiple of six."

Is Phil correct? NO.

Explain how you decided.

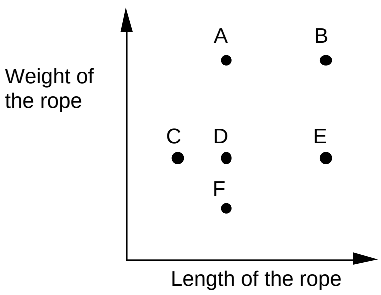
multiples of 6 are every other multiple of 3 (starting from 6). You could take a number that was a multiple of just 3 and add it w/ a number that was a multiple of 3 and 6, and it would only be a multiple of 3, because you would be 3 short / 3 over a multiple of 3.

Assessment Task Focused on Variable and Function

ROPE

This problem gives you the chance to:
- Interpret information from a graph

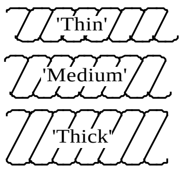
The six points on the graph represent pieces of climbing rope.



(a) Which pieces are the same length as rope D? _____________________

(b) Which have the same weight as rope D? _____________________

The ropes are made from the same material, but there are 3 different thicknesses:

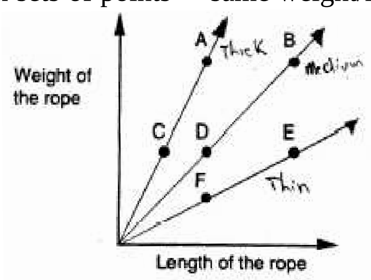


(c) Which points represent 'thin' rope? _____________________

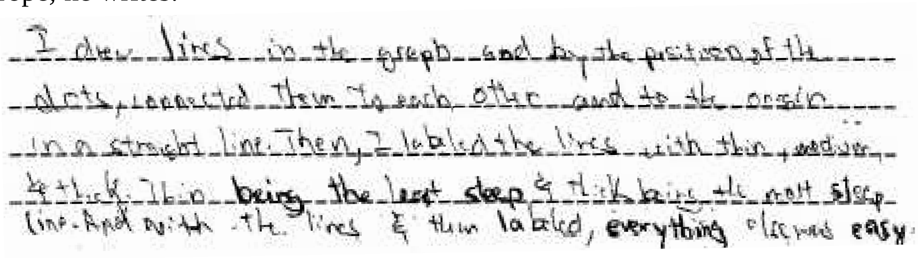
(d) Which points represent 'thick' rope ? _____________________

Explain how you know.

This student illustrates his knowledge of the functional relationship between the length of the rope and the weight of the rope. He is able to identify sets of points corresponding to same weight, same length, and — using lines drawn from the origin through sets of points — same weight/length ratio.



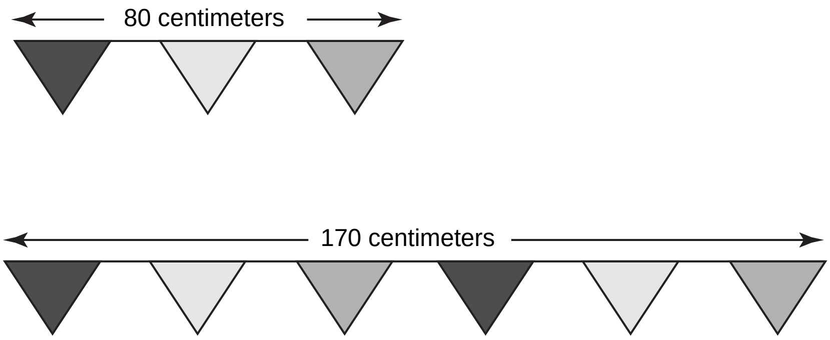
Thus he correctly completes parts (a)–(d) of the task. To explain how he determined the pairs of points that represent thick, medium and thin pieces of rope, he writes:


I drew lines in the graph and by the position of the dots, connected them to each other and to the origin in a straight line. Then, I labeled the lines with thin, medium & thick. Thin being the least steep & thick being the most steep line. And with the lines & then labeled, everything seemed easy.

Assessment Task Focused on Equality

Party Flags

This problem gives you the chance to:
- find sizes by interpreting a diagram
- express a function by a formula

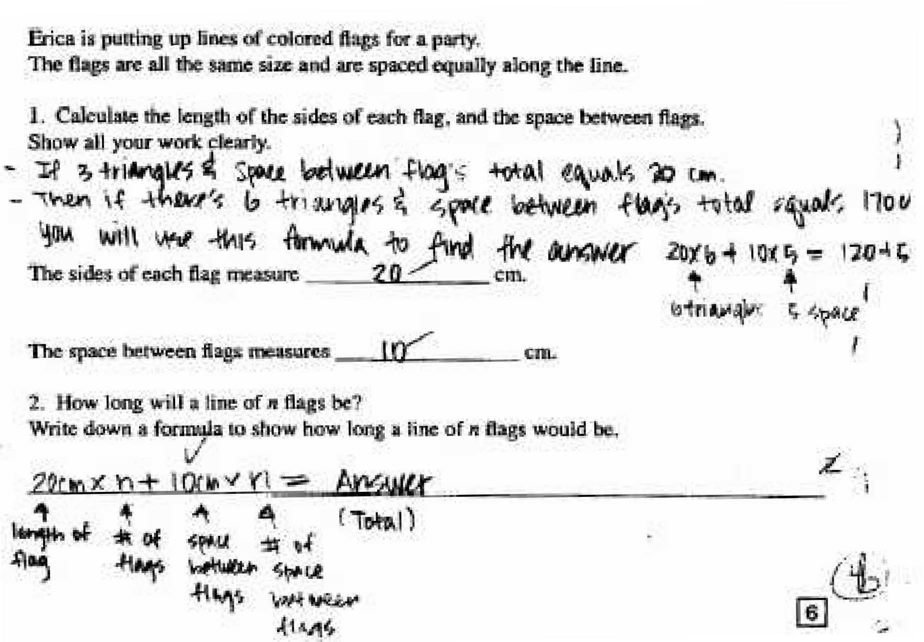


Erica is putting up lines of colored flags for a party.
The flags are all the same size and are spaced equally along the line.

1. Calculate the length of the sides of each flag, and the space between flags.
Show all your work clearly.

2. How long will a line of *n* flags be?
Write down a formula to show how long a line of *n* flags would be.

The student uses a visual strategy to make sense of the given information, formulating an algebraic expression and solving for the missing values for the space between the flags and their sides. The focus on the physical model also allows the student to generalize a formula for any size string of flags.


Erica is putting up lines of colored flags for a party.
The flags are all the same size and are spaced equally along the line.

1. Calculate the length of the sides of each flag, and the space between flags.
Show all your work clearly.

- If 3 triangles & space between flag's total equals 30 cm.
- Then if there's 6 triangles & space between flag's total equals 170 you will use this formula to find the answer 20x6 + 10x5 = 120+5

6 triangles 5 space

The sides of each flag measure 20 cm.

The space between flags measures 10 cm.

2. How long will a line of *n* flags be?
Write down a formula to show how long a line of *n* flags would be.

20cm x n + 10cm x n = Answer (Total)

length of flag; # of flags; space between flags; # of space between flags

6

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